On the topology of wandering Julia components [1]

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Abstract

It is known that for a rational map f with a disconnected Julia set, the set of wandering Julia components is uncountable. We prove that all but countably many of them have a simple topology, namely having one or two complementary components. We show that the remaining countable subset Σ is backward invariant. Conjecturally Σ does not contain an infinite orbit. We give a very strong necessary condition for Σ to contain an infinite orbit, thus proving the conjecture for many different cases. We provide also two sufficient conditions for a Julia component to be a point. Finally we construct several examples describing different topological structures of Julia components.

1 Introduction

Let f be a rational map from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself of degree $d \geq 2$. By the classical theory of iterated rational maps initiated by Fatou and Julia around 1920, we know that the Riemann sphere $\widehat{\mathbb{C}}$ is divided into two sets: one is called the Fatou set $\mathcal{F} := \{z \in \widehat{\mathbb{C}} \mid \text{the sequence } \{f^n\}_{n\geq 1} \text{ is normal in a neighborhood of } z\}$ and the other is called the Julia set $\mathcal{J} := \widehat{\mathbb{C}} \backslash \mathcal{F}$. For the Fatou set, the following is a well-known theorem of Sullivan:

Theorem A. ([Sul]) *Every Fatou domain of the rational map f is preperiodic.*

Furthermore, Shishikura ([Shi]) proved that there are at most $2d-2$ periodic cycles of Fatou domains of the rational map f.

Suppose the Julia set $\mathcal{J}$ is disconnected. A **Julia component** of f is a connected component of $\mathcal{J}$. The map f maps each Julia component onto a Julia component. A Julia component J is **periodic** if $f^p(J) = J$ for some $p \geq 1$, **preperiodic** if $f^n(J)$ is periodic for some $n \geq 0$, and **wandering** otherwise, that is if $f^i(J) \cap f^j(J) = \emptyset$ for all $i \neq j \geq 0$. Thus a component J is wandering if and only if $\mathrm{orb}(J) = \{f^n(J) \mid n \in \mathbb{N}\}$ is infinite. Since $\mathcal{J}$ is disconnected, there are countably many preperiodic Julia components and uncountably many wandering Julia components (see [Bea], [Mil1], [McM], [PT1]).

Let J be a periodic Julia component with period p containing more than one point. Then by a theorem of McMullen (see [McM]), we know that (f^p, J) is quasi-conformally conjugate to $(g, \mathcal{J}(g))$ where g is a rational map with a connected Julia set $\mathcal{J}(g)$. More precisely, there exists a quasi-conformal map ϕ from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself such that $\phi(J) = \mathcal{J}(g)$ and $\phi \circ f^p = g \circ \phi$ on J. So the study of the preperiodic components is in some sense converted to the study to rational maps with connected Julia sets.

For wandering Julia components, so far, we don't know much about them. We know some topological properties of them for special rational maps.

Theorem B. ([QY]) *Every wandering Julia component of a polynomial is a point.*

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The result is generalized to a rational map with a completely invariant Fatou domain, see [Z].

A non-empty connected compact set $J \subset \widehat{\mathbb{C}}$ is **simply** (resp. **doubly**) connected if $\widehat{\mathbb{C}} \setminus J$ has exactly one (resp. two) connected components. We say that a rational map is **geometrically finite**, resp. **nice**, if every critical point in the Julia set (if any) is preperiodic, resp. if every Julia component containing a critical point (if any) is preperiodic. We have

Theorem C. ([PT1]) *1. Every wandering Julia component of a geometrically finite rational map is a Jordan curve or a point.*

2. Every wandering Julia component of a nice map is either simply or doubly connected.

Furthermore, they depicted when a simply connected Julia component of a nice map is a point.

Theorem D. ([Mil2]) *The Julia set of a rational map with two critical points is either connected or totally disconnected.*

In this paper, we will investigate the topological structure of wandering Julia components. First, we define a function

$$C : \{\text{non-empty connected compact sets} \subsetneq \widehat{\mathbb{C}}\} \to \{1, 2, 3\}$$

as follows: for $J \subsetneq \widehat{\mathbb{C}}$ a connected compact set with $J \neq \emptyset$,

$$C(J) = \left\{ \begin{array}{ll} 1 & \text{if } J \text{ is simply connected} \\ 2 & \text{if } J \text{ is doubly connected} \\ 3 & \text{otherwise.} \end{array} \right\} = \min\{3, \#\{\text{complementary components of } J\}\}.$$

In case that J is a Julia component of the rational map f, we can show that $n \mapsto C(f^n(J))$ is weakly decreasing (see Lemma 2.1 below). For a wandering J, define $C(\text{orb}(J))$ as $\lim_{n\to\infty} C(f^n(J))$.

For a Julia component of the rational map f, we refer to it as a **buried** component if it has no intersection with the boundary of any Fatou domain of f, otherwise we call it **exposed**. One can refer to section 7 in [McM], section 2 in [PT1] and section 4 in this paper for examples of buried components.

Our main result here is the following:

Theorem 1.1. *Let f be a rational map with a disconnected Julia set.*

(a) A wandering Julia component J with $C(\text{orb}(J)) \geq 2$ is necessarily buried, that is, it has no intersection with the boundary of any Fatou domain of f.

(b) The set $\Sigma := \{J \mid J \text{ a Julia component}, C(J) = 3\}$ is backward invariant, and is either empty or countable.

(c) If Σ contains the forward orbit of at least one wandering Julia component, then there are two distinct Julia components $J_1 \neq J_2$ satisfying simultaneously:

$$(*) \left\{ \begin{array}{l} \textit{they are both wandering and they both contain a critical value;} \\ \textit{they are both buried;} \\ C(J_1) = C(J_2) = C(\text{orb}(J_1)) = C(\text{orb}(J_2)) = 1; \\ \text{orb}(J_s) \textit{ accumulates to } J_t \textit{ for any pair } (s,t) \in \{1,2\}^2; \\ J_1 \notin \text{orb}(J_2) \textit{ and } J_2 \notin \text{orb}(J_1). \end{array} \right.$$

We will give an example showing that the condition $(*)$ is not sufficient for having a wandering Julia component J with $C(\mathrm{orb}(J)) = 3$.

These results support the conjecture that there is no Julia component J with infinite orbit and with $C(\mathrm{orb}(J)) = 3$. We mention two easy to check corollaries of Theorem 1.1 (with the first one containing Theorem C as a particular case):

Corollary 1.2. *Let f be a rational map with disconnected Julia set. If all but at most one Julia components containing critical points are either exposed or preperiodic, then every wandering Julia component is eventually simply connected or doubly connected.*

Corollary 1.3. *Let f be a rational map with disconnected Julia set. If all but one Julia components containing critical points are doubly or multiply connected, then every wandering Julia component is eventually simply connected or doubly connected.*

The paper is organized as follows. In Section 2, we prove Theorem 1.1 and the main tool we used is Theorem A. In Section 3, we provide two sufficient conditions for a Julia component to be a point. Finally we use quasi-conformal surgery to construct several examples describing different topological structures of Julia components in Section 4.

2 Proof of Theorem 1.1

2.1 Preliminary results

For a non-empty compact connected set $K \subset \widehat{\mathbb{C}}$, we say that an open set W is a complementary component of K if W is a connected component of $\widehat{\mathbb{C}} \setminus K$.

Lemma 2.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a branched covering. Let $K \subset \widehat{\mathbb{C}}$ be a non-empty compact connected set and K' be a connected component of $f^{-1}(K)$.*

(i) We have $C(K') \geq C(K)$. Furthermore, if K' contains no critical points of f, then $C(K') = C(K)$.

(ii) The map f induces a correspondence

$$f_* : \{\textit{complementary components of } K'\} \to \{\textit{complementary components of } K\}$$

so that $f_*(W') = W$ if f maps the prime ends of W' onto those of W. Moreover either $f(W') = f_*(W')$ or $f(W') = \widehat{\mathbb{C}}$. The latter occurs if and only if $W' \cap f^{-1}(K) \neq \emptyset$.

Proof. (i) Set $Q = \{$complementary components of $K\}$ and

$$R = \{\text{complementary components of } K \text{ containing critical values}\}.$$

Then $\#R < \infty$.

Take a finitely connected domain $U \subset \widehat{\mathbb{C}}$ with smooth boundary satisfying that

(1) $K \subset U$,
(2) $\overline{U} \setminus K$ contains no critical values of f,
(3) for q_U the number of boundary components of U, we have $q_U = 1$ if $\#R \leq 1$ and $q_U = \#R$ otherwise.

Then U can be decomposed into three sets: $A \sqcup K \sqcup D$, where A is the union of finitely many disjoint annuli and D is the union of disjoint simply connected domains. The sets A and D have the following properties:

(a) $\#\{\text{connected components of } A\} = q_U$.

(b) $A \cup D$ contain no critical values.

(c) For each component of A, one of its boundary components is contained in ∂K while the other one is a component of ∂U. For each component of D, its boundary is contained in ∂K.

(d) For a component A_i of A, each connected component A_i' of $f^{-1}(A_i)$ is an annulus disjoint from $f^{-1}(K)$ with one boundary lying in $\partial f^{-1}(U)$ and the other lying in $\partial f^{-1}(K)$. The map $f : A_i' \to A_i$ is a covering. For a component D_i of D, each connected component D_i' of $f^{-1}(D_i)$ is a simply connected domain disjoint from $f^{-1}(K)$ with $\partial D_i' \subset \partial f^{-1}(K)$ and with $f : D_i' \to D_i$ a homeomorphism.

There exists a connected components U' of $f^{-1}(U)$ containing K'. Clearly $f^{-1}(K) \cap U' = U' \setminus f^{-1}(A \cup D)$. It follows from (d) that $f^{-1}(K) \cap U'$ is connected (for this one can use approximations of $f^{-1}(K \cup D) \cap U'$ by domains with Jordan curve boundaries, and then the Jordan curve theorem). Therefore $f^{-1}(K) \cap U' = K'$ and $U' = A' \sqcup K' \sqcup D'$, where $A' = f^{-1}(A) \cap U'$, $D' = f^{-1}(D) \cap U'$. Furthermore the boundary curves of U' are in pairwise distinct complementary components of K'. Denote by $q_{U'}$ the number of these boundary curves. We have $q_{U'} = \#\{\text{connected components of } A'\} \geq 1$.

