Surgeries and combinatorics
Joint work with Cui Guizhen

Tan Lei, Université de Cergy-Pontoise

July 13, 2007

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$

A repellor $F = (f, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ f : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$ if $\exists\ \boxed{\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id}$

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$ if $\exists\ \boxed{\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id}$

s.t. $F \circ \theta^{-1} \sim_{conj} F'$;

A repellor $F = (F, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$ if $\exists\ \boxed{\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id}$

s.t. $F \circ \theta^{-1} \sim_{conj} F'$;

i.e. if $\exists$

A repellor $F = (f, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$ if $\exists\ \boxed{\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id}$

s.t. $F \circ \theta^{-1} \sim_{conj} F'$;

i.e. if $\exists\ \ \mathcal{L} \xrightarrow[\approx]{\theta} \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ with

A repellor $F = (f, \mathcal{E}, \mathcal{L})$:

$$\begin{cases} \mathcal{E} \subset\subset \mathcal{L} & \text{two nested bordered Riemann surfaces in } \overline{\mathbb{C}} \\ F : \mathcal{E} \to \mathcal{L} & \text{a topological covering (preserving the orientation)} \end{cases}$$

Conjugacy $F \sim_{conj} F'$

if $\exists\ \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ s.t. $\boxed{F' \circ \Phi|_{\mathcal{E}} = \Phi \circ F.}$

$$\begin{array}{ccc} \mathcal{L} \supset \mathcal{E} & \xrightarrow[\approx]{\Phi|_{\mathcal{E}}} & \mathcal{E}' \subset \mathcal{L}' \\ F \downarrow & & \downarrow F' \\ \mathcal{L} & \xrightarrow[\Phi]{} & \mathcal{L}' \end{array} .$$

Comb-equivalence $F \sim_{comb} F'$ if $\exists\ \boxed{\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id}$

s.t. $F \circ \theta^{-1} \sim_{conj} F'$;

i.e. if $\exists\ \ \mathcal{L} \xrightarrow[\approx]{\theta} \mathcal{L} \xrightarrow[\approx]{\Phi} \mathcal{L}'$ with $\begin{cases} \boxed{F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F} \\ \theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial\mathcal{L}}{\sim}} id. \end{cases}$

Questions :

1. A complete invariant for each $\sim_{comb}$-class?

Questions :

1. A complete invariant for each $\sim_{comb}$-class?

2. Given a repellor F, is $F \sim_{comb}$ a hol.-model? A criterion?

Questions :

1. A complete invariant for each $\sim_{comb}$-class?

2. Given a repellor F, is $F \sim_{comb}$ a hol.-model? A criterion?

interests : key step in the generalization of Thurston's theorem to non-postcritically finite maps.

Questions :

1. A complete invariant for each $\sim_{comb}$-class?

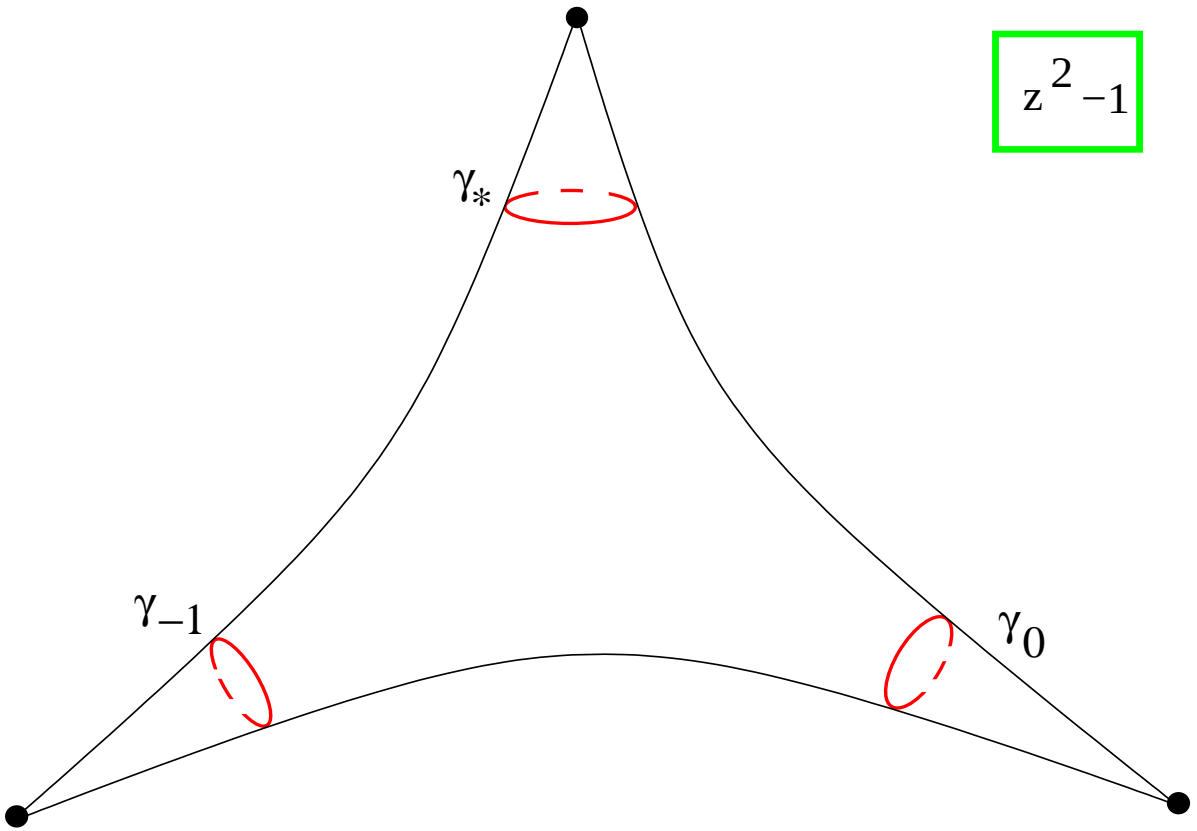
2. Given a repellor F, is $F \sim_{comb}$ a hol.-model? A criterion?

interests : key step in the generalization of Thurston's theorem to non-postcritically finite maps.

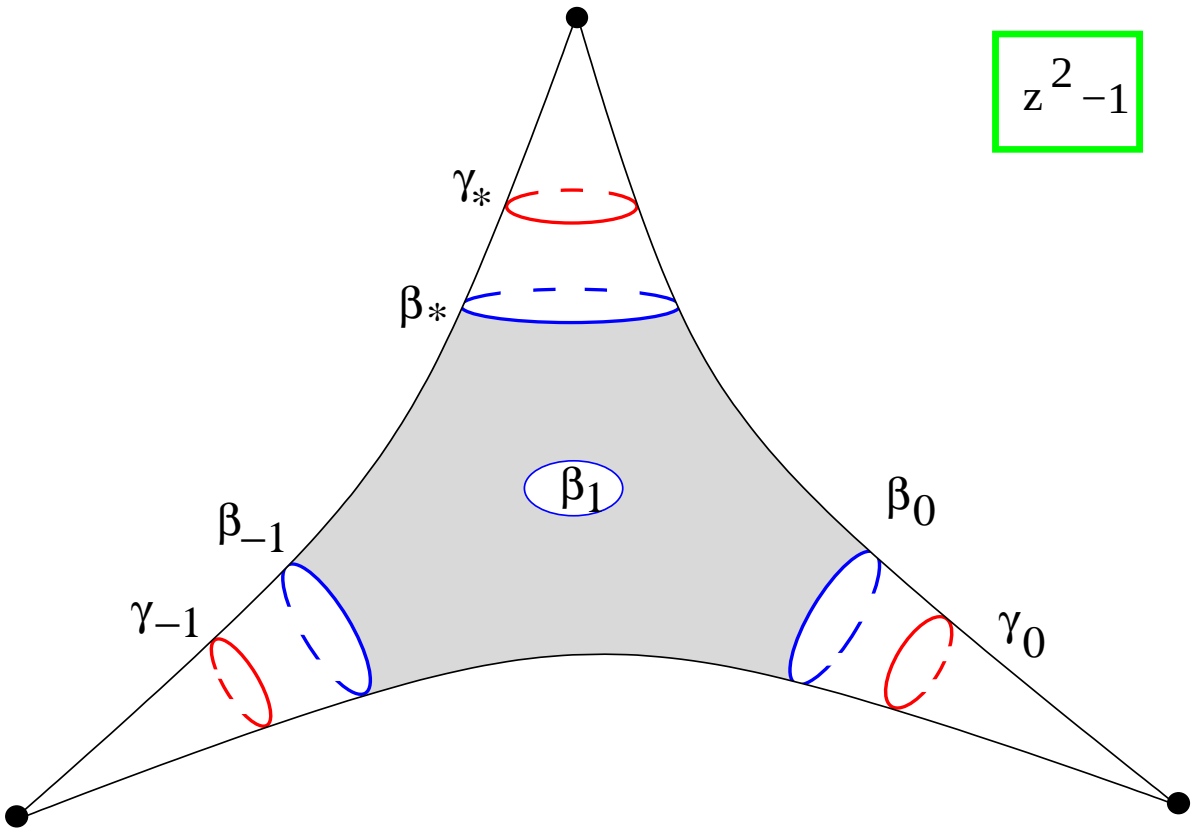
Applications : landing properties of parameter rays, cusp points on the boundary of hyperbolic components, hyperbolic components having a commun boundary point...

(the method is intrinsic and provides dynamical informations).

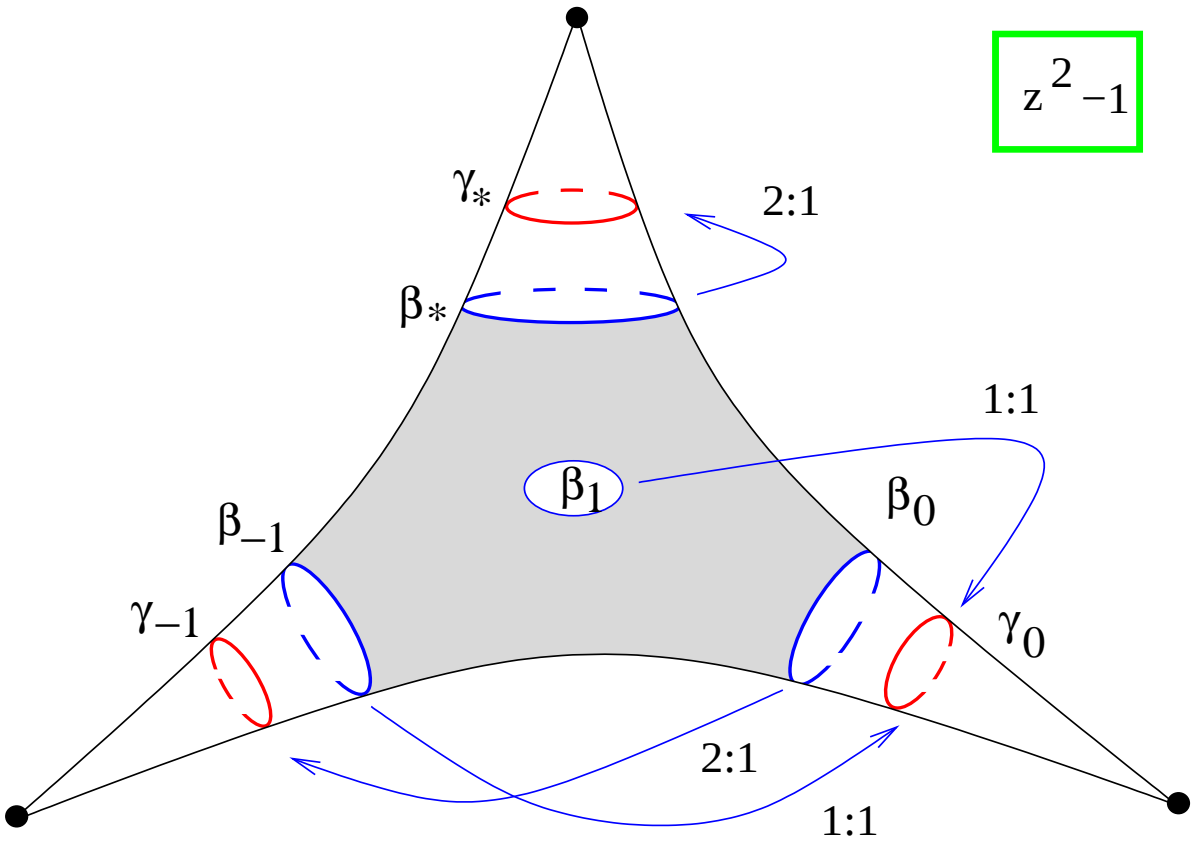
Examples



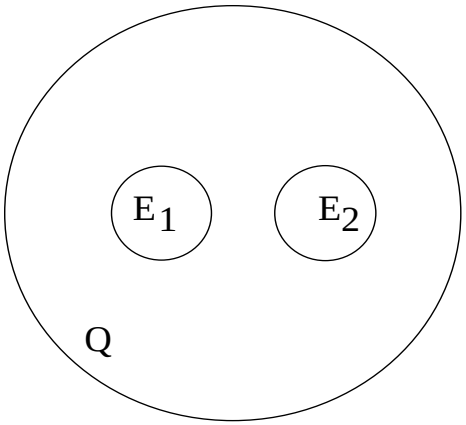
Examples



Examples

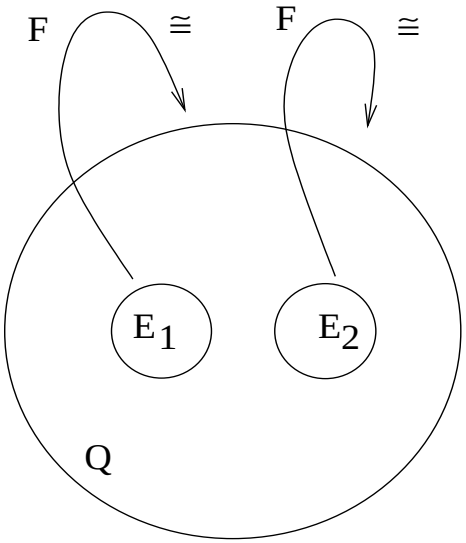


Examples



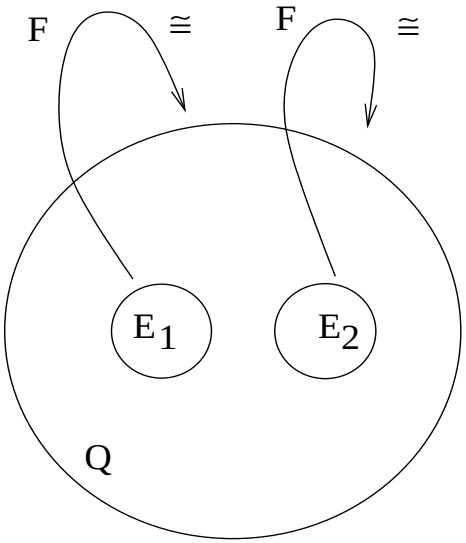
$\mathcal{L} = Q$, $\mathcal{E} = E_1 \cup E_2$, with Q, E_1, E_2 Jordan discs.

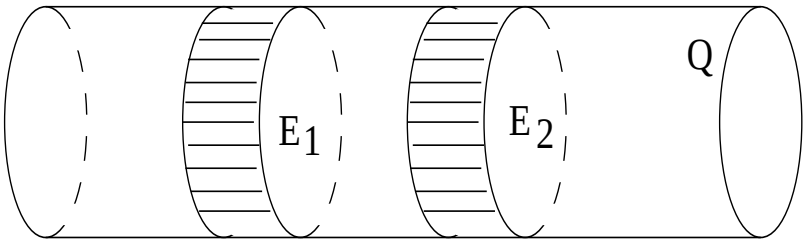
Examples



$\mathcal{L} = Q$, $\mathcal{E} = E_1 \cup E_2$, with Q, E_1, E_2 Jordan discs.
$F : E_i \longrightarrow Q$ homeo.

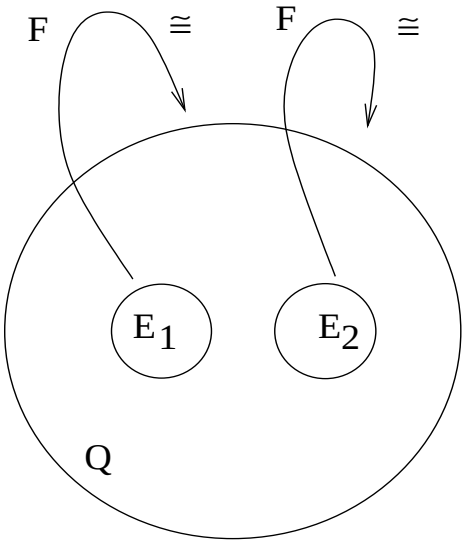
Examples

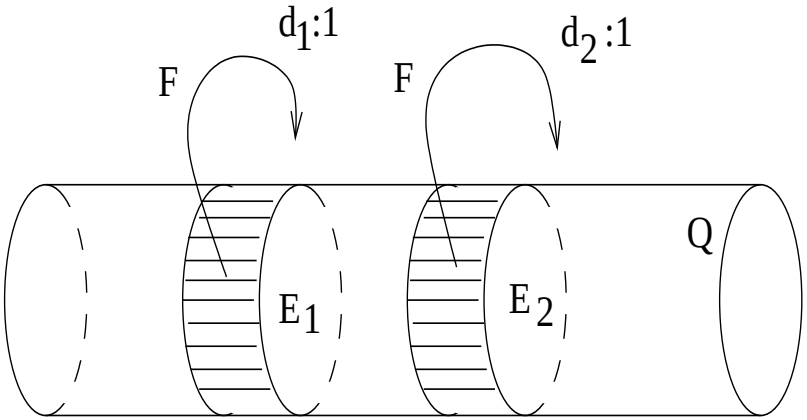




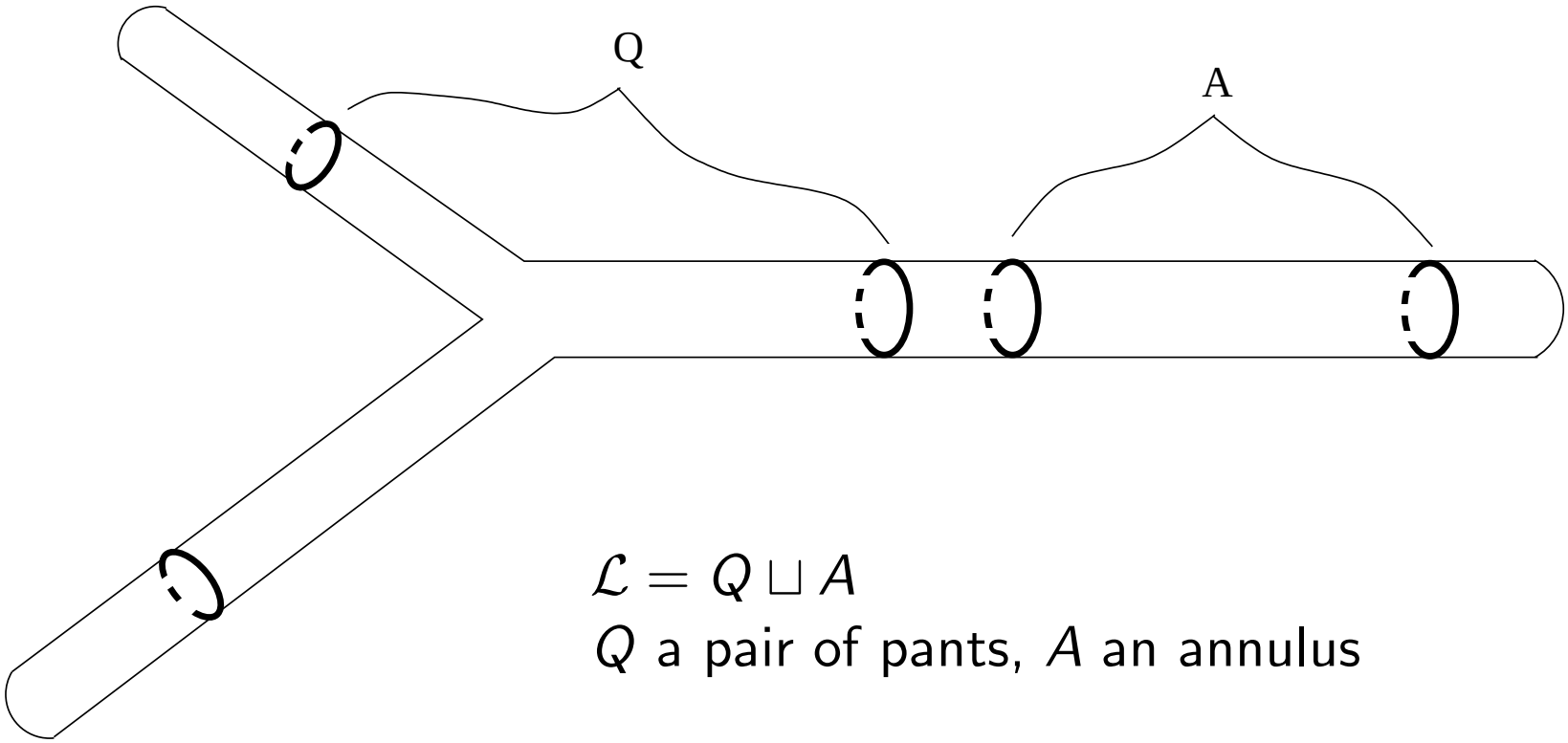
$\mathcal{L} = Q$, $\mathcal{E} = E_1 \cup E_2$, with Q, E_1, E_2 closed annuli.

Examples



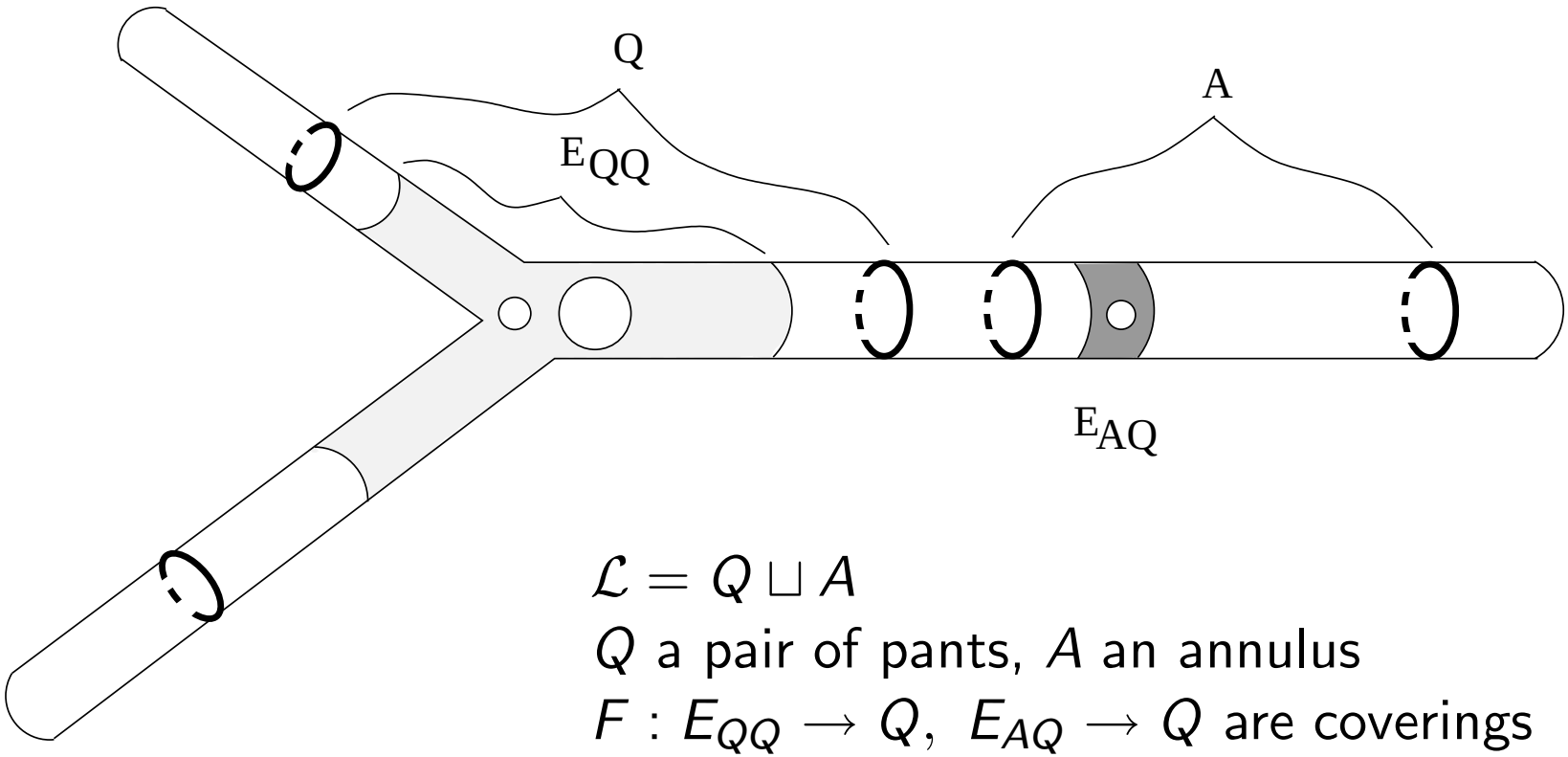


$\mathcal{L} = Q$, $\mathcal{E} = E_1 \cup E_2$, with Q, E_1, E_2 closed annuli.
$F : E_i \to Q$ are $d_i : 1$ coverings



