TOPOLOGICAL SIMPLICITY OF THE CREMONA GROUPS
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Abstract. The Cremona group is topologically simple when endowed with the Zariski or Euclidean topology, in any dimension $\geq 2$ and over any infinite field. Two elements are always connected by an affine line, so the group is path-connected.

1. Introduction

Fixing a field $k$ and an integer $n$, the Cremona group of rank $n$ over $k$ can be described algebraically as the group of automorphisms of the $k$-algebra $\text{Cr}_n(k) = \text{Aut}_k(k(x_1, \ldots, x_n))$ or geometrically as the group $\text{Bir}_{\mathbb{P}^n}(k)$ of birational transformations of $\mathbb{P}^n$ that are defined over the field $k$.

In an open problem session held at the international congress (see [Mumfo1974]), D. Mumford asked the following: "Let $G = \text{Aut}_k \mathbb{C}(X,Y)$ be the Cremona group [...]. The problem is to topologize $G [...]$ Is $G$ simple?".

As described in [Serre2010] (see section 2.1 below), one can endow the Cremona group with a natural Zariski topology, which is induced by morphisms $A \to \text{Bir}_{\mathbb{P}^n}$, where $A$ is an algebraic variety (see §2). In [Blanc2010], it is shown that the group $\text{Bir}_{\mathbb{P}^2}(k)$ is topologically simple when endowed with this topology (i.e. it does not contain any non-trivial closed normal strict subgroup), when $k$ is algebraically closed. In this text, we generalise this result and give a simple proof of the following:

Theorem 1. For each infinite field $k$ and each $n \geq 1$, the group $\text{Bir}_{\mathbb{P}^n}(k)$ is topologically simple when endowed with the Zariski topology (i.e. it does not contain any non-trivial closed normal strict subgroup).

Remark 1.1. For each field $k$, the group $\text{Bir}_{\mathbb{P}^2}(k)$ is not simple as an abstract group [CanLam2013, Lonjo2015]. If $k = \mathbb{R}$, it contains normal subgroups of index $2^m$ for each $m \geq 1$ [Zimme2015]. For each $n \geq 3$ and each field $k$, deciding whether the abstract group $\text{Bir}_{\mathbb{P}^n}(k)$ is simple or not is a still wide open question.

Remark 1.2. If $k$ is a finite field, the Zariski topology on $\text{Bir}_{\mathbb{P}^n}(k)$ is the discrete topology (see Lemma 2.8), so the topological simplicity is equivalent to the simplicity as an abstract group, and is therefore false for $n = 2$, and open for $n \geq 3$. For $n = 1$, this is true if and only if $k = \mathbb{F}_{2^a}, a \geq 2$ (see Lemma 2.14).

Recall that a local field is a locally compact topological field with respect to a non-discrete valuation. All examples are $\mathbb{R}, \mathbb{C}$ and finite extensions of $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$. If $k$ is
a local field then there exists a natural topology on $\text{Bir}_{\mathbb{P}^n}(k)$, which makes it a Hausdorff topological group, and whose restriction on any algebraic subgroup (for instance on $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k)$ and $(\text{PGL}_2(k))^n \subset \text{Aut}_{\mathbb{P}^1}(k)$) is the Euclidean topology (the classical topology given by distances between matrices) [BlaFur2013, Theorem 3]. This topology was called Euclidean topology of $\text{Bir}_{\mathbb{P}^n}(k)$. We will show the following analogue of Theorem 1, for this topology:

**Theorem 2.** For each local field $k$ and each $n \geq 2$, the topological group $\text{Bir}_{\mathbb{P}^n}(k)$ is simple when endowed with the Euclidean topology (i.e. it does not contain any non-trivial closed normal strict subgroup).

**Remark 1.3.** The result is, of course, false for $n = 1$, since $\text{PSL}_2(\mathbb{R})$ is a non-trivial normal strict subgroup of $\text{PGL}_2(\mathbb{R})$, closed for the Euclidean topology.

In the 1000-th Bourbaki Seminar [Serre2010], J.-P. Serre asked whether the group $\text{Bir}_{\mathbb{P}^n}(k)$ is connected with respect to the Zariski topology. When $k$ is algebraically closed, a positive answer is given in [Blanc2010, Théorème 5.1]. We generalise this result (and give a simpler proof of it) as follows:

**Theorem 3.** For each infinite field $k$, each $n \geq 2$ and each $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$, there is a morphism $\rho : \mathbb{A}^1 \rightarrow \text{Bir}_{\mathbb{P}^n}$, defined over $k$, such that $\rho(0) = f$ and $\rho(1) = g$. In particular, the group $\text{Bir}_{\mathbb{P}^n}(k)$ is connected with respect to the Zariski topology.

The second property is also true for $n = 1$, although the first one is false.

For each $n \geq 2$, the groups $\text{Bir}_{\mathbb{P}^n}(\mathbb{R})$ and $\text{Bir}_{\mathbb{P}^n}(\mathbb{C})$ are path-connected, and thus connected with respect to the Euclidean topology.

The authors thank the referee for his careful reading and his suggestions for improving the exposition of this article.

