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Combinatorial rigidity of unicritical maps

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract In this paper, we combine the KSS nest constructed by Kozlovski, Shen and van Strien, and the analytic method proposed by Avila, Kahn, Lyubich and Shen to prove the combinatorial rigidity of unicritical maps.

Keywords combinatorial rigidity, unicritical maps, KSS nest

MSC(2000): 37F10, 37F20

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1 Introduction

Rigidity is one of the fundamental and remarkable phenomena in holomorphic dynamics. The general rigidity problem can be posed as

Rigidity problem [11]. Any two combinatorially equivalent rational maps are quasi-conformally equivalent. Except for the Lattès examples, the quasi-conformal deformations come from the dynamics of the Fatou set.

In the quadratic case, the rigidity problem is equivalent to the famous hyperbolic conjecture. The MLC conjecture asserting that the Mandelbrot set is locally connected is stronger than the hyperbolic conjecture (cf. [4]). In 1990, Yoccoz [7] proved MLC for all parameter values which are at most finitely renormalizable. Lyubich [11] proved MLC for infinitely renormalizable quadratic polynomials of bounded type. In [9], Kozlovski, Shen and van Strien gave the proof of the rigidity for real polynomials with all critical points real. In [1], Avila, et al. proved that any unicritical polynomial $f_c : z \mapsto z^d + c$ which is at most finitely renormalizable and has only repelling periodic points is combinatorially rigid which implies that the connectedness locus (the Multibrot set) is locally connected at the corresponding parameter values. The rigidity problem for the rational maps with Cantor Julia sets is totally solved (cf. [18, 19]). In [19], Zhai took advantage of a length-area method introduced by Kozlovski et al. (cf. [9]) to prove the quasi-conformal rigidity for rational maps with Cantor Julia sets. Kozlovski and van Strien proved that topologically conjugate non-renormalizable polynomials are quasi-conformally conjugate (cf. [10]).

In the following, we list some other cases in which the rigidity problem is researched (see also [19]).

(i) Robust infinitely renormalizable quadratic polynomials [12];

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(ii) Summable rational maps with small exponents [5];

(iii) Holomorphic Collet-Eckmann repellers [10];

(iv) Uniformly weakly hyperbolic rational maps [6].

In this paper, we will give an alternative proof of the combinatorial rigidity of unicritical maps (see the definition in Subsection 1.1). This is the first step to prove the combinatorial rigidity of multi-critical maps.

In the proof, we will exploit the powerful combinatorial tool called "puzzle" and a sophisticated choice of puzzle pieces called the KSS nest constructed in [9]. We use the estimate obtained in [15] as a starting point of the proof. To get the quasi-conformal conjugation, we adapt the analytic method in [1] (see Lemma 3.2) to the KSS nest.

The paper is organized as follows. In Subsection 1.1, we present the main result of this paper, Theorem 1.1. In Subsection 1.2, we introduce the definitions of the puzzle and the tableau. In Subsection 1.3, we propose Theorem 1.2. We summarize the construction of the KSS nest and prove Theorem 2.1 in Section 2. The proof of Theorem 1.2 is given in Section 3. We deduce Theorem 1.2 to Theorem 1.1 in the appendix.

1.1 Set up

$\mathbf{V} = \sqcup_{i\in I} V_i$ is the disjoint union of finitely many Jordan domains with disjoint and quasi-circle boundaries, $\mathbf{U}$ is compactly contained in $\mathbf{V}$, and is the union of finitely many Jordan domains with disjoint closures; $f : \mathbf{U} \to \mathbf{V}$ is a proper holomorphic map with a unique critical point c, with $\deg f|_c = \delta$, and with c contained in $\mathbf{K}_f := \{z \in \mathbf{U},\ f^n(z) \in \mathbf{U},\ \forall n\}$.

Denote by $\mathbf{P}_m := \{f^j(c),\ j = 1, \ldots, m\}$ and by $\mathbf{P} := \overline{\bigcup_{m\geqslant 1} \mathbf{P}_m}$ the closure of the orbit of c (the postcritical set).

For another such map $\tilde f : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}}$, we mark the same objects with a tilde.