The restricted map $f : U' \to U$ is a branched covering. Denote its degree by d. Denote by $q_D, q_{D'}$ the number of connected components of D, D' respectively (they may be zero or $+\infty$). Let $k = \#\{\text{critical points in } U' \text{ counted with multiplicity}\}$. We have, by definition, the point (d) above and the Riemann-Hurwitz Formula:

$$C(K) = \min\{3, q_U + q_D\} \;; C(K') = \min\{3, q_{U'} + q_{D'}\} \;;$$

$$q_{D'} = d \cdot q_D \;; (2 - q_{U'}) + k = d(2 - q_U).$$

$C(K') = 1 \Longrightarrow q_{U'} = 1, q_{D'} = 0 \Longrightarrow q_U = 1, q_D = 0 \Longrightarrow C(K) = 1.$

$$C(K') = 2 \Longrightarrow \begin{cases} q_{U'} = 1, q_{D'} = 1 \Longrightarrow q_U = 1, q_D = 1, d = 1 \Longrightarrow C(K) = 2 \\ \text{or} \\ q_{U'} = 2, q_{D'} = 0 \Longrightarrow \begin{cases} q_U = 1 \text{ (if } k \geq 1), q_D = 0 \Longrightarrow C(K) = 1 \\ \text{or} \\ q_U = 2 \text{ (if } k = 0), q_D = 0 \Longrightarrow C(K) = 2. \end{cases} \end{cases}$$

$$C(K') = 3, k = 0 \Longrightarrow \begin{cases} q_{U'} = 1, q_{D'} \geq 2 \Longrightarrow q_U = 1, d = 1, q_D \geq 2 \Longrightarrow C(K) = 3 \\ \text{or} \\ q_{U'} = 2, q_{D'} \geq 1 \Longrightarrow q_U = 2, q_D \geq 1 \Longrightarrow C(K) = 3 \\ \text{or} \\ q_{U'} \geq 3 \Longrightarrow q_U \geq 3 \Longrightarrow C(K) = 3. \end{cases}$$

Obviously, $C(K') = 3 \geq C(K)$, if $k > 0$.

(ii) Let W' be a complementary component of K'.

Assume at first $W' \cap f^{-1}(K) = \emptyset$. Then $f(W')$ is open, connected, and is disjoint from K. So $f(W')$ is contained in a unique complementary component W of K. Then $f^{-1}(W)$ has a unique connected component U' containing W'. But U' is open, connected, and is disjoint from K'. So U' must be contained in a complementary component of K'. Consequently $U' = W'$ and $f(W') = W$. Clearly $f_*(W') = W$ as well.

Assume now $W' \cap f^{-1}(K) \neq \emptyset$. Then $W' \setminus f^{-1}(K)$ has a unique component E with $\partial E \supset \partial W'$. One can show as above that $f(E) =: W$ is a complementary component of K, and $f_*(W') = W$. To show $f(W') = \widehat{\mathbb{C}}$ we choose a Jordan curve $\eta \subset W$ close to ∂W so that the annular component of $W \setminus \eta$, as well as η, does not contain critical values. We now use the following fact:

Fact: let $\delta \subset \widehat{\mathbb{C}}$ be a Jordan curve containing no critical values of f. Then
(a) $f^{-1}(\delta)$ is the union of finitely many pairwise disjoint Jordan curves;
(b) f maps every component of $\widehat{\mathbb{C}} \setminus f^{-1}(\delta)$ properly onto a complementary component of δ;
(c) any two components of $\widehat{\mathbb{C}} \setminus f^{-1}(\delta)$ sharing a common boundary curve are mapped onto distinct components of $\widehat{\mathbb{C}} \setminus \delta$.

For our curve η, we know that $W' \cap f^{-1}(\eta)$ has at least two curves, and one of them (the one close to $\partial W'$) separates the others from $\partial W'$. This implies that $f(W') = \widehat{\mathbb{C}}$. □

Let now I be a subset of $\widehat{\mathbb{C}}$. We define a separating number relative to I

$$S_I : \{\text{non-empty connected compact sets in } \widehat{\mathbb{C}} \text{ disjoint from } I\} \to \{1, 2, 3\}$$

as follows:

$$S_I(J) = \begin{cases} 1 & \text{if exactly one complementary component of } J \text{ intersects } I; \\ 2 & \text{if exactly two complementary components of } J \text{ intersects } I; \\ 3 & \text{if more than two complementary components of } J \text{ intersects } I. \end{cases}$$

Clearly $S_I(J) \leq C(J)$.

Let f be a rational map with disconnected Julia set $\mathcal{J}$.

We fix a normalization (up to conformal conjugacy) so that $f(\infty) = \infty$. Set

$$I = \{\infty\} \cup \{\text{the critical values of } f\}\ .$$

The set I is finite. It is not difficult to see $\{J \text{ a Julia component } \mid J \cap I = \emptyset,\ S_I(J) = 3\}$ is a finite set. Let

$$\begin{aligned} \mathcal{J}_{prep} &= \{\text{preperiodic Julia components of } f\}, \\ \mathcal{J}_I &= \{\text{comp. of } f^{-n}(J) \mid n \geq 0, J \text{ a Julia component}, J \cap I \neq \emptyset, \} \setminus \mathcal{J}_{prep}, \\ \mathcal{S}_{pre-3} &= \{\text{comp. of } f^{-n}(J) \mid \\ &\qquad n \geq 0,\ J \text{ a wandering Julia component}, J \cap I = \emptyset,\ S_I(J) = 3\}. \end{aligned}$$

For $i = 1, 2$,

$$\mathcal{S}_i = \{J \mid J \text{ a wandering Julia component}, J \cap I = \emptyset,\ S_I(J) = i\} \setminus (\mathcal{S}_{pre-3} \cup \mathcal{J}_I).$$

There are countably many Julia components in $\mathcal{J}_{prep} \cup \mathcal{J}_I \cup \mathcal{S}_{pre-3}$. But $\mathcal{S}_1 \cup \mathcal{S}_2$ consists of uncountably many Julia components.

Set $\mathcal{A} = \mathcal{S}_1 \cup \mathcal{S}_2$. In the following, we will focus on the set $\mathcal{A}$. From the definitions of these sets, we can see that $f(\mathcal{A}) \subset \mathcal{A}$ (by abuse of notation, we use f to denote the map induced by f on the set of Julia components).

The set $\mathcal{A}$ has the following decomposition. For $i = 1, 2, 3$, set

$$\mathcal{C}_i = \{J \in \mathcal{A} \mid C(J) = i\},$$

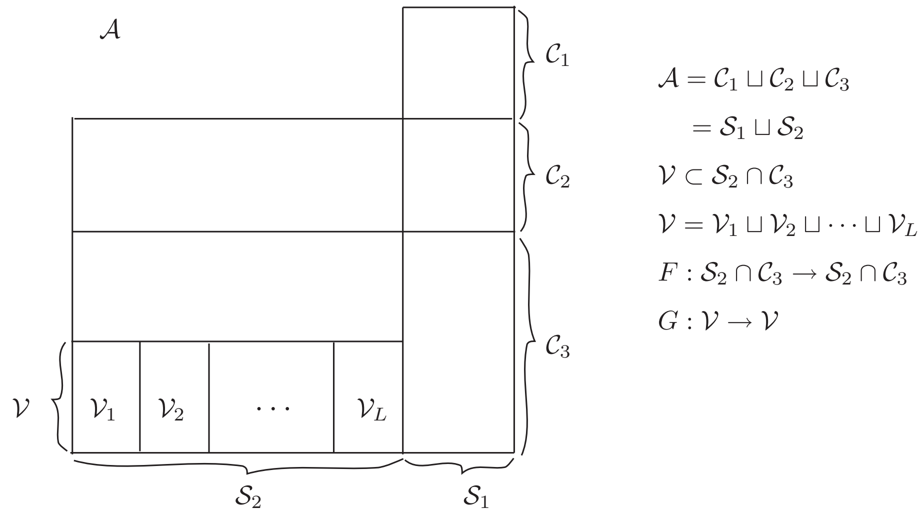

Figure 1: Decompositions of $\mathcal{A}$

Then by Lemma 2.1, $f^{-1}(\mathcal{C}_i) \cap \mathcal{A} \subset \mathcal{C}_i$ and $f(\mathcal{C}_i) \subset \mathcal{C}_i$. Furthermore $\mathcal{C}_1 \subset \mathcal{S}_1$.
Definition. (a) For a bounded compact connected set K in $\mathbb{C}$, denote by $\widehat{K}$, the fill-in, to be the union of K with all the bounded complementary components of K.

(b) For every $J \in \mathcal{S}_2$, i.e. J is a wandering Julia component with a unique bounded complementary component containing critical values, denote by $V(J)$ this complementary component .

For J', J'' two distinct components in $\mathcal{S}_2$, there are only three possible configurations:

$$\widehat{J''} \subset\subset V(J'), \quad \widehat{J'} \subset\subset V(J''), \quad \text{or} \quad \widehat{J''} \cap \widehat{J'} = \emptyset \ . \tag{1}$$

Lemma 2.2. *(a) For $J \in \mathcal{S}_1$ and for J' a component of $f^{-1}(J)$, the set $\widehat{J'}$ is mapped homeomorphically onto $\widehat{J}$ by f. In particular $f(W') = f_*(W')$ for every complementary component W' of J'.*

(b) For $J \in \mathcal{A}$, if $\#\mathrm{orb}(J) \cap \mathcal{S}_2 < \infty$, then $J \in \mathcal{C}_1$.

(c) If $\mathcal{C}_3 \neq \emptyset$ then $\mathcal{S}_2 \cap \mathcal{C}_3 \neq \emptyset$, and for any $J \in \mathcal{C}_3$, $\#\mathrm{orb}(J) \cap (\mathcal{S}_2 \cap \mathcal{C}_3) = \infty$.

Proof. (a) follows from the fact that for $J \in \mathcal{S}_1$, the set $\widehat{J}$ is simply connected and contains no critical values.