$\mathcal{L} = Q \sqcup A$

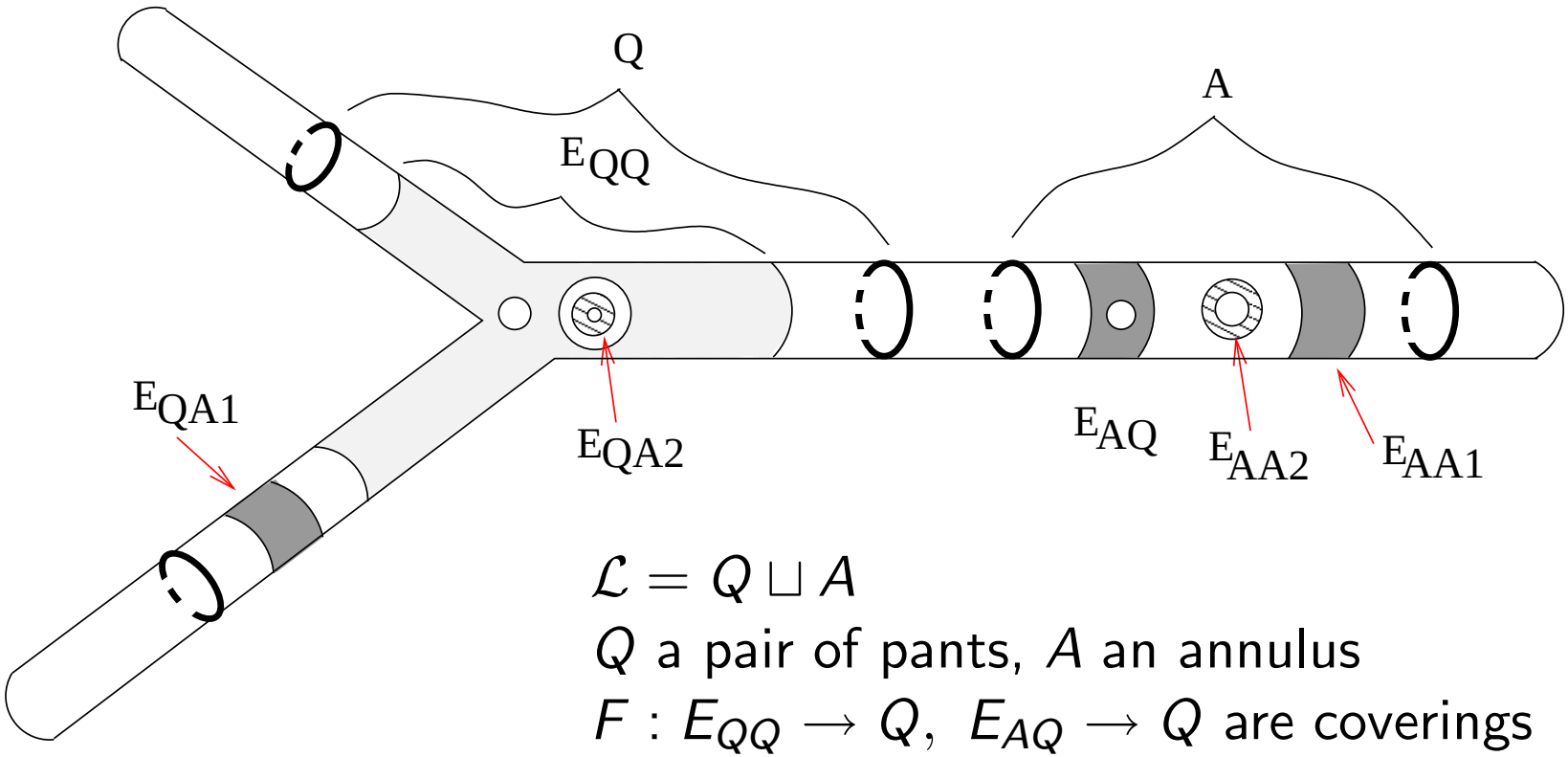
Q a pair of pants, A an annulus



$\mathcal{L} = Q \sqcup A$

Q a pair of pants, A an annulus

$F : E_{QQ} \to Q,\ E_{AQ} \to Q$ are coverings

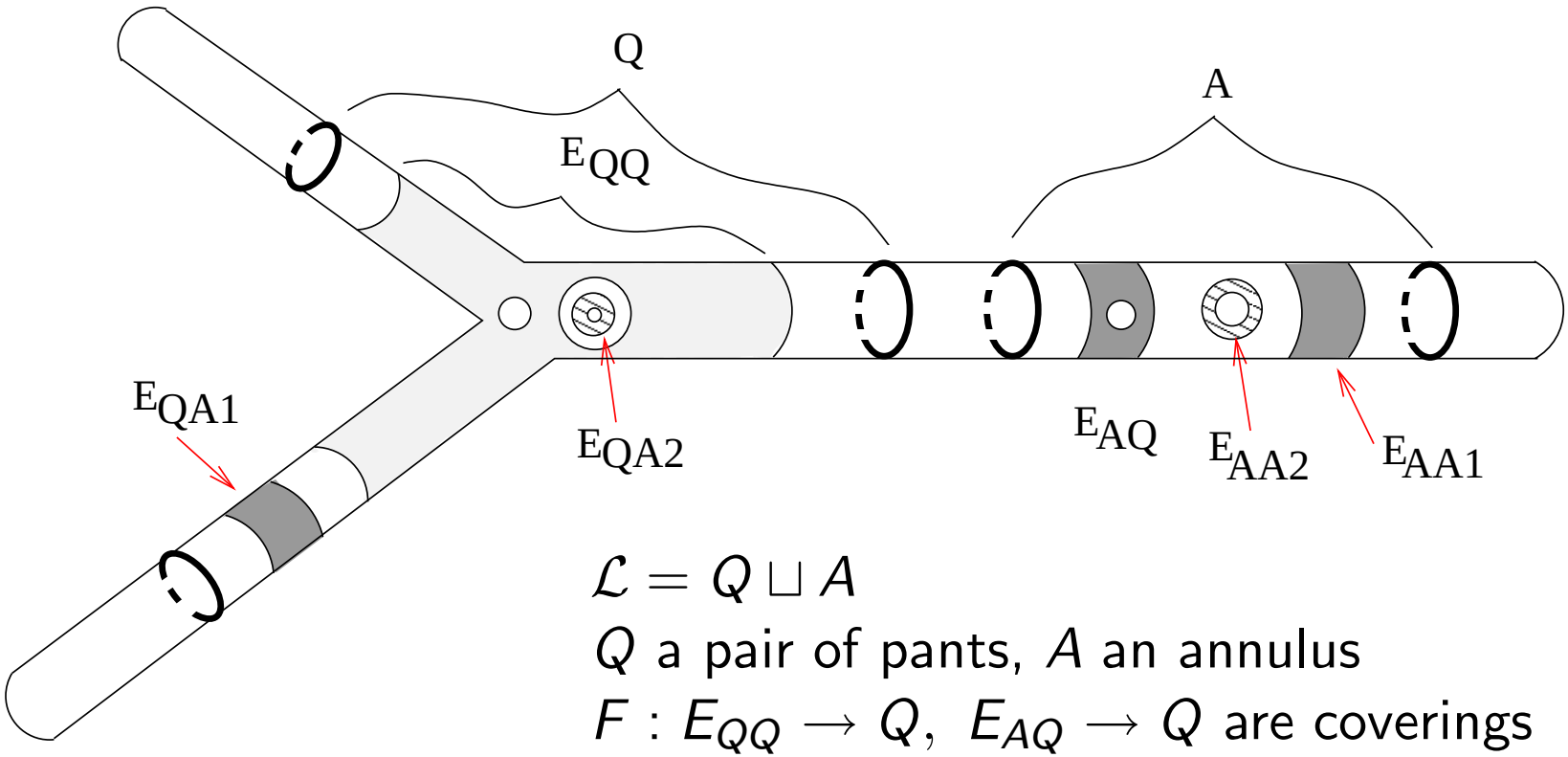


$\mathcal{L} = Q \sqcup A$

Q a pair of pants, A an annulus

$F : E_{QQ} \to Q$, $E_{AQ} \to Q$ are coverings

$F : E_{*Aj} \to A$ are coverings



$\mathcal{L} = Q \sqcup A$

Q a pair of pants, A an annulus

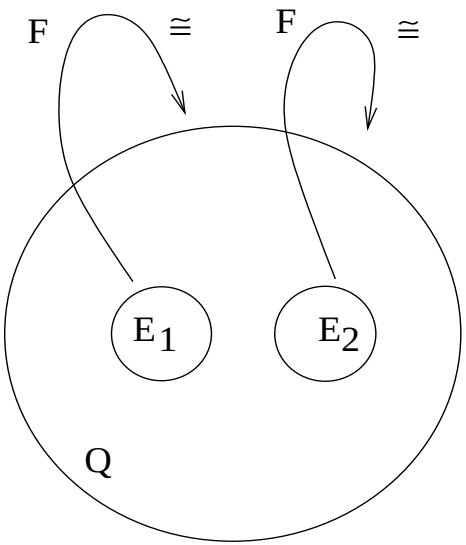
$F : E_{QQ} \to Q$, $E_{AQ} \to Q$ are coverings

$F : E_{*Aj} \to A$ are coverings

$F : \bigcup E_* \to \mathcal{L} = Q \sqcup A$ a repellor

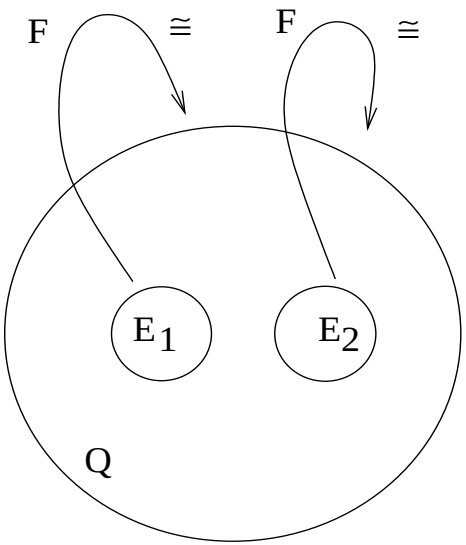
Is $F \sim_{comb}$ hol.-model?

Is $F \sim_{comb}$ hol.-model?



Usually $F \not\sim_{conj}$ a hol.-model

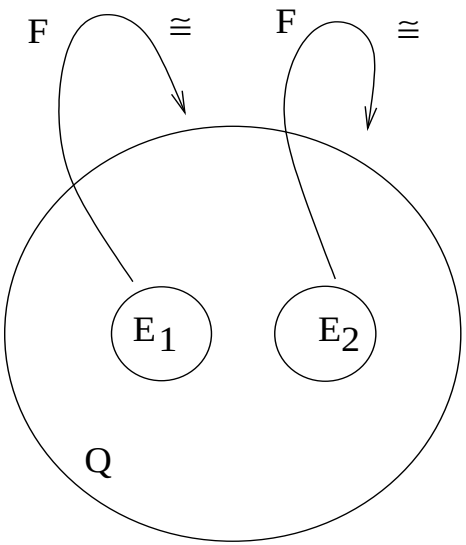
Is $F \sim_{comb}$ hol.-model?



Usually $F \not\sim_{conj}$ a hol.-model

But one has always

Is $F \sim_{comb}$ hol.-model?

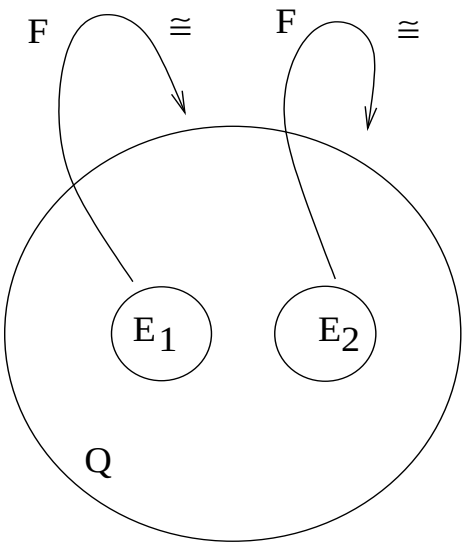


Usually $F \not\sim_{conj}$ a hol.-model

But one has always

$F \sim_{comb} F'$ holomorphic.

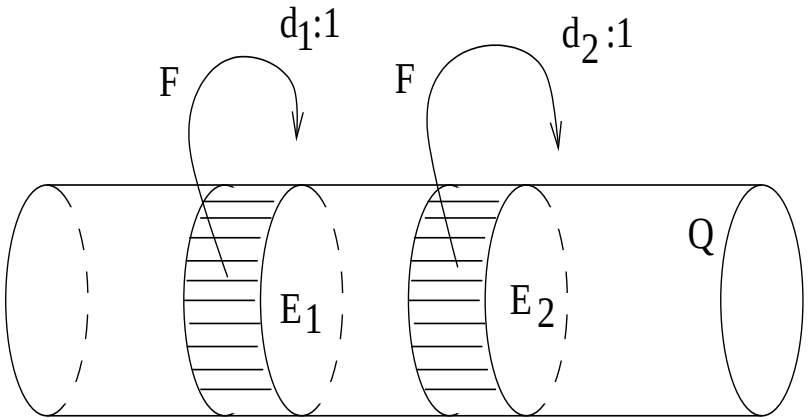
Is $F \sim_{comb}$ hol.-model?



Usually $F \not\sim_{conj}$ a hol.-model

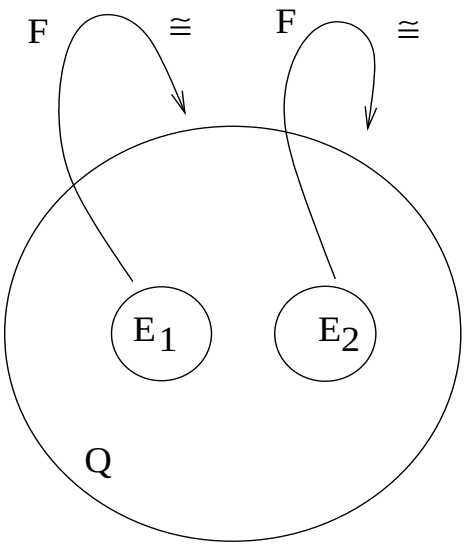
But one has always

$F \sim_{comb} F'$ holomorphic.



$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

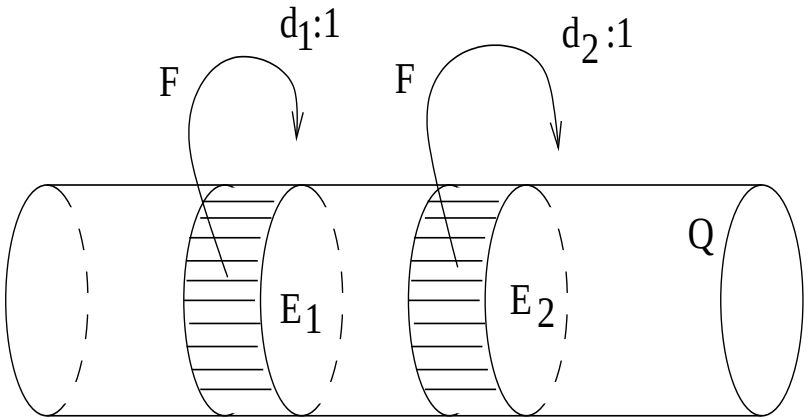
Is $F \sim_{comb}$ hol.-model?



Usually $F \not\sim_{conj}$ a hol.-model

But one has always

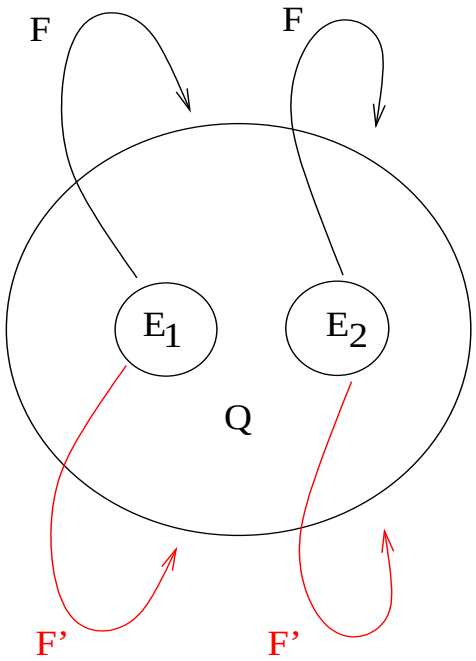
$F \sim_{comb} F'$ holomorphic.



$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

$\sum_i \frac{1}{d_i} < 1$ (**moduli** prob.)

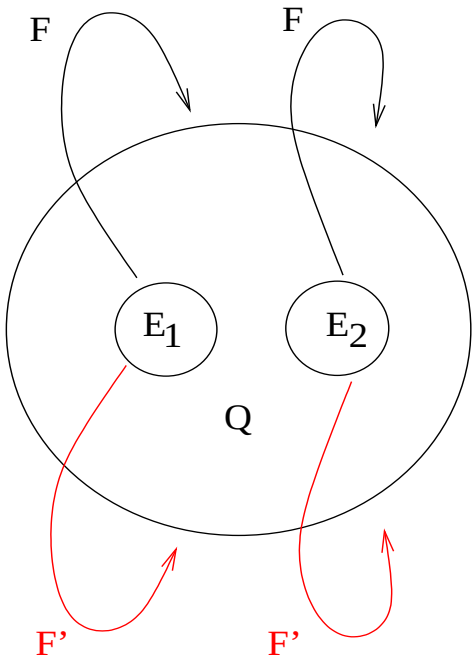
To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



1. Construct at first the model F':

$$\begin{array}{ccccc} F' : & E_i & \longrightarrow & Q & \\ & || & & || & \text{holomorphic .} \\ & E_i' & & Q' & \end{array}$$

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$

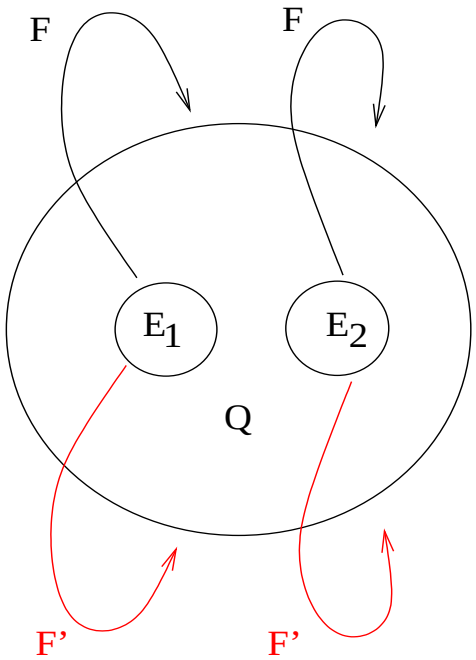


1. Construct at first the model F':

$$\begin{array}{ccccc} F' : & E_i & \to & Q & \\ & || & & || & \text{holomorphic} \\ & E_i' & & Q' & \end{array}$$

2. Set $\Phi = id : Q \to Q$.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



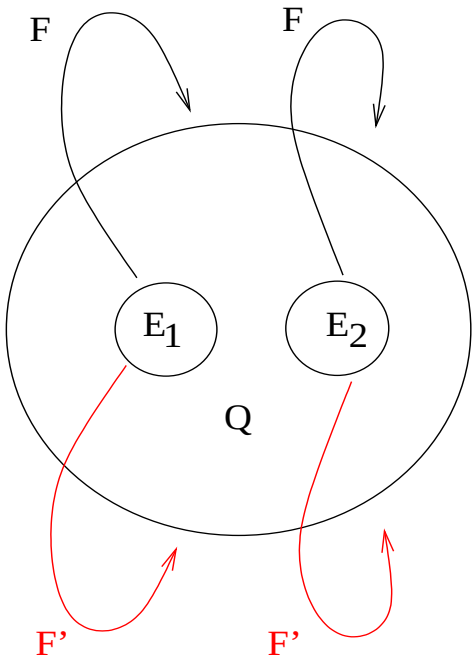
1. Construct at first the model F':

$$\begin{array}{ccccc} F' : & E_i & \to & Q & \\ & || & & || & \text{holomorphic} \\ & E_i' & & Q' & \end{array}$$

2. Set $\Phi = id : Q \to Q$.

3. Set $\begin{cases} \theta|_{E_i} &= (F')^{-1} \circ F \\ \theta|_{\partial Q} &= id \end{cases}$

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



1. Construct at first the model F':

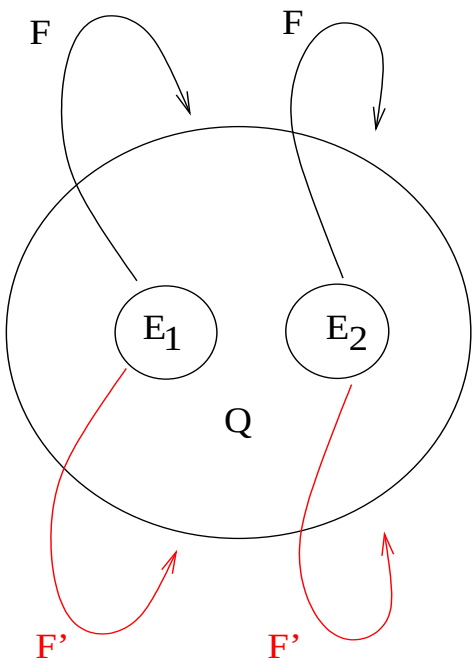
$$\begin{array}{ccccc} F' : & E_i & \to & Q & \\ & || & & || & \text{holomorphic} \\ & E_i' & & Q' & \end{array}$$

2. Set $\Phi = id : Q \to Q$.

3. Set $\begin{cases} \theta|_{E_i} &= (F')^{-1} \circ F \\ \theta|_{\partial Q} &= id \end{cases}$

then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



1. Construct at first the model F':

$$\begin{array}{ccccc} F' : & E_i & \to & Q & \\ & || & & || & \text{holomorphic} \\ & E_i' & & Q' & \end{array}$$

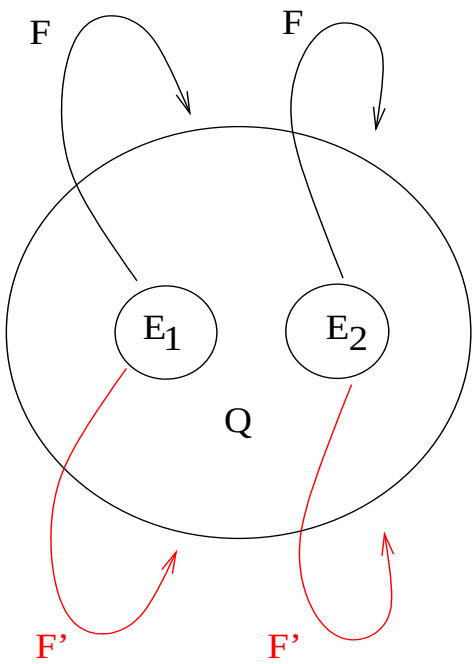
2. Set $\Phi = id : Q \to Q$.

3. Set $\begin{cases} \theta|_{E_i} &= (F')^{-1} \circ F \\ \theta|_{\partial Q} &= id \end{cases}$

   then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.

4. Extend θ as a homeo.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



1. Construct at first the model F':

$$\begin{array}{ccccc} F' : & E_i & \to & Q & \\ & || & & || & \text{holomorphic} \\ & E_i' & & Q' & \end{array}$$

2. Set $\Phi = id : Q \to Q$.

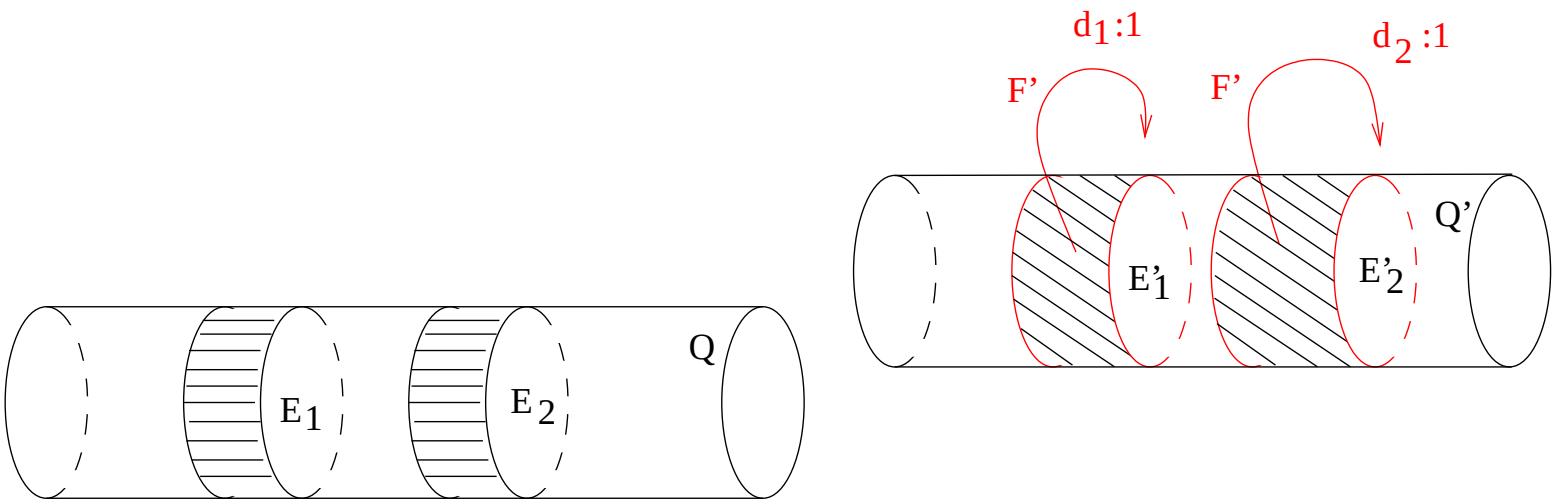
3. Set $\begin{cases} \theta|_{E_i} &= (F')^{-1} \circ F \\ \theta|_{\partial Q} &= id \end{cases}$

   then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.