Two such maps $(f : \mathbf{U} \to \mathbf{V}), (\tilde f : \tilde{\mathbf{U}} \to \tilde{\mathbf{V}})$ are said to be *c-equivalent* (combinatorially equivalent), if there is a pair of orientation preserving homeomorphisms $h_0, h_1 : \mathbf{V} \to \tilde{\mathbf{V}}$ such that

$$\begin{cases} h_1(\mathbf{U}) = \tilde{\mathbf{U}} \text{ and } h_1(\mathbf{P}) = \tilde{\mathbf{P}}, \\ h_1 \text{ is isotopic to } h_0 \text{ rel } \partial\mathbf{V} \cup \mathbf{P}, \\ h_0 \circ f \circ h_1^{-1}|_{\tilde{\mathbf{U}}} = \tilde f, \\ h_1|_{\overline{\mathbf{V}\setminus\mathbf{U}}} \text{ is } C_0\text{-qc (qc is an abbreviation of quasi-conformal) for some } C_0 \geqslant 1, \end{cases} \tag{1.1}$$

$$\text{in particular}\quad \begin{array}{ccc} \mathbf{V}\supset\mathbf{U} & \xrightarrow{h_1} & \tilde{\mathbf{U}}\subset\tilde{\mathbf{V}} \\ f\downarrow & & \downarrow\tilde f \\ \mathbf{V} & \xrightarrow[h_0]{} & \tilde{\mathbf{V}} \end{array} \quad\text{commutes.}$$

This definition is to be compared with the notion of combinatorial equivalence introduced by McMullen in [13]. Notice that this definition is slightly different from the definitions of combinatorial eguivalence in [1] and [10], since we define it without using the external rays and angles.

We say that f and $\tilde f$ are *qc-conjugate off* $\mathbf{K}_f$ if there is a qc map $H : \mathbf{V}\setminus\mathbf{K}_f \to \tilde{\mathbf{V}}\setminus\mathbf{K}_{\tilde f}$ so that $H \circ f = \tilde f \circ H$ on $\mathbf{U}\setminus\mathbf{K}_f$,

$$\text{i.e.}\quad \begin{array}{ccc} \mathbf{U}\setminus\mathbf{K}_f & \xrightarrow{H} & \tilde{\mathbf{U}}\setminus\mathbf{K}_{\tilde f} \\ f\downarrow & & \downarrow\tilde f \\ \mathbf{V}\setminus\mathbf{K}_f & \xrightarrow[H]{} & \tilde{\mathbf{V}}\setminus\mathbf{K}_{\tilde f} \end{array} \quad\text{commutes.}$$

We say that f and $\tilde f$ are *qc-conjugate* if there is a qc map $H' : \mathbf{V} \to \tilde{\mathbf{V}}$ so that $H' \circ f = \tilde f \circ H'$ on $\mathbf{U}$,

$$\text{i.e.}\quad \begin{array}{ccc} \mathbf{V}\supset\mathbf{U} & \xrightarrow{H'} & \tilde{\mathbf{U}}\subset\tilde{\mathbf{V}} \\ f\downarrow & & \downarrow\tilde f \\ \mathbf{V} & \xrightarrow[H']{} & \tilde{\mathbf{V}} \end{array} \quad\text{commutes.}$$

Theorem 1.1. *Assume that f is a map in the set up with the component of $\mathbf{K}_f$ containing c non-periodic, then $\mathbf{K}_f$ is totally disconnected. Assume that $\tilde{f}$ is another map in the set up. Then the following statements are equivalent to each other*:

A. *f and $\tilde{f}$ are c-equivalent*; B. *f and $\tilde{f}$ are qc-conjugate off $\mathbf{K}_f$*; C. *f and $\tilde{f}$ are qc-conjugate.*

1.2 Puzzle and tableau

The connected components of $f^{-m}(\mathbf{V})$ are called puzzle pieces of depth m for every non-negative integer m.

For any $x \in \mathbf{K}_f$, the tableau $\mathcal{T}(x)$, following Branner-Hubbard [2], is the graph embedded in $\{(u, v), u \in \mathbb{R}^-, v \in \mathbb{R}\}$ with the axis of u pointing upwards and the axis of v pointing rightwards (this is the standard $\mathbb{R}^2$ with reversed orientation), with vertices indexed by $-\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, \ldots\}$, with the vertex at $(-m, 0)$ being $P_m(x)$, the puzzle piece of depth m containing x, and with $f^j(P_m(x))$ occupying the $(-m+j, j)$-th entry of $\mathcal{T}(x)$. Therefore a given puzzle piece Q may appear at different entries of $\mathcal{T}(x)$ for different x's, but will always be on the same row, denoted by row(Q). There are three types of edges: vertical, horizontal and diagonal. It is also equipped with the graph metric so that each edge is isometric to the unit interval $[0, 1]$.