(b) Take an integer n_0 such that $\mathrm{orb}(f^{n_0}(J)) \cap \mathcal{S}_2 = \emptyset$. Then f maps $\widehat{f^n(J)}$ homeomorphically onto $\widehat{f^{n+1}(J)}$ for $n \geq n_0$.

We want to show that $\widehat{f^n(J)} \cap \widehat{f^m(J)} = \emptyset$ for all $n \neq m$, $n, m \geq n_0$.

Otherwise one is nested in the other.

If $\widehat{f^{n+p}(J)} \subset \widehat{f^n(J)}$ for some $n \geq n_0$ and $p \geq 1$, then by Schwarz Lemma, f^p has an attracting fixed point in $\mathrm{int}\widehat{f^n(J)}$ and $\mathrm{int}\widehat{f^n(J)}$ is contained in the attracting basin which contradicts with the fact that $f^{n+p}(J) \subset \mathrm{int}\widehat{f^n(J)}$. If $\widehat{f^{n+p}(J)} \supset \widehat{f^n(J)}$, then $\mathrm{int}\widehat{f^{n+p}(J)}$ is an open set intersecting $\mathcal{J}$. So $\mathrm{int}\widehat{f^{n+lp}(J)} \supset \mathcal{J}$ for some $l > 1$ which contradicts with the fact $f^{n+lp}(J) \subset \mathcal{J}$. Hence we know that $\widehat{f^n(J)} \cap \widehat{f^m(J)} = \emptyset$ for all $n \neq m \geq n_0$. We conclude that $f^{n_0}(J) \in \mathcal{C}_1$ for otherwise we could get a wandering Fatou domain in $\mathrm{int}\widehat{f^{n_0}(J)}$.

(c) Assume there is $J \in \mathcal{C}_3$. Then from $f(\mathcal{C}_3) \subset \mathcal{C}_3$ we have $\mathrm{orb}(J) \subset \mathcal{C}_3$. By (b), $\#\mathrm{orb}(J) \cap \mathcal{S}_2 = \infty$. So $\#\mathrm{orb}(J) \cap (\mathcal{S}_2 \cap \mathcal{C}_3) = \infty$, and $\mathcal{S}_2 \cap \mathcal{C}_3 \neq \emptyset$. □

Assume $J \in \mathcal{S}_2 \cap \mathcal{C}_3$. Let J' be a connected component of $f^{-1}(J)$. Take an annulus A containing J so that $\overline{A} \cap I = \emptyset$. In particular A contains no critical value.

Then there exists a connected component A' of $f^{-1}(A)$ which is also an annulus and which contains J'. Denote the inner boundary of A' by γ' and the complementary component of J' containing γ' by U'.

Denote by $\gamma^{\pm}$ the outer/inner boundary of A. The complementary component of J containing γ^- is simply $V(J)$.

We want to determine $f(W')$ explicitly for any bounded complementary component W' of J' :

Lemma 2.3. *(0) $f(\gamma')$ is one of $\gamma^{\pm}$.*

(1) Assume $f(\gamma') = \gamma^-$. Then $f_*(U') = V(J)$. If $\text{int}(\widehat{\gamma'}) \cap f^{-1}(\gamma) = \emptyset$, then $f(\widehat{\gamma'}) = \widehat{\gamma}$ and $f(U') = V(J)$. Otherwise $f(\widehat{\gamma'}) = f(U') = \widehat{\mathbb{C}}$.

(2) Assume $f(\gamma') = \gamma^+$. If $\text{int}(\widehat{\gamma'}) \cap f^{-1}(\gamma^+) = \emptyset$, then $f(\widehat{\gamma'}) = \widehat{\mathbb{C}} \backslash \text{int}(\widehat{\gamma^+})$, and $f(U')$ is equal to the unbounded complementary component of J. Otherwise, $f(\widehat{\gamma'}) = f(U') = \widehat{\mathbb{C}}$.

(3) U' is the unique bounded complementary component of J' intersecting $f^{-1}(I)$.

(4) For any bounded complementary component W' of J' with $W' \neq U'$, we have $f(W') = f_*(W')$, that $f(W')$ is a bounded complementary component of J, is distinct from $V(J)$, and $f : W' \to f(W')$ is a homeomorphism.

Proof. This lemma follows directly from Lemma 2.1. □

2.2 Proof of Theorem 1.1.(a).

Notice that the set of buried (resp. exposed) Julia components is fully invariant by f.

Let J be an exposed wandering Julia component. Then $J \cap \partial U \neq \emptyset$ for some Fatou domain U. Note that by Sullivan's theorem (see Theorem D) U is eventually periodic and $f^n(J) \cap \partial f^n(U) \neq \emptyset$ for all $n \geq n_0$. So replacing f by an iterate of f if necessary, we may assume $f(U) = U$. This implies that U is infinitely connected. By Fatou's classification of periodic Fatou domains, we have U is either attracting or parabolic (see [Mil1], [Bea]).

We now normalize f so that ∞ is the fixed point with U as a basin.

For this normalization we define I and the sets $\mathcal{J}_*, \mathcal{S}_*$ as above. Define the filled-in set $\widehat{J}$ accordingly.

We claim that $\widehat{f^n(J)}$ is a connected component of $\mathbb{C} \backslash U$ and $\widehat{f^n(J)} \cap \widehat{f^m(J)} = \emptyset$ for all $n \neq m$, $n, m \geq n_0$. In fact,

$$\begin{aligned} \widehat{f^n(J)} &\triangleq f^n(J) \cup (\text{the bounded complementary components of } f^n(J)) \\ &= \mathbb{C} \backslash (\text{the unbounded component of } \mathbb{C} \backslash f^n(J)) \\ &= \mathbb{C} \backslash (\text{the unbounded component of } \mathbb{C} \backslash B_n) \\ &\subset \mathbb{C} \backslash U, \end{aligned}$$

where B_n is a connected component of ∂U. Note that all the boundary components are pairwise disjoint, thus the claim holds. Combining with the fact that the set I is finite, we know $\text{orb}(f^{n_0}(J)) \cap \mathcal{S}_2$ is a finite set and then by Lemma 2.2, $f^n(J) \in \mathcal{C}_1$ for $n \geq n_1$ for some $n_1 \geq n_0$. That is, $C(\text{orb}(J)) = 1$. □

2.3 Proof of Theorem 1.1.(b).

Assume that the Julia set $\mathcal{J}$ of f is disconnected.

The fact that the set $\Sigma := \{J \mid J \text{ a Julia component}, \ C(J) = 3\}$ is backward invariant follows from Lemma 2.1.(i).

Set $\mathbb{Q}[i] = \{a + ib \in \mathbb{C} \mid a, b \in \mathbb{Q}\}$. This is a countable dense subset of $\widehat{\mathbb{C}}$. Define a map

$$\phi : \Sigma \to (\mathbb{Q}[i])^3, \quad J \mapsto (z_1, z_2, z_3) \in (\mathbb{Q}[i])^3$$

so that z_i and z_j are in distinct complementary components of J whenever $i \neq j$.

It is easily seen that ϕ is injective. So Σ is at most countable.

Assume $\Sigma \neq \emptyset$ and that Σ is finite. As Σ is backward invariant, one can define a multivalued map $f^{-1} : \Sigma \to \Sigma$ mapping J to a component of $f^{-1}(J)$. As Σ is finite, this map is actually single valued, and is bijective. This implies that for any $J \in \Sigma$ we have $f(J) \in \Sigma$ and that the grand orbit $\widetilde{J}$ of J consists of finitely many Julia components. It follows that the iterative sequence $\{f^n\}_{n \geq 1}$ is normal outside $\widetilde{J}$ and consequently $\mathcal{J} = \widetilde{J}$.

We then reach a contradiction as $\mathcal{J}$, being disconnected by assumption, must have uncountably many connected components.

It follows that Σ is either empty or countable. □

2.4 Preliminary results for the proof of Theorem 1.1.(c).

Assume for this entire subsection that $\mathcal{C}_3 \neq \emptyset$. It then follows from Lemma 2.2.(c) that

$$\mathcal{C}_3 \neq \emptyset \Longrightarrow \mathcal{S}_2 \cap \mathcal{C}_3 \neq \emptyset \ . \tag{2}$$

Fix a choice of a normalization so that ∞ is a fixed point. Then the Julia component containing ∞ (if any) is a fixed component. In any case every wandering Julia component is bounded.

Define now $I, \mathcal{J}_*, \mathcal{S}_*, \mathcal{A}, \widehat{J}$ as above for this normalization.

By Lemma 2.2.(c), every Julia component J in $\mathcal{S}_2 \cap \mathcal{C}_3$ returns eventually to $\mathcal{S}_2 \cap \mathcal{C}_3$ under the iterations of f. Denote by $s(J) > 0$ the minimal return time. This induces a first-return time function together with a first-return map:

$$s : \mathcal{S}_2 \cap \mathcal{C}_3 \to \mathbb{N}^*, \ J \to s(J);$$
$$F : \mathcal{S}_2 \cap \mathcal{C}_3 \to \mathcal{S}_2 \cap \mathcal{C}_3, \ J \to f^{s(J)}(J).$$

Notice that $\{f^n(J) \mid f^n(J) \in \mathcal{S}_2\} = \text{orb}(J) \cap \mathcal{S}_2 = \text{orb}_F(J)$ and F is injective on $\text{orb}_F(J)$, thus F is injective on $\text{orb}(J) \cap \mathcal{S}_2$.

Let now $J \in \mathcal{S}_2 \cap \mathcal{C}_3$. Applying Lemma 2.3 we know that there exists only one bounded complementary component of $f^{s(J)-1}(J)$ intersecting $f^{-1}(I)$. Notice that $f^{s(J)-1}$ maps $\widehat{J}$ onto $\widehat{f^{s(J)-1}(J)}$ homeomorphically. Hence we conclude that there exists only one bounded complementary component of J intersecting $f^{-s(J)}(I)$. Set

$$U(J) := \text{the unique bounded complementary component of } J \text{ intersecting } f^{-s(J)}(I) \ .$$

It may or may not occur that $U(J) = V(J)$.