4. Extend θ as a homeo.

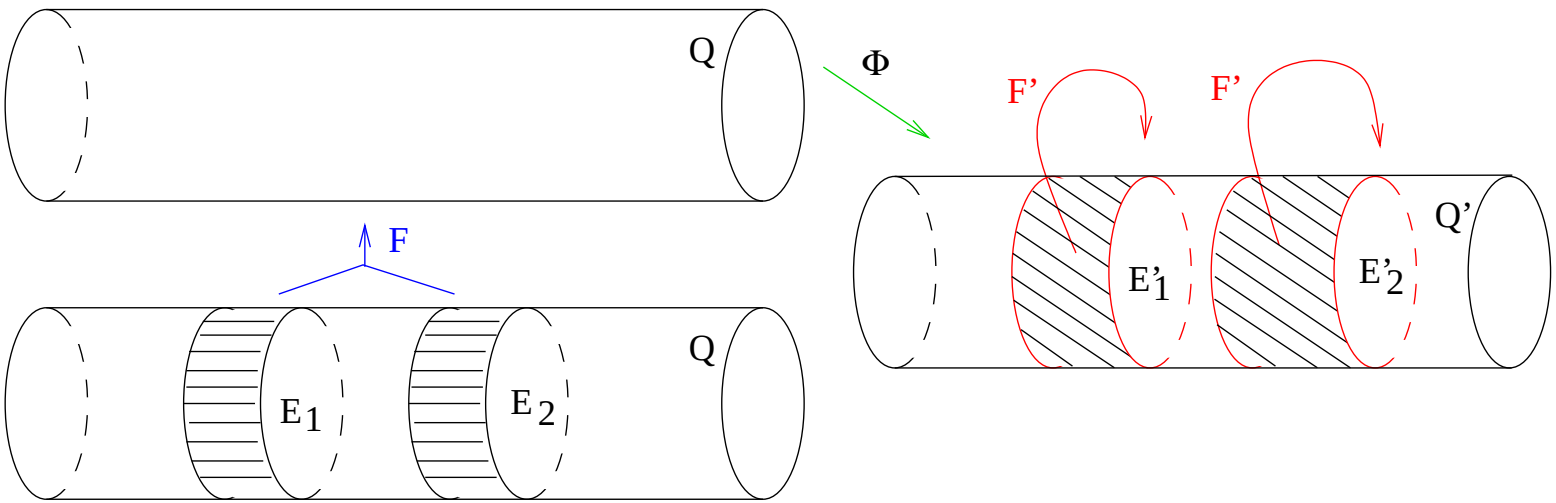
5. Check $\theta \overset{\text{isotop.}}{\underset{\text{rel}\,\partial Q}{\sim}} id$

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



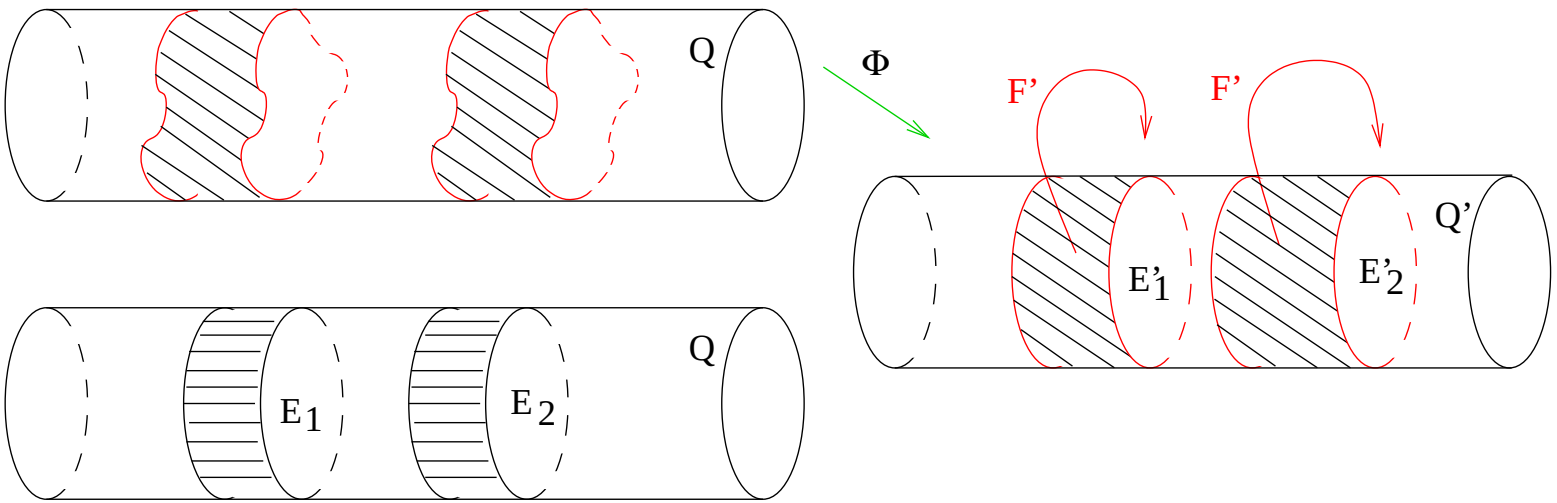
1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



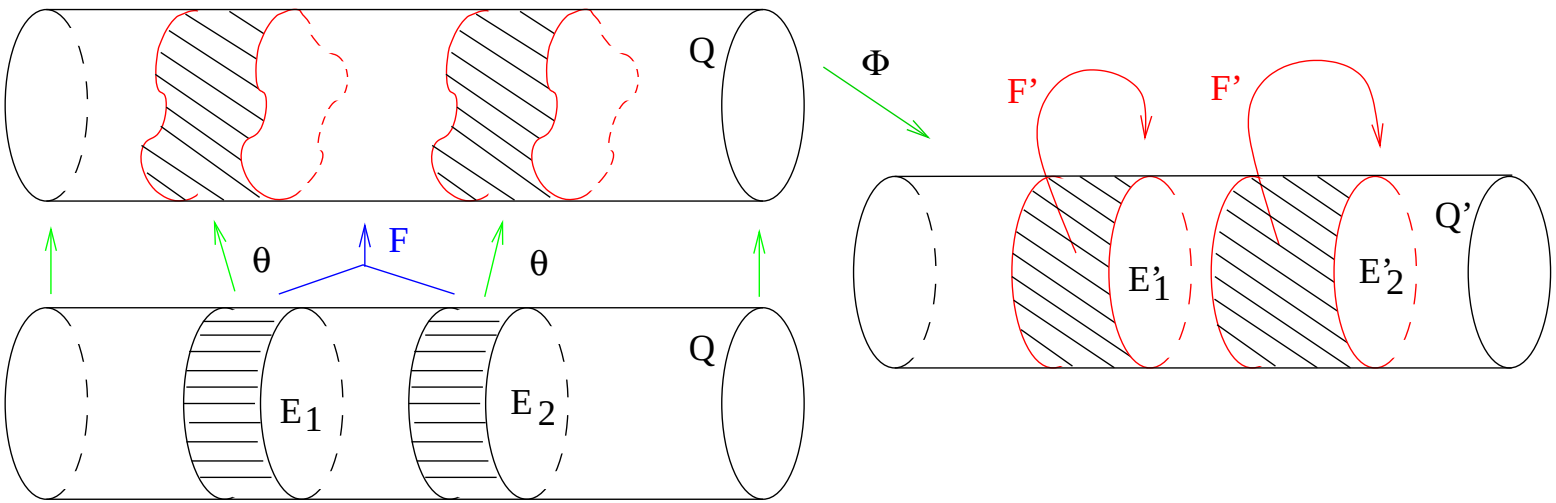
1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.
2. Set $\Phi : Q \to Q'$ a homeo.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



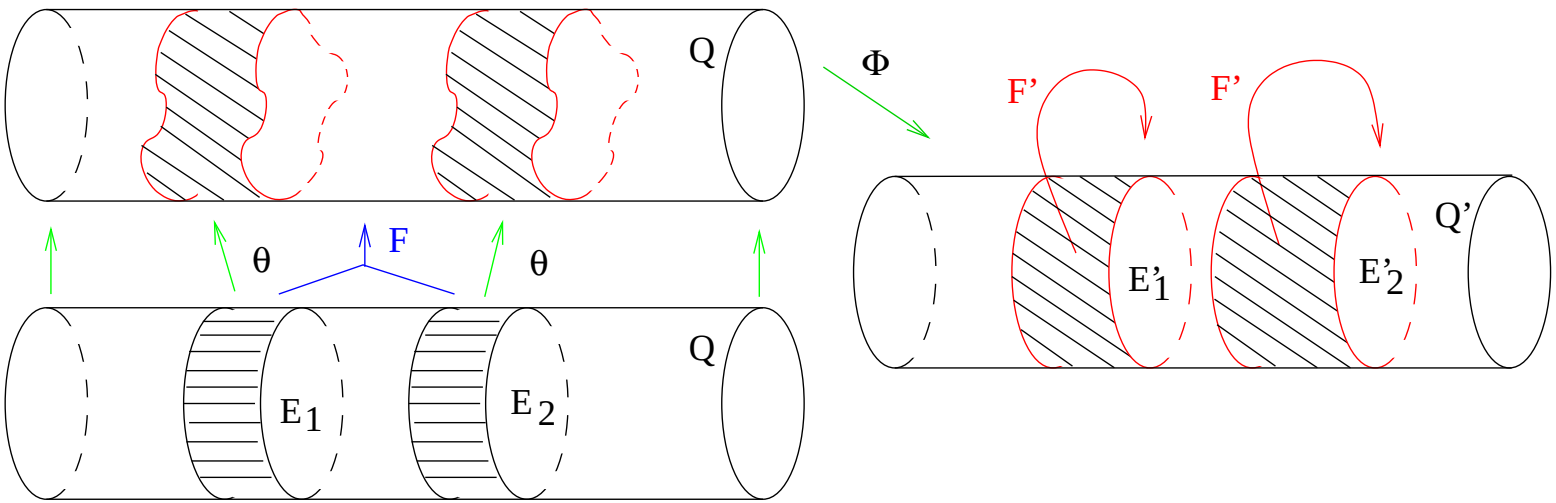
1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.
2. Set $\Phi : Q \to Q'$ a homeo. 3. Pull-back E_i' by Φ

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$



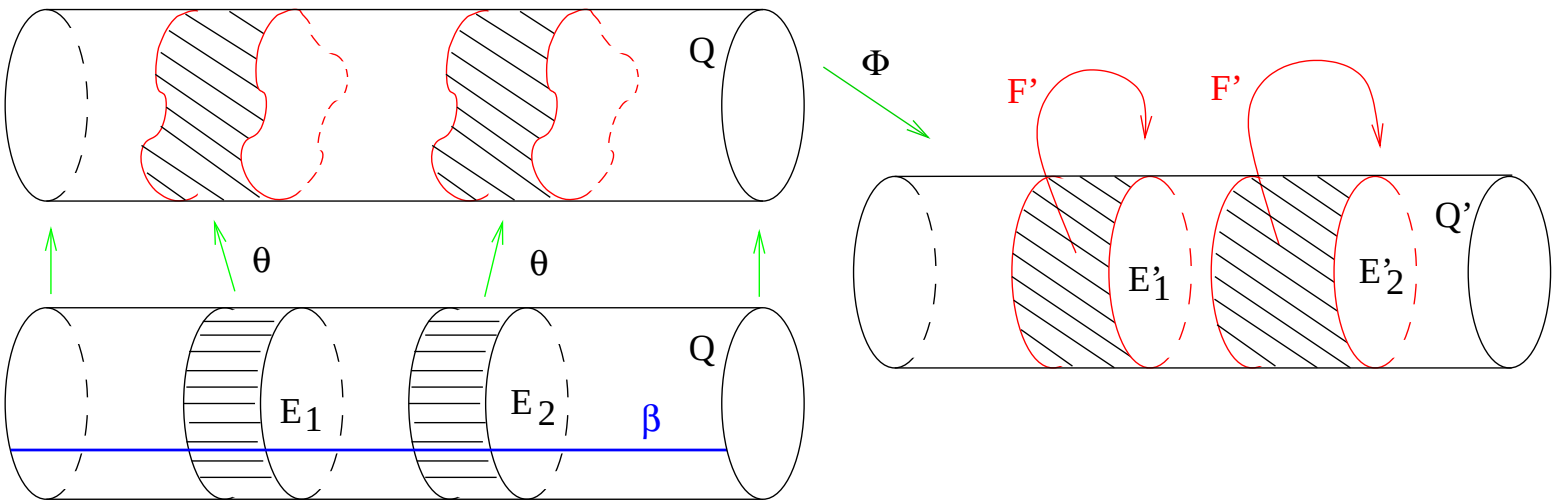
1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.
2. Set $\Phi : Q \to Q'$ a homeo. 3. Pull-back E_i' by Φ
4. Set $\begin{cases} \theta|_{E_i} := (\Phi^{-1} \circ F' \circ \Phi)^{-1} \circ F : E_i \longrightarrow \Phi^{-1}(E_i) \\ \theta|_{\partial Q} = id \end{cases}$

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$

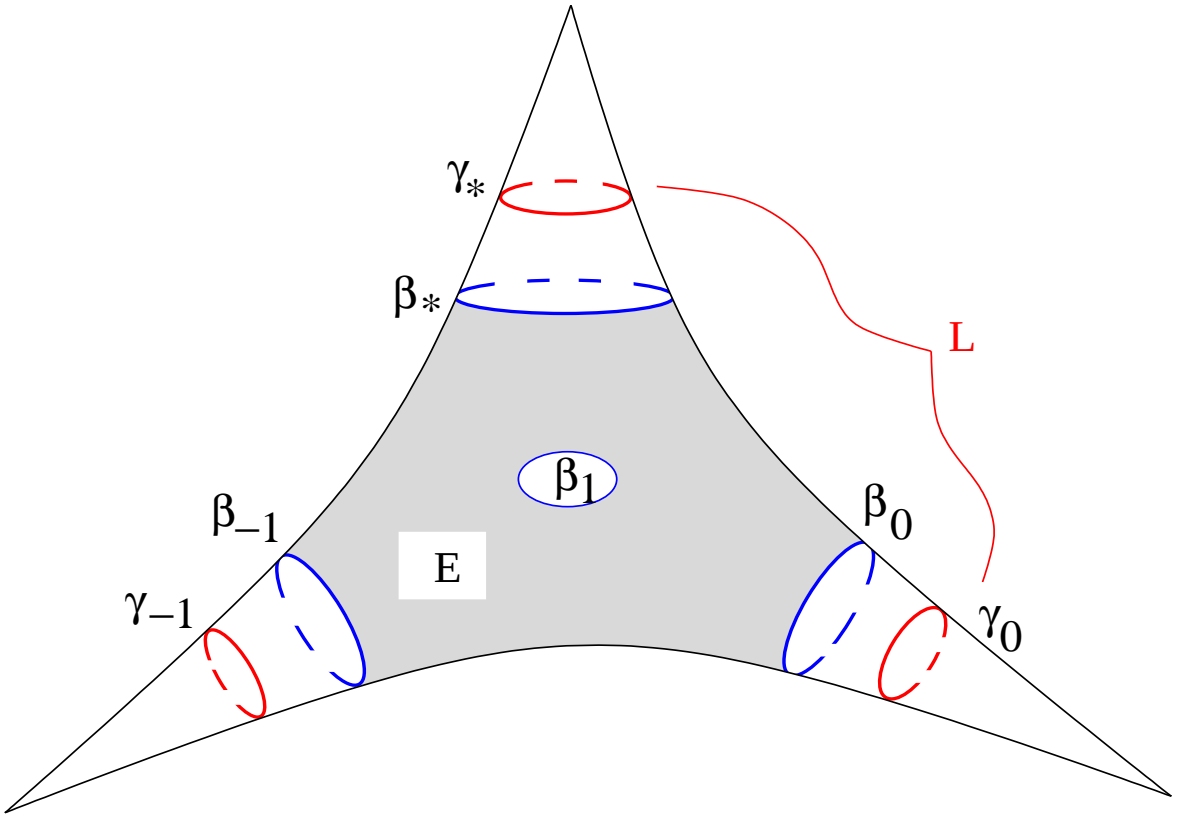


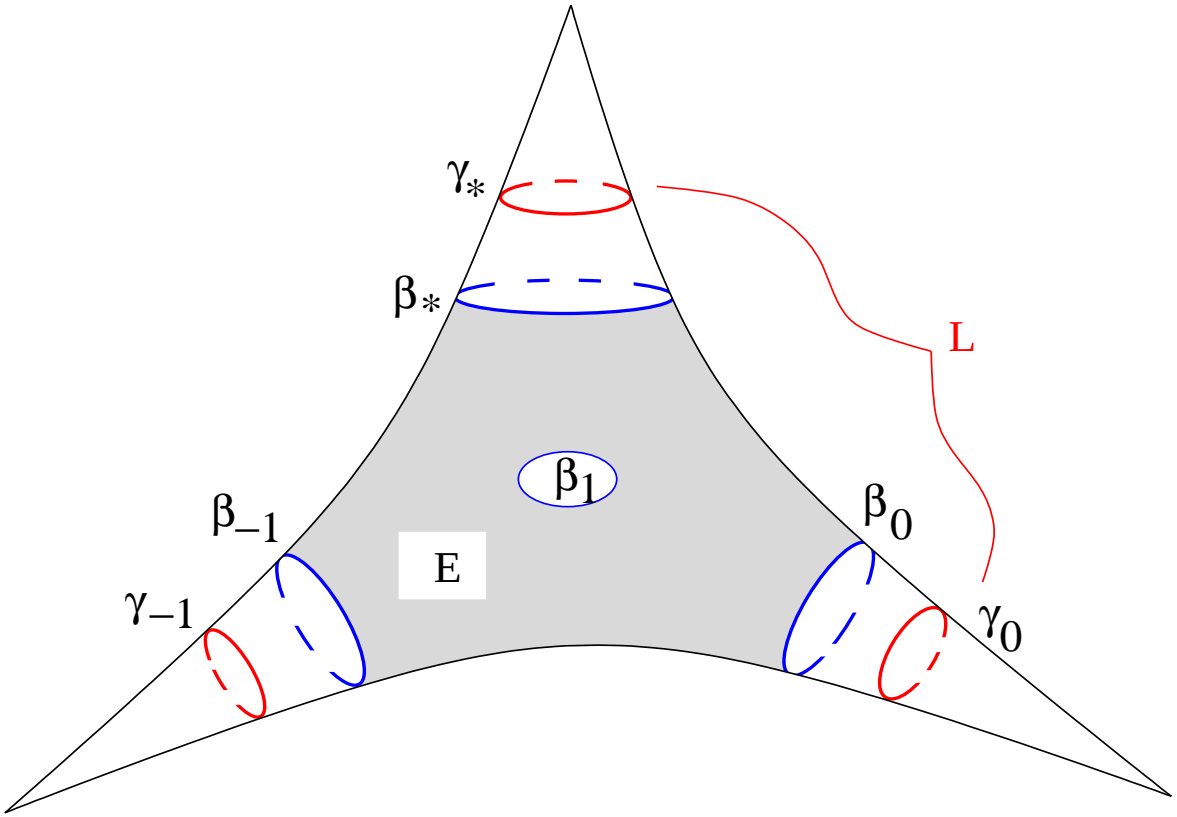
1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.
2. Set $\Phi : Q \to Q'$ a homeo. 3. Pull-back E_i' by Φ
4. Set $\begin{cases} \theta|_{E_i} := (\Phi^{-1} \circ F' \circ \Phi)^{-1} \circ F : E_i \longrightarrow \Phi^{-1}(E_i) \\ \theta|_{\partial Q} = id \end{cases}$
5. Extend θ as a homeo. $Q \to Q$. Then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim id$

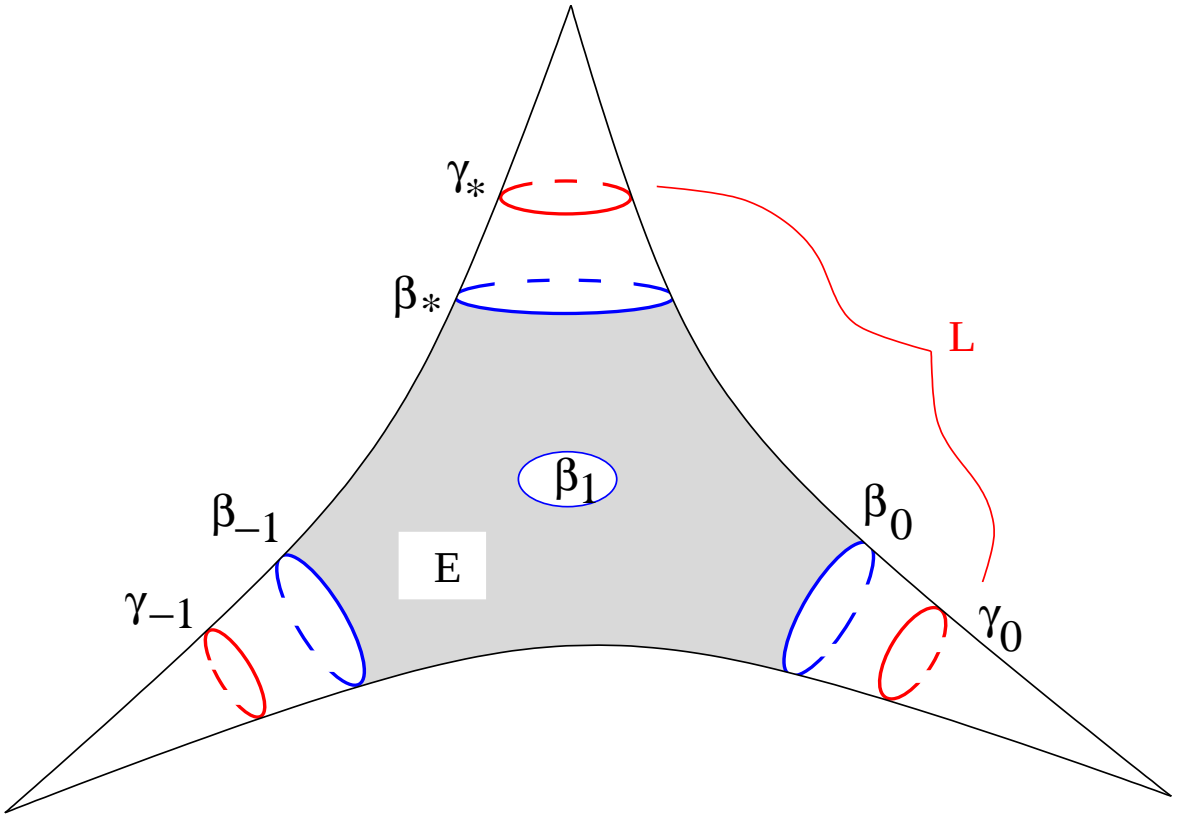


1. Construct at first a model hol. repellor $F' : E_i' \to Q'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.
2. Set $\Phi : Q \to Q'$ a homeo. 3. Pull-back E_i' by Φ
4. Set $\begin{cases} \theta|_{E_i} : = (\Phi^{-1} \circ F' \circ \Phi)^{-1} \circ F : E_i \longrightarrow \Phi^{-1}(E_i) \\ \theta|_{\partial Q} = id \end{cases}$
5. Extend θ as a homeo. $Q \to Q$. Then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.
6. Via Dehn twist to modify the extension so that $\theta(\beta) \sim \beta$.
   This guarantees: $\theta \overset{\text{isotop.}}{\sim} id$ rel. ∂Q.

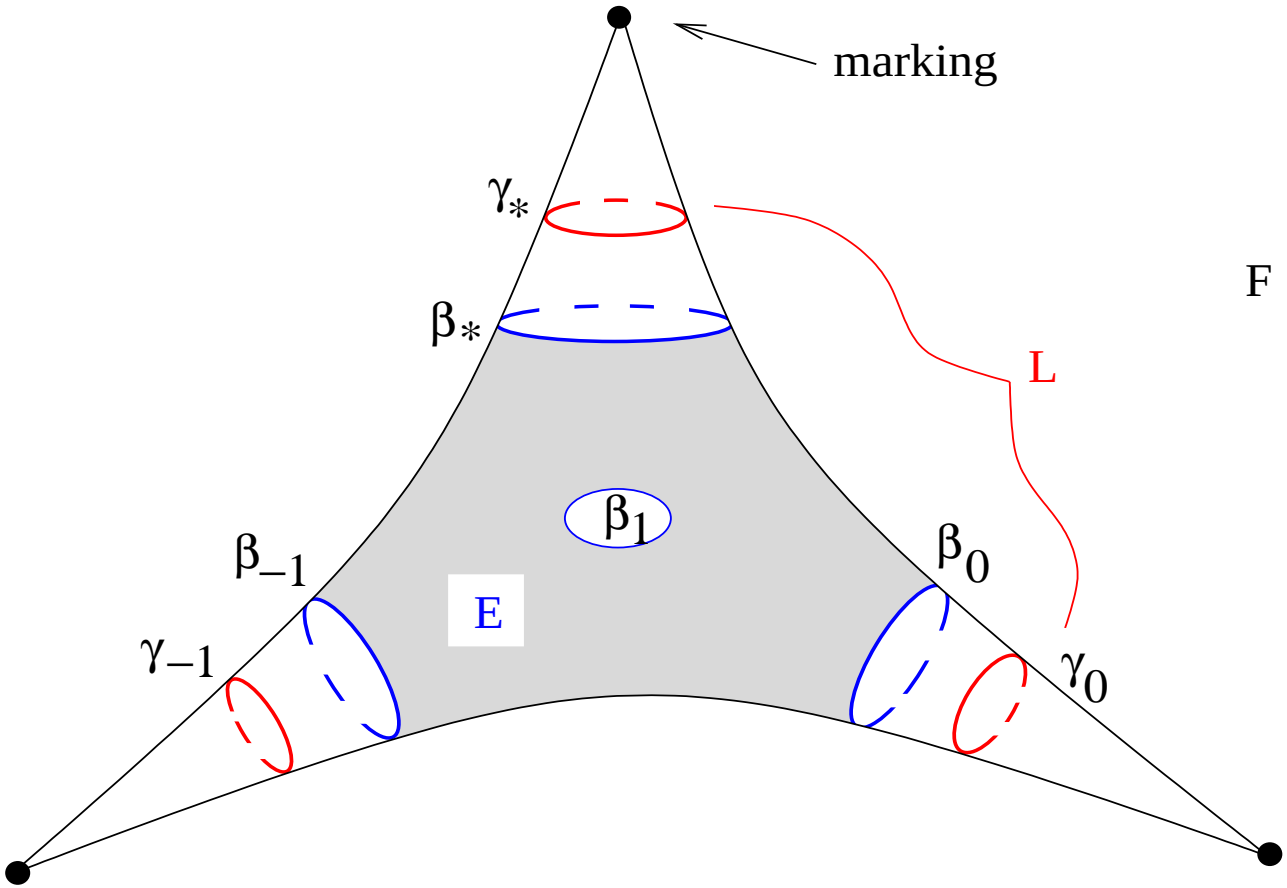

γ_*
β_*
L
β_1
β_{-1}
E
β_0
γ_{-1}
γ_0

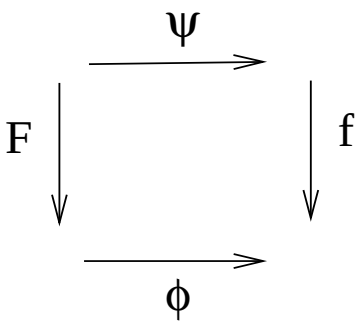


Let $F : E \to L$ be a covering of degree 2. This induces a covering $F : \{\beta_0, \beta_{-1}, \beta_*, \beta_1\} \to \{\gamma_0, \gamma_{-1}, \gamma_*\}$ of degree 2. Then
$F \sim_{comb}$ a hol. model $\iff$

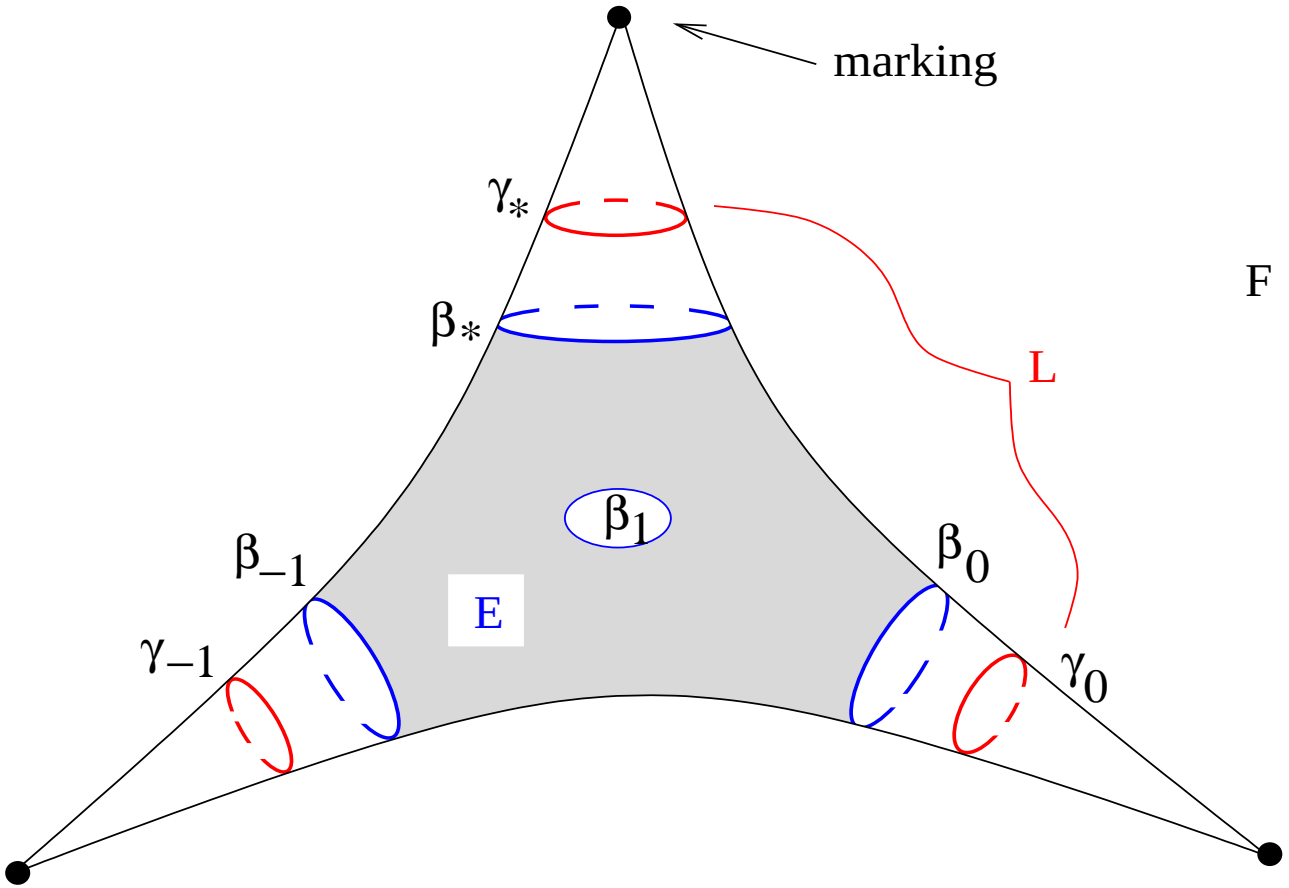


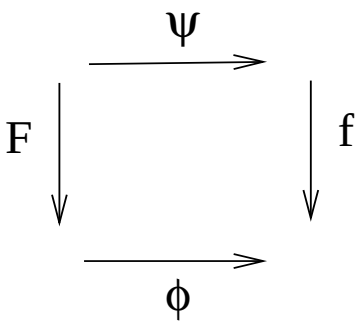
Let $F : E \to L$ be a covering of degree 2. This induces a covering $F : \{\beta_0, \beta_{-1}, \beta_*, \beta_1\} \to \{\gamma_0, \gamma_{-1}, \gamma_*\}$ of degree 2. Then $F \sim_{comb}$ a hol. model $\iff$ the transition matrix of F on $\{\gamma_0, \gamma_{-1}, \gamma_*\}$ has a leading eigenvalue $\lambda < 1$.