For a vertex Q with index $(-m, n)$, we say that m is **the depth**, or the row number of Q.

A vertex in a tableau is said to be critical if it represents a critical puzzle piece. A vertex in a tableau is marked by $\circ$ if it is non-critical, by a solid $\bullet$ if it is a critical puzzle piece.

A vertical segment, say bounded by two vertices E and F, is assigned a length, denoted by $\left|\begin{smallmatrix} E \\ | \\ F \end{smallmatrix}\right|$, which is simply the depth difference between F and E. It is also assigned a modulus, equaling to $\mathrm{mod}(E \backslash \overline{F})$. Recall that modulus of an annulus A is a conformal invariant, and is defined to be $\frac{1}{2\pi} \log R$ if A is mapped conformally onto $\{z \in \mathbb{C} \mid 1 < |z| < R\}$ (see e.g. [14]).

The map f induces a (partial) dynamical system, indicated by the diagonal edges, mapping $\mathcal{T}(x) \backslash \{0\text{-th row}\}$ onto $\mathcal{T}(x) \backslash \{0\text{-th column}\} = \mathcal{T}(f(x))$. See Figure 1.

All tableaus satisfy the following three rules (see Figure 1).

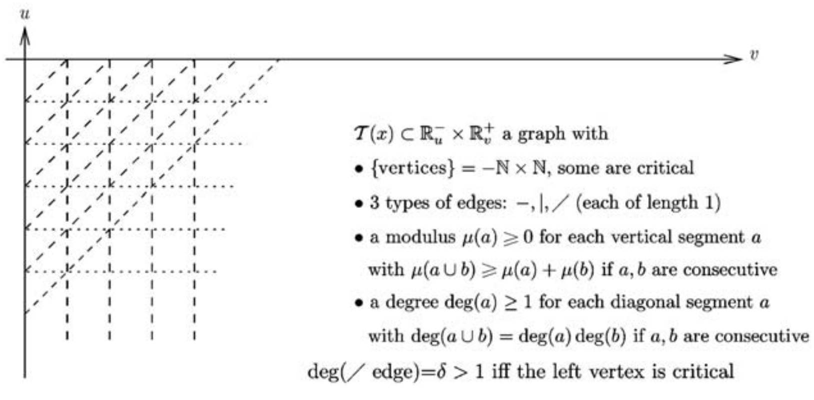


Figure 1 Tableau

> Rule 1 (vertical segment rule): in each column either there is no critical vertex or the critical vertices form a single vertical segment with one end on the top of the column.

An **upper triangle** T in a tableau is by the definition a sides-included filled triangle bounded by a vertical segment on the left, a horizontal segment on the top and a diagonal segment as the third edge. Its **size**, denoted by $|T|$, is simply the length of any of its edges, and its **depth** is the depth of its lowest vertex.

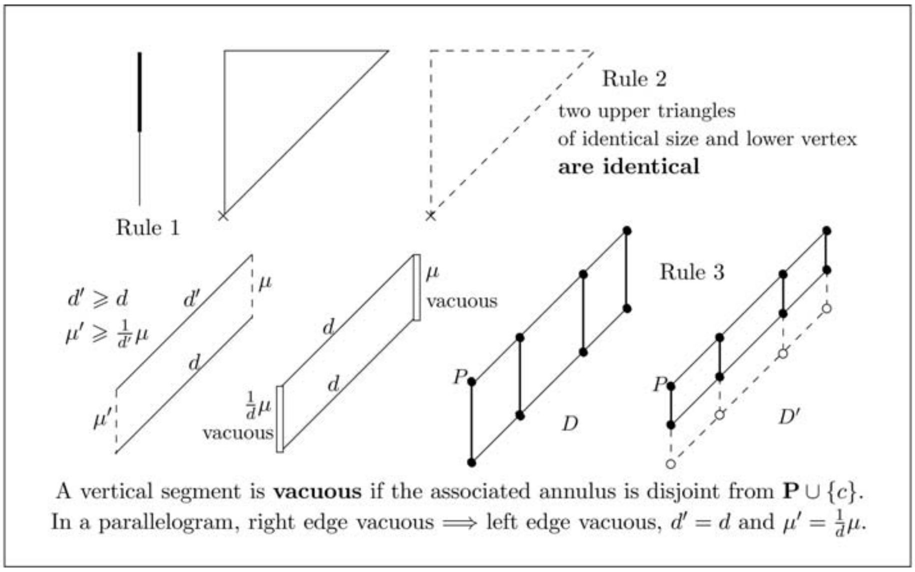


Figure 2 Tableau rules, upper triangles and parallelograms, with D critically full

> Rule 2 (double triangle rule): Two upper triangles in $\mathcal{T}(x)$ and $\mathcal{T}(x')$ with identical size and lower vertex are identical.