## 2. Preliminaries

### 2.1. The families of birational maps and the Zariski topology induced.

In [Demaz1970], M. Demazure introduced the following functor (that he called $\text{Psaut}$, for pseudo-automorphisms, the name he gave to birational transformations):

**Definition 2.1.** Let $k$ be an algebraically closed field, $X$ be an irreducible algebraic variety and $A$ a noetherian scheme, both defined over $k$. We define

$$\text{Bir}_X(A) = \left\{ \text{A-birational transformations of } A \times X \text{ inducing an isomorphism } U \rightarrow V, \text{ where } U, V \text{ are open subsets of } A \times X, \text{ whose projections on } A \text{ are surjective} \right\},$$

$$\text{Aut}_X(A) = \left\{ \text{A-automorphisms of } A \times X \right\} = \text{Bir}_X(A) \cap \text{Aut}(A \times X).$$

**Remark 2.2.** When $A = \text{Spec}(k)$, we see that $\text{Bir}_X(A)$ corresponds to the group of birational transformations of $X$ defined over $k$, which we will denote by $\text{Bir}_X(k)$. Similarly, $\text{Aut}_X(k)$ corresponds to the group of automorphisms of $X$ defined over $k$.

For each field $k$ over which $X$ is defined, we will similarly denote by $\text{Bir}_X(k)$ and $\text{Aut}_X(k)$ the group of birational transformations and automorphisms of $X$ defined over $k$.

Definition 2.1 implicitly gives rise to the following notion of families, or morphisms $A \rightarrow \text{Bir}_X(k)$ (as in [Serre2010, Blanc2010, BlaFur2013]):
**Definition 2.3.** Taking $A, X$ as above, an element $f \in \text{Bir}_X(A)$ and a $k$-point $a \in A(k)$, we obtain an element $f_a \in \text{Bir}_X(k)$ given by $x \mapsto p_2(f(a, x))$, where $p_2 : A \times X \to X$ is the second projection.

The map $a \mapsto f_a$ represents a map from $A$ (more precisely from the $A(k)$-points of $A$) to $\text{Bir}_X(k)$, and will be called a $k$-morphism (or morphism defined over $k$) from $A$ to $\text{Bir}_X$. If moreover $f \in \text{Aut}_X(A)$, then $f$ also yields a morphism from $A$ to $\text{Aut}_X$.

If $k \subset k'$ is a subfield over which $X$, $A$ and $f$ are defined, we will also say that the $k$-morphism above is a $k'$-morphism.

**Remark 2.4.**

1. If $X, Y$ are two irreducible algebraic varieties and $\psi : X \to Y$ is a birational map, all of them defined over an algebraically closed field $k$, the two functors $\text{Bir}_X$ and $\text{Bir}_Y$ are isomorphic, via $\psi$. In other words, morphisms $A \to \text{Bir}_X$ corresponds, via $\psi$, to morphisms $A \to \text{Bir}_Y$. The same holds with $\text{Aut}_X$ and $\text{Aut}_Y$, if $\psi$ is an isomorphism. We further get a bijection between $k$-morphisms to $\text{Bir}_X$ and $\text{Bir}_Y$ if $X, Y$ and $\psi$ are defined over a subfield $k \subset k$.

2. If $X$ is projective, the connected component $\text{Aut}^o_X$ of $\text{Aut}_X$ is an algebraic group, so there is a natural notion of morphism from $A$ to $\text{Aut}^o_X$ in this case, and this one coincides with the above definition.

3. Just like with morphisms of algebraic varieties, for any field extension $k \subset k'$, any $k$-morphism $A \to \text{Bir}_X$ is also a $k'$-morphism, and thus yields a map $A(k') \to \text{Bir}_X(k')$.

Even if $\text{Bir}_X$ is not representable by an algebraic variety or an ind-algebraic variety in general [BlaFur2013], we can define a topology on the group $\text{Bir}_X(k)$, given by this functor. This topology is called Zariski topology by J.-P. Serre in [Serre2010]:

**Definition 2.5.** Let $X$ be an irreducible algebraic variety defined over a field $k$. A subset $F \subseteq \text{Bir}_X(k)$ is closed in the Zariski topology if for any $k$-algebraic variety $A$ and any $k$-morphism $A \to \text{Bir}_X$ the preimage of $F$ in $A(k)$ is closed.

**Remark 2.6.** In this definition one can of course replace “any algebraic variety $A$” with “any irreducible algebraic variety $A$”.

Endowed with this topology, $\text{Bir}_P^n(k)$ is connected for each $n \geq 1$, and $\text{Bir}_P^2(k)$ is topologically simple for each algebraically closed field $k$ [Blanc2010].

Let us make the following observation, whose statement and proof is analogue to classical statements for algebraic varieties:

**Lemma 2.7.** Let $k$ be a field and $X$ a geometrically irreducible algebraic variety defined over $k$. The Zariski topology on $\text{Bir}_X(k)$ is finer than the topology on $\text{Bir}_X(k)$ induced by the Zariski topology of $\text{Bir}_X(k)$, where $k$ is the algebraic closure of $k$.

**Proof.** We show that for each closed subset $F' \subset \text{Bir}_X(k)$, the set $F = F' \cap \text{Bir}_X(k)$ is closed with respect to the Zariski topology.

To do this, we need to show that the preimage of $F$ by any $k$-morphism $\rho : A \to \text{Bir}_X$ is closed. By definition of the Zariski topology of $\text{Bir}_X(k)$, the set $C = \{a \in A(k) \mid \rho(a) \in F'\}$ is Zariski closed in $A(k)$. The closure $R$ of $C \cap A(k)$ in $A(k)$ is defined over $k$. [Blanc2010]
Since \( R(k) \subset C(k) \), we have \( R \cap A(k) = R(k) \subset C \cap A(k) \subset R \cap A(k) \), so \( C \cap A(k) = R(k) \) is closed in \( A(k) \).

It remains to observe that the equality \( F = F' \cap \text{Bir}_X(k) \) implies that \( C \cap A(k) = \{a \in A(k) \mid \rho(a) \in F'\} = \{a \in A(k) \mid \rho(a) \in F\} = \rho^{-1}(F) \).