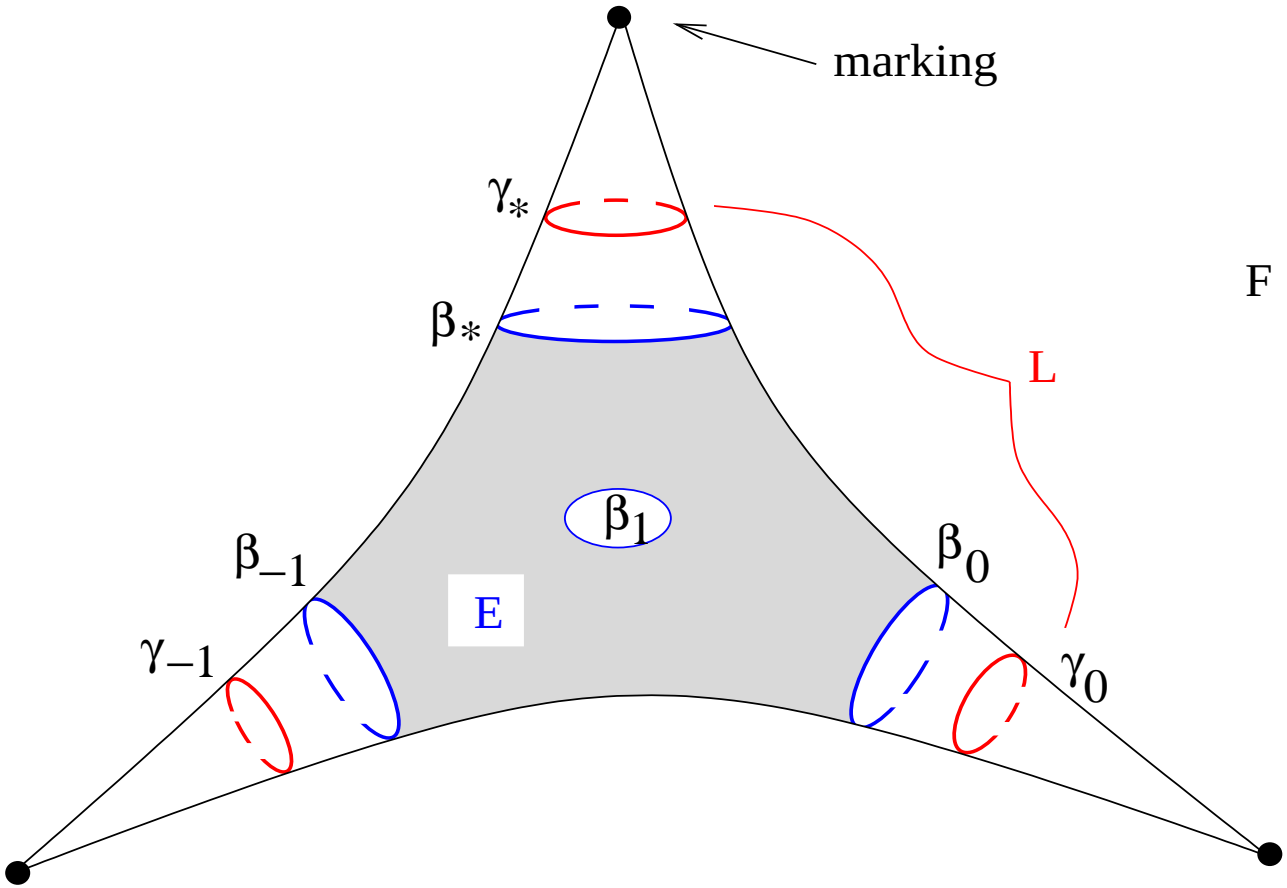


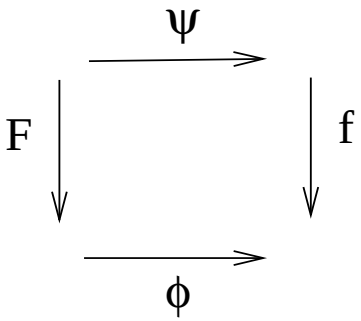
Proof. Mark one point on each corner.



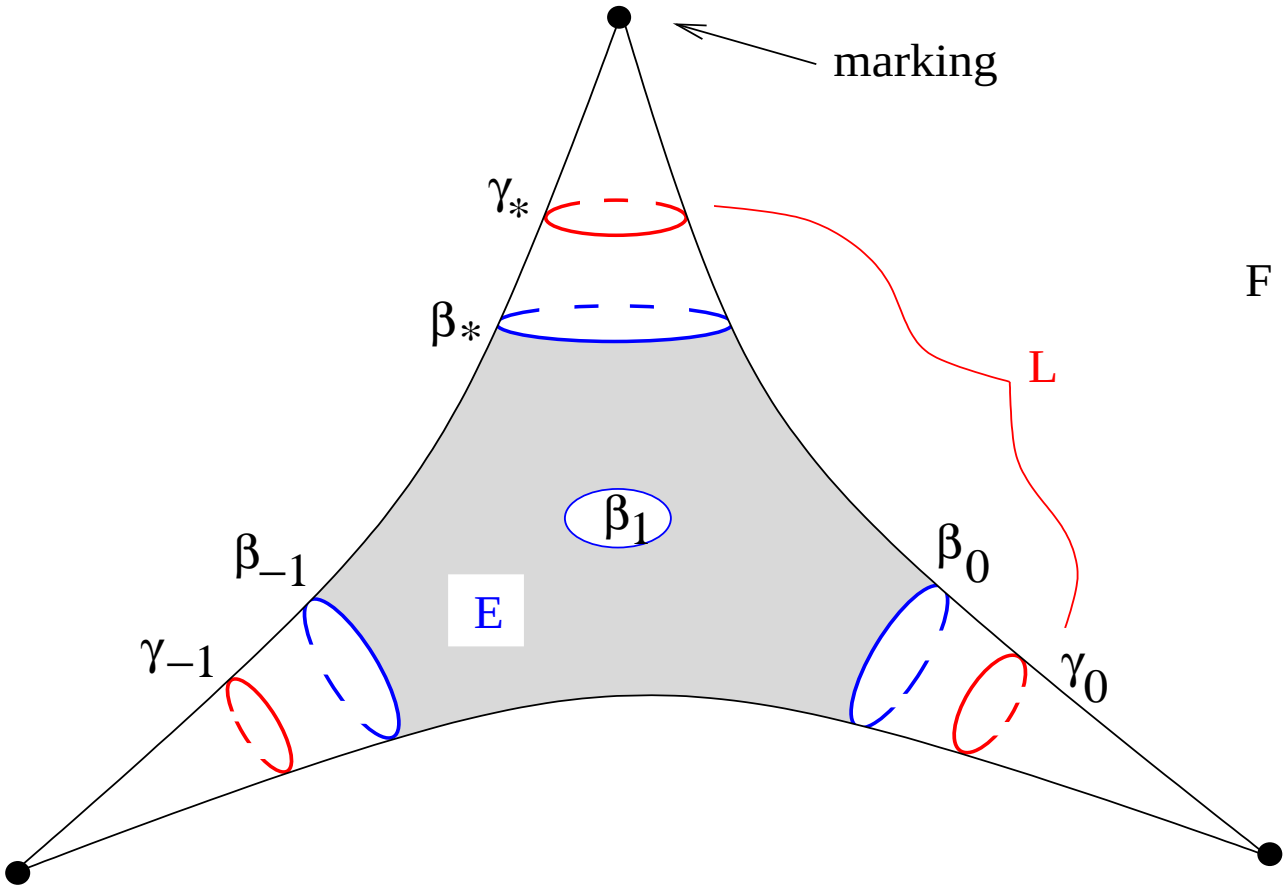


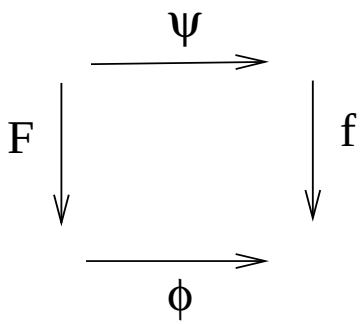
Proof. Mark one point on each corner. Extend F as a branched cover with





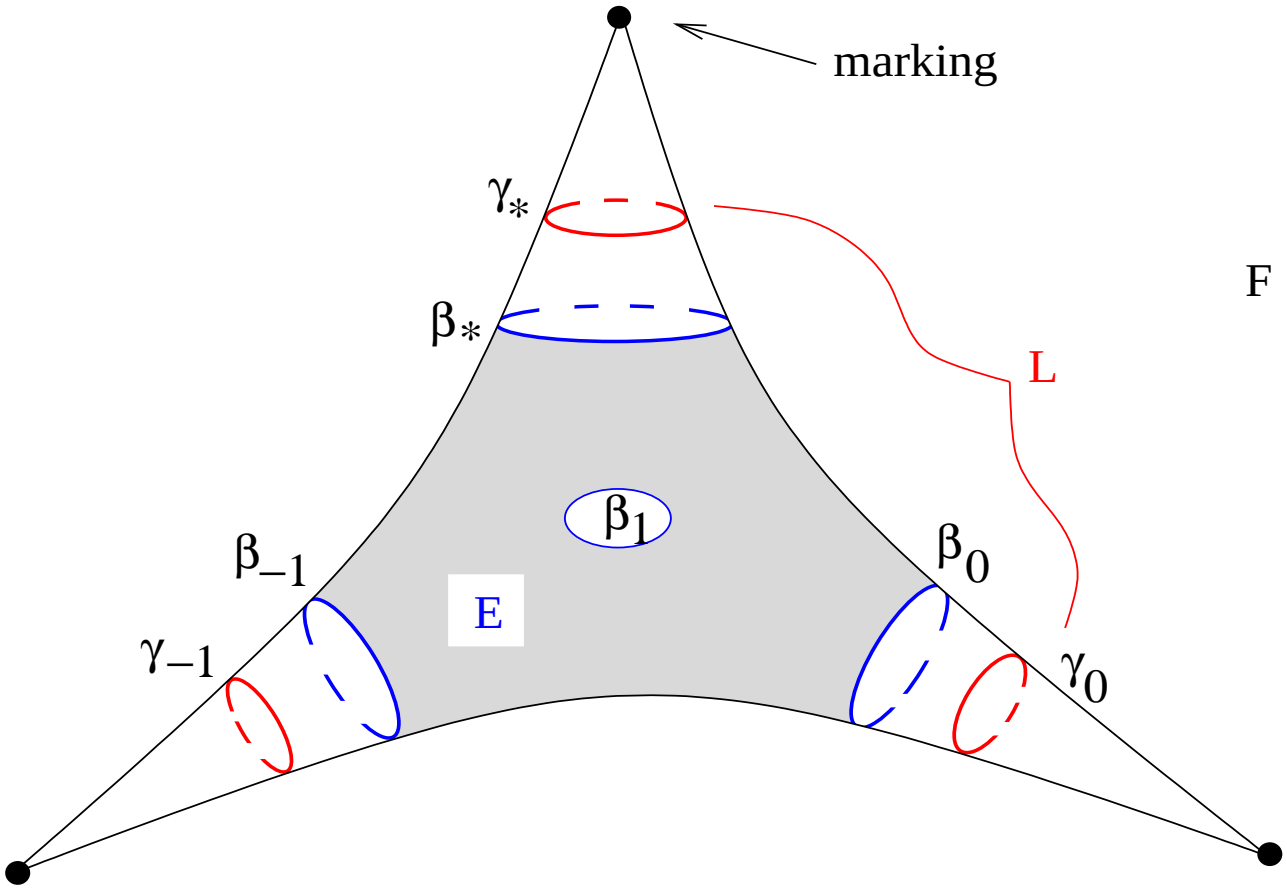
Proof. Mark one point on each corner. Extend F as a branched cover with $P_F \subset \mathcal{P}$, and F

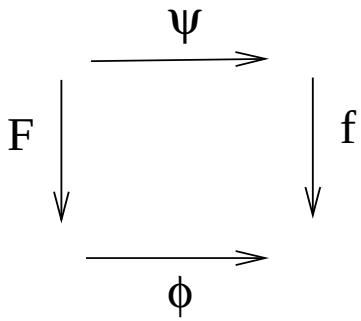




Proof. Mark one point on each corner. Extend F as a branched cover with $P_F \subset \mathcal{P}$, and F holomorphic outside L.

$\#\mathcal{P} = 3 \overset{\text{Thurston}}{\Longrightarrow}$ there are $\phi = id$, ψ homeomorphism, and f a rational map, such that $f \circ \psi = \phi \circ F$, and ψ isotopic to ϕ rel $\mathcal{P}$.

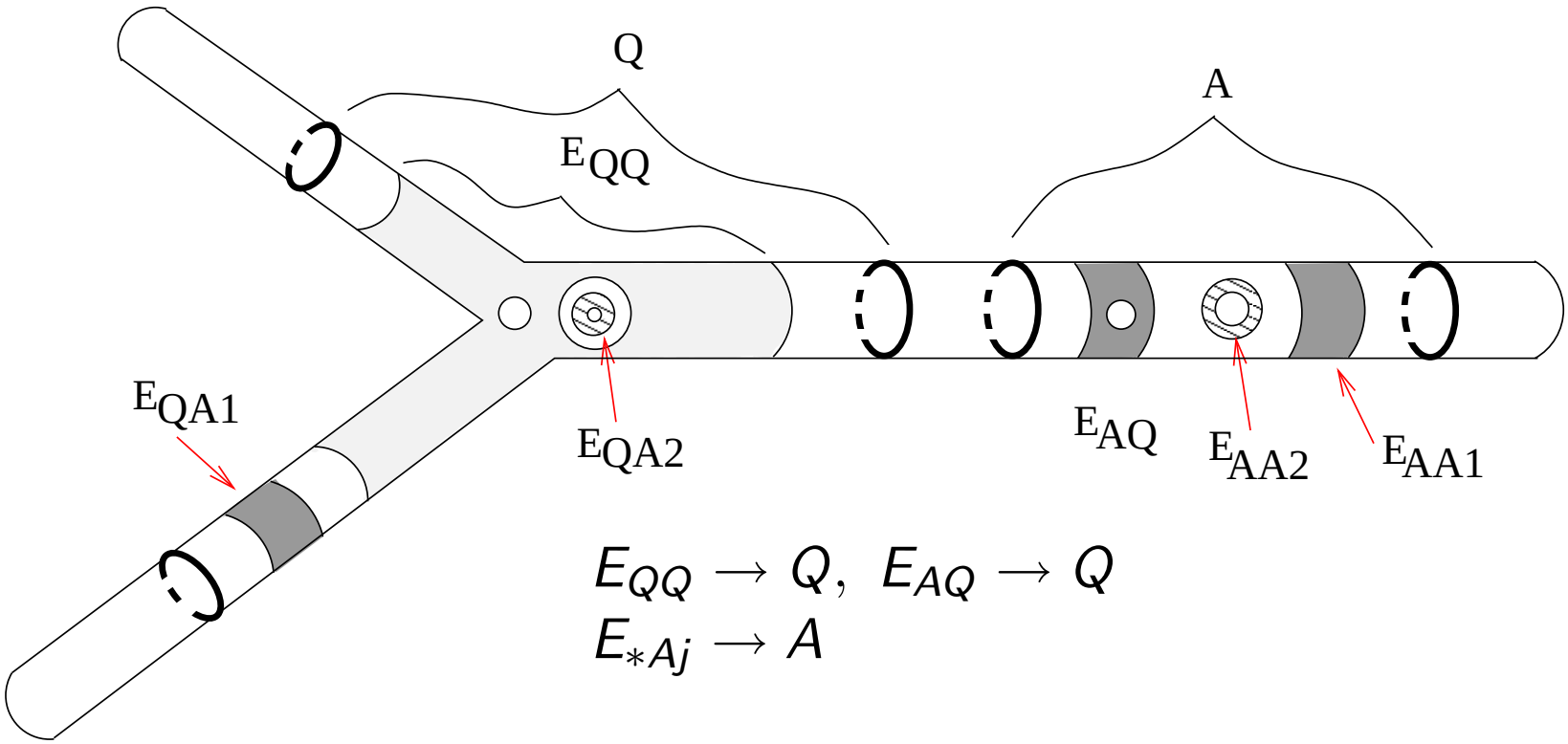




Proof. Mark one point on each corner. Extend F as a branched cover with $P_F \subset \mathcal{P}$, and F holomorphic outside L.

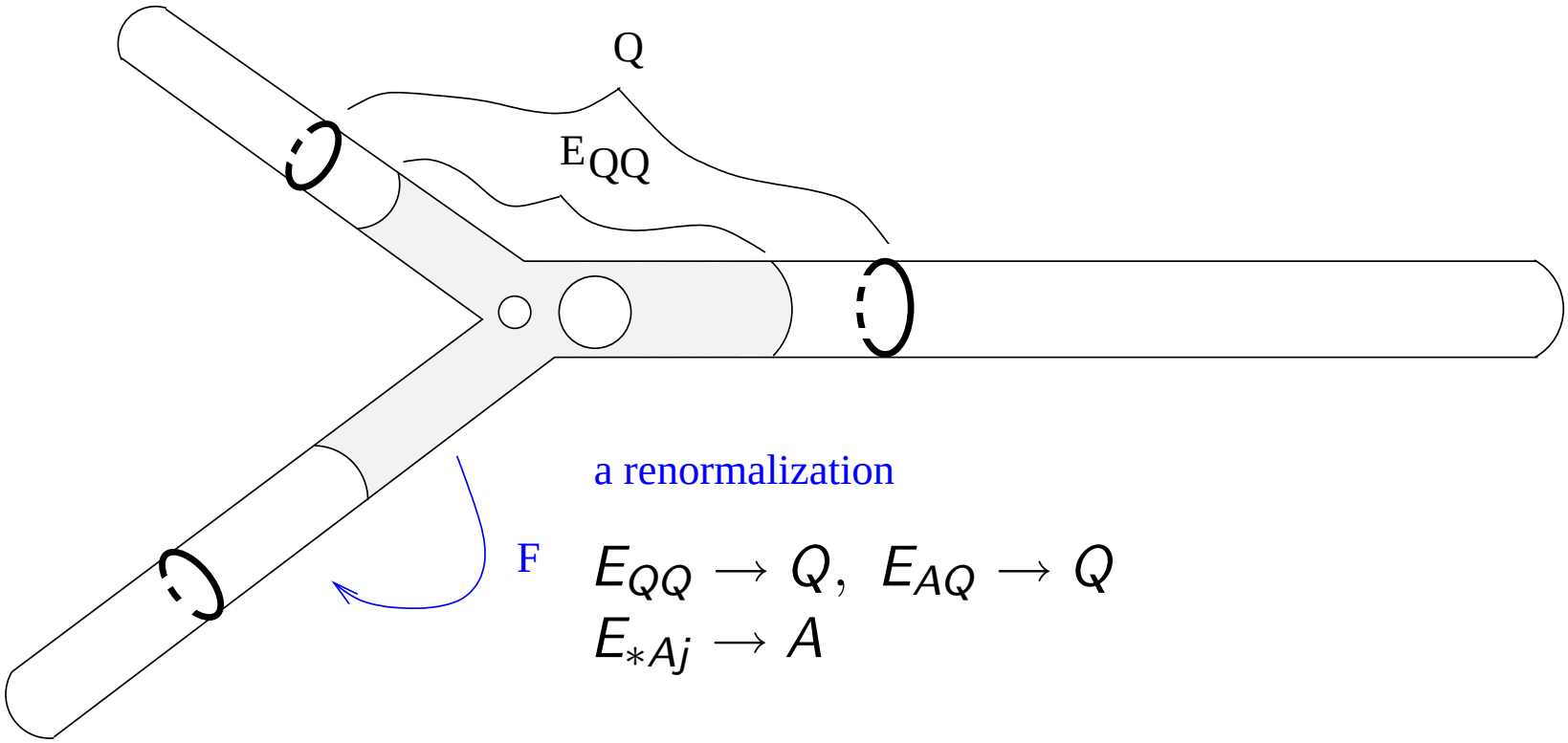
$\#\mathcal{P} = 3 \overset{\text{Thurston}}{\Longrightarrow}$ there are $\phi = id$, ψ homeomorphism, and f a rational map, such that $f \circ \psi = \phi \circ F$, and ψ isotopic to ϕ rel $\mathcal{P}$.

$\lambda < 1 \overset{\text{Shishikura}}{\Longrightarrow}$ ϕ, ψ can be modified so that $\phi = \psi$ outside L, and ψ isotopic to ϕ rel $\overline{\mathbb{C}} \setminus L$.