A **vertical parallelogram** in a tableau is a sides-included filled parallelogram bounded by two vertical segments and two diagonal segments.

A vertical parallelogram D is said **critically full** if its two vertical edges are entirely critical, and if every vertical critical segment that touches the top diagonal edge of D (if any) remains critical at least until it reaches the lower diagonal edge of D.

> Rule 3a (double parallelogram rule): Let $D \subset \mathcal{T}(x), D' \subset \mathcal{T}(x')$ be two vertical parallelograms with identical size and upper left vertex, such that D is critically full. Then D' is either identical to D, or has its lower diagonal edge completely non critical.

A vertical segment $\overset{I}{\underset{J}{|}}$ is called **vacuous** if $(I \backslash \overline{J}) \cap (\mathbf{P} \cup \{c\}) = \emptyset$. In other words, in any column of any critical tableau $\mathcal{T}(c)$, whenever I appears the full vertical segment from I to J appears (otherwise the orbit of c would visit $I \backslash \overline{J}$).

> Rule 3b: A vertical parallelogram with right edge vacuous has every of its vertical segment vacuous, and furthermore is critically full if its two top vertices are both critical.

For two puzzle pieces Q and I such that $f^k(Q) = I$ for some $k > 0$, we use $_{[Q}\diagup^{I]}$ to denote the diagonal segment from Q to I. Thus the consecutive vertices on this segment are $Q, f(Q), f^2(Q), \ldots, f^k(Q) = I$. We use also $_{[Q}\diagup^{I[}$ if we exclude I. Define $_{]Q}\diagup^{I]}$ and $_{]Q}\diagup^{I[}$ accordingly. This diagonal segment (whether it is closed, half open or open) is assigned a length, denoted by $|_Q\diagup^I|$, which is simply the depth difference between Q and I. It is also assigned a degree, equal to $\deg(f^k : Q \to I)$.

For the critical point c, the nest $(P_m(c))_m$ is called a **critical nest**.

A **child** of a critical puzzle piece I is a critical puzzle piece J that is both a sub-piece of I and a pullback of I, such that $_{]J}\diagup^{I[}$ does not meet any critical puzzle piece. See Figure 3.

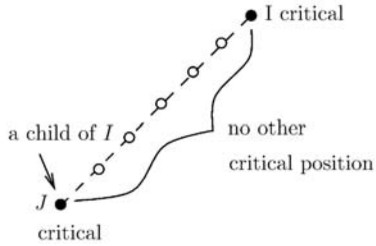


Figure 3 Child

1.3 Statement

Hypothesis of recurrence. *Let f be as in the set up. In this paper we assume*

- *marching horizontally to the right of any vertex in $\mathcal{T}(c)$, one will meet a critical vertex; and $\mathcal{T}(c)$ does not contain a full column of critical vertices* (*except of course on the* 0*-th column*);
- *a critical piece of any depth has at most finitely many children.*

(This is the commonly called *persistently recurrent* condition).

The objective of this note is to present a proof of the following.

Theorem 1.2. *Let $f, \tilde{f}$ be two maps satisfying the hypothesis of recurrence and that $\mathbf{K}_f$ has no interior. Then the following statements are equivalent to each other*:

A. *f and $\tilde{f}$ are c-equivalent*; B. *f and $\tilde{f}$ are qc-conjugate off $\mathbf{K}_f$*; C. *f and $\tilde{f}$ are qc-conjugate.*

We will then show in the appendix how to deduce Theorem 1.1 from Theorem 1.2.