Lemma 2.4. *Let $J \in \mathcal{S}_2 \cap \mathcal{C}_3$. Let P be a bounded complementary component of J with $P \neq U(J)$. Then there is a minimal integer k such that P is mapped by some iterate of f onto $U(F^k(J))$. Moreover for this k, $U(F^k(J)) \neq V(F^k(J))$.*

Proof. Set $J_i = F^i(J)$ for $i \geq 1$.

As $P \neq U(J)$, $f^{s(J)}(P)$ is a bounded complementary component of J_1, and is distinct from $V(J_1)$.

If $f^{s(J)}(P) \neq U(J_1)$, then $f^{s(J)+s(J_1)}(P)$ is a bounded complementary component of J_2, and is distinct from $V(J_2)$. If again $f^{s(J)+s(J_1)}(P) \neq U(J_2)$, then $f^{s(J)+s(J_1)+s(J_2)}(P)$ is a bounded complementary component of J_3, and is distinct from $V(J_3)$, and so on.

$$\begin{array}{ccccccccccc} J & & J_1 & & J_2 & & \cdots & & J_{k-1} & & \cdots \\ P & \xrightarrow{f^{s(J)}} & * & \xrightarrow{f^{s(J_1)}} & * & \longrightarrow & \cdots & \longrightarrow & * & \longrightarrow & \cdots \\ & & \neq V(J_1) & & \neq V(J_2) & & \cdots & & \neq V(J_{k-1}) & & \cdots \\ \neq U(J) & & \neq U(J_1) & & \neq U(J_2) & & \cdots & & \neq U(J_{k-1}) & & \cdots \end{array}$$

Assume that the lemma is not true. Then for any $n \geq 1$, $f^{s(J)+s(J_1)+s(J_2)+\cdots+s(J_{n-1})}(P)$ is a bounded complementary component of J_n, and is distinct from $U(J_n)$ and from $V(J_n)$. If follows from (1) and the fact that $J_n \neq J_m$ if $n \neq m$, that the open sets $f^{s(J)+s(J_1)+s(J_2)+\cdots+s(J_{n-1})}(P)$, $n \geq 1$ are pairwise disjoint. Consequently P is a wandering Fatou domain. Impossible.

There is therefore a minimal k such that $f^{s(J)+s(J_1)+s(J_2)+\cdots+s(J_{k-1})}(P) = U(J_k)$. Due the minimality of k and the above diagram, we see that $U(J_k) \neq V(J_k)$ for this k. □

Set

$$\mathcal{V} = \{J \in \mathcal{S}_2 \cap \mathcal{C}_3 \mid V(J) \neq U(J)\}.$$

The above lemma proves:

$$\mathcal{S}_2 \cap \mathcal{C}_3 \neq \emptyset \Longrightarrow \mathcal{V} \neq \emptyset\ . \tag{3}$$

Lemma 2.5. *Every F-orbit visits $\mathcal{V}$ infinitely many times.*

Proof. Recall that F maps $\mathcal{S}_2 \cap \mathcal{C}_3$ into $\mathcal{S}_2 \cap \mathcal{C}_3$. Let $J \in \mathcal{S}_2 \cap \mathcal{C}_3$. Then there is a bounded complementary component P of J with $P \neq U(J)$. By above there is a minimal k such that P is mapped onto $U(F^k(J))$, and for this k, $F^k(J) \in \mathcal{V}$. □

Start now from a $J \in \mathcal{V}$. We will try to follow the orbit of $V(J)$. By definition of $\mathcal{V}$, we know that $V(J) \neq U(J)$ so one can apply Lemma 2.4 to $P = V(J)$. Let now k to be the minimal integer so that $V(J)$ is mapped onto $U(F^k(J))$. We know also $F^k(J) \in \mathcal{V}$. Set $J_i = F^i(J)$. Here is a schematic picture:

$$\begin{array}{ccccccccccc} J & & J_1 & & J_2 & & \cdots & & J_{k-1} & & J_k = G(J) \\ V(J) & \xrightarrow{f^{s(J)}} & * & \xrightarrow{f^{s(J_1)}} & * & \longrightarrow & \cdots & \longrightarrow & * & \xrightarrow{f^{s(J_{k-1})}} & U(J_k) \\ & & \neq V(J_1) & & \neq V(J_2) & & \cdots & & \neq V(J_{k-1}) & & \neq V(J_k) \\ \neq U(J) & & \neq U(J_1) & & \neq U(J_2) & & \cdots & & \neq U(J_{k-1}) & & \end{array} \tag{4}$$

This induces a pair of maps:

$$\begin{array}{l} n : \mathcal{V} \to \mathbb{N}^*,\ J \to n(J) := s(J) + s(J_1) + \cdots + s(J_{k-1}); \\ G : \mathcal{V} \to \mathcal{V},\ J \to F^k(J) = f^{n(J)}(J)\ . \end{array}$$

Therefore we have the following commutative diagram:

$$\begin{array}{cccccc} \mathcal{V} \ni & J & \xrightarrow{G} & G(J) & & \in \mathcal{V} \\ & V \downarrow & & \downarrow U & & \\ U(J) \neq & V(J) & \underset{f^{n(J)}}{\longrightarrow} & U(G(J)) & \neq V(G(J)). & \end{array} \tag{5}$$

We define an equivalence relation on $\mathcal{V}$: two distinct components J, J' of $\mathcal{V}$ are said to be equivalent if $V(J)$ and $V(J')$ contain the same set critical values.

Since there are only finitely many critical values, the set $\mathcal{V}$ is decomposed into finitely many equivalence classes: $\mathcal{V} = \cup_{i=1}^{L} \mathcal{V}_i$.

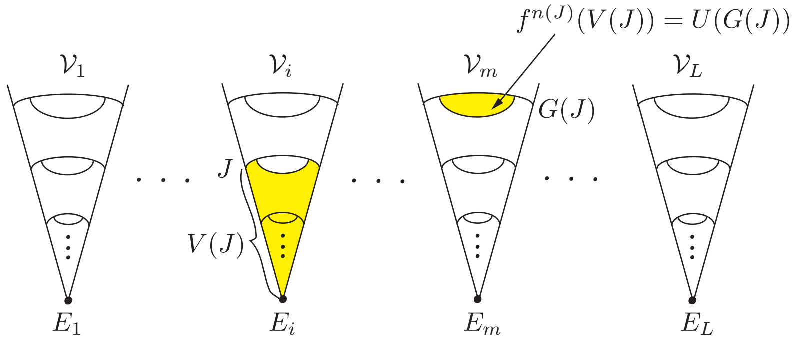

Figure 2: The equivalence classes in $\mathcal{V}$

Definition (ordering). We define a nesting ordering in each class $\mathcal{V}_i$ by:

$$J \prec J' \text{ if } J' \text{ is more deeply nested, that is, } V(J) \supset\supset \widehat{J'}.$$

Lemma 2.6. *Consider any equivalence class $\mathcal{V}_i$.*

(a) The relation $\prec$ is a total ordering.

(b) Both maps $s : \mathcal{V}_i \to \mathbb{N}^*$ and $n : \mathcal{V}_i \to \mathbb{N}^*$ are strictly increasing.

(c) The set $\mathcal{V}_i$ is at most countable.

(d) Every infinite sequence of distinct elements in $\mathcal{V}_i$ (if any) admits an increasing subsequence with respect to $\prec$.

(e) For any $J \in \mathcal{V}_i$, either $G(J) \in \mathcal{V}_l$ for some $l \neq i$ or $G(J) \in \mathcal{V}_i$ and $G(J) \prec J$.

Proof. (a) is evident by the definition of the equivalence relation in $\mathcal{V}$.

(b) Fix a pair $J, J' \in \mathcal{V}_i$ such that $J' \succ J$, we will prove a stronger statement:

$$n(J') \geq s(J') > n(J) \geq s(J). \tag{6}$$

The fact $n(J) \geq s(J)$ follows directly from the equation $n(J) = s(J) + s(J_1) + \cdots + s(J_{k-1})$. Now we show that $s(J') > n(J)$ holds. Notice that $f^i(\widehat{J'}) \subset f^i(V(J))$ for $1 \leq i \leq n(J)$. So by above $f^i(J')$ does not separate I for $i \leq n(J)$. Hence $s(J') > n(J)$. This proves (6) as well as (b).

(c) follows from the injectivity of $s : \mathcal{V}_i \to \mathbb{N}^*$ (as s is strictly increasing). Alternatively, one can obtain this from the fact that $\mathcal{V}_i \subset \mathcal{C}_3 \subset \Sigma$ and by part (b) of Theorem 1.1 the set Σ is at most countable.

(d) By the injectivity of s, we have

$$\forall J \in \mathcal{V}_i, \quad \#\{J' \in \mathcal{V}_i \mid J' \prec J\} < \infty . \tag{7}$$

Property (d) follows.

(e) Assume the contrary, i.e. $G(J) \in \mathcal{V}_i$ and $J \prec G(J)$. Then $V(J) \supset\supset \widehat{G(J)}$. But then

$$f^{n(J)}(V(J)) \overset{(5)}{=} U(G(J)) \subset \widehat{G(J)} \subset\subset V(J) .$$

By Schwarz Lemma $f^{n(J)}$ has an attracting fixed point in $V(J)$ and every point in $V(J)$ is in the attracting basin. This contradicts that $V(J)$ contains Julia points, for instance $G(J)$. Property (e) follows. □

For each $i \in \{1, \cdots, L\}$, set

$$E_i = \cap\{V(J') \mid J' \in \mathcal{V}_i\}.$$

Clearly E_i contains critical values and is simply connected. It follows from above that for every infinite sequence $J_1, J_2, \cdots$ of distinct elements of $\mathcal{V}_i$ (if any), we have $E_i = \cap_{m=1}^{\infty} V(J_m)$.

Lemma 2.7. *If an equivalence class $\mathcal{V}_i$ receives infinitely many visits of a single orbit $\mathrm{orb}_G(J)$, $J \in \mathcal{V}$, then*
(i) E_i is a Julia component that is wandering, buried, simply connected, and that contains critical values,
(ii) $E_i \notin \mathrm{orb}_f(E_i) \backslash \{E_i\}$, and for any other $\mathcal{V}_j$ such that $\mathrm{orb}_G(J)$ visits $\mathcal{V}_j$ infinitely many times, $E_j \notin \mathrm{orb}_f(E_i)$.