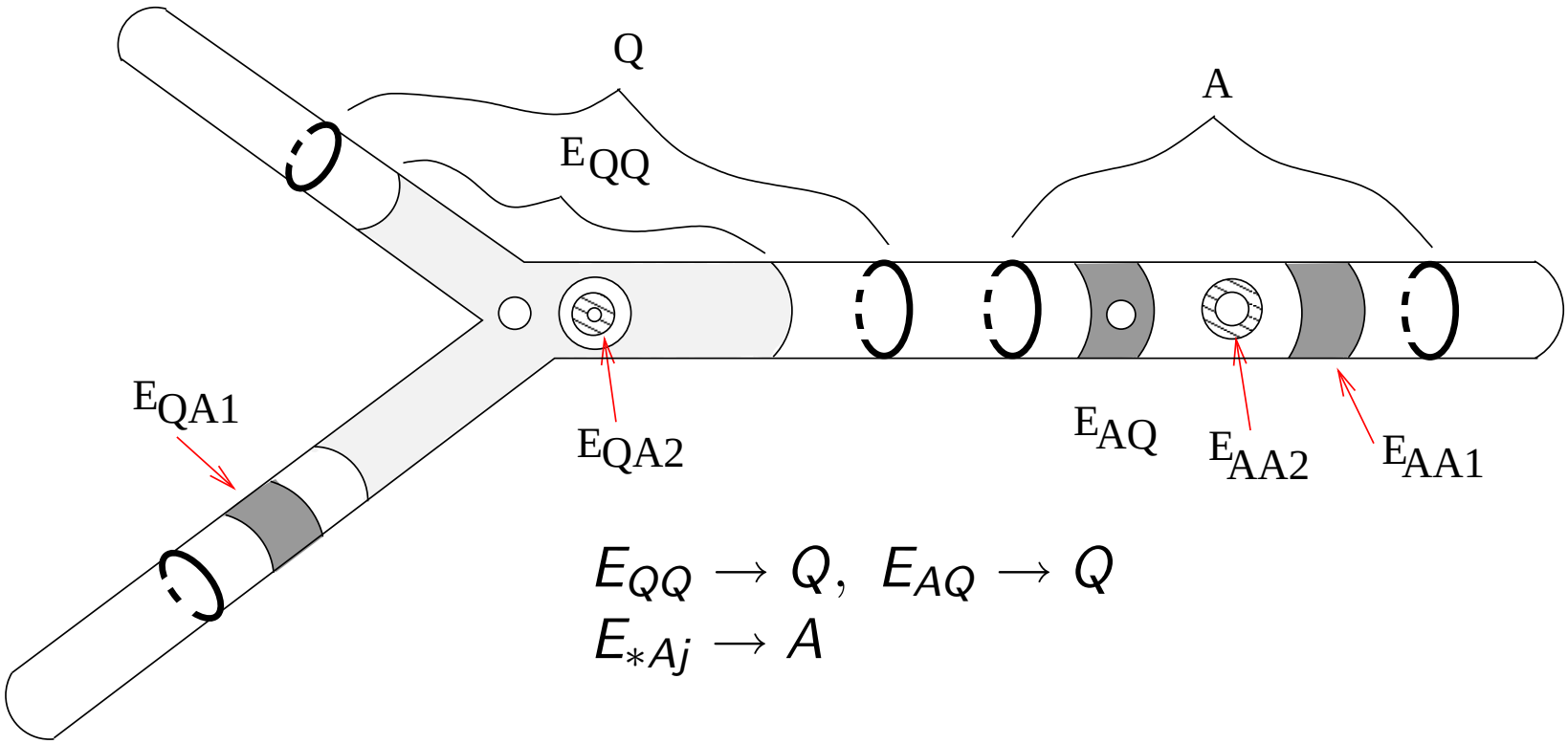
$E_{QQ} \to Q,\ E_{AQ} \to Q$
$E_{*Aj} \to A$

$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$



$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

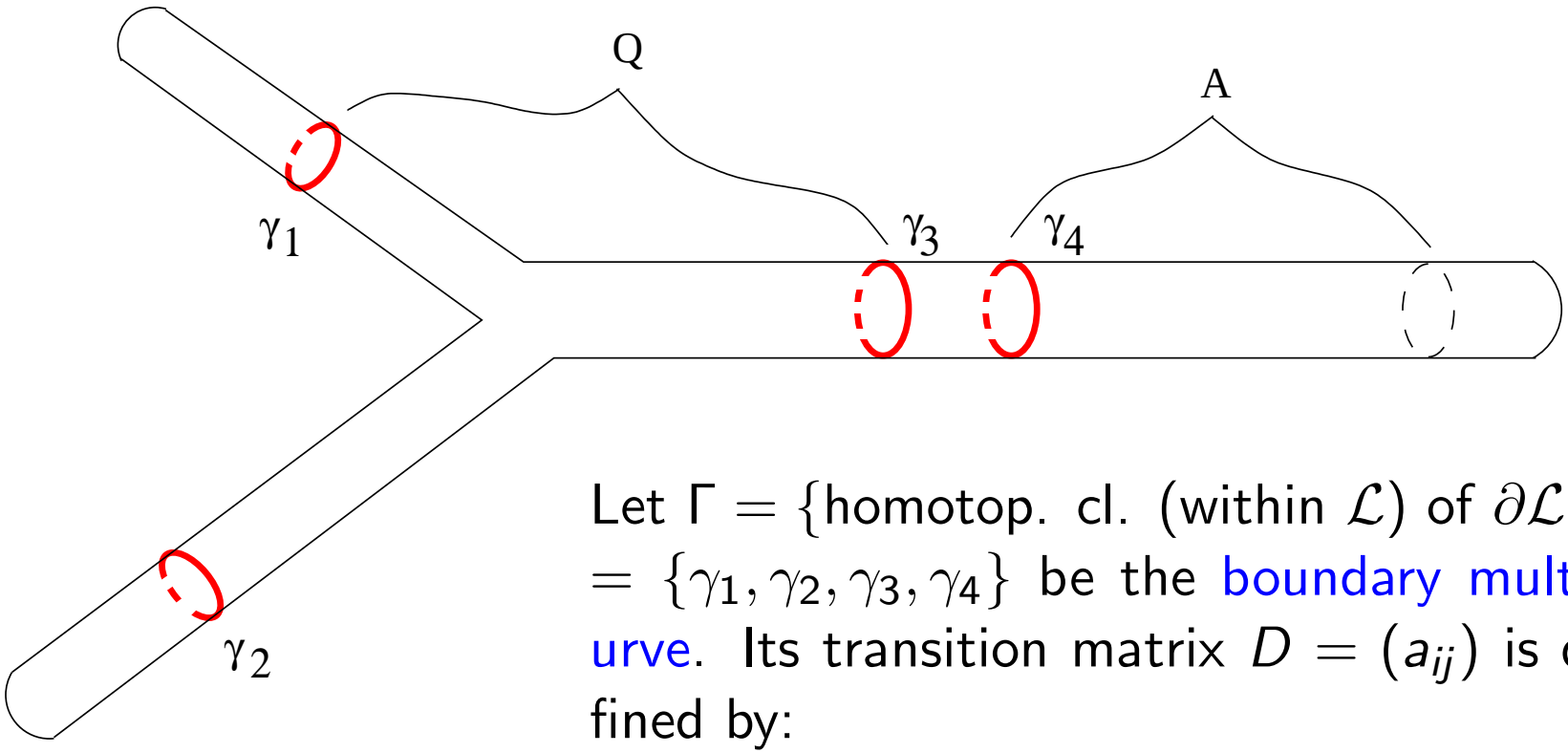
$\begin{cases} \bullet\ (F, E_{QQ}, Q) \text{ has no Thurston obstructions} \\ \quad ((F, E_{QQ}, Q) \text{ is called a renormalization of } F). \\ \bullet \end{cases}$



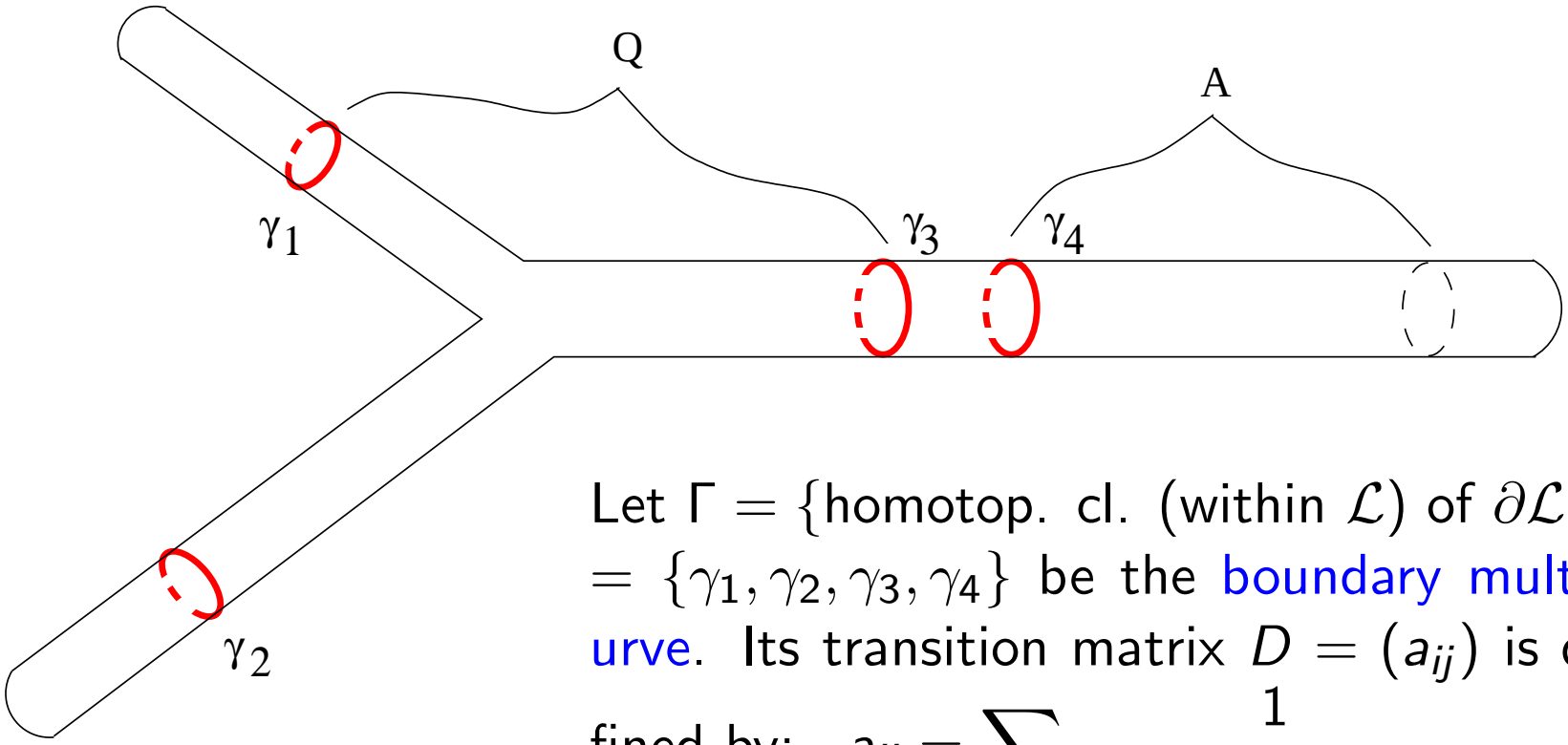
$E_{QQ} \to Q,\ E_{AQ} \to Q$
$E_{*Aj} \to A$

$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

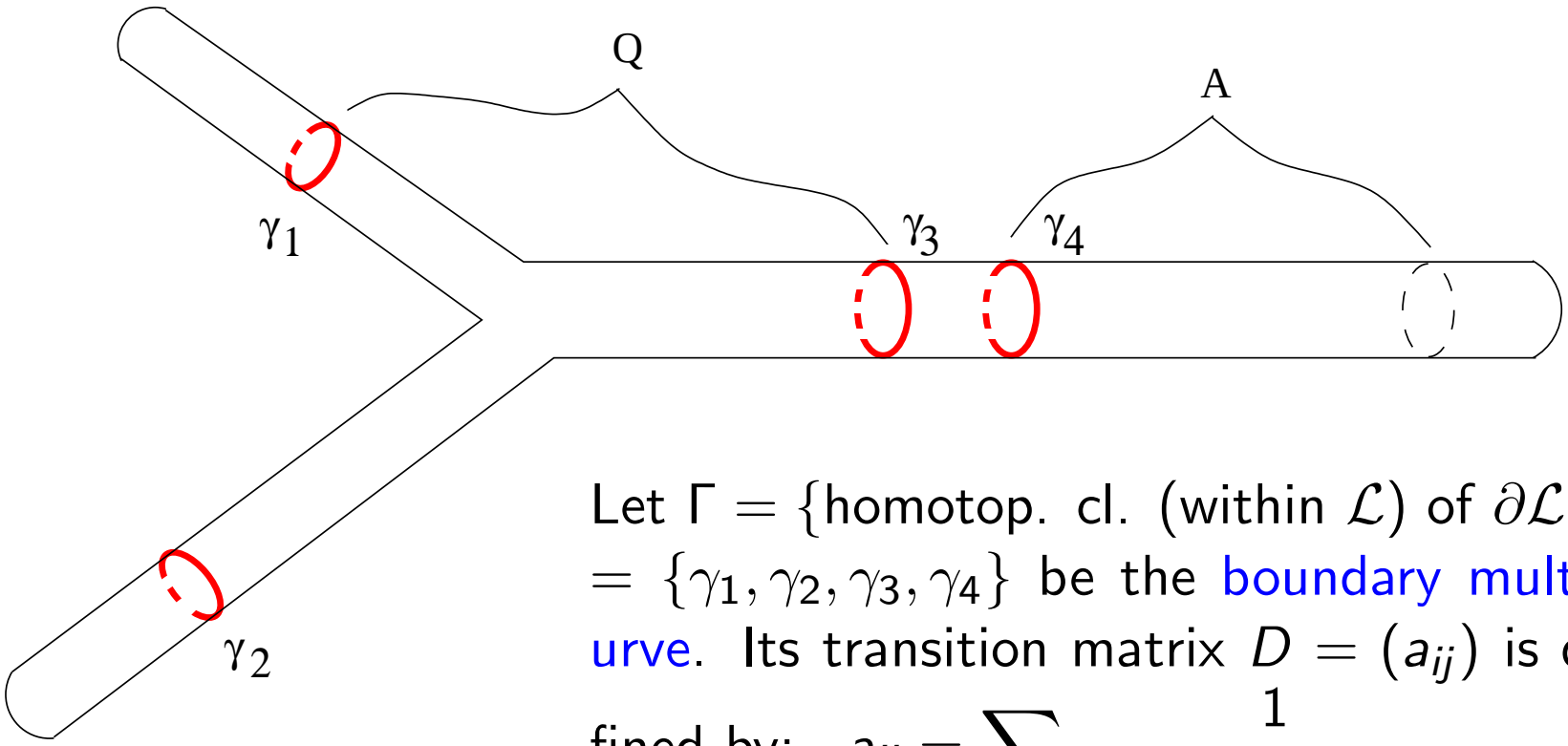
- (F, E_{QQ}, Q) has no Thurston obstructions
  ((F, E_{QQ}, Q) is called a renormalization of F).
- and F has no boundary obstructions.



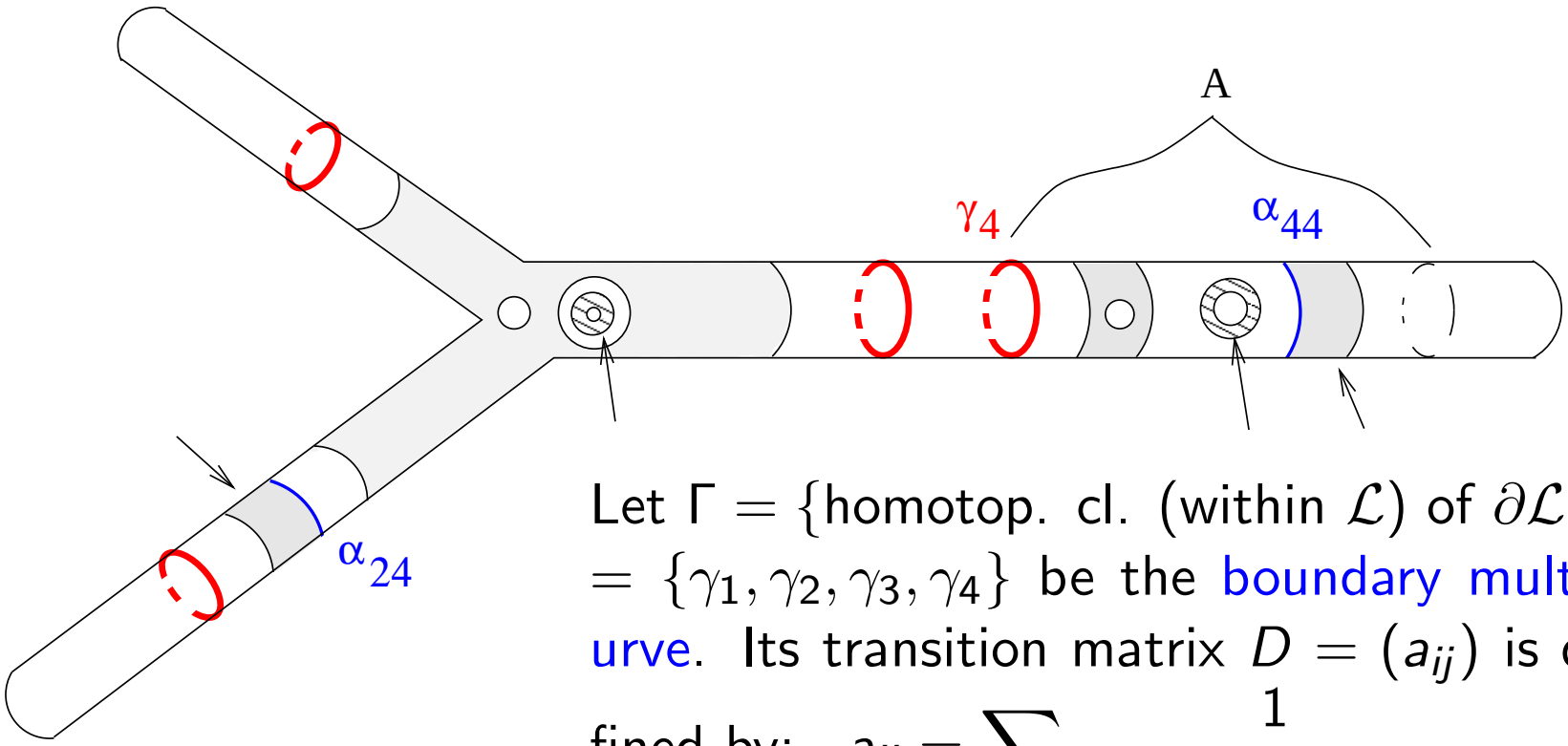
Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by:



Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by: $a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$,

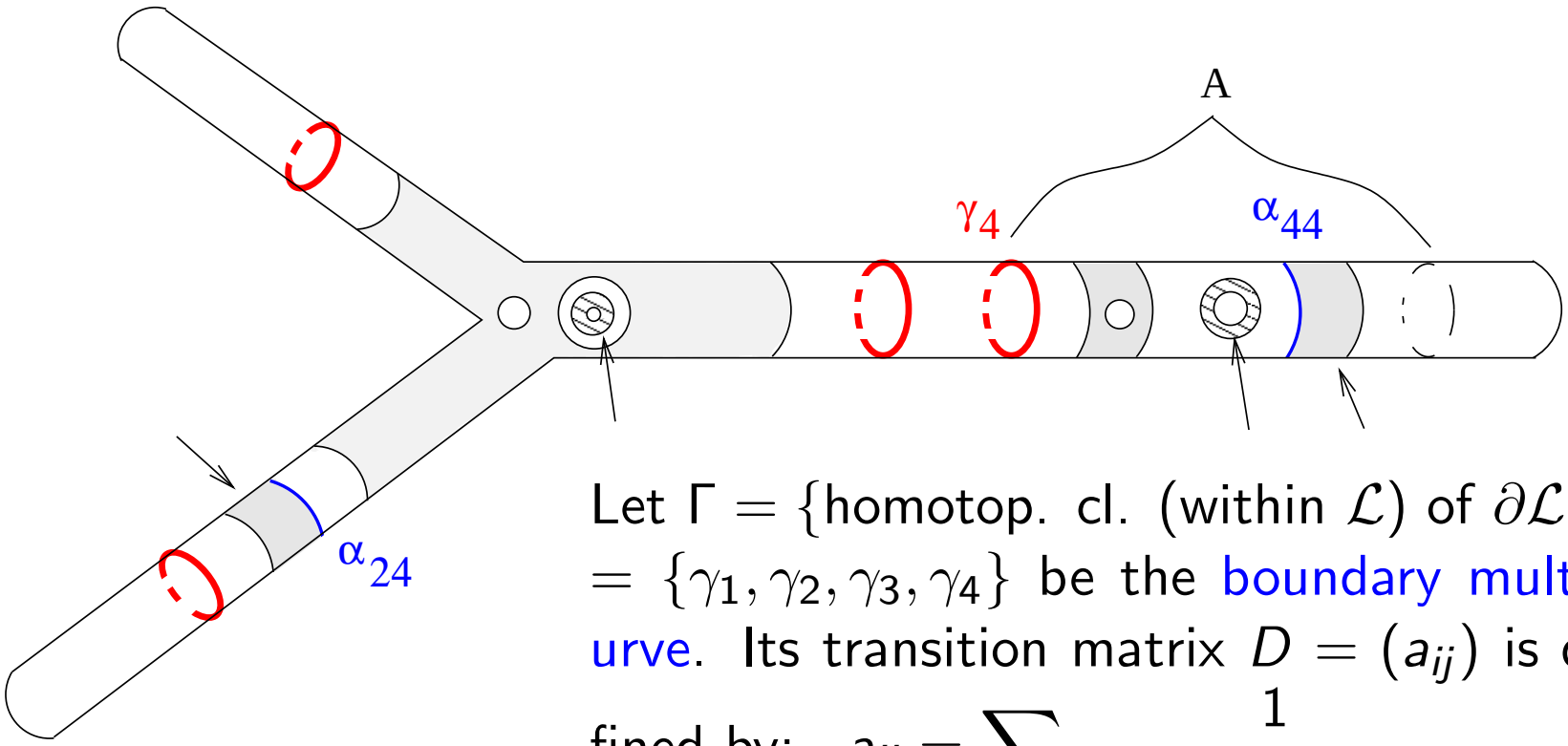


Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by: $a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.



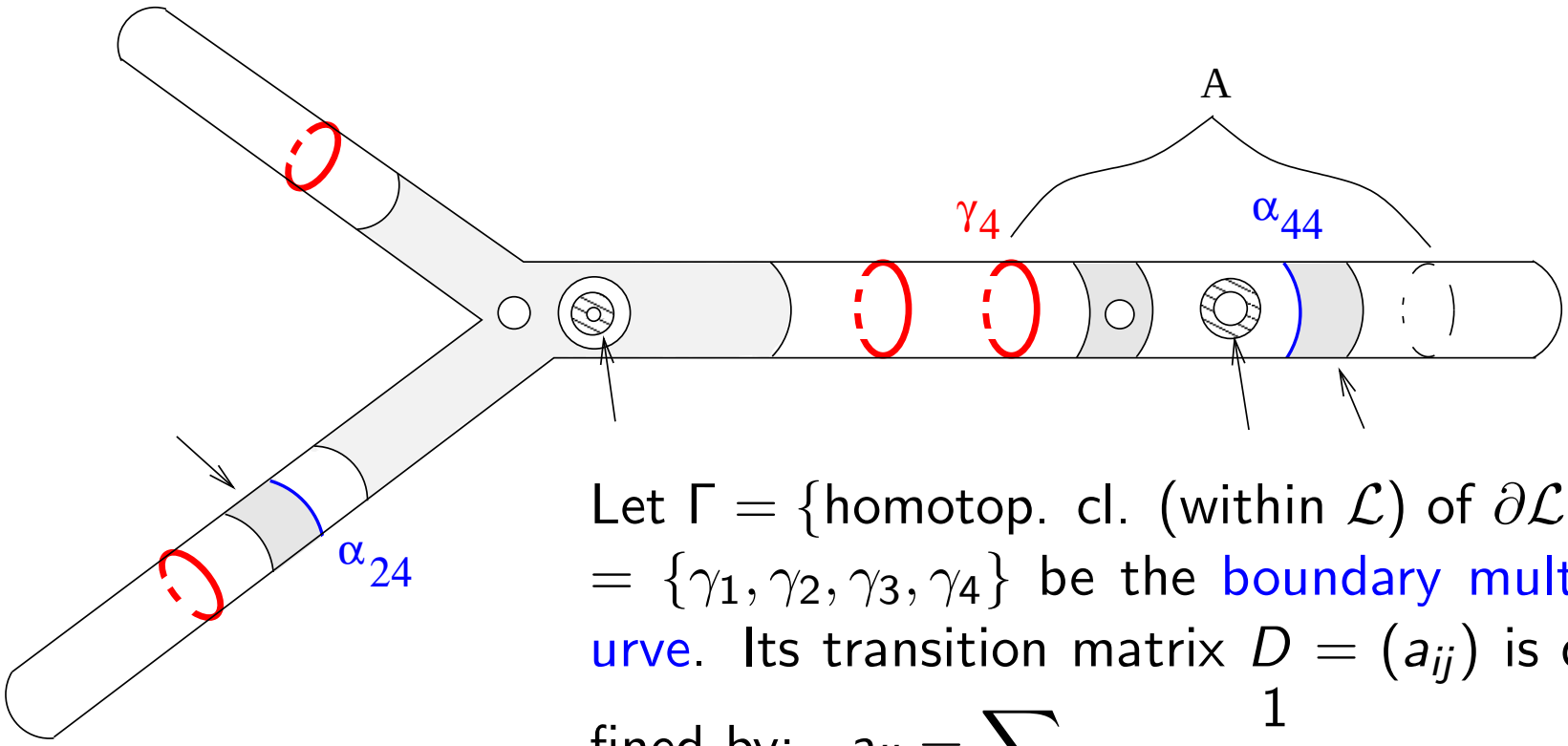
Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by: $a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.

For example: $a_{44} = \frac{1}{deg(F : \alpha_{44} \to \gamma_4)}$, $a_{24} = \frac{1}{deg(F : \alpha_{24} \to \gamma_4)}$.



Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by: $a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.

$\boxed{\lambda(D) < 1}$ corresponds to $\sum \frac{1}{d_i} < 1$, deals with moduli problems.



Let $\Gamma = \{\text{homotop. cl. (within } \mathcal{L}) \text{ of } \partial\mathcal{L}\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D = (a_{ij})$ is defined by: $a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.

$\boxed{\lambda(D) < 1}$ corresponds to $\sum \frac{1}{d_i} < 1$, deals with moduli problems.
F has a boundary obstruction if $\lambda(D) \geq 1$.

The proof of this example uses Thurston's theorem and Grötzsch inequalities.

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$ then $F \sim_{comb}$ a hol.-model.

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$ then $F \sim_{comb}$ a hol.-model.

(Combining this theorem with quasi-conformal surgery, we get a generalization of Thurston's theorem for hyperbolic maps.)

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$ then $F \sim_{comb}$ a hol.-model.

(Combining this theorem with quasi-conformal surgery, we get a generalization of Thurston's theorem for hyperbolic maps.)

What is constant complexity?

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$ then $F \sim_{comb}$ a hol.-model.

(Combining this theorem with quasi-conformal surgery, we get a generalization of Thurston's theorem for hyperbolic maps.)

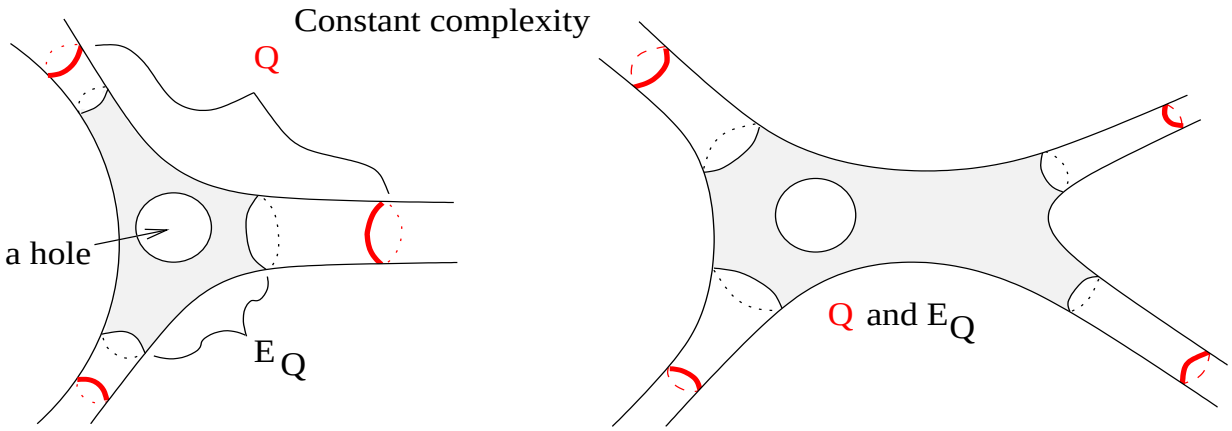
What is constant complexity?
renormalization?