**Lemma 2.8.** Let \( k \) be a finite field and \( X \) be an algebraic variety defined over \( k \). The Zariski topology on \( \text{Bir}_X(k) \) is the discrete topology.

**Proof.** Let us show that any subset \( F \subset \text{Bir}_X(k) \) is closed. For this, we take a \( k \)-algebraic variety \( A \), a \( k \)-morphism \( \rho : A \to \text{Bir}_X \), and observe that \( \rho^{-1}(F) \) is finite in \( A \), hence is closed. \( \square \)

### 2.2. The varieties \( H_d \)

The following algebraic varieties are useful to study morphisms to \( \text{Bir}_{\mathbb{P}^n} \).

**Definition 2.9.** [BlaFur2013, Definition 2.3] Let \( d, n \) be positive integers.

1. We define \( W_d \) to be the projective space parametrising, for each field \( k \), equivalence classes of non-zero \((n+1)\)-uples \((h_0, \ldots, h_n)\) of homogeneous polynomials \( h_i \in k[x_0, \ldots, x_n] \) of degree \( d \), where \((h_0, \ldots, h_n)\) is equivalent to \((\lambda h_0, \ldots, \lambda h_n)\) for any \( \lambda \in k^* \). The equivalence class of \((h_0, \ldots, h_n)\) will be denoted by \([h_0 : \cdots : h_n]\).

2. We define \( H_d \subset W_d \) to be the set of elements \( h = [h_0 : \cdots : h_n] \in W_d \) such that the rational map \( \psi_h : \mathbb{P}^n 
\to \mathbb{P}^n \) given by
   \[
   [x_0 : \cdots : x_n] \mapsto [h_0(x_0, \ldots, x_n) : \cdots : h_n(x_0, \ldots, x_n)]
   \]
   is birational. We denote by \( \pi_d \) the map \( H_d(k) \to \text{Bir}_{\mathbb{P}^n}(k) \) which sends \( h \) onto \( \psi_h \).

**Proposition 2.10.** Let \( d, n \) be positive integers.

1. The set \( H_d \) is locally closed in the projective space \( W_d \) and thus inherits the structure of an algebraic variety.

2. The map \( \pi_d \) corresponds to a morphism \( H_d \to \text{Bir}_{\mathbb{P}^n} \), defined over any field. For each field \( k \), the image of the corresponding map \( H_d(k) \to \text{Bir}_{\mathbb{P}^n}(k) \) consists of all birational maps of degree \( \leq d \).

**Proof.** Follows from [BlaFur2013, Lemma 2.4]. \( \square \)

### 2.3. The Euclidean topology

Suppose that \( k \) is a local field.

The Euclidean topology of \( \text{Bir}_{\mathbb{P}^n}(k) \) described in [BlaFur2013, Section 5] is defined as follows: on \( W_d(k) \simeq \mathbb{P}^{n(n+1)/2} \) we put the classical Euclidean topology, on \( H_d(k) \subset W_d(k) \) the induced topology and on \( \pi_d(H_d(k)) = \{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d\} \) the quotient topology induced by \( \pi_d \). The Euclidean topology on \( \text{Bir}_{\mathbb{P}^n}(k) \) is then the inductive limit topology induced by the inclusions

\[
\{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d\} \to \{f \in \text{Bir}_{\mathbb{P}^n}(k) \mid \deg(f) \leq d + 1\}.
\]

**Lemma 2.11.** Let \( k \) be a local field, let \( A \) be an algebraic variety defined over \( k \), and let \( \rho : A \to \text{Bir}_{\mathbb{P}^n} \) be a \( k \)-morphism. Then the map

\[
A(k) \to \text{Bir}_{\mathbb{P}^n}(k)
\]

is continuous for the Euclidean topologies.
Proof. There exists an open affine covering $(A_i)_{i \in I}$ of $A$, with respect to the Zariski topology, with the following property: for each $i \in I$ there exists an integer $d_i$ and a morphism of algebraic varieties $\rho_i : A_i \to H_{d_i}$, such that the restriction of $\rho$ to $A_i$ is $\pi_{d_i} \circ \rho_i$ [BlaFur2013, Lemma 2.6]. It follows from the construction that the $A_i$ and $\rho_i$ can be assumed to be defined over $k$.

We take a subset $U \subset \text{Bir}_{P^n}(k)$, open with respect to the Euclidean topology, and want to show that $\rho^{-1}(U) \subset A(k)$ is open with respect to the Euclidean topology. As all $A_i(k)$ are open in $A(k)$, it suffices to show that $\rho^{-1}(U) \cap A_i(k)$ is open in $A_i(k)$ for each $i$. This follows from the fact that $\rho|_{A_i} = \pi_{d_i} \circ \rho_i$ and that both $\pi_{d_i}$ and $\rho_i$ are continuous with respect to the Euclidean topology. \hfill $\square$

2.4. The projective linear group. Note that $\text{Bir}_{P^n}(k)$ contains the algebraic group $\text{Aut}_{P^n}(k) = \text{PGL}_{n+1}(k)$ and that the restriction of the Zariski topology to this subgroup corresponds to the usual Zariski topology of the algebraic variety $\text{PGL}_{n+1}(k)$, which can be viewed as the open subset of $\mathbb{P}^{(n+1)^2-1}(k)$, more precisely as complement of the hypersurface given by the vanishing of the determinant.