Proof. (i) For any $f^k(J) \in \mathcal{V}_i$, set $s_k = s(f^k(J))$. Take an infinite increasing sequence $k_1 < k_2 < \cdots$ such that

$$\{f^{k_1}(J), f^{k_2}(J), \cdots\} \subset \mathcal{V}_i\ .$$

By Lemma 2.6.(d) and (b), passing to a subsequence if necessary (denoted again by $k_1, k_2, \cdots$), we may assume $s_{k_1} < s_{k_2} < \cdots$.

Then $E_i = \bigcap_{m=1}^{\infty} V(f^{k_m}(J))$. From $E_i \subset V(f^k(J))$ for $k = k_1, k_2, \cdots$ and the relation (4) , we have

$$f^{s_k}(E_i) \subset f^{s_k}(V(f^k(J))) \neq V(f^{s_k+k}(J)) \quad \text{for each } k = k_1, k_2, \cdots\ .$$

Since F is injective on $\mathrm{orb}_F(J)$, we have $\{F(f^{k_m}(J))\}_{m \geq 1}$ are pairwise distinct.

It follows from the relation (1) in section 2.1 that the open sets $f^{s_{k_1}}(V(f^{k_1}(J)))$, $f^{s_{k_2}}(V(f^{k_2}(J))), \cdots$ are pairwise disjoint. Consequently $f^{s_{k_1}}(E_i), f^{s_{k_2}}(E_i), \cdots$ are pairwise disjoint. Hence the set E_i contains no points in the Fatou set for otherwise we would get a wandering Fatou domain in E_i. So it must be contained in a Julia component $\tilde{J}$ with $C(\tilde{J}) = 1$.

If $E_i \neq \tilde{J}$, then $\tilde{J}$ will intersect some $J' \in \mathcal{V}_i$. Note that J' and $\tilde{J}$ are both Julia components, so $\tilde{J} = J'$. But E_i is contained in a complementary component of J'. A contradiction. So $E_i = \tilde{J}$, it is a wandering Julia component, and is buried.

(ii) Notice that for each $1 \leq l \leq s_k$, $f^l(E_i) \subset f^l(V(f^k(J)))$ and $f^l(V(f^k(J))) \cap I = \emptyset$. On the other hand, $E_i \cap I \neq \emptyset$ and $E_j \cap I \neq \emptyset$. Hence $f^l(E_i) \neq E_i, E_j$ for each $1 \leq l \leq s_k$. Since $\lim_{m\to\infty} s_{k_m} = +\infty$, we conclude that (ii) holds. □

Definition. For a pair E_i, E_m (they may be equal), we say that E_i *accumulates to* E_m, written as $E_i \to E_m$, if any neighborhood of E_m contains infinitely many of $f^k(E_i)$, $k > 0$.

Lemma 2.8. *Fix $J \in \mathcal{V}$, and i such that the equivalence class $\mathcal{V}_i$ receives infinitely many visits of $\mathrm{orb}_G(J)$ (such class exists always).*
(a) There is then $m \neq i$ satisfying the following two properties:
- *$\mathcal{V}_m$ receives infinitely many visits of $\mathrm{orb}_G(J)$;*
- *the set E_i accumulates to E_m, i.e. $E_i \to E_m$.*

(b) There is $q \neq m$ (but q may be equal to i) such that $E_m \to E_q$ and $E_i \to E_q$.

Proof. (a). By Lemma 2.6.(e) and formula (7) , we can find an infinite sequence

$$t_1 < t_2 < \cdots$$

such that, for any l, $f^{t_l}(J) \in \mathcal{V}_i \cap \mathrm{orb}_G(J)$ and $G(f^{t_l}(J)) \in \mathcal{V}_{j_l}$ for some $j_l \neq i$. As there are only finitely many choices for j_l, there are $m \neq i$ so that $j_l = m$ for infinitely many l.

Fix any such m. By passing to a subsequence of $\{t_l\}$, denoted still by $\{t_l\}$, we may assume $j_l \equiv m$, i.e. $G(f^{t_l}(J)) \in \mathcal{V}_m$ for all l. In particular $\mathcal{V}_m$ receives infinitely many visits of $\mathrm{orb}_G(J)$.

We apply now Lemma 2.6.(d) to the infinite sequence $\{f^{t_l}(J), l \in \mathbb{N}\} \subset \mathcal{V}_i$. Thus after passing to a subsequence of $\{t_l\}$, denoted still by $\{t_l\}$, we have

$$f^{t_1}(J) \prec f^{t_2}(J) \prec \cdots \ .$$

For any $f^t(J) \in \mathcal{V}_i$, set $n_t = n(f^t(J))$. As the map n is increasing with respect to $\prec$ (Lemma 2.6.(b)), we have

$$n_{t_1} < n_{t_2} < \cdots \ .$$

Note that G is injective on the orbit $\mathrm{orb}_G(J)$. So $\{G(f^{t_l}(J)), l \in \mathbb{N}\}$ are pairwise distinct.

We apply now Lemma 2.6.(d) to the infinite sequence $\{G(f^{t_l}(J)), l \in \mathbb{N}\} \subset \mathcal{V}_m$. Thus after passing to subsequences of $\{t_l\}$, denoted still by $\{t_l\}$, we have

$$G(f^{t_1}(J)) \prec G(f^{t_2}(J)) \prec \cdots \ .$$

In other words G restricted to this sequence preserves the nesting order from $\mathcal{V}_i$ to $\mathcal{V}_m$.

On the other hand, for any $l \geq 1$, we have $E_i \subset V(f^{t_l}(J))$ and $f^{n_{t_l}}(V(f^{t_l}(J))) = U(G(f^{t_l}(J)))$ by (5) . Therefore, for any $l \geq 2$,

$$\begin{aligned} f^{n_{t_l}}(E_i) \subset f^{n_{t_l}}(V(f^{t_l}(J))) = U(G(f^{t_l}(J))) \quad &\subset \quad \widehat{G(f^{t_l}(J))} \backslash V(G(f^{t_l}(J))) \\ &\subset\subset \quad V(G(f^{t_{l-1}}(J))) \backslash E_m \ . \end{aligned}$$

But $\bigcap_l V(G(f^{t_{l-1}}(J))) = E_m$. Hence E_i accumulates to E_m.

(b) Applying part (a) but with $\mathcal{V}_i$ replaced by $\mathcal{V}_m$, we find $\mathcal{V}_q$ for $q \neq m$ so that $\mathcal{V}_q$ contains infinitely many $G(f^v(J))$, $f^v(J) \in \mathrm{orb}(J) \cap \mathcal{V}_m$, and $E_m \to E_q$.

We want to show $E_i \to E_q$. In fact, for any $J^* \in \mathcal{V}_q$, there exists $f^v(J) \in \mathcal{V}_m$ such that

$$G(V(f^v(J))) = U(G(f^v(J))) \subset \widehat{J}^*.$$

Since $E_i \to E_m$, there exists $f^\xi(E_i) \subset V(f^v(J))$. Then $G(f^\xi(E_i)) \subset G(V(f^v(J))) \subset \widehat{J}^*$ and therefore $E_i \to E_q$. □

2.5 Proof of Theorem 1.1.(c).

Assume that a rational map f has a wandering Julia component J so that $C(\mathrm{orb}(J)) = 3$, in other words $C(f^n(J)) = 3$ for any $n \geq 0$.

Replacing J by a forward iterate of it if necessary, we may assume that for any $n \geq 0$, $f^n(J) \cap I = \emptyset$ and $S_I(f^n(J)) \leq 2$. Therefore $J \in \mathcal{S}_1 \cup \mathcal{S}_2$. As $C(J) = 3$ it follows from the definition that $J \in \mathcal{C}_3$, and therefore $\mathcal{C}_3 \neq \emptyset$.

It follows then from (2) and (3) that $\mathcal{S}_2 \cap \mathcal{C}_3 \neq \emptyset$ and $\mathcal{V} \neq \emptyset$. So $\mathcal{V}$ is decomposed into finitely many (non-empty) equivalence classes $\mathcal{V}_1, \cdots, \mathcal{V}_L$. The map $G : \mathcal{V} \to \mathcal{V}$ is also well defined.

Fix $J \in \mathcal{V}$. Then there is a subset Λ of the index set $\{1, \cdots, L\}$ such that each $\mathcal{V}_i$, $i \in \Lambda$ receives infinitely many visits of $\text{orb}_G(J)$. By Lemma 2.8 and the finiteness of Λ one can find a cycle $\mathcal{V}_{i_1} = \mathcal{V}_{i_{p+1}}, \mathcal{V}_{i_2}, \cdots, \mathcal{V}_{i_p}$ such that

$$E_{i_1} \to E_{i_2} \to \cdots \to E_{i_p} \to E_{i_1},$$

and $E_{i_1} \rightleftarrows E_{i_p}$. Set $J_1 = E_{i_1}$ and $J_2 = E_{i_p}$. By Lemmas 2.7 and 2.8 they satisfy $(*)$ required by Theorem 1.1.(c). ☐

This ends the proof of Theorem 1.1.

Corollary 2.9. *Each equivalence class $\mathcal{V}_i$ can be represented by a Jordan curve γ_i in the same homotopic class (rel I) of a Julia component in $\mathcal{V}_i$. Then $\Gamma = \{\gamma_i \mid i = 1, \cdots, L\}$ is a multicurve. If $\gamma_k \in \Gamma$ such that both complementary components of γ_k contains curves of Γ up to homotopy, then the corresponding class $\mathcal{V}_k$ can only be visited at most finitely many times by an orbit in $\mathcal{C}_3$.*

3 Point Julia components

In this section, we state two sufficient conditions for a Julia component to be a point. Let f be a rational map with disconnected Julia component $\mathcal{J}$.

The following proposition follows directly from Shrinking Lemma of Fatou.