The proof of this example uses Thurston's theorem and Grötzsch inequalities. More generally (with a similar proof):

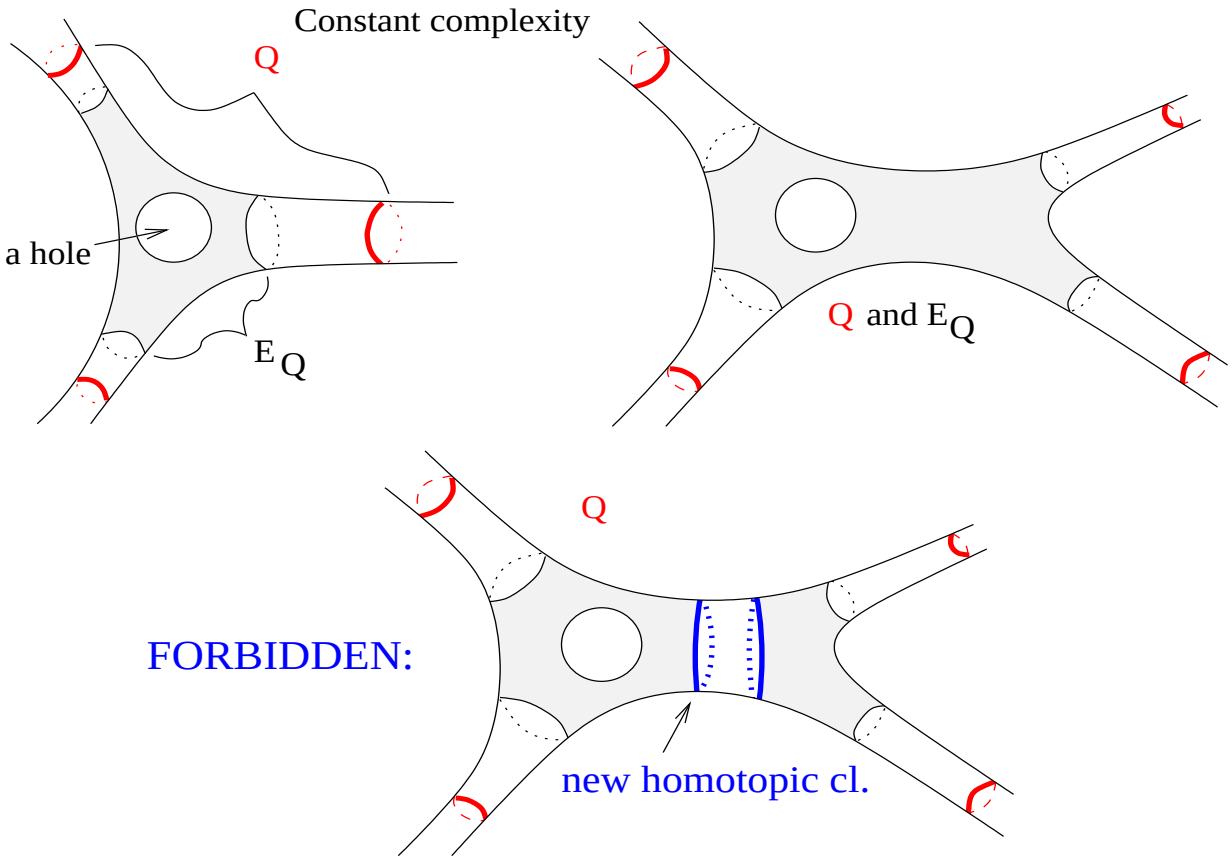
Theorem 1. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity (see below). If $\begin{cases} \text{every renormalization has no Thurston obstructions;} \\ \text{and } F \text{ has no boundary obstructions,} \end{cases}$ then $F \sim_{comb}$ a hol.-model.

(Combining this theorem with quasi-conformal surgery, we get a generalization of Thurston's theorem for hyperbolic maps.)

What is constant complexity?
renormalization?
Thurston obstructions? Boundary obstructions?



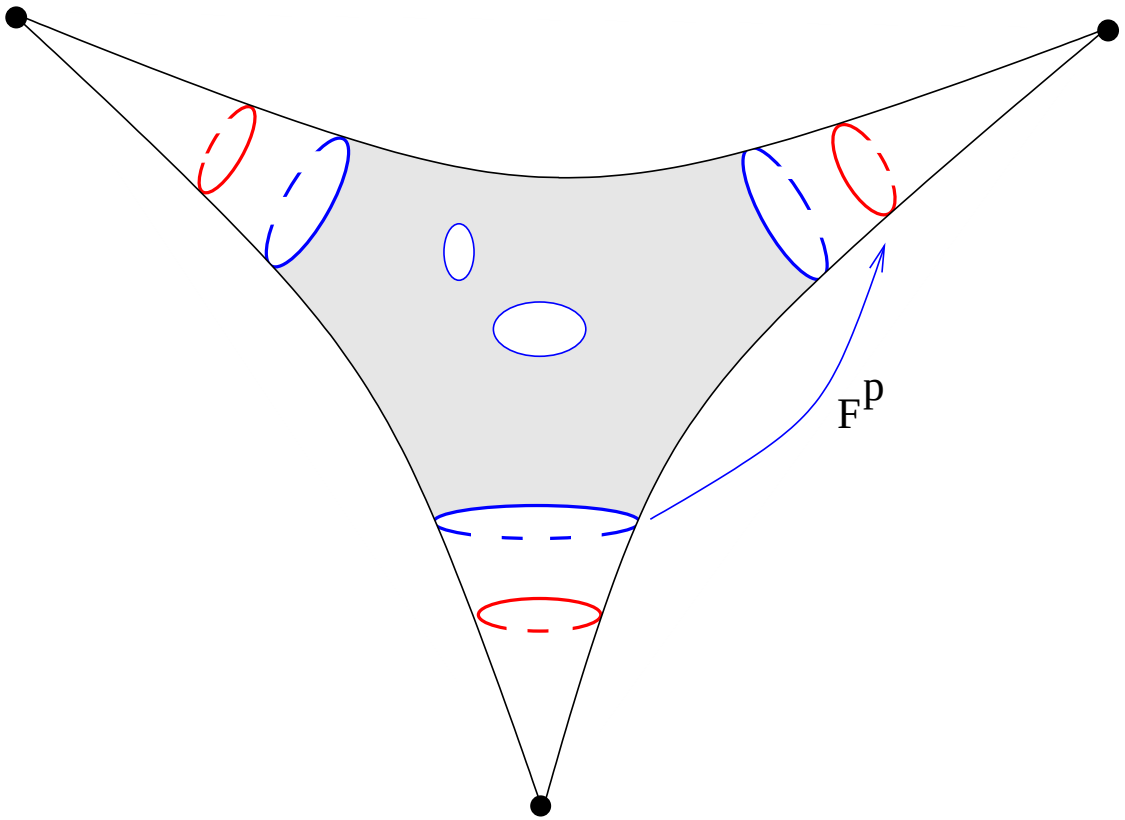
A repellor $(F, \mathcal{E}, \mathcal{L})$ is of constant complexity, if,
$\forall$ non-(disc-annular) $\mathcal{L}$-piece Q,
$\exists !$ $\mathcal{E}$-piece $E_Q \subset Q$ s.t each curve in ∂Q is homotopic (within Q) to a curve in ∂E_Q.



A repellor $(F, \mathcal{E}, \mathcal{L})$ is of constant complexity, if,
$\forall$ non-(disc-annular) $\mathcal{L}$-piece Q,
$\exists !$ $\mathcal{E}$-piece $E_Q \subset Q$ s.t each curve in ∂Q is homotopic (within Q) to a curve in ∂E_Q.

This $F : \bigcup E_Q \to \bigcup_{\#\partial Q \geq 3} Q$ induces $F_* : \{Q, \#\partial Q \geq 3\} \circlearrowleft$. A F_*-periodic Q induces a renormalization (F^p, E_Q^p, Q).

Extension. Let $\mathcal{P} = \{\text{one point in each component of } \overline{\mathbb{C}} \setminus Q\}$. Let h be a postcritically finite branched cover extension of (F^p, E_Q^p, Q) with $\mathcal{P}_h \subset \mathcal{P}$ and $h(\mathcal{P}) \subset \mathcal{P}$.



This $F : \bigcup E_Q \to \bigcup_{\#\partial Q \geq 3} Q$ induces $F_* : \{Q, \#\partial Q \geq 3\} \circlearrowleft$. A F_*-periodic Q induces a renormalization (F^p, E_Q^p, Q).

Extension. Let $\mathcal{P} = \{\text{one point in each component of } \overline{\mathbb{C}} \setminus Q\}$. Let h be a postcritically finite branched cover extension of (F^p, E_Q^p, Q) with $\mathcal{P}_h \subset \mathcal{P}$ and $h(\mathcal{P}) \subset \mathcal{P}$.

A Jordan curve γ is said non-peripheral if
$\gamma \cap \mathcal{P} = \emptyset$ *and each component of* $\overline{\mathbb{C}} \setminus \gamma$ *contains* ≥ 2 *points of* $\mathcal{P}$.

This $F : \bigcup E_Q \to \bigcup_{\#\partial Q \geq 3} Q$ induces $F_* : \{Q, \#\partial Q \geq 3\} \circlearrowleft$. A F_*-periodic Q induces a renormalization (F^p, E_Q^p, Q).

Extension. Let $\mathcal{P} = \{\text{one point in each component of } \overline{\mathbb{C}} \setminus Q\}$. Let h be a postcritically finite branched cover extension of (F^p, E_Q^p, Q) with $\mathcal{P}_h \subset \mathcal{P}$ and $h(\mathcal{P}) \subset \mathcal{P}$.

A Jordan curve γ is said non-peripheral if
$\gamma \cap \mathcal{P} = \emptyset$ *and each component of* $\overline{\mathbb{C}} \setminus \gamma$ *contains* ≥ 2 *points of* $\mathcal{P}$.

$\Gamma = \{\gamma_1, \cdots, \gamma_k\}$ is said a multicurve, if the γ_i's are *non-peripheral, mutually disjoint and mutually non-homotopic*.

This $F : \bigcup E_Q \to \bigcup_{\#\partial Q \geq 3} Q$ induces $F_* : \{Q, \#\partial Q \geq 3\} \circlearrowleft$. A F_*-periodic Q induces a renormalization (F^p, E_Q^p, Q).

Extension. Let $\mathcal{P} = \{\text{one point in each component of } \overline{\mathbb{C}} \setminus Q\}$. Let h be a postcritically finite branched cover extension of (F^p, E_Q^p, Q) with $\mathcal{P}_h \subset \mathcal{P}$ and $h(\mathcal{P}) \subset \mathcal{P}$.

A Jordan curve γ is said non-peripheral if
$\gamma \cap \mathcal{P} = \emptyset$ *and each component of* $\overline{\mathbb{C}} \setminus \gamma$ *contains* ≥ 2 *points of* $\mathcal{P}$.

$\Gamma = \{\gamma_1, \cdots, \gamma_k\}$ is said a multicurve, if the γ_i's are *non-peripheral, mutually disjoint and mutually non-homotopic*.

Γ induces a transition matrix $D_\Gamma = (b_{ij})$ with
$b_{ij} = \sum_{\delta} \frac{1}{deg(h : \delta \to \gamma_j)}$, where δ runs through the curves in $h^{-1}(\gamma_j)$ homotopic (rel $\mathcal{P}$) to γ_i.

This $F : \bigcup E_Q \to \bigcup_{\#\partial Q \geq 3} Q$ induces $F_* : \{Q, \#\partial Q \geq 3\} \circlearrowleft$. A F_*-periodic Q induces a renormalization (F^p, E_Q^p, Q).

Extension. Let $\mathcal{P} = \{\text{one point in each component of } \overline{\mathbb{C}} \setminus Q\}$. Let h be a postcritically finite branched cover extension of (F^p, E_Q^p, Q) with $\mathcal{P}_h \subset \mathcal{P}$ and $h(\mathcal{P}) \subset \mathcal{P}$.

A Jordan curve γ is said non-peripheral if
$\gamma \cap \mathcal{P} = \emptyset$ *and each component of* $\overline{\mathbb{C}} \setminus \gamma$ *contains* ≥ 2 *points of* $\mathcal{P}$.

$\Gamma = \{\gamma_1, \cdots, \gamma_k\}$ is said a multicurve, if the γ_i's are *non-peripheral, mutually disjoint and mutually non-homotopic*.

Γ induces a transition matrix $D_\Gamma = (b_{ij})$ with
$b_{ij} = \sum_{\delta} \frac{1}{deg(h : \delta \to \gamma_j)}$, where δ runs through the curves in $h^{-1}(\gamma_j)$ homotopic (rel $\mathcal{P}$) to γ_i.

(F^p, E_Q^p, Q) has *a Thurston obstruction* if $\exists \Gamma$ such that $\lambda(D_\Gamma) \geq 1$.

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

$Y =$

$\{$curves is $\partial\mathcal{L}$, non null-homotopic within $\mathcal{L}\}$/homotopy within $\mathcal{L}$.

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

$Y =$

$\{$curves is $\partial\mathcal{L}$, non null-homotopic within $\mathcal{L}\}$/homotopy within $\mathcal{L}$.

The corresponding transition matrix $D = (a_{ij})$ is defined by:

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

$Y =$

$\{$curves is $\partial\mathcal{L}$, non null-homotopic within $\mathcal{L}\}$/homotopy within $\mathcal{L}$.

The corresponding transition matrix $D = (a_{ij})$ is defined by:

$$a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)},$$

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

$Y =$

$\{$curves is $\partial\mathcal{L}$, non null-homotopic within $\mathcal{L}\}$/homotopy within $\mathcal{L}$.

The corresponding transition matrix $D = (a_{ij})$ is defined by:

$a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.

Boundary multicurve and boundary obstructions

Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor of constant complexity. Its boundary multicurve is:

$Y =$

$\{$curves is $\partial\mathcal{L}$, non null-homotopic within $\mathcal{L}\}$/homotopy within $\mathcal{L}$.

The corresponding transition matrix $D = (a_{ij})$ is defined by:

$a_{ij} = \sum_{\alpha} \frac{1}{\deg(F : \alpha \to \gamma_j)}$, with α running through the curves in $F^{-1}(\gamma_j)$ homotopic (within $\mathcal{L} = Q \sqcup A$) to γ_i.

We say that F has a boundary obstruction if $\lambda(D) \geq 1$.

Thurston-like theorems

Thurston-like theorems

For renormalizations. If a renormalization (F^p, E^p_Q, Q)

$$\begin{cases} \text{has no boundary obstructions, and} \\ \text{has no Thurston obstructions,} \end{cases}$$

then it is $\sim_{comb}$ a hol.-model (with any prescribed equipotentials).

(Proof: Thurston+Shishikura.)

Thurston-like theorems

For renormalizations. If a renormalization (F^p, E_Q^p, Q)

$$\begin{cases} \text{has no boundary obstructions, and} \\ \text{has no Thurston obstructions,} \end{cases}$$

then it is $\sim_{comb}$ a hol.-model (with any prescribed equipotentials).

(Proof: Thurston+Shishikura.)

For a periodic cycle of renormalization. If $(F, \bigcup_{i=1}^p E_i, \bigcup_{i=1}^p Q_i)$ is a periodic cycle of renormalization satisfying

$$\begin{cases} F \text{ has no boundary obstructions, and} \\ \text{has no Thurston obstructions,} \end{cases}$$

then it is $\sim_{comb}$ a hol.-model (with any prescribed equipotentials).

Thurston-like theorems

For renormalizations. If a renormalization (F^p, E_Q^p, Q)

$$\begin{cases} \text{has no boundary obstructions, and} \\ \text{has no Thurston obstructions,} \end{cases}$$

then it is $\sim_{comb}$ a hol.-model (with any prescribed equipotentials).
(Proof: Thurston+Shishikura.)

For a periodic cycle of renormalization. If $(F, \bigcup_{i=1}^p E_i, \bigcup_{i=1}^p Q_i)$ is a periodic cycle of renormalization satisfying

$$\begin{cases} F \text{ has no boundary obstructions, and} \\ \text{has no Thurston obstructions,} \end{cases}$$

then it is $\sim_{comb}$ a hol.-model (with any prescribed equipotentials).

This, + Grötzsch '$\leq$', proves:
For repellors. Let $(F, \mathcal{E}, \mathcal{L})$ be a repellor, of constant complexity. If

$$\begin{cases} \text{it has no boundary obstructions, and} \\ \text{every renormalization has no Thurston obstructions,} \end{cases}$$

then $F \sim_{comb}$ a hol.-model.

+ Measurable Riemann Mapping theorem:

For hyperbolic maps. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic on $U(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a hyperbolic rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic. Moreover R is unique.

+ Measurable Riemann Mapping theorem:

For hyperbolic maps. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic on $U(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a hyperbolic rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic. Moreover R is unique.

+ distortion controls, plumbing and pinching surgeries:

For geometrically finite maps Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a geometrically type branched coverings, holomorphic on $U^*(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a unique geometrically finite rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic.

+ Measurable Riemann Mapping theorem:

For hyperbolic maps. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic on $U(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a hyperbolic rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic. Moreover R is unique.

+ distortion controls, plumbing and pinching surgeries:

For geometrically finite maps Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a geometrically type branched coverings, holomorphic on $U^*(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a unique geometrically finite rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic.

with the same price:

Application for parabolic perturbations A parabolic map is stably accessible from hyperbolic components with any given stable combinatorics.

+ Measurable Riemann Mapping theorem:

For hyperbolic maps. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic on $U(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a hyperbolic rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic. Moreover R is unique.

+ distortion controls, plumbing and pinching surgeries:

For geometrically finite maps Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a geometrically type branched coverings, holomorphic on $U^*(acc(\mathcal{P}_G))$, without Thurston obstructions. Then G is comb-equivalent to a unique geometrically finite rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_{\phi-\text{conj}} R$, with $\theta \sim id$ rel $\overline{U} \cup \mathcal{P}_G$, and $\phi|_U$ holomorphic.

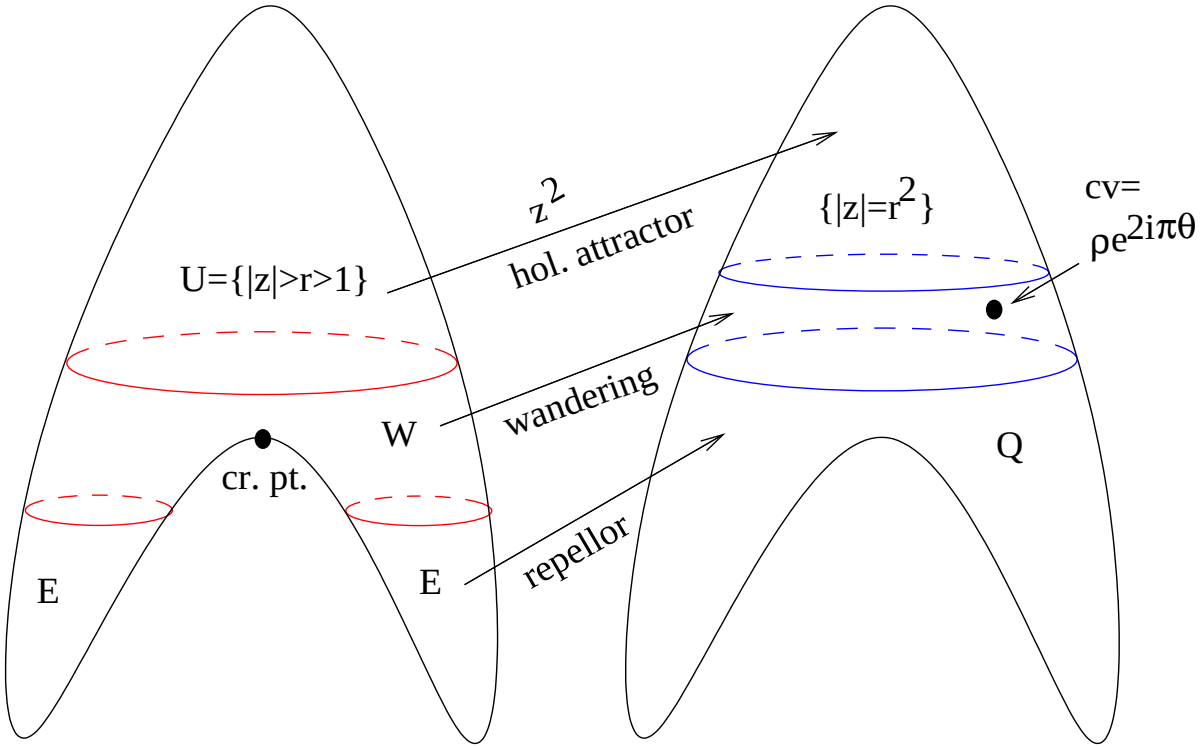
with the same price:

Application for parabolic perturbations A parabolic map is stably accessible from hyperbolic components with any given stable combinatorics.

Pinching paths converge to cusp boundary points.

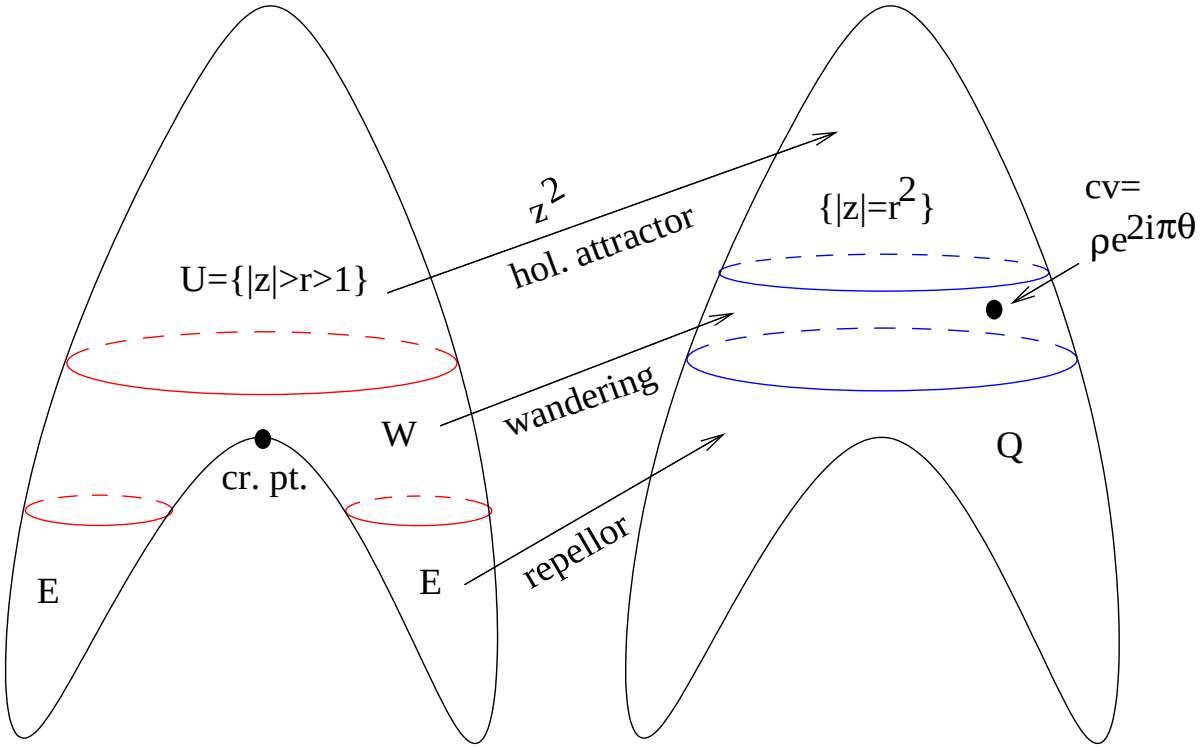
An example: $G \sim_{comb} z^2 + c$ with $\ddot{\text{B}}\text{ott}(c) = \rho e^{2\pi i\theta}$

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



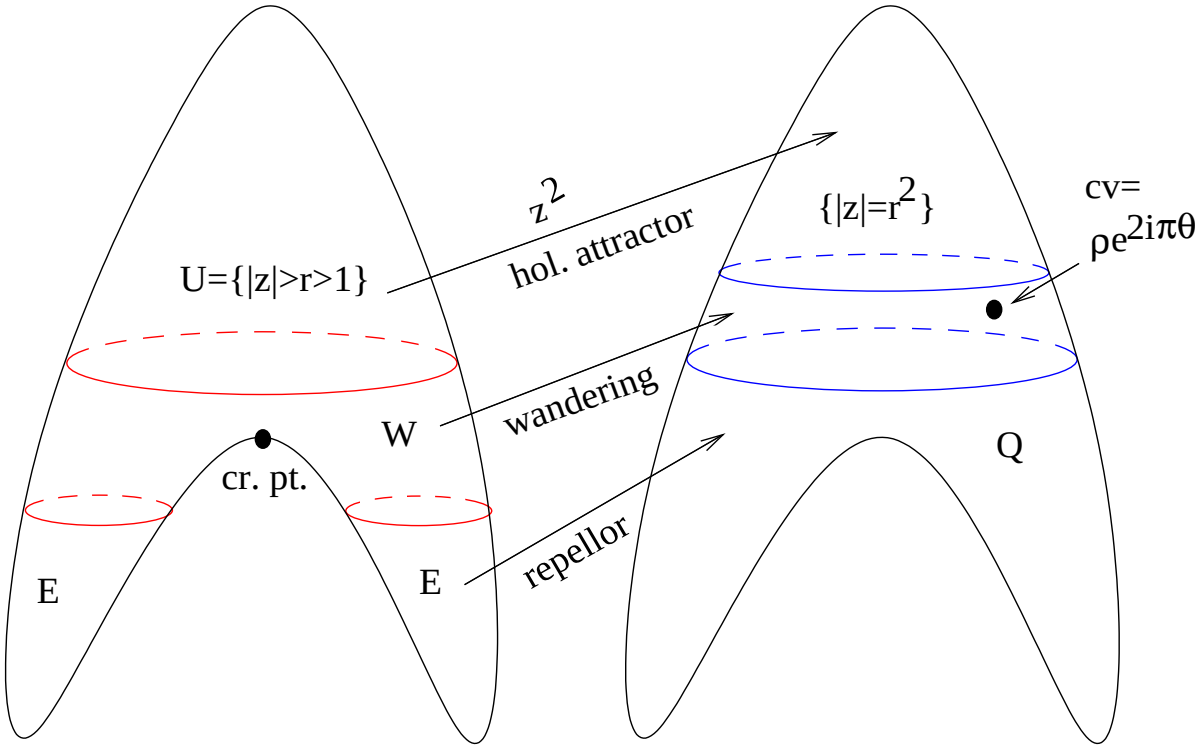
$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover,