Let us make the following two observations:

**Lemma 2.12.** If $k$ is an infinite field and $n \geq 2$, then $\text{PSL}_n(k)$ is dense in $\text{PGL}_n(k)$ with respect to the Zariski topology. Moreover, every non-trivial normal subgroup of $\text{PGL}_n(k)$ contains $\text{PSL}_n(k)$. In particular, $\text{PGL}_n(k)$ does not contain any non-trivial normal strict subgroup which is closed with respect to the Zariski topology.

**Proof.** (1) Observe that the group homomorphism $\det : \text{GL}_n(k) \to k^*$ yields a group homomorphism

$$\det : \text{PGL}_n(k) \to (k^*)/\{f^n \mid f \in k^*\},$$

whose kernel is the group $\text{PSL}_n(k)$. We consider the morphism

$$\rho : \mathbb{A}^1(k) \setminus \{0\} \to \text{PGL}_n(k)$$

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & I \end{pmatrix}$$

where $I$ is the identity matrix of size $(n-1) \times (n-1)$, and observe that $\rho^{-1}(\text{PSL}_n(k))$ contains $\{t^n \mid t \in \mathbb{A}^1(k)\}$, which is an infinite subset of $\mathbb{A}^1(k)$ and is therefore dense in $\mathbb{A}^1(k)$. The closure of $\text{PSL}_n(k)$ contains thus $\rho(\mathbb{A}^1(k) \setminus \{0\})$. As every element of $\text{PGL}_n(k)$ is equal to some $\rho(t)$ modulo $\text{PSL}_n(k)$, we obtain that $\text{PSL}_n(k)$ is dense in $\text{PGL}_n(k)$.

(2) Let $N \subset \text{PGL}_n(k)$ be a normal subgroup with $N \neq \{\text{id}\}$, and let $f \in N$ be a non-trivial element. We want to show that $N$ contains $\text{PSL}_n(k)$. Since the center of $\text{PGL}_n(k)$ is trivial, one can replace $f$ with $\alpha f \alpha^{-1} f^{-1}$, where $\alpha \in \text{PGL}_n(k)$ does not commute with $f$, and assume that $f \in N \cap \text{PSL}_n(k)$. Then, as $\text{PSL}_n(k)$ is a simple group [Dieud1971, Chapitre II, §2], we obtain $\text{PSL}_n(k) \subset N$.

(1) and (2) imply that $\text{PGL}_n(k)$ does not contain any non-trivial normal strict subgroup which is closed with respect to the Zariski topology. \hfill $\square$

**Remark 2.13.** Lemma 2.12 does not work for the Euclidean topology. For instance, for each $n \geq 1$, the group $\text{PSL}_{2n}(\mathbb{R}) = \{A \in \text{PGL}_{2n}(\mathbb{R}) \mid \det(A) > 0\}$ is a normal strict subgroup of $\text{PGL}_{2n}(\mathbb{R})$ which is closed with respect to the Euclidean topology.
Lemma 2.14. Let $k$ be a finite field. Then

1. $\text{PGL}_2(k) = \text{PSL}_2(k)$ if and only if $\text{char}(k) = 2$,
2. $\text{PGL}_2(k)$ is a simple group if and only if $k = \mathbb{F}_2$, $a \geq 2$.

Proof. (1): As explained before, $\text{PSL}_2(k) = \text{PGL}_2(k)$ if and only if every element of $k^*$ (or equivalently of $k$) is a square. As $k$ is finite, the group homomorphism

$$k^* \to k^*$$

$$x \mapsto x^2$$

is surjective if and only if it is injective, and this corresponds to ask that the characteristic of $k$ is 2.

(2): If $\text{char}(k) \neq 2$, then $\text{PSL}_2(k) \nsubseteq \text{PGL}_2(k)$ is a non-trivial normal subgroup.

If $\text{char}(k) = 2$, then $\text{PGL}_2(k) = \text{PSL}_2(k)$ is a simple group if and only if $k \neq \mathbb{F}_2$ ([Dieudonne1971, Chapitre II, §2]).

3. PROOF OF THE RESULTS

3.1. The construction associated to fixed points. Let us explain the following simple construction that will be often used in the sequel.

Example 3.1. Let $f \in \text{Bir}_{\mathbb{P}^n}(k)$ be an element fixing the point $p = [1 : 0 : \cdots : 0]$ and that induces a local isomorphism at $p$.

In the chart $x_0 = 1$, we can write $f$ locally as

$$x = (x_1, \ldots, x_n) \mapsto \left( \frac{p_{1,1}(x) + \cdots + p_{1,m}(x)}{1 + q_{1,1}(x) + \cdots + q_{1,m}(x)}, \ldots, \frac{p_{n,1}(x) + \cdots + p_{n,m}(x)}{1 + q_{n,1}(x) + \cdots + q_{n,m}(x)} \right),$$

where the $p_{i,j}, q_{i,j} \in k[x_1, \ldots, x_n]$ are homogeneous of degree $j$. For each $t \in k \setminus \{0\}$, the element

$$\theta_t : (x_1, \ldots, x_n) \mapsto (tx_1, \ldots, tx_n)$$

extends to a linear automorphism of $\mathbb{P}^n(k)$ fixing $p$. Then the map $t \mapsto (\theta_t)^{-1} \circ f \circ \theta_t$ gives rise to a morphism $F : \mathbb{A}^1 \setminus \{0\} \to \text{Bir}_{\mathbb{P}^n}(k)$ whose image contains only conjugates of $f$ by linear automorphisms.