2 The KSS nest

In this section we will prove

Theorem 2.1. *Assume that f is a map satisfying the hypothesis of recurrence. Then there are two constants C_K (in fact $C_K = \delta^3$) and $\Delta > 0$, depending on δ and $\widehat{\mu}$ (see below), and a nested sequence of critical puzzle pieces $K_n \subset\subset K_{n-1}$, $n \geqslant 1$, with K_0 being the critical puzzle piece of depth* 0*, satisfying*

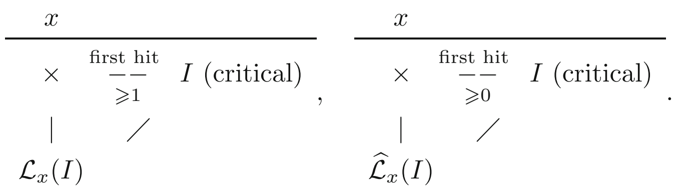
(i) *each K_n, $n \geqslant 1$, is a pullback of K_{n-1} and* $\deg\left({}_{K_n}\diagup^{K_{n-1}}\right) \leqslant C_K$;

(ii) *each K_n, $n \geqslant 1$, contains a sub-critical piece K_n^- such that* $\mathrm{mod}(K_n \backslash \overline{K_n^-}) \geqslant \Delta$ *and* $(K_n \backslash \overline{K_n^-}) \cap \mathbf{P} = \emptyset$.

Here

$$\widehat{\mu} = \min\{\mathrm{mod}(P_0(c)\backslash \overline{W}) \mid W \textit{ a component of } \mathbf{U} \textit{ contained in } P_0(c)\}. \tag{2.1}$$

2.1 First hits have bounded degree



For any pair (I, x) such that I is a critical puzzle piece, and $x \in \mathbf{K}_f$, we produce a puzzle piece $\mathcal{L}_x(I)$ (called *the pullback of the first $\geqslant 1$ hit* of x to I) as follows: start from the 0-th column of $\mathcal{T}(x)$ at row(I), march right $k \geqslant 1$ steps until the first hit of a critical spot[1] (if any, this exists always if $x \in \mathbf{P}$). Then that spot represents I and $\mathcal{L}_x(I)$ is the pulled-back piece by f^k of I containing x. It is the lower vertex of an upper triangle whose left edge is on the 0-th column of $\mathcal{T}(x)$, whose right vertex is I and whose top edge does not contain critical spots (except at the ends).

[1] We will frequently use the word 'spot' for a vertex in a tableau.

Similarly we define *the pullback of the first* $\geqslant 0$ *hit* of x to I to be a puzzle piece $\widehat{\mathcal{L}}_x(I)$ containing x which is equal to I if $x \in I$, otherwise is equal to $\mathcal{L}_x(I)$.

Lemma 2.2. *For I a critical piece and $x \in \mathbf{K}_f$, the open diagonal ${}_{]\mathcal{L}_x(I)}\diagup^{I\,[}$ does not contain any critical spot. And* $\deg\left({}_{\mathcal{L}_x(I)}\diagup^{I}\right) = \delta$ *or* $= 1$ *depending whether $\mathcal{L}_x(I)$ is critical or not.*

Proof. Otherwise we get two upper triangles of different sizes, both with I as the right vertex, and both have a vertical left edge entirely critical. Now moving the smaller triangle to the left and applying Rule 2 would imply an earlier hit of critical spots on the top edge of the larger triangle.

2.2 The last child operator Γ

Lemma 2.3. (*A consequence of the hypothesis of recurrence*): *Let J be a critical puzzle piece. Then* $2 \leqslant \sharp\{\textit{children of } J\} < +\infty$.

The proof can be found in [17]. Denote by $\Gamma(J)$ the **last child** of J, i.e. the child with the greatest depth.

Definition. *For J a critical puzzle piece, denote by $r(J)$ (resp. $R(J)$) the minimal (resp. maximal) length of a horizontal segment linking two consecutive J vertices in $\mathcal{T}(c)$.*

We might have $R(J) = +\infty$ *a priori*, but the following corollary of Lemma 2.3 will exclude this possibility. The estimates here play an essential role in the sequel.

Corollary 2.4. *Let J be a critical puzzle piece.*

(i) *If S is a critical sub-piece of J then $R(S) \geqslant R(J)$ and $r(S) \geqslant r(J)$;*

(ii) *If J' ($\neq J$) is a critical sub-piece of J and is also a pullback of J, then for k the number of critical spots on the half open diagonal ${}_{[J'}\diagup^{J\,]}$, we have,* $r(J) \leqslant |{}_{J'}\diagup^{J}| \leqslant k \cdot r(J')$;

(iii) $\begin{array}{r} R(J) < \\ 2\cdot r(J) \leqslant \end{array} \left\{ |{}_{\Gamma(J)}\diagup^{J}| \leqslant r(\Gamma(J)), \right.$ *in particular* $r(\Gamma(J)) \geqslant 2r(J)$.

Proof. (i) Obvious, by Rule 1.

(ii) Denote