Proposition 3.1. *Suppose W is a complementary component of a Julia component such that $W \cap \mathcal{J} \neq \emptyset$ and W is disjoint from the postcritical set. Then any Julia component J with the f-orbit of J visiting W infinitely many times is a point.*

Let $J_1, J_2, \cdots, J_L$ be preperiodic Julia components and $V_1, V_2, \cdots, V_L$ be pairwise disjoint complementary components of $J_1, J_2 \cdots, J_L$ respectively. Denote $\mathbf{V} = \cup_{i=1}^{L} V_i$. Let $\widetilde{\mathbf{U}} = \{z \in \mathbf{V} \mid \exists\ l \geq 1 \text{ such that } f^l(z) \in \mathbf{V}\}$. Suppose $\widetilde{\mathbf{U}} \neq \emptyset$. Define the first-return map

$$\widetilde{R} : \widetilde{\mathbf{U}} \to \mathbf{V},\ z \to f^{u(z)}(z),$$

where $u(z) \geq 1$ is the smallest integer with $f^{u(z)}(z) \in \mathbf{V}$. Let W be a component of $\widetilde{\mathbf{U}}$. Then there exists an integer u such that $u = u(z)$ for all $z \in W$ and $f^u(W)$ is some V_i. Choose $\mathbf{U}$ to be the union of finitely many components of $\widetilde{\mathbf{U}}$ which are simply connected and compactly contained in $\mathbf{V}$. Define $R : \mathbf{U} \to \mathbf{V}$ as $R = \widetilde{R}|_{\mathbf{U}}$. Define the filled-in Julia set of the map R

$$K_R := \{z \in \mathbf{U} \mid R^n(z) \in \mathbf{U}, n \geq 1\}.$$

We have the following proposition.

Proposition 3.2. *A component K of K_R is a point if and only if the R-orbit of K contains no periodic components with critical points of R.*

Proof. The "only if" part is easy: suppose K is a point, then so is each $R^n(K), n \geq 1$. If $R^{n_0}(K)$ is a periodic critical point for some $n_0 \geq 1$, that is it is a superattracting periodic point of R, then it must contain a superattracting Fatou domain of R. It is impossible.

For the "if" part, we will only give a sketch of the proof. One can consult [QY], [PQRTY] for details.

The connected components of $R^{-n}(\mathbf{U})$ are called puzzle pieces of depth n. For a component $K_R(x)$ of K_R containing x such that the R-orbit of it contains no periodic components with critical points of R, we shall find a nested sequence containing it

$$P_{n_1}(x) \supset P_{n_1'}(x) \supset P_{n_2}(x) \supset P_{n_2'}(x) \supset \cdots$$

such that $\sum_{i=1}^{\infty}(\mathrm{mod}(P_{n_i}(x)\backslash\overline{P_{n_i'}(x)})) = \infty$. Then by Grötzsch inequality,

$$\mathrm{mod}(P_{n_1}(x)\backslash\overline{K_R(x)}) = \infty$$

and hence $K_R(x) = \{x\}$ is a point.

For any $x \in K_R$, the definition of tableau $T(x)$ can be found in [QY], [PQRTY]. Briefly speaking, it is a two dimensional array $P_n(R^l(x))$ associated with the R-orbit of x. For $x, y \in K_R$, we say the R-orbit of x combinatorially accumulates to y, written as $x \to y$, if for any $n \geq 0$, there exists an integer $j > 0$ such that $f^j(x) \in P_n(y)$ (this definition follows Definition 1 in [QY]). One can find the definitions that the tableau $T(c)$ for a critical point c is non-critical or periodic or reluctantly recurrent or persistently recurrent in [QY] (ref. Definitions 1 and 2 in [QY]).

Let

$$\mathrm{Crit}(x) = \{c \in K_R \mid x \to c\}.$$

There are the following five possibilities.

Case a. $\mathrm{Crit}(x) = \emptyset$.
Case b. $\exists\ c \in \mathrm{Crit}(x)$ such that $T(c)$ is non-critical.
Case c. $\exists\ c \in \mathrm{Crit}(x)$ such that $T(c)$ is reluctantly recurrent.
Case d. $\exists\ c \in \mathrm{Crit}(x)$ such that $T(c)$ is periodic.
Case e. $\exists\ c \in \mathrm{Crit}(x)$ such that $T(c)$ is persistently recurrent but not periodic.

In Cases a, b, c, d, we can easily construct the desired sequence (see Propositions 1 and 2 in [QY]). In Case e, applying Theorem 2.1 in [PQRTY] together with the proof of Main Proposition in [QY], we can find the desired sequence. □

Corollary 3.3. *A wandering Julia component J with $R^n(J) \subset \mathbf{U}$ for all $n \geq 0$ is a point.*

4 Examples

Wandering Julia components are classified either by their topology (simply, doubly or multiply connected), or by their positions relative to the Fatou components (exposed or buried). They may or may not contain critical orbits. One may question about the existence and the co-existence of various types of such Julia components. The examples here are built to answer some of these questions.

4.1 Preliminary results

We have shown in Theorem 1.1(a) that if a wandering Julia component is exposed, it must be eventually simply connected.

We have also:

Lemma 4.1. *Let f be a rational map. There are eventually simply connected and exposed wandering Julia components if and only if there is an infinitely connected Fatou domain.*

Proof. Let U be an infinitely connected Fatou domain. Then it must contain uncountably many boundary components. Note that each boundary component is contained in a unique Julia component and there are only countably many preperiodic Julia components. Thus there exists a wandering exposed Julia component. By Theorem 1.1(a), it is eventually simply connected.

Now let J be an eventually simply connected and exposed wandering Julia component. Then there exists a Fatou domain U such that $f^n(J) \cap \partial f^n(U) \neq \emptyset$ for all $n \geq 0$, and there exists an integer $n_0 \geq 0$ such that $f^n(U)$ is periodic and $f^n(J)$ is simply connected for $n \geq n_0$. So we can assume that J itself is simply connected and $f(U) = U$. Each $f^n(J)$ contains a unique boundary component for $n \geq 0$ because $f^n(J) \cap \partial U \neq \emptyset$. Combining with the condition that J is wandering and the fact that f maps each boundary component onto another, we can find a boundary component B with $f^i(B) \cap f^j(B) = \emptyset$ and hence U has infinitely many boundary components. □

Lemma 4.2. ([PT1]). *Let f be a nice rational map with the Julia set $\mathcal{J}_f$ and the postcritical set P_f . Let E be the union of Julia components J of f so that either $J \cap P_f \neq \emptyset$, or $S_{P_f}(J) \geq 3$, or $S_{P_f}(J) = 2$ but J does not separate any pair of Julia components in the same homotopy class. If $\widehat{\mathbb{C}} \backslash E$ has no annular components that are disjoint from P_f, then all wandering components of $\mathcal{J}_f$ are simply connected.*

4.2 The disc-annulus surgery

The following surgery exposed in [PT2] will be our essential tool to construct various examples.

We say that $V \subset \widehat{\mathbb{C}}$ is a *smooth disc* if ∂V is a real-analytic Jordan curve. By a *branched covering* we mean a proper C^1 map between smooth, oriented (real) 2-manifolds, possibly with boundary, such that the boundary map is a covering map of (real) 1-manifolds, and such that on the interior, the map is given in appropriate local (complex) coordinates by $z \mapsto z^d$ for some d.

The following two lemmas are the technical ingredients for the definition of our disc-annulus surgery.

Lemma 4.3. *Let $A \subset \widehat{\mathbb{C}}$ be an open annulus bounded by two C^1 Jordan curves $\gamma^{\pm}$, and let W be an open disc bounded by a C^1 Jordan curve η. Give orientations to the curves such that A and W lie to the left of their boundaries. Let $f^{\pm} : \gamma^{\pm} \to \eta$ be two orientation-preserving C^1-coverings with degree $d^{\pm} \geq 1$. Then there exists a branched covering $a : \overline{A} \to \overline{W}$ satisfying the following properties:*

1. $a|_{\gamma^{\pm}} = f^{\pm}$

2. $a(A) = W$ and the degree of a is $d^+ + d^-$

3. a can be chosen to be C^1 in $\overline{A}$ and holomorphic and proper in a union of any collection of finitely many disjoint smooth discs.

We call the map a a *covering extension* of the boundary maps $f^{\pm}$. In practice, we will take a to be holomorphic in a neighborhood of its critical points.

Lemma 4.4. *Let D, D' be two smooths discs in $\widehat{\mathbb{C}}$. Then a holomorphic proper mapping $F : D \to D'$ extends to a holomorphic map in a neighborhood of $\overline{D}$. In particular $F : \partial D \to \partial D'$ is a C^1 covering.*

A branched covering $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *quasi-regular* if $F = h \circ f \circ g$ where f is a rational map and h, g are quasi-conformal homeomorphisms. A branched covering

F is *quasi-rational* if it is quasi-conformally conjugate to a rational map. The Julia set of a quasi-rational map is thus well defined, and has the same qualitative metric and measure-theoretic properties as the Julia set of a rational map. Denote the Julia set of F by $\mathcal{J}_F$.

Given a rational map or branched covering f, let

$$P_f = \text{the } \textbf{postcritical set} = \overline{\cup_{n>0} f^n(C_f)}$$

where C_f is the set of critical points of f.

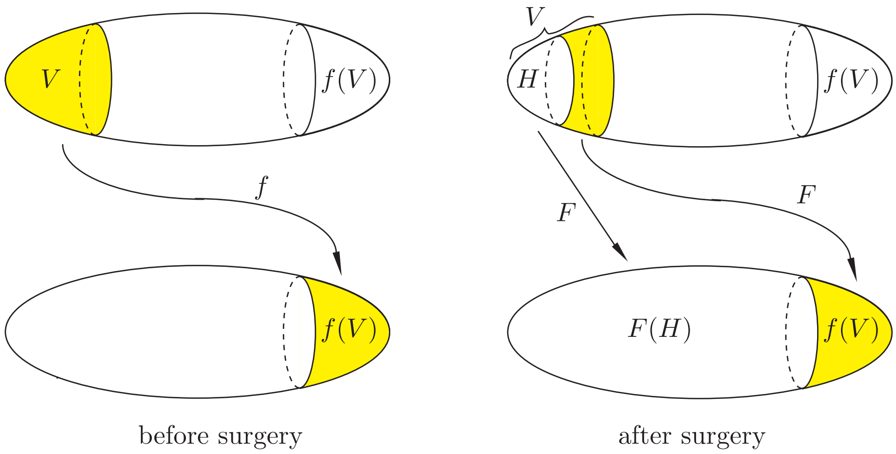

Figure 3: surgery

The following is a particular case of results in [PT2].