Writing $F$ locally, we can observe that $F$ extends to a morphism $\mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}(k)$ such that $F(0)$ is linear. Indeed, $F(t)$ can be written locally as follows:

$$F(t)(x) = F(t)(x_1, \ldots, x_n) = \left( \frac{p_{1,1}(x) + tp_{1,2}(x) + \cdots + t^{m-1}p_{1,m}(x)}{1 + tq_{1,1}(x) + \cdots + t^{m}q_{1,m}(x)}, \ldots, \frac{p_{n,1}(x) + tp_{n,2}(x) + \cdots + t^{m-1}p_{n,m}(x)}{1 + tq_{n,1}(x) + \cdots + t^{m}q_{n,m}(x)} \right),$$

and $F(0)$ corresponds to the derivative (linear part) of $F$ at $p$, which is locally given by

$$(x_1, \ldots, x_n) \mapsto (p_{1,1}(x), \ldots, p_{n,1}(x))$$

and which is an element of $\text{Aut}_{\mathbb{P}^n}(k) \subset \text{Bir}_{\mathbb{P}^n}(k)$ since $f$ was chosen to be a local isomorphism at $p$.

Using the example above, one can construct $k$-morphisms $\mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$.

Proposition 3.2. Let $k$ be a field, $n \geq 1$, let $g \in \text{Bir}_{\mathbb{P}^n}(k)$ and $p \in \mathbb{P}^n(k)$ be a point such that $g$ fixes $p$ and induces a local isomorphism at $p$. Then there exist $k$-morphisms $\nu : \mathbb{A}^1 \setminus \{0\} \to \text{Aut}_{\mathbb{P}^n}$ and $\rho : \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ such that the following hold:
(1) For each field extension \(k \subset k'\) and each \(t \in \mathbb{A}^1(k') \setminus \{0\}\), we have
\[
\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t).
\]
Moreover, \(\nu(1) = \text{id}\), so \(\rho(1) = g\).

(2) The element \(\rho(0)\) belongs to \(\text{Aut}_{\mathbb{P}^n}(k)\). It is the identity if and only if the action of \(g\) on the tangent space \(T_p(\mathbb{P}^n)\) is trivial.

**Proof.** Conjugating by an element of \(\text{Aut}_{\mathbb{P}^n}(k)\), we can assume that \(p = [1 : 0 : \cdots : 0]\). We then choose \(\nu\) to be given by
\[
\nu(t): [x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : tx_1 : \cdots : tx_n],
\]
and define \(\rho: \mathbb{A}^1 \setminus \{0\} \to \text{Bir}_{\mathbb{P}^n}\) by \(\rho(t) = \nu(t)^{-1} \circ g \circ \nu(t)\). As it was shown in Example 3.1, the \(k\)-morphism \(\rho\) extends to a \(k\)-morphism \(\mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}\) such that \(\rho(0) \in \text{Aut}_{\mathbb{P}^n}(k)\). Moreover, this element is trivial if and only if the action of \(g\) on the tangent space \(T_p(\mathbb{P}^n)\) is trivial. \(\square\)

3.2. Closed normal subgroups of the Cremona groups. As a consequence of Proposition 3.2, we obtain the following result:

**Proposition 3.3.** Let \(k\) be an infinite field. Let \(n\) be a positive integer. Let \(N \subset \text{Bir}_{\mathbb{P}^n}(k)\). If \(N\) is closed with respect to the Zariski topology or to the Euclidean topology (if \(k\) is a local field), then \(N \cap \text{Aut}_{\mathbb{P}^n}(k)\) is not the trivial group.

**Proof.** We can assume that \(n \geq 2\), as the result is trivial for \(n = 1\) (in which case \(\text{Bir}_{\mathbb{P}^n}(k) = \text{Aut}_{\mathbb{P}^n}(k)\) ) . Let us choose a non-trivial element \(f \in N\). As \(f\) is a birational transformation, it induces an isomorphism \(U \to V\), where \(U, V \subset \mathbb{P}^n\) are two non-empty open subsets defined over \(k\). Since \(k\) is infinite, \((U(k)\) and \((V(k)\) are not empty, so we can find \(p \in U(k)\), and \(q = f(p) \in V(k)\). We can moreover choose \(p \neq q\), since \(\{p \in U \mid f(p) = p\}\) is open and non-empty in \(U\). Let us take an element \(\alpha \in \text{Aut}_{\mathbb{P}^n}(k)\) that fixes \(p\) and \(q\). The element \(g = \alpha^{-1} f^{-1} \alpha\) fixes \(p\) and is a local isomorphism at this point. We can choose \(\alpha\) such that the derivative \(D_p(g)\) of \(g\) at this point is not trivial, since
\[
D_p(g) = D_p(\alpha^{-1}) \circ D_q(f^{-1}) \circ D_q(\alpha) \circ D_p(f).
\]
Indeed, changing coordinates one can assume that \(q = [1 : 0 : \cdots : 0]\), \(p = [0 : 1 : 0 : \cdots : 0]\) and can for instance choose \(\alpha: [x_0 : \cdots : x_n] \mapsto [x_0 + \xi x_2 : x_1 : x_2 : \cdots : x_n]\), for some \(\xi \in k\). This choice yields \(D_q(\alpha) = \text{id}\) and gives infinitely many possibilities for \(D_p(\alpha^{-1})\).