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



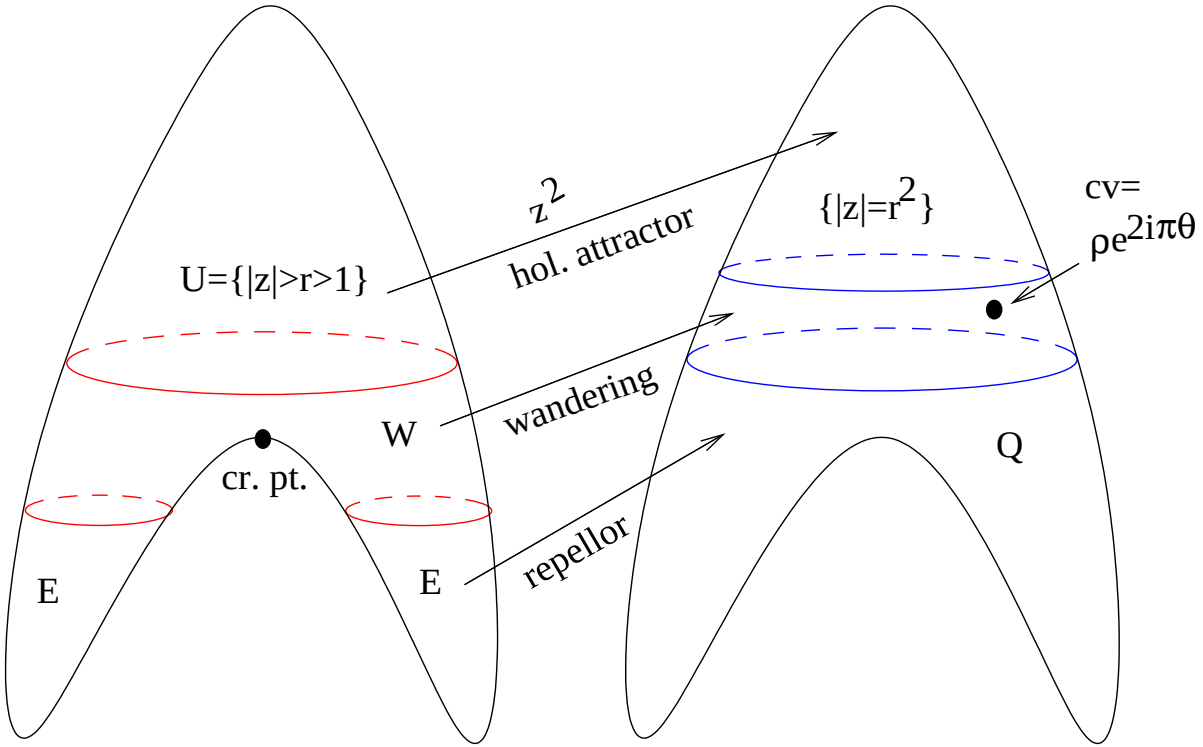
$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor,

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



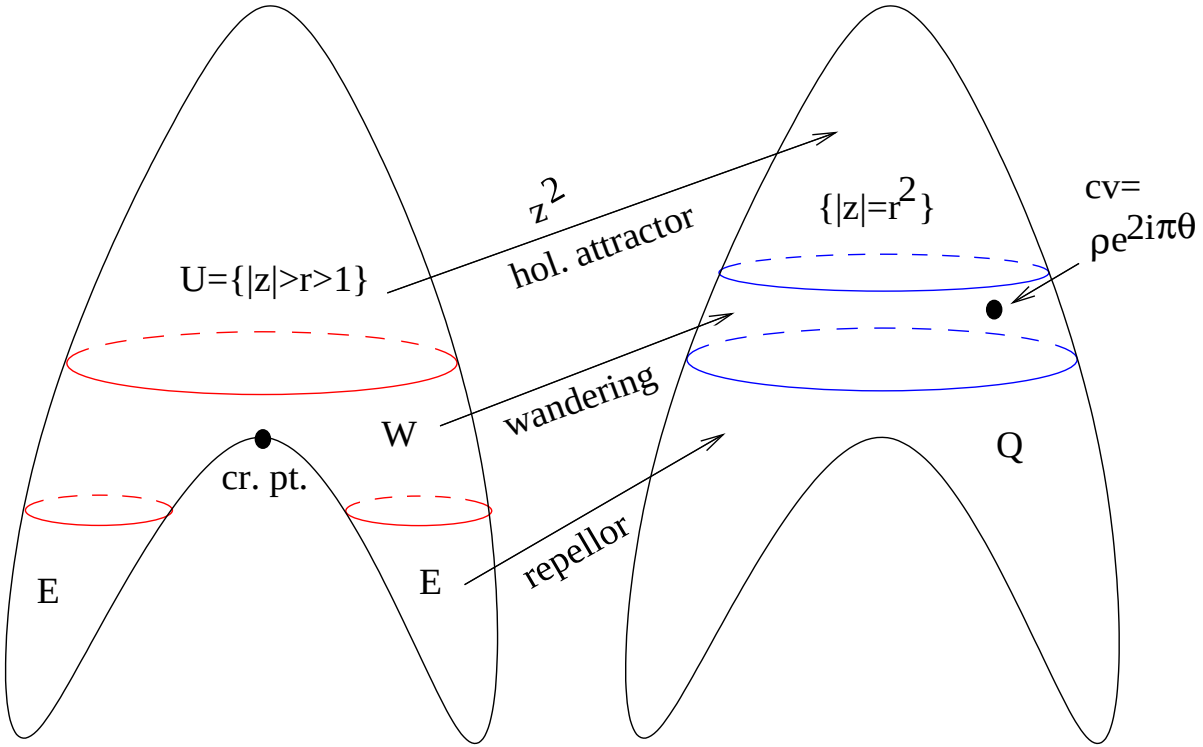
$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor,

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



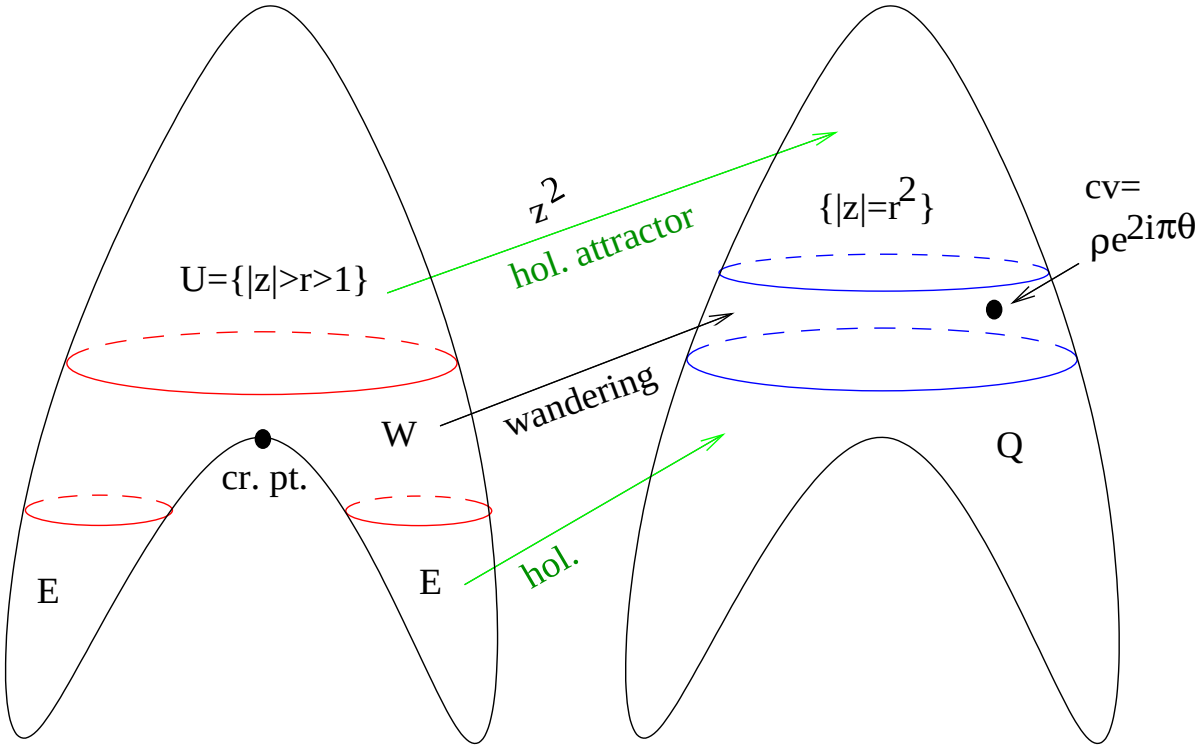
$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor, $G(W) \subset U$,

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor, $G(W) \subset U$, and $cv(G) = \rho e^{2\pi i\theta}$.

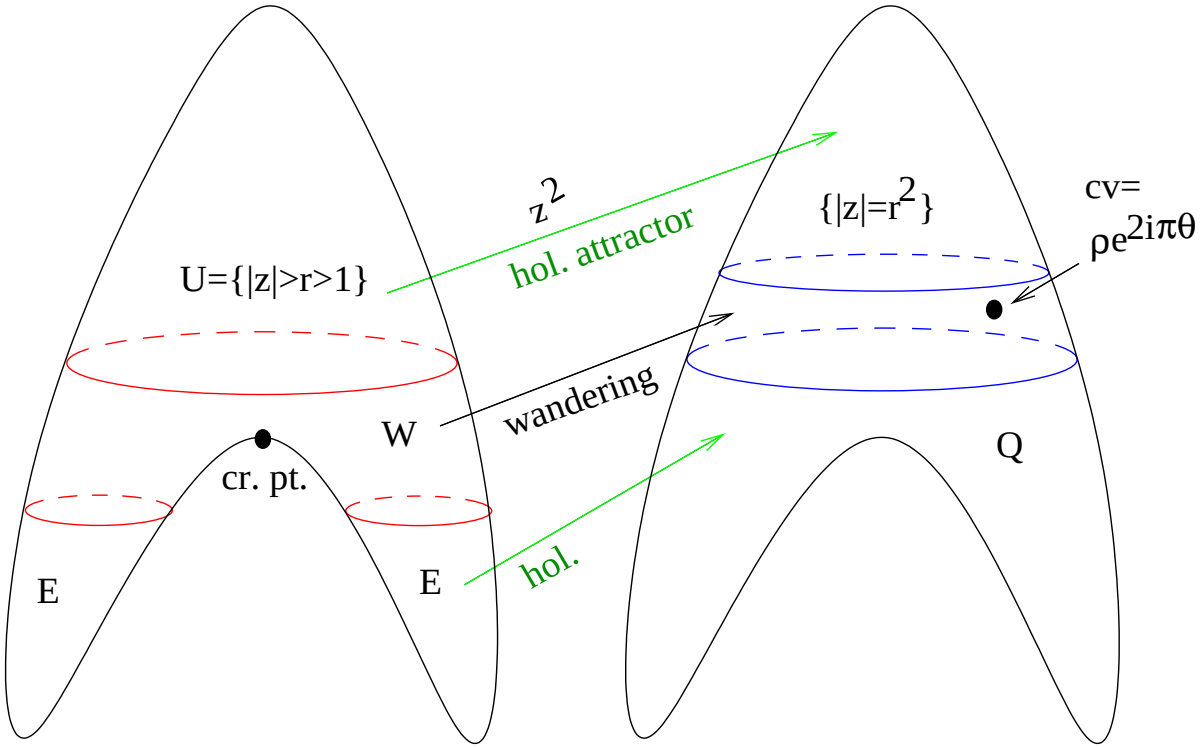
An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor, $G(W) \subset U$, and $cv(G) = \rho e^{2\pi i\theta}$.

1. Replace G by $G \circ \theta^{-1}$ with $\theta(Q) = Q$ and $\theta \overset{\text{isotop.}}{\underset{\text{rel } \partial Q}{\sim}} id$, we may assume $G|_{U \cup E}$ holomorphic, and $G|_W$ quasi-regular.

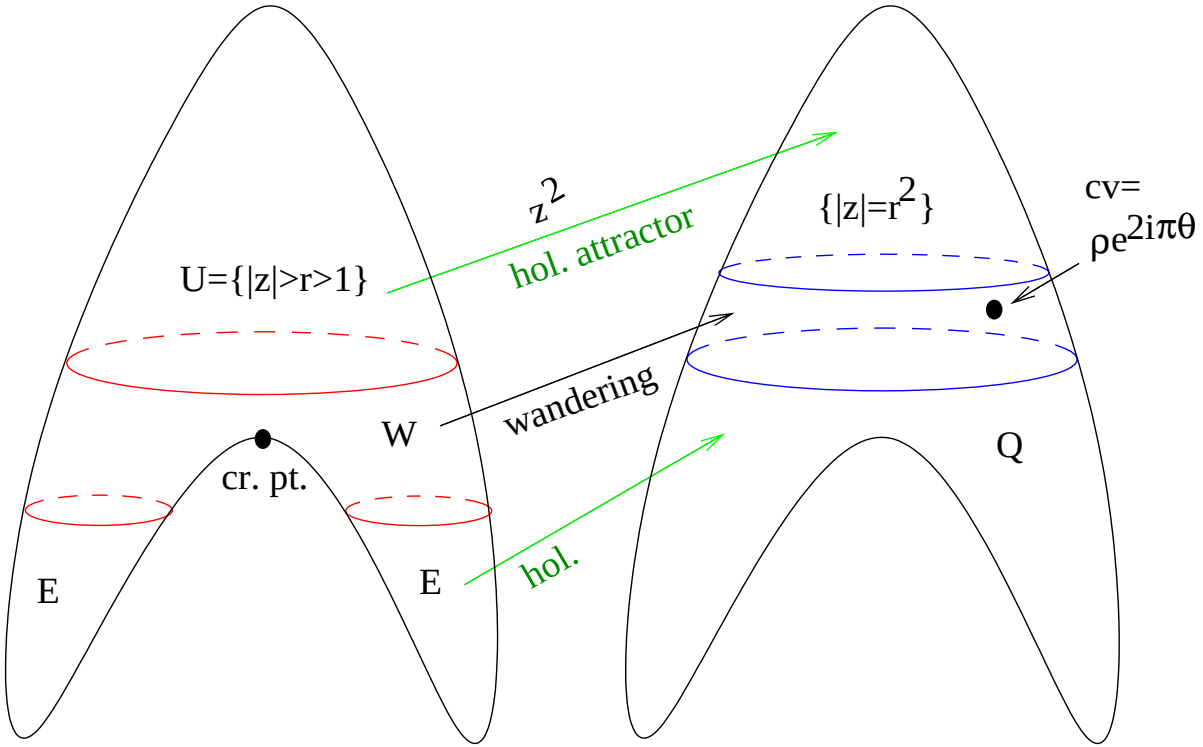
An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$



$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor, $G(W) \subset U$, and $cv(G) = \rho e^{2\pi i\theta}$.

1. Replace G by $G \circ \theta^{-1}$ with $\theta(Q) = Q$ and $\theta \overset{\text{isotop.}}{\underset{\text{rel } \partial Q}{\sim}} id$, we may assume $G|_{U \cup E}$ holomorphic, and $G|_W$ quasi-regular.
2. $\exists \phi$ qc., and hol. on U such that $\phi \circ G \circ \phi^{-1}$ is $z^2 + c$ (quasi-conformal surgery).

An example: $G \sim_{comb} z^2 + c$ with Bött$(c) = \rho e^{2\pi i\theta}$

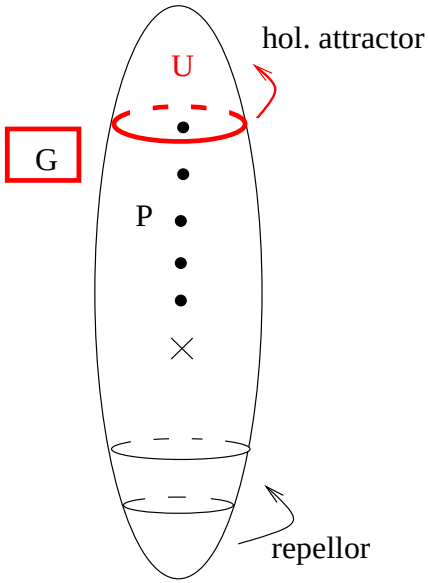


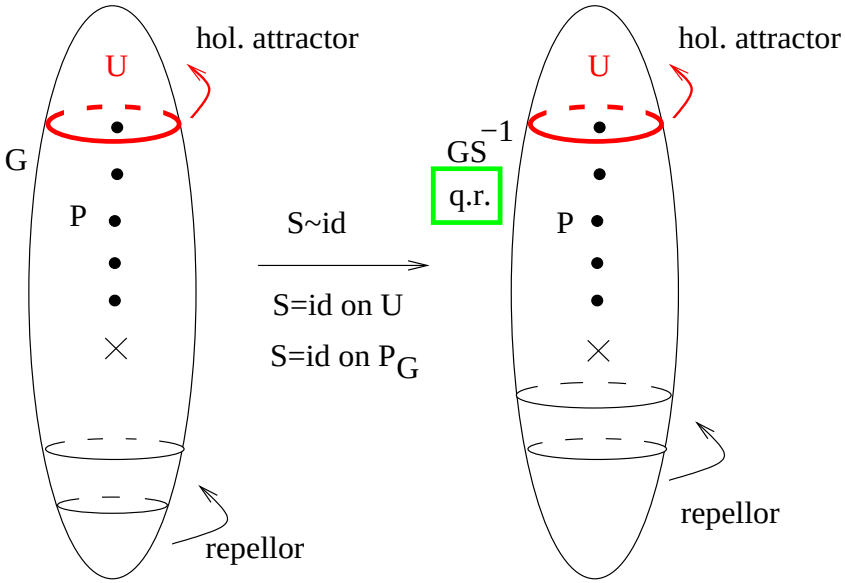
$G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a branched cover, with (G, U) a hol. attractor, (G, E, Q) a repellor, $G(W) \subset U$, and $cv(G) = \rho e^{2\pi i\theta}$.

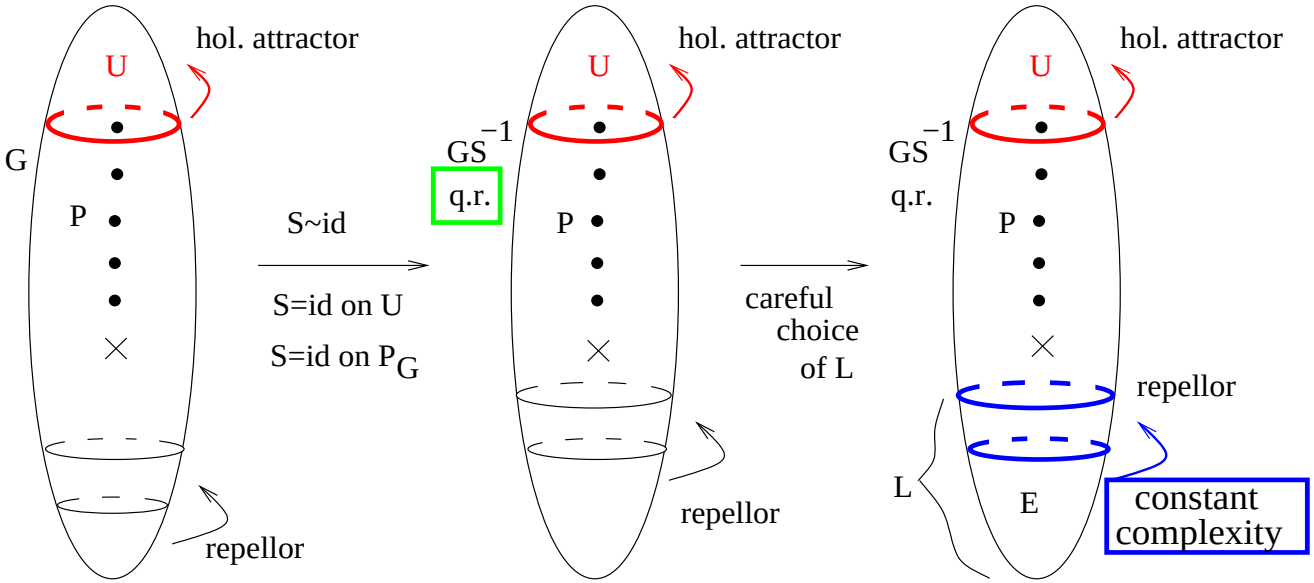
1. Replace G by $G \circ \theta^{-1}$ with $\theta(Q) = Q$ and $\theta \overset{\text{isotop.}}{\underset{\text{rel } \partial Q}{\sim}} id$, we may assume $G|_{U \cup E}$ holomorphic, and $G|_W$ quasi-regular.
2. $\exists \phi$ qc., and hol. on U such that $\phi \circ G \circ \phi^{-1}$ is $z^2 + c$ (quasi-conformal surgery).
3. Actually Bött$(c) = \rho e^{2\pi i\theta}$.

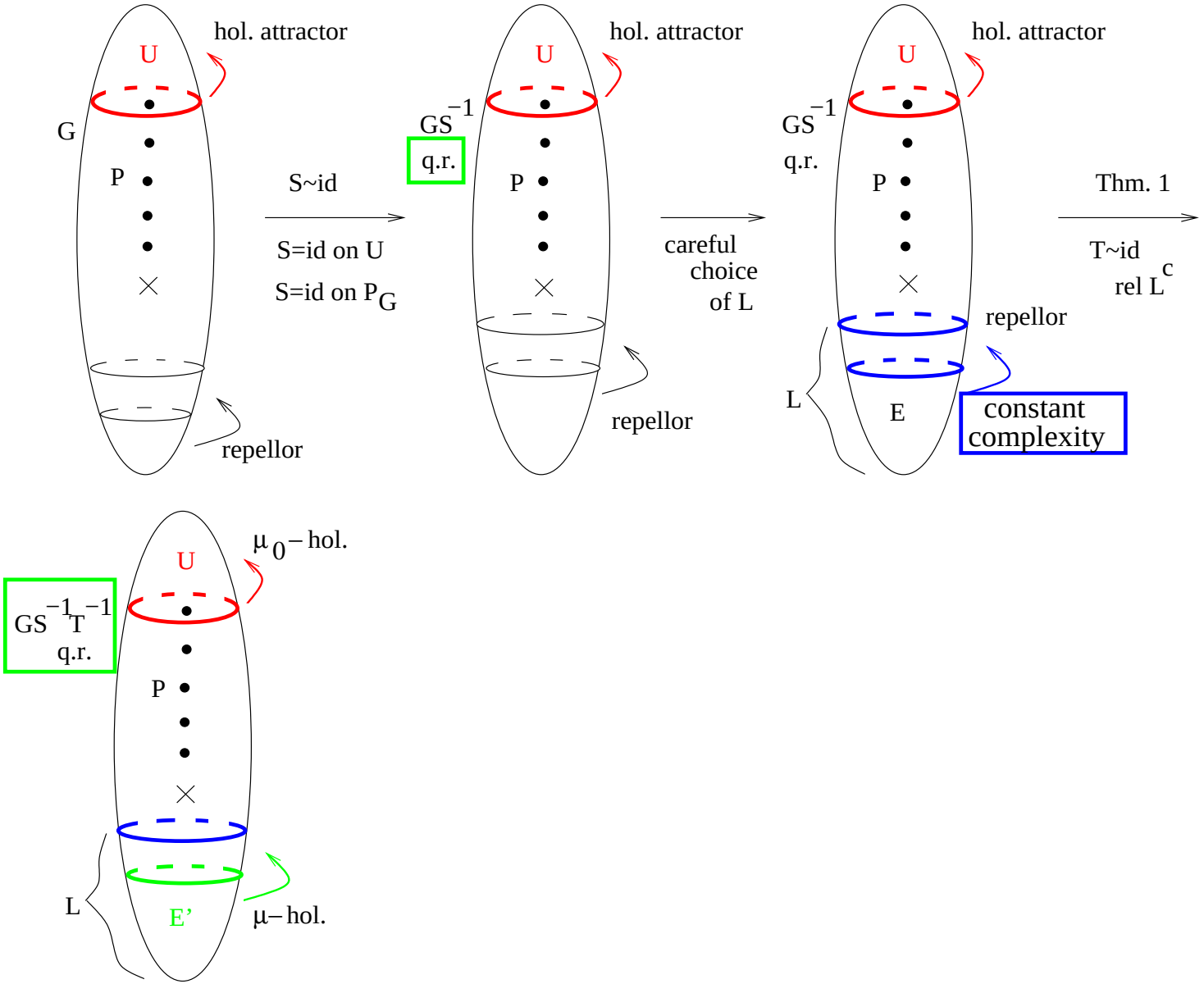
Theorem 2. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic near $acc(\mathcal{P}_G)$, without Thurston obstructions. Then G is comb-equivalent to a hyperbolic rational map R, i.e. $\exists \theta, \phi$ such that $G \circ \theta^{-1} \sim_\phi R$.