Theorem 4.5. Disc-annulus surgery. *Let f be a rational map with Julia set $\mathcal{J}_f$. Let z_0 be a point in the Fatou set such that z_0 is neither a periodic point nor contained in a rotation domain. Let (V, H, h, a) satisfy the following conditions:*

- *V is a smooth disc containing z_0 in the Fatou set such that*
  - *∂V contains no critical points;*
  - *$f : V \to f(V)$ is proper;*
  - *$f^j(\overline{V}) \cap V = \emptyset$ for $0 < j < \infty$,*
- *H is a smooth disc with $\overline{H} \subset V$;*
- *$h : H \to \widehat{\mathbb{C}} \backslash \overline{f(V)}$ is bi-holomorphic, and*
- *$a : V \backslash \overline{H} \to f(V)$ is a covering extension of the boundary maps such that a is holomorphic near the critical points.*

Then the map

$$F : \begin{cases} \widehat{\mathbb{C}} \backslash V \to f(\widehat{\mathbb{C}} \backslash V) & F(z) = f(z) \\ H \to \widehat{\mathbb{C}} \backslash f(\overline{V}) & F(z) = h(z) \\ \overline{V} \backslash H \to \overline{f(V)} & F(z) = a(z) \end{cases}$$

is quasi-rational.

We refer to F as a *disc-annulus surgery of f supported on a neighborhood of z_0.* Note that $F(V) = \widehat{\mathbb{C}}$ and $\deg(F) = \deg(f) + \deg(h)$.

Corollary 4.6. *In the above setting, denote by W_f, resp. W_F the Fatou components of f. resp. F, containing ∂V. If W_f is periodic, then W_F is periodic and infinitely connected. If W_f is strictly preperiodic, then the connectivity of W_F is equal to $m_0 + m_1$, where m_0 and m_1 are the connectivity's of W_f and $f(W_f)$. Furthermore*

1. *$\mathcal{J}_f \subset \mathcal{J}_F$, $\mathcal{J}_F$ is disconnected, and every connected component of $\mathcal{J}_f$ is a connected component of $\mathcal{J}_F$.*

2. *If $H \cap P_f = \emptyset$, then also $H \cap P_F = \emptyset$. In this case,*
   - *every Julia component of F passing through H infinitely many times is a point, and*
   - *every other Julia component of F is conformally homeomorphic to a Julia component of f.*

3. *$H \cap \mathcal{J}_F \neq \emptyset$. If $H \cap P_f = \emptyset$ and if W_f is not fixed under the iteration of f, then there is a cantor set L contained in H with each point of L being a buried Julia component for F.*

Proof. The proofs of 1 and 2 can be found in [PT2]. Here we just prove 3.

3. As h is bi-holomorphic, one may define $\xi : \widehat{\mathbb{C}}\backslash f(\overline{V}) \to H$ as the inverse map of h. It is also a univalent branch of F^{-1} on $\widehat{\mathbb{C}}\backslash f(\overline{V})$. By Schwarz Lemma ξ has a unique attracting fixed point w_0 in H. Consequently w_0 is a repelling fixed point of F, thus is contained in $\mathcal{J}_F$.

Assume now $H \cap P_f = \emptyset$ and $f(W_f) \neq W_f$. Consequently $H \cap P_F = \emptyset$ and $f(W_f) \cap W_f = \emptyset$. There is therefore a Julia component J' separating W_f and $f(W_f)$ (i.e. with W_f and $f(W_f)$ contained in distinct complementary components of J). As $f(\overline{V}) \subset f(W_f)$, the component J' separates $\overline{V}$ from $f(\overline{V})$. In other words, J' is essentially contained in the annulus $\widehat{\mathbb{C}}\backslash(\overline{H} \cup f(\overline{V}))$.

Set $U_i = \xi^i(H)$, $i = 1, 2$. They are Jordan domains. The annulus $U_1\backslash\overline{U}_2$ contains Julia points as it contains $\xi^2(J')$. By classical properties of the Julia set, the set of preimages of w_0 is dense in $\mathcal{J}_F$. In particular one can find a point $w' \in U_1\backslash\overline{U}_2$ with $F^{i_0}(w') = w_0$ for some $i_0 \geq 1$.

Let $k \geq 1$ be the first time such that $F^k(w') \in H$. Let W be the component of $F^{-k}(H)$ containing w'. As H is a Jordan domain with $\overline{H} \cap P_F = \emptyset$, we know that W is also a Jordan domain and F^k maps W univalently onto H. From the choice of w' we know that $W \cap (U_1\backslash\overline{U}_2) \neq \emptyset$. If there is $z \in (\partial U_1 \cup \partial U_2) \cap \overline{W}$, then $F^j(z) \subset f(\overline{V})$ for $j = 2$ or 3. Since $f^n(\overline{V}) \cap \overline{V} = \emptyset$, W can not be mapped into H by the forward iterations of F. It is impossible. Hence $\overline{W} \subset U_1\backslash\overline{U}_2$.

Define $G : U_2 \cup W \to H$ by $G|_{U_2} = F^2$ and $G|_W = F^k$. Then G is a conformal repellor. Denote by L its non-escaping set. It is a Cantor set by Schwarz Lemma, and is a subset of $\mathcal{J}_F$.

Now $\xi(J')$ is a Julia component for F and is contained essentially in the annulus $H\backslash\overline{U}_1$. But $\overline{U_2 \cup W} \subset U_1$. It follows that $\xi(J')$ is contained essentially in the annuli $H\backslash\overline{U}_2$ and $H\backslash\overline{W}$.

As every pullback by G of $\xi(J')$ is again a Julia component for F, we conclude that every point of L is a buried Julia component for F. □

4.3 Examples of Julia components with different topological structures

In this part, we will first construct examples (examples 1-6) where the Julia sets are hyperbolic and there can be no exposed doubly wandering Julia components in these examples. Hence there are seven possible combinations of "empty set" and "uncountably many" to be filled in our scheme, and we will give concrete examples of all (examples 1-6) but one which is shown as below:

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	uncountably many	$\emptyset$
{buried}	$\emptyset$	uncountably many

Here are the six examples:
Example 1, a polynomial with a disconnected Julia set satisfying that the filled-in Julia set contains no critical points.

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	uncountably many	$\emptyset$
{buried}	$\emptyset$	$\emptyset$

Example 2, McMullen, resp. Pilgrim-Tan, $f(z) = \frac{z^2}{1+bz^2} + \frac{1}{10^{11}z^3}$; with $b = 0$, resp. $b = -1$. See [McM, PT1].

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	$\emptyset$	$\emptyset$
{buried}	$\emptyset$	uncountably many

Example 3, disc-annulus surgery in a non periodic Fatou domain.

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	$\emptyset$	$\emptyset$
{buried}	uncountably many	$\emptyset$

Let $f(z) = z^2 - 1$. Set $z_0 = +1$. Perform a univalent disc-annulus surgery in a neighborhood V of z_0 to get a quasi-rational map F. Apply Lemma 4.1 to F we know that F has no exposed wandering Julia components. We apply then Lemma 4.2 to prove that F has no doubly connected wandering Julia components. As $\mathcal{J}_F$ is not connected, it has uncountably many buried simply connected wandering Julia components. All of them are point components by [PT1].
Example 4, disc-annulus surgery in a periodic Fatou domain.

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	uncountably many	$\emptyset$
{buried}	uncountably many	$\emptyset$

Let $f(z) = z^2 - 1$. Set $z_0 = 0.25$. Perform a univalent disc-annulus surgery in $V = D(0.25, 0.01)$ to get a quasi-rational map F. Apply Lemma 4.1 to F we know that F has uncountably many exposed wandering Julia components. We apply then

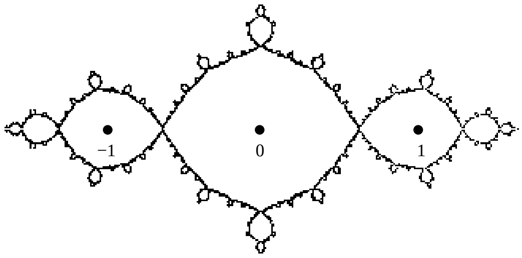

Figure 4: The Julia set of $z \mapsto z^2 - 1$

Lemma 4.2 to prove that F has no doubly connected wandering Julia components. From Corollary 4.6.3, we get a Cantor set of buried point Julia components for F.

Example 5, disc-annulus surgery in a non periodic Fatou domain of a map in Example 2.

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	$\emptyset$	$\emptyset$
{buried}	uncountably many	uncountably many

Let $f(z) = z^2 + \dfrac{1}{10^{11}z^3}$. Set $z_0 = 0$. It is mapped by f to the superattracting fixed point ∞. Perform a univalent disc-annulus surgery in $V = D(0, \epsilon)$ to get a quasi-rational map F. Apply Lemma 4.1 to F we know that F has no exposed wandering Julia components. We then apply Corollary 4.6 to prove that F has uncountably doubly connected wandering Julia components (all quasi-circles). From Corollary 4.6.3, we get a Cantor set of buried point Julia components of F.

Example 6, double disc-annulus surgery of a map in Example 2.

{wandering Julia comp.}	{simply connected}	{doubly connected}
{exposed}	uncountably many	$\emptyset$
{buried}	uncountably many	uncountably many

Let $f(z) = z^2 + \dfrac{1}{10^{11}z^3}$. Set z_0 to be a point in the Fatou component ∞, disjoint from P_f. Perform a univalent disc-annulus surgery on a small neighborhood V of z_0 disjoint from P_f to get a quasi-rational map F. Apply Lemma 4.1 to F we know that F has uncountably many exposed wandering Julia components. We then apply Corollary 4.6.1 to prove that F has uncountably doubly connected wandering Julia components (all quasi-circles).