By Proposition 3.2, there exists a \(k\)-morphism \(\rho:\mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}\) such that \(\rho(0) \in \text{Aut}_{\mathbb{P}^n}(k) \setminus \{\text{id}\}\) and such that \(\rho(t) \in N\) for each \(t \in \mathbb{A}^1(k) \setminus \{0\}\). Since \(N\) is closed (with respect to the Zariski or to the Euclidean topology), \(\rho^{-1}(N) \subset \mathbb{A}^1(k)\) is closed (with respect to the Zariski or to the Euclidean topology respectively, see Lemma 2.11 in the latter case) and contains \(\mathbb{A}^1(k) \setminus \{0\}\). For the Zariski topology, one uses the fact that \(k\) is infinite to get \(\rho^{-1}(N) = \mathbb{A}^1(k)\). For the Euclidean topology, one uses the fact that \(k\) is non-discrete to get the same result. In each case, we find that \(\rho(0) \in N \cap \text{Aut}_{\mathbb{P}^n}(k)\). \(\square\)

**Lemma 3.4.** Let \(k\) be an infinite field, \(n \geq 2\) an integer and \(N \subset \text{Bir}_{\mathbb{P}^n}(k)\) be a normal subgroup, with \(N \cap \text{Aut}_{\mathbb{P}^n}(k) \neq \{\text{id}\}\). Then \(\text{PGL}_{n+1}(k) = \text{Aut}_{\mathbb{P}^n}(k) \subset N\).
Proof. The group $N \cap \text{Aut}_{\mathbb{P}^n}(k)$ is a non-trivial normal subgroup of $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k)$, so contains $\text{PSL}_{n+1}(k)$ by Lemma 2.12.

For each $a \in k^*$, we define $g_a \in N$ and $h \in \text{Bir}_{\mathbb{P}^n}(k)$ by

\[
g_a : [x_0 : \cdots : x_n] \mapsto [x_0 : ax_1 : \frac{1}{x_0} x_2 : x_3 : \cdots : x_n] \quad \text{and} \quad h : [x_0 : \cdots : x_n] \mapsto [x_0 : x_1 : x_2 : \frac{x_3}{x_0} : x_3 : \cdots : x_n].
\]

Then, $g_a' = hg_a h^{-1} \in N$ is given by

\[
g_a' : [x_0 : \cdots : x_n] \mapsto [x_0 : ax_1 : x_2 : x_3 : \cdots : x_n].
\]

As every element of $\text{PGL}_n(k)$ is equal to some $g_a'$ modulo $\text{PSL}_{n+1}(k)$, we obtain that $\text{PGL}_{n+1}(k) \subset N$. \hfill \Box

**Proposition 3.5.** Let $k$ be an infinite field, $n \geq 2$ an integer and consider $\text{Bir}_{\mathbb{P}^n}(k)$ endowed with the Zariski topology or the Euclidean topology (if $k$ is a local field). Then the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(k)$ generated by $\text{Aut}_{\mathbb{P}^n}(k)$ is dense in $\text{Bir}_{\mathbb{P}^n}(k)$.

In particular, $\text{Bir}_{\mathbb{P}^n}(k)$ does not contain any non-trivial closed normal strict subgroup.

**Proof.** (1) Let $f \in \text{Bir}_{\mathbb{P}^n}(k)$, $f \neq \text{id}$. It induces an isomorphism $U \to V$, where $U, V \subset \mathbb{P}^n$ are two non-empty open subsets, defined over $k$. Since $k$ is infinite, we can find $p \in U(k)$. There exist $\alpha_1, \alpha_2 \in \text{Aut}_{\mathbb{P}^n}(k)$ such that $g := \alpha_1 f \alpha_2$ fixes $p$, is a local isomorphism at this point and such that $D_\rho(g)$ is not trivial. By Proposition 3.2, there exist $k$-morphisms $\nu : \mathbb{A}^1 \setminus \{0\} \to \text{Aut}_{\mathbb{P}^n}(k)$ and $\rho_1 : \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}(k)$ such that $\rho_1(t) = \nu(t)^{-1} \circ g^{-1} \circ \nu(t)$ for each $t \in \mathbb{A}^1(k) \setminus \{0\}$ and $\rho_1(0) \in \text{Aut}_{\mathbb{P}^n}(k)$. We define a $k$-morphism

\[
\rho_2 : \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}(k), \quad \rho_2(t) = \alpha_1^{-1} \circ g \circ \rho_1(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}.
\]

Since $\alpha_1, \alpha_2, \rho_1(0), \nu(t) \in \text{Aut}_{\mathbb{P}^n}(k)$ for all $t \in \mathbb{A}^1 \setminus \{0\}$, the map

\[
\rho_2(t) = \alpha_1^{-1} \circ (g \circ \nu(t)^{-1} \circ g^{-1}) \circ \nu(t) \circ \rho_1(0)^{-1} \circ \alpha_2^{-1}
\]

is contained in the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(k)$ generated by $\text{Aut}_{\mathbb{P}^n}(k)$, for each $t \in \mathbb{A}^1 \setminus \{0\}$. Therefore, $f = \rho_2(0)$ is contained in the closure of the normal subgroup of $\text{Bir}_{\mathbb{P}^n}(k)$ generated by $\text{Aut}_{\mathbb{P}^n}(k)$.

(2) Let $\{\text{id}\} \neq N \subset \text{Bir}_{\mathbb{P}^n}(k)$ be a closed normal subgroup (with respect to the Zariski or to the Euclidean topology). It follows from Proposition 3.3 and Lemma 3.4 that $\text{Aut}_{\mathbb{P}^n}(k) \subset N$. Since $N$ is closed, it contains the closure of the normal subgroup generated by $\text{Aut}_{\mathbb{P}^n}(k)$, which is equal to $\text{Bir}_{\mathbb{P}^n}(k)$.

Note that Proposition 3.5, together with Lemma 2.12 (for dimension 1 in the case of the Zariski topology), yields Theorems 1 and 2.