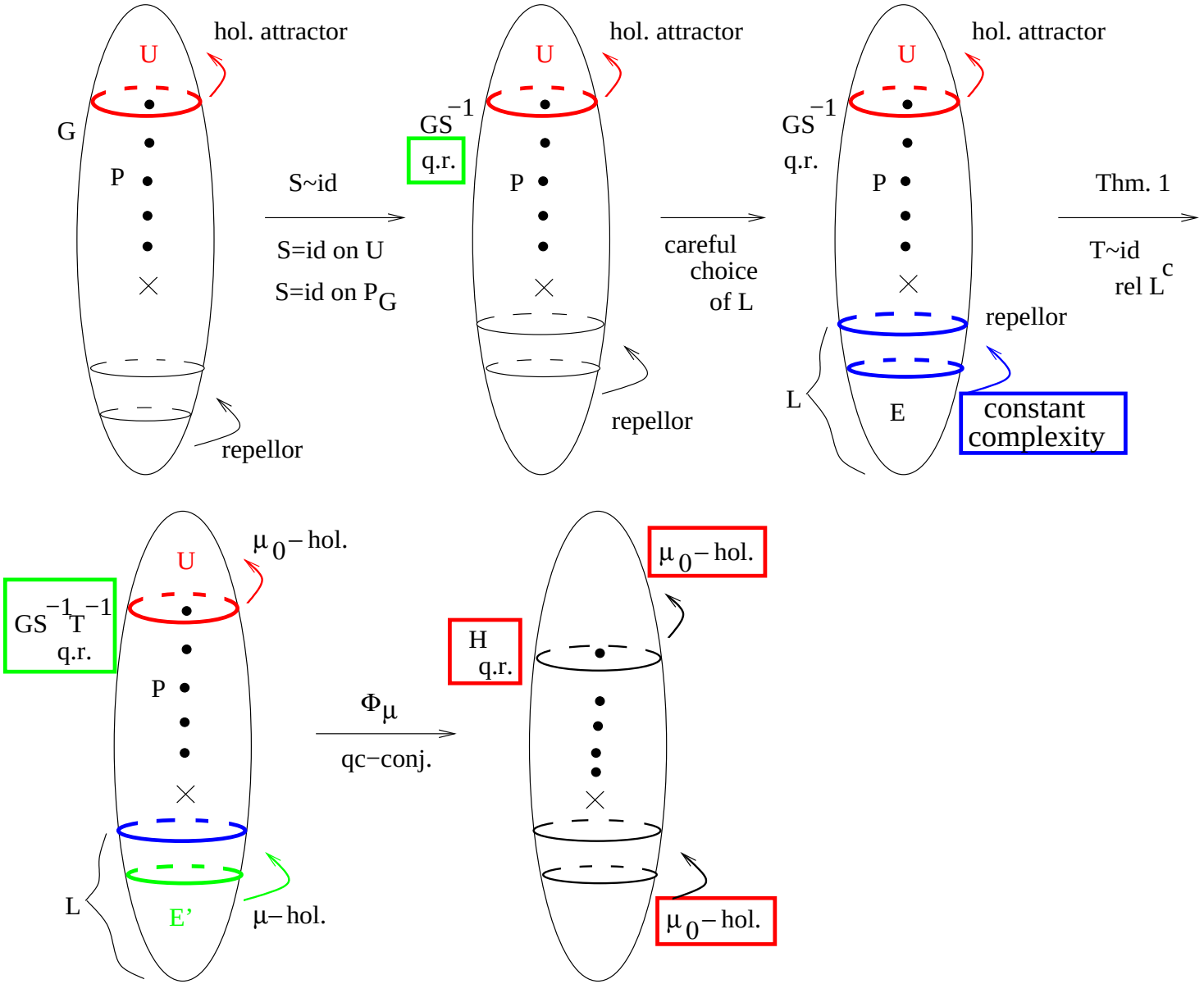
Sketch of the proof.

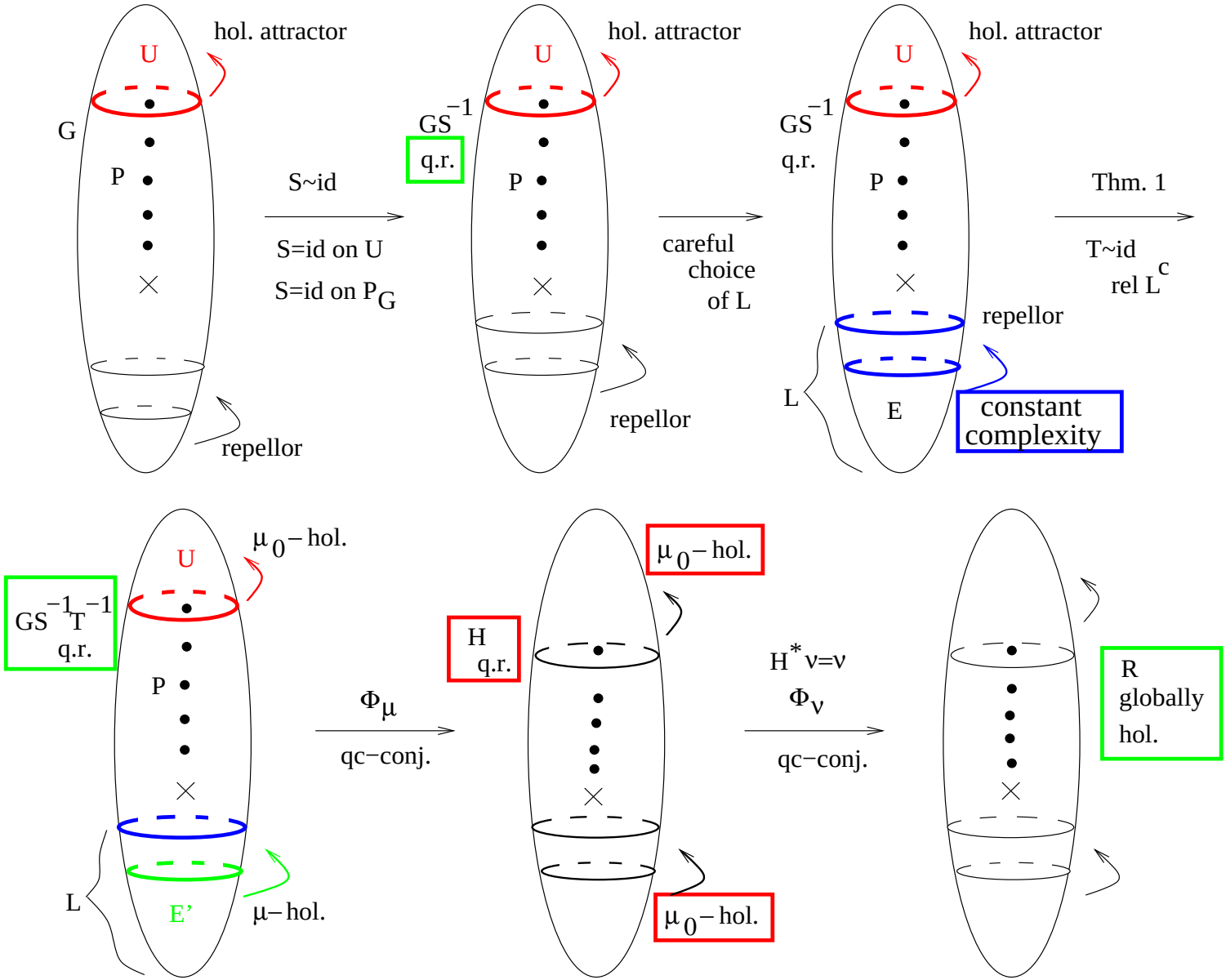

hol. attractor
U
G
P
repellor


U
hol. attractor
G
P
repellor
S~id
S=id on U
S=id on P_G
GS^{-1}
q.r.
U
hol. attractor
P
repellor


U
hol. attractor
G
P
repellor
S~id
S=id on U
S=id on P_G
GS^{-1}
q.r.
U
hol. attractor
P
repellor
careful
choice
of L
GS^{-1}
q.r.
U
hol. attractor
P
repellor
L
E
constant
complexity


U
hol. attractor
G
P
repellor
S~id
S=id on U
S=id on P_G
GS^{-1}
q.r.
careful
choice
of L
Thm. 1
T~id
rel L^c
L
E
constant
complexity
GS^{-1}T^{-1}
q.r.
μ_0 − hol.
E'
μ− hol.


U
hol. attractor
G
P
repellor
S~id
S=id on U
S=id on P_G
GS^{-1}
q.r.
careful
choice
of L
Thm. 1
T~id
rel L^c
repellor
L
E
constant
complexity
μ_0 − hol.
GS^{-1}T^{-1}
q.r.
E'
μ − hol.
Φ_μ
qc−conj.
H
q.r.


U
hol. attractor
G
P
repellor
S~id
S=id on U
S=id on P_G
GS^{-1}
q.r.
careful
choice
of L
Thm. 1
T~id
rel L^c
repellor
L
E
constant
complexity
μ_0 − hol.
GS^{-1}T^{-1}
q.r.
E'
μ − hol.
Φ_μ
qc−conj.
H
q.r.
H^*ν=ν
Φ_ν
R
globally
hol.

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points $=$ attrac. / parabolic

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points $=$ attrac. / parabolic
- with no obstructions (generalized Thurston obstructions)

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points = attrac. / parabolic
- with no obstructions (generalized Thurston obstructions)

$\Longrightarrow \exists!$ g rational map $\sim_{\text{comb}} G$, i.e. for

$U = \sqcup(\text{round attract. discs}) \sqcup (\text{attract. petals}) = \textit{basic set}$

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points $=$ attrac. / parabolic
- with no obstructions (generalized Thurston obstructions)

$\Longrightarrow \exists!$ g rational map $\sim_{\text{comb}} G$, i.e. for

$U = \sqcup(\text{round attract. discs}) \sqcup (\text{attract. petals}) = \textit{basic set}$

$\exists\ \theta, \phi$ homeo., s.t. $G \circ \theta^{-1} \sim_{\phi-\text{conj}} g$:

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points $=$ attrac. / parabolic
- with no obstructions (generalized Thurston obstructions)

$\Longrightarrow \exists!$ g rational map $\sim_{\text{comb}} G$, i.e. for

$U = \sqcup(\text{round attract. discs}) \sqcup (\text{attract. petals}) = \textit{basic set}$

$\exists\ \theta, \phi$ homeo., s.t. $G \circ \theta^{-1} \sim_{\phi-\text{conj}} g$:

$$\begin{array}{rcll} U, \overline{\mathbb{C}} & \xrightarrow{\phi, \approx} & V, \overline{\mathbb{C}} & \theta|_U = id \\ \theta, \approx \uparrow\uparrow G & & \circlearrowleft\ g & \theta \sim_{P_G} id \\ U, \overline{\mathbb{C}} & & & \phi|_U \text{ hol.} \end{array}$$

Theorem A. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. points $=$ attrac. / parabolic
- with no obstructions (generalized Thurston obstructions)

$\Longrightarrow \exists!$ g rational map $\sim_{\text{comb}} G$, i.e. for

$U = \sqcup(\text{round attract. discs}) \sqcup (\text{attract. petals}) = \textit{basic set}$

$\exists\ \theta, \phi$ homeo., s.t. $G \circ \theta^{-1} \sim_{\phi-\text{conj}} g$:

$$\begin{array}{rcll} U, \overline{\mathbb{C}} & \xrightarrow{\phi, \approx} & V, \overline{\mathbb{C}} & \theta|_U = id \\ \theta, \approx \uparrow\uparrow G & & \circlearrowleft\ g & \theta \sim_{P_G} id \\ U, \overline{\mathbb{C}} & & & \phi|_U \text{ hol.} \end{array}$$

Moreover $g = \lim_{n\to\infty} g_n$ *(Thurston algorithm of successive liftings converges).*

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

$V_x = \textit{disc}$ around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

$V_x = \textit{disc}$ around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$

Then $\exists\ C(r) \searrow 0$ as $r \searrow 0$ such that

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

$V_x = \textit{disc}$ around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$

Then $\exists\ C(r) \searrow 0$ as $r \searrow 0$ such that

$$\forall\ N, \forall\ \phi : \overline{\mathbb{C}} \smallsetminus \bigcup_{x \in g^{-N}(X)} \overline{V_x(r)} \to \overline{\mathbb{C}} \qquad \text{univalent}$$

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

$V_x = \textit{disc}$ around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$

Then $\exists\ C(r) \searrow 0$ as $r \searrow 0$ such that

$$\forall\ N, \forall\ \phi : \overline{\mathbb{C}} \smallsetminus \bigcup_{x \in g^{-N}(X)} \overline{V_x(r)} \to \overline{\mathbb{C}} \qquad \text{univalent}$$

$$\boxed{\text{distortion}(\phi) \leq C(r) \searrow 0.}$$

Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbb{C}}$), $X \subset J_g$ a periodic cycle,

$V_x = \textit{disc}$ around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$

Then $\exists\ C(r) \searrow 0$ as $r \searrow 0$ such that

$$\forall\ N, \forall\ \phi : \overline{\mathbb{C}} \smallsetminus \bigcup_{x \in g^{-N}(X)} \overline{V_x(r)} \to \overline{\mathbb{C}} \qquad \text{univalent}$$

$$\boxed{\text{distortion}(\phi) \leq C(r) \searrow 0.}$$

Applications of Theorem A + Theorem B + *easy surgeries* :

I. (Douady-Hubbard) convergence of preperiodic periodic rays of M.

II. Parabolic maps are accessible from various hyperbolic components.

III. Pinching deformations converge.

Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }

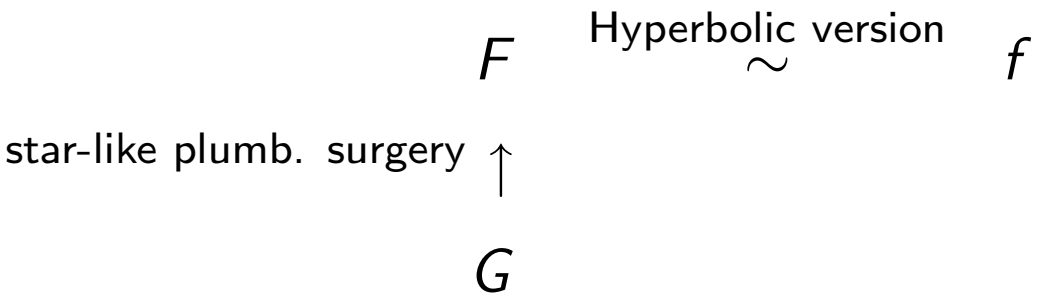
Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }

$$F$$

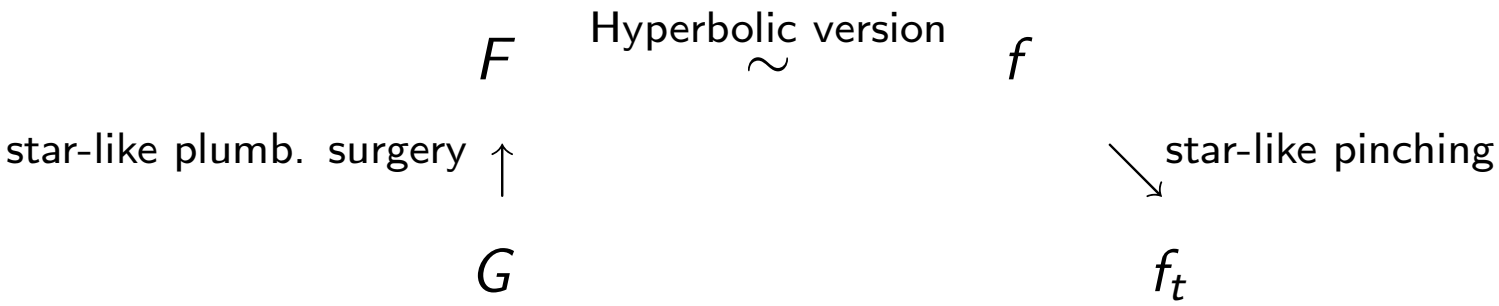
star-like plumb. surgery $\uparrow$

$$G$$

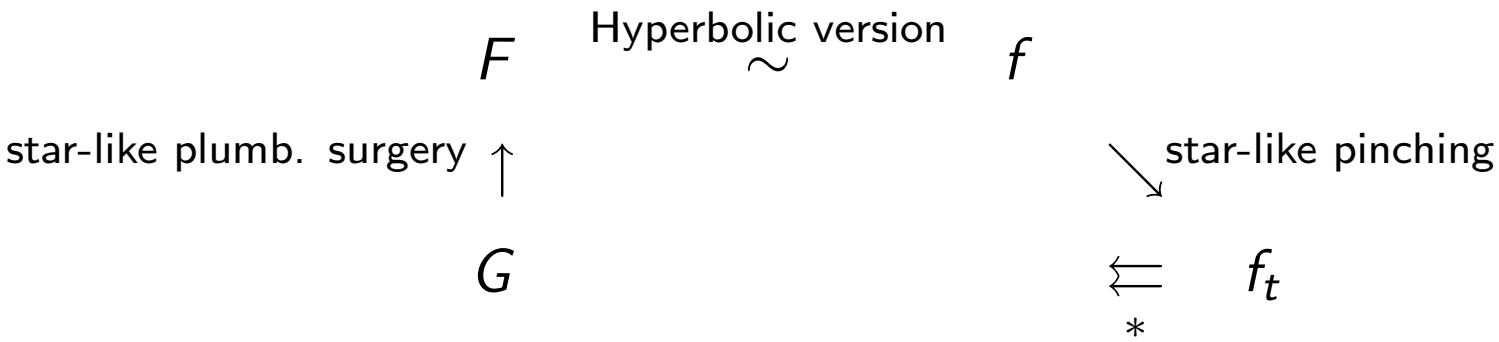
Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }



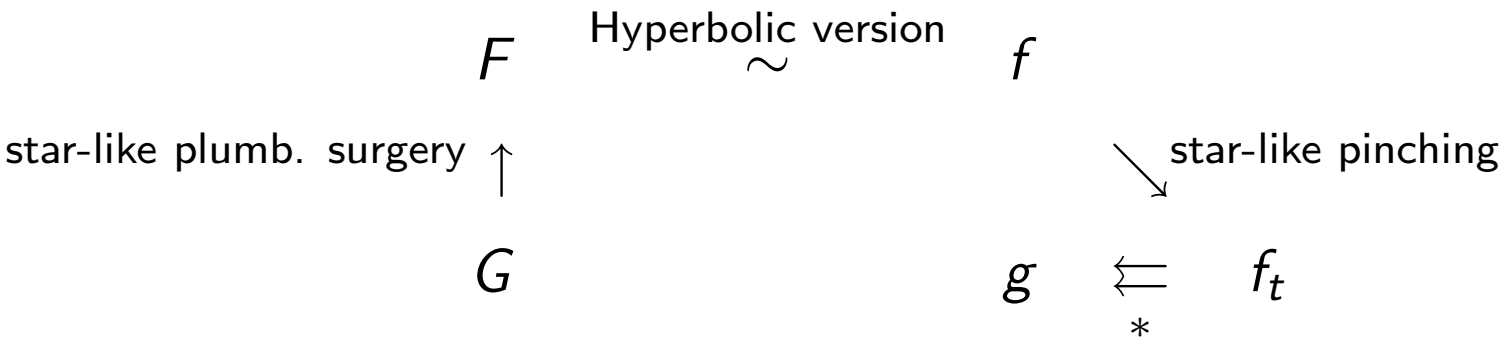
Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }



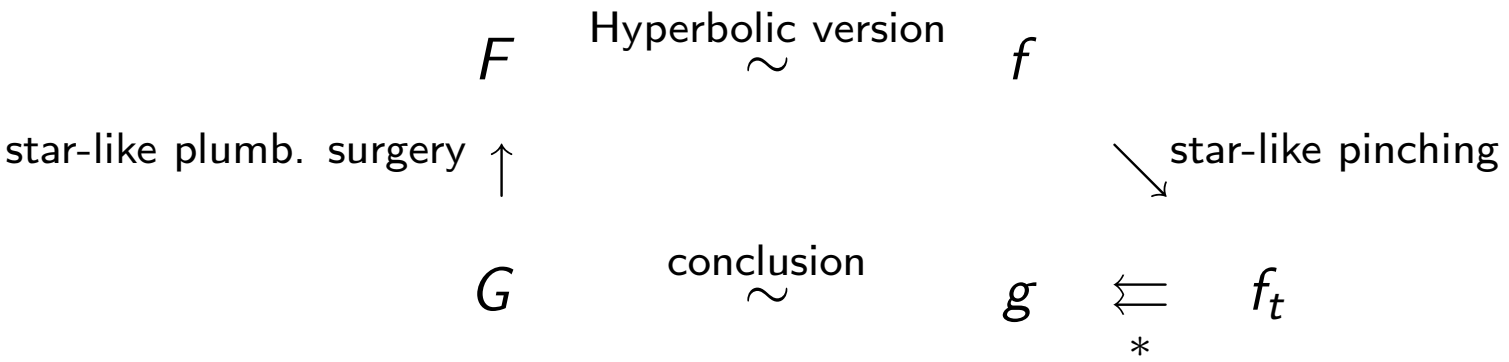
Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }



Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }



Existence part of Theorem A with parabolics $\Longleftarrow$ { Hyperbolic version; Control of distortions * }



Convergence of preperiodic external rays of M (Douady-Hubbard)

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$.

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$.Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$.Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n})$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t)$$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0,$$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0, \text{ so}$$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0$, so

$|c(0) - c(t)| = \lim_{n\to\infty} |c(0) - c(t,n)| \leq \varepsilon(t)$

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$. Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$Q_{c(0)} \overset{\text{cut-glue}}{\longrightarrow} F_t \overset{\text{Thm A}}{\longrightarrow} \begin{array}{c} Q_{c(t)} \\ \uparrow_{n\to\infty} \\ Q_{c(t,n)} \end{array}$$

$$\begin{array}{ccc} \vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\ F_t \downarrow & & \downarrow Q_{c(t,2)} \\ & \overset{\phi_{t,1}}{\longrightarrow} & \\ F_t \downarrow & & \downarrow Q_{c(t,1)} \\ & \overset{\phi_{t,0}=id}{\longrightarrow} & \end{array}$$

$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0$, so

$|c(0) - c(t)| = \lim_{n\to\infty} |c(0) - c(t,n)| \leq \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0.$

Parabolic perturbation:
given g, $\exists$ f_t with
prescribed combinatorics
such that $f_t \longrightarrow g$

$$F_t$$

$$\text{plumbing} \uparrow$$

$$g$$

Parabolic perturbation: given g, $\exists$ f_t with prescribed combinatorics such that $f_t \longrightarrow g$

$$\begin{array}{ccc} F_t & \overset{\text{Theorem A}}{\sim} & f_t \\ \text{plumbing}\ \uparrow & & \uparrow_{n\to\infty} \\ g & & f_{t,n} \end{array}$$

$$\begin{array}{ccccc} \vdots & \xleftarrow{\widehat{q}_2} & \vdots & \xleftarrow{\eta_2} & \vdots \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,2} \\ & \xleftarrow{\widehat{q}_1} & & \xleftarrow{\eta_1} & \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,1} \\ & \xleftarrow{\widehat{q}_0} & & \xleftarrow{\eta_0=id} & \end{array}$$

which is

$$\begin{array}{ccc} \vdots & \xrightarrow{\phi_{t,2}} & \vdots \\ g\downarrow & & \downarrow f_{t,2} \\ & \xrightarrow{\phi_{t,1}} & \\ g\downarrow & & \downarrow f_{t,1} \\ & \xrightarrow{\phi_{t,0}} & \end{array}$$

Parabolic perturbation: given g, $\exists$ f_t with prescribed combinatorics such that $f_t \longrightarrow g$

$$\begin{array}{ccc} F_t & \overset{\text{Theorem A}}{\sim} & f_t \\ \text{plumbing}\ \uparrow & & \uparrow_{n\to\infty} \\ g & \underset{t\to\infty}{\overset{\text{Thm. B}}{\leftleftarrows}} & f_{t,n} \end{array}$$

$$\begin{array}{ccccc} \vdots & \xleftarrow{\widehat{q}_2} & \vdots & \xleftarrow{\eta_2} & \vdots \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,2} \\ & \xleftarrow{\widehat{q}_1} & & \xleftarrow{\eta_1} & \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,1} \\ & \xleftarrow{\widehat{q}_0} & & \xleftarrow{\eta_0=id} & \end{array}$$

which is

$$\begin{array}{ccc} \vdots & \xrightarrow{\phi_{t,2}} & \vdots \\ g\downarrow & & \downarrow f_{t,2} \\ & \xrightarrow{\phi_{t,1}} & \\ g\downarrow & & \downarrow f_{t,1} \\ & \xrightarrow{\phi_{t,0}} & \end{array}$$

Parabolic perturbation: given g, $\exists$ f_t with prescribed combinatorics such that $f_t \longrightarrow g$

$$\begin{array}{ccc} F_t & \overset{\text{Theorem A}}{\sim} & f_t \\ \text{plumbing}\ \uparrow & \swarrow_{t\to\infty} & \uparrow_{n\to\infty} \\ g & \underset{t\to\infty}{\leftleftarrows} \text{ Thm. B} & f_{t,n} \end{array}$$

$$\begin{array}{ccccc} \vdots & \xleftarrow{\widehat{q}_2} & \vdots & \xleftarrow{\eta_2} & \vdots \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,2} \\ & \xleftarrow{\widehat{q}_1} & & \xleftarrow{\eta_1} & \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,1} \\ & \xleftarrow{\widehat{q}_0} & & \xleftarrow{\eta_0=id} & \end{array}$$

which is

$$\begin{array}{ccc} \vdots & \xrightarrow{\phi_{t,2}} & \vdots \\ g\downarrow & & \downarrow f_{t,2} \\ & \xrightarrow{\phi_{t,1}} & \\ g\downarrow & & \downarrow f_{t,1} \\ & \xrightarrow{\phi_{t,0}} & \end{array}$$

Pinching deformations of f

Let Γ be a multicurve in $\mathcal{R}_f$ whose lifts are Julia non-separating. μ_t : pinching structure supported on collar(Γ). Then $(f, \widetilde{\mu_t})$ creates f_t and $f_t \underset{t\to\infty}{\to}$ some g.

$$\begin{array}{ccc} f & \overset{\text{pinch } \Gamma,\mu_t}{\longrightarrow} & f_t \\ \text{surgery} \downarrow & & \\ G & & \end{array}$$

Pinching deformations of f

Let Γ be a multicurve in $\mathcal{R}_f$ whose lifts are Julia non-separating. μ_t : pinching structure supported on collar(Γ). Then $(f, \widetilde{\mu_t})$ creates f_t and $f_t \underset{t\to\infty}{\to}$ some g.

$$\begin{array}{ccc} f & \overset{\text{pinch } \Gamma,\mu_t}{\longrightarrow} & f_t \\ \text{surgery} \downarrow & & \\ G & \underset{\sim}{\text{Theorem A}} & g \end{array}$$

Pinching deformations of f

Let Γ be a multicurve in $\mathcal{R}_f$ whose lifts are Julia non-separating. μ_t : pinching structure supported on collar(Γ). Then $(f, \widetilde{\mu_t})$ creates f_t and $f_t \underset{t\to\infty}{\to}$ some g.

$$\begin{array}{ccccc} f & \overset{\text{pinch } \Gamma,\mu_t}{\longrightarrow} & f_t & & \hat{f}_t \\ \text{surgery}\ \downarrow & & & & \uparrow\downarrow\downarrow \ \text{Parabolic perturbation} \\ G & & \underset{\sim}{\text{Theorem A}} & & g \end{array}$$

Pinching deformations of f

Let Γ be a multicurve in $\mathcal{R}_f$ whose lifts are Julia non-separating. μ_t : pinching structure supported on collar(Γ). Then $(f, \widetilde{\mu_t})$ creates f_t and $f_t \underset{t\to\infty}{\to}$ some g.

$$
\begin{array}{ccccc}
f & \overset{\text{pinch } \Gamma,\mu_t}{\longrightarrow} & f_t & \underset{\textit{conclusion}}{\overset{\text{unicity}}{=}} & \hat{f}_t \\
\text{surgery}\ \downarrow & & & & \uparrow\downarrow\downarrow \ \text{Parabolic perturbation} \\
G & & \overset{\text{Theorem A}}{\sim} & & g
\end{array}
$$