Perform a second disc-annulus surgery, but this time on a small Jordan domain in the Fatou component of 0 for F, disjoint from P_F. Denote the resulting map by $\widetilde{F}$. Applying then Corollary 4.6.1 and 3, we conclude that $\widetilde{F}$ has now also uncountably many buried point Julia components.

The above examples are hyperbolic ones. In the end of this subsection, we will construct a non-hyperbolic example.

Example 7, critical values on buried wandering Julia components.

Let g be a rational map quasi-conformally conjugate to the quasi-rational map F constructed in Example 6. Assume that the quasi-conformal conjugacy maps ∞ to ∞.

Then the immediate basin of ∞ of g contains three distinct critical values ∞, v_1, v_2.

Deform slightly g if necessary so that v_1, v_2 are in distinct grand orbits.

Now make a double disc-annulus surgery on g as follows:

For $i = 1, 2$ choose a smooth neighborhood V_i of v_i such that for all $j > 0$, $g^j(\overline{V}_i) \cap V_i = \emptyset$, and that $V_1 \cap g^j(\overline{V}_2) = \emptyset$, $V_2 \cap g^j(\overline{V}_1) = \emptyset$.

Choose then a smooth disc $H_i \ni v_i$ so that $V_i \supset \overline{H}_i$.

Choose two distinct points w_1, w_2 each in a wandering Julia component (of any type: exposed or buried, simply connected or doubly connected) outside $V_1 \cup V_2$.

Choose univalent maps $h_i : H_i \to \widehat{\mathbb{C}} \backslash g(V_i)$ so that $h_i(v_i) = w_i$, $i = 1, 2$.

Choose covering extension of the boundary maps $a_i : V_i \backslash \overline{H_i} \to g(V_i)$, $i = 1, 2$ such that each a_i is holomorphic near the critical points.

Then the map

$$G : \begin{cases} \widehat{\mathbb{C}} \backslash (V_1 \cup V_2) \to g(\widehat{\mathbb{C}} \backslash (V_1 \cup V_2)) & G(z) = g(z) \\ H_i \to \widehat{\mathbb{C}} \backslash \overline{g(V_i)} & G(z) = h_i(z), \ i = 1, 2 \\ \overline{V_i} \backslash H_i \to \overline{g(V_i)} & G(z) = a_i(z), \ i = 1, 2 \end{cases}$$

is quasi-rational.

Note that for $i = 1, 2$, the point v_i is a critical value of G, is contained in a wandering Julia component of the same type as w_i, and is non recurrent.

By choosing w_1, w_2 buried we can construct an example showing that our condition in Theorem 1.1.(c) is not sufficient to have wandering Julia components with $C(\text{orb}(J)) = 3$.

By choosing w_1 on a Jordan curve wandering Julia component, we obtain an example that, for c_1 a critical point so that $G(c_1) = v_1$ (and thus to $G^2(c_1) = w_1$), the point c_1 is contained in a wandering Julia component J with $C(J) = 3$ (but $C(\text{orb}(J)) = 2$)).

For this example, the set Σ is countable (in particular non-empty) but it does not contain any infinite Julia orbit.

4.4 Recurrent critical points in buried simply connected wandering Julia components

Example 8. We will construct in four steps an example of a rational map with a recurrent critical point in a buried and wandering Julia component. We then explain how to construct examples with several such critical points.

Step 1. It is known from the work of Branner-Hubbard that there are (many) cubic polynomials with a Cantor Julia set containing a recurrent critical point.

Let f be one of them. For this map, there is a Jordan disc V' bounded by an equipotential, so that $f^{-1}(V')$ has two connected components U_1' and U_2' with disjoint closure and compactly contained in V', and that $f : U_i' \to V'$ is proper of degree i for $i = 1, 2$.

Step 2. Let g be a degree 5 polynomial with one escaping critical point satisfying the following properties: there is a Jordan disc V, so that $g^{-1}(V)$ has three connected components U_1 and U_2 and U_3, all compactly contained in V and have disjoint closures, and that $g : U_1 \cup U_2 \to V$ is quasi-conformally conjugate to $f : U_1' \cup U_2' \to V'$, and $g : U_3 \to V$ is hybrid equivalent to $z \to z^2$. Thus two critical points of g escape to ∞, one other critical point is in a point Julia component and is recurrent, while the fourth critical point is a fixed point. We may furthermore assume that 0 is the fixed critical point.

The construction of such g is easy: Take a point $w \in V' \backslash \overline{U_1' \cup U_2'}$, and a conformal representation $\phi : V' \to \mathcal{D}$ (where $\mathcal{D}$ denotes the unit disc) so that $\phi(w) = 0$. Let $R > 1$ so that $\{|z| \leq 1/R\} \cap \phi(\overline{U_1' \cup U_2'}) = \emptyset$. Set $\Phi(z) = R^2 \cdot \phi(z)$. Then Φ has an C^1 extension to the boundary, with $\Phi(\overline{V'}) = \{|z| \leq R^2\}$ and $\Phi(\overline{U_1' \cup U_2'}) \subset \{R < |z| < R^2\}$.

We can then define

$$G : \begin{cases} \Phi(\overline{U_1' \cup U_2'}) \to \{|z| \leq R^2\} & \text{by } G(z) = \Phi \circ f \circ \Phi^{-1}(z) \\ \{|z| \leq R\} \to \{|z| \leq R^2\} & \text{by } G(z) = z^2 \\ \{|z| \geq R^2\} \to \{|z| \geq R^{10}\} & \text{by } G(z) = z^5 \\ \{R \leq |z| \leq R^2\} \backslash \Phi(U_1' \cup U_2') \to \{R^2 \leq |z| \leq R^{10}\} & \text{by quasi-regular interpolation.} \end{cases}$$

The last line above is similar to the construction in Lemma 4.3.

This G is clearly quasi-rational, and is conjugate to a degree 5 polynomial g by a quasi-conformal change of coordinates ψ so that $\psi(0) = 0$, $\psi(\infty) = \infty$. This is our desired polynomial. We may then set $V = \psi \circ \Phi(V') = \psi(\{|z| < R^2\})$, $U_3 = \psi(\{|z| < R\})$ and $U_i = \psi \circ \Phi(U_i')$ for $i = 1, 2$.

Step 3 (folding). This is a technique that has been used to construct examples such as that of McMullen and of Pilgrim-Tan, shown by formula in Example 2 above.

For $K, K' \subset \widehat{\mathbb{C}}$ two continua we use the notation $A(K, K')$ to denote the empty set in case $K \cap K' \neq \emptyset$ and to be the unique annular complementary component of $(K \cup K')$ if $K \cap K' = \emptyset$.

For L a Jordan domain in $\widehat{\mathbb{C}}$ denote by γ_L the boundary curve of L.

For our polynomial g, choose a small disc neighborhood D of the superattracting fixed point 0 so that $g(D) \subset\subset D$.

Thus γ_D, γ_V and $\gamma_{g(V)}$ denote the corresponding boundary curves.

Notice that the escaping critical values v_1, v_2 of g are contained in $A(\gamma_V, \gamma_{g(V)})$. Choose now a smooth essential Jordan curve $\eta \subset A(\gamma_V, \gamma_{g(V)})$ so that $v_1, v_2 \in A(\gamma_V, \eta)$ and $\text{mod}A(\gamma_D, \gamma_{g(V)}) = d \cdot \text{mod}A(\eta, \gamma_{g(V)})$ for some integer d. Denote by D_∞ the disc containing ∞ bounded by $\gamma_{g(V)}$.

Now define

$$H : \begin{cases} \overline{V} \to \overline{g(V)} & \text{by } H(z) = g(z); \\ \overline{A}(\eta, \gamma_{g(V)}) \to \overline{A}(\gamma_D, \gamma_{g(V)}),\ \eta \to \gamma_{g(V)} & \text{by a covering of degree } d \\ & \quad \text{holomorphic in the interior;} \\ D_\infty \to D & \text{by quasi-regular interpolation;} \\ A(\gamma_V, \eta) \to D_\infty & \text{by quasi-regular interpolation.} \end{cases}$$

One can prove then this H is quasi-rational, with only one periodic Fatou component: the immediate basin of 0 (it is simply connected), with one recurrent critical point as a Julia component, that is buried and wandering, and with all the other critical points attracted by 0. This H has no exposed wandering Julia components.

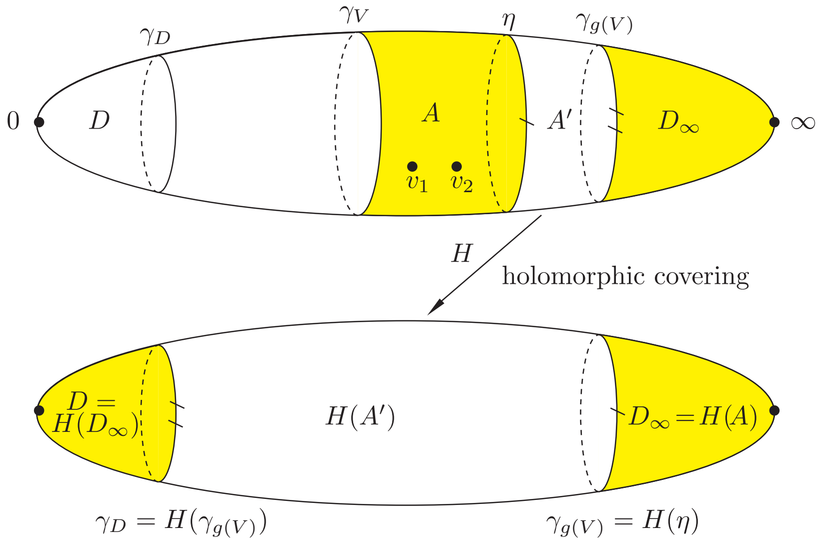

Figure 5: folding

Step 4. Get a rational map h from H through a quasi-conformal conjugacy. The Julia set of h has the same topological properties as that of H. Furthermore our Theorem 1.1 proves that h has no wandering Julia component J with $C(\text{orb}(J)) = 3$.
Further examples. Using a similar folding surgery on a more complicated polynomial g one can get a rational map with several recurrent critical points each being a buried and wandering Julia component.

By adding a disc-annulus surgery in the immediate basin of 0 one can get a map with exposed wandering Julia components as well.

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