### 3.3. Connectedness of the Cremona groups.

The group $\text{Bir}_{\mathbb{P}^n}$ is connected with respect to the Zariski topology [Blanc2010]. More precisely, we have the following:

**Proposition 3.6.** [Blanc2010, Théorème 5.1] Let $k$ be an algebraically closed field and $n \geq 1$. For each $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$ there is an open subset $U \subset \mathbb{A}^1(k)$ that contains 0 and 1, and a morphism $\rho : U \to \text{Bir}_{\mathbb{P}^n}(k)$ such that $\rho(0) = f$ and $\rho(1) = g$.

This corresponds to saying that $\text{Bir}_{\mathbb{P}^n}(k)$ is “rationally connected”. We will generalise this for any field $k$, and provide a morphism from the whole $\mathbb{A}^1$ (Proposition 3.11 below), showing then that $\text{Bir}_{\mathbb{P}^n}(k)$ is “$\mathbb{A}^1$-uniruled”.
Let us recall the following classical fact.

**Lemma 3.7.** For each field $k$ and each integer $n \geq 2$, there is an integer $m$ and a $k$-morphism $\rho: \mathbb{A}^m \to \text{SL}_n$ such that $\rho(\mathbb{A}^m(k)) = \text{SL}_n(k)$.

**Proof.** Using Gauss-Jordan elimination, every element of $\text{SL}_n(k)$ is a product of a diagonal matrix and $r$ elementary matrices of the first kind: matrices of the form $I + \lambda e_{i,j}$, $\lambda \in k$, $i \neq j$, where $(e_{i,j})_{i,j=1,\ldots,n}$ is the canonical basis of the vector space of $n \times n$-matrices. Moreover, the number $r$ can be chosen to be the same for all elements of $\text{SL}_n(k)$. We then observe that

$$
\begin{pmatrix}
1 & \lambda - 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \lambda^{-1} - 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\lambda & 1
\end{pmatrix} =
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}
$$

for each $\lambda \in k^*$. Using finitely many such products, we obtain then all diagonal elements. This gives the existence of $s \in \mathbb{N}$, only dependent on $n$, such that every element of $\text{SL}_n(k)$ is a product of $s$ elementary matrices of the first kind.

Denoting by $\nu_{i,j}: \mathbb{A}^1 \to \text{SL}_n(k)$ the $k$-morphism sending $\lambda$ to $I + \lambda e_{i,j}$, this shows that every element of $\text{SL}_n(k)$ is in the image of a product morphism $\mathbb{A}^m \to \text{SL}_n(k)$ of finitely many $\nu_{i,j}$. The number of such maps being finite, we can enlarge $m$ and obtain one morphism for all maps. \hfill \Box

**Corollary 3.8.** For each field $k$, each integer $n \geq 2$ and all $f, g \in \text{PSL}_n(k)$, there exists a $k$-morphism $\nu: \mathbb{A}^1 \to \text{PSL}_n$ such that $\nu(0) = f$ and $\nu(1) = g$.

**Proof.** It suffices to take a morphism $\rho: \mathbb{A}^m \to \text{SL}_n$ as in Lemma 3.7, to choose $v, w \in \mathbb{A}^m(k)$ such that $\rho(v) = f$, $\rho(w) = g$ in $\text{PSL}_n(k)$, and to define $\nu(t) = \rho(v + t(w - v))$. \hfill \Box

**Remark 3.9.** By construction, Corollary 3.8 also works for $\text{SL}_n(k)$, but is in fact false for $\text{GL}_n(k)$. Indeed, every $k$-morphism $\nu: \mathbb{A}^1 \to \text{GL}_n$ gives rise to a morphism $\det \circ \nu: \mathbb{A}^1 \to \mathbb{A}^1 \setminus \{0\}$, which is necessarily constant. As every morphism $\mathbb{A}^1 \to \text{PGL}_n$ lifts to a morphism $\mathbb{A}^1 \to \text{GL}_n$, the same holds for $\text{PGL}_n$.

**Example 3.10.** Let $k$ be a field, $n \geq 2$ and $\lambda \in k^*$. We consider $g \in \text{Bir}_{P^m}(k)$ given by

$$g: [x_0 : \ldots : x_n] \mapsto \left[ \frac{x_0(x_1 + \lambda x_2) + x_1 x_2}{x_1 + x_2} : x_1 : \ldots : x_n \right]$$

We observe that $p_1 = [0 : 1 : 0 : \ldots : 0]$ and $p_2 = [0 : 0 : 1 : 0 : \ldots : 0]$ are both fixed by $g$. In local charts $x_1 = 1$ and $x_2 = 1$, the map $g$ becomes:

$$[x_0 : 1 : x_2 : x_3 : \ldots : x_n] \mapsto \left[ \frac{x_0(1 + \lambda x_2) + x_2}{x_2 + 1} : 1 : x_2 : x_3 : \ldots : x_n \right]$$

$$[x_0 : x_1 : 1 : x_3 : \ldots : x_n] \mapsto \left[ \frac{x_0(x_1 + \lambda) + x_1}{x_1 + 1} : 1 : x_1 : x_3 : \ldots : x_n \right]$$

Applying Proposition 3.2 to the two fixed points, we get two $k$-morphisms $\rho_1, \rho_2: \mathbb{A}^1 \to \text{Bir}_{P^m}$ such that $\rho_1(1) = g = \rho_2(1)$ and $\rho_1(0), \rho_2(0) \in \text{Aut}_{P^m}(k)$. The two elements are provided by the construction Example 3.1. Choosing for this one the affine coordinates $x_1 \neq 0$ and $x_2 \neq 0$ using permutations of the coordinates, we obtain the following maps corresponding to the linear parts in these affine spaces:

$$\rho_1(0): \ [x_0 : x_1 : x_2 : x_3 : \ldots : x_n] \mapsto [x_0 + x_2 : x_1 : x_2 : x_3 : \ldots : x_n],$$

$$\rho_2(0): \ [x_0 : x_1 : x_2 : x_3 : \ldots : x_n] \mapsto [x_0 \lambda + x_1 : x_1 : x_2 : x_3 : \ldots : x_n].$$
We can now give the following generalisation of [Blanc2010, Théorème 5.1] (Proposition 3.6):

**Proposition 3.11.** For each infinite field $k$, each integer $n \geq 2$ and all $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$, there exists a $k$-morphism $\nu: \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ such that $\nu(0) = f$ and $\nu(1) = g$.

**Proof.** Multiplying the morphism with $f^{-1}$, we can assume that $f = \text{id}$. We denote by $N \subset \text{Bir}_{\mathbb{P}^n}(k)$ the subset given by

$$N = \left\{ g \in \text{Bir}_{\mathbb{P}^n}(k) \mid \text{there exists a } k\text{-morphism } \nu: \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n} \text{ such that } \nu(0) = \text{id} \text{ and } \nu(1) = g \right\}.$$ 

If $f, g \in N$ are associated to $k$-morphisms $\nu_f, \nu_g$, we define a $k$-morphism $\nu_{fg}: \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ by $\nu_{fg}(t) = \nu_f(t)\nu_g(t)$, which satisfies $\nu_{fg}(0) = \text{id}$ and $\nu_{fg}(1) = fg$. For each $h \in \text{Bir}_{\mathbb{P}^n}(k)$, we can also define a morphism $t \mapsto h\nu_f(t)h^{-1}$. Thus, $N$ is a normal subgroup of $\text{Bir}_{\mathbb{P}^n}(k)$ and it contains $\text{PSL}_{n+1}(k)$ by Corollary 3.8. As $N$ is a priori not closed, we cannot apply Theorem 1. However, we will apply Proposition 3.2 and Example 3.10 to obtain the result.

First, taking $\lambda, g, \rho_1, \rho_2$ as in Example 3.10, the morphisms $t \mapsto \rho_i(t)\circ \rho_i(0)^{-1}$, $i = 1, 2$, show that $g \circ (\rho_1(0))^{-1} \circ g \circ (\rho_2(0))^{-1} \in N$, which implies that $\rho_1(0) \circ (\rho_2(0))^{-1} \in N$. Since $\rho_1(0) \in \text{PSL}_{n+1}(k) \subset N$, this implies that

$$\rho_2(0): [x_0 : x_1 : x_2 : x_3 : \cdots : x_n] \mapsto [x_0\lambda + x_1 : x_2 : x_3 : \cdots : x_n]$$

belongs to $N$, for each $\lambda \in k^*$. Hence, $\text{Aut}_{\mathbb{P}^n}(k) = \text{PGL}_{n+1}(k) \subset N$.

Second, we take any $g \in \text{Bir}_{\mathbb{P}^n}(k)$ of degree $d \geq 2$, take a point $p \in \mathbb{P}^n(k)$ such that $g$ induces a local isomorphism at $p$, choose $\alpha \in \text{PSL}_{n+1}(k)$ such that $\alpha \circ g$ fixes $p$. Proposition 3.2 yields the existence of a $k$-morphism $\rho: \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ with $\rho(1) = \alpha \circ g$ and $\rho(0) \in \text{Aut}_{\mathbb{P}^n}(k)$. Choosing $\rho': \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ given by $\rho'(t) = \rho(t) \circ \rho(0)^{-1}$, we obtain that $\rho'(1) = \alpha \circ g \circ \rho(0)^{-1} \in N$. Since $\alpha, \rho(0) \in \text{Aut}_{\mathbb{P}^n}(k) \subset N$, this shows that $g \in N$ and concludes the proof. 

**Corollary 3.12.** For each infinite field $k$ and each $n \geq 1$, the group $\text{Bir}_{\mathbb{P}^n}(k)$ is connected with respect to the Zariski topology.

**Proof.** For $n = 1$, the result follows from the fact that $\text{Bir}_{\mathbb{P}^1} = \text{Aut}_{\mathbb{P}^1} = \text{PGL}_2$ is an open subvariety of $\mathbb{P}^3$. For $n \geq 2$, this follows from Proposition 3.11. 

**Corollary 3.13.** For each $n \geq 2$, the groups $\text{Bir}_{\mathbb{P}^n}(\mathbb{R})$ and $\text{Bir}_{\mathbb{P}^n}(\mathbb{C})$ are path-connected, and thus connected with respect to the Euclidean topology.

**Proof.** Let us fix $k = \mathbb{R}$ or $k = \mathbb{C}$. For each $f, g \in \text{Bir}_{\mathbb{P}^n}(k)$ there is a $k$-morphism $\nu: \mathbb{A}^1 \to \text{Bir}_{\mathbb{P}^n}$ such that $\nu(0) = f$ and $\nu(1) = g$ (Proposition 3.11). The corresponding map $k = \mathbb{A}^1(k) \to \text{Bir}_{\mathbb{P}^n}(k)$ is continuous with respect to the Euclidean topologies (Lemma 2.11). The restriction of this map to the interval $[0, 1] \subset \mathbb{R} \subset \mathbb{C}$ yields a map $[0, 1] \to \text{Bir}_{\mathbb{P}^n}(k)$, continuous with respect to the Euclidean topologies and sending 0 to $f$ and 1 to $g$. 

Theorem 3 is now proven, as a consequence of Proposition 3.11 and Corollaries 3.12 and 3.13.
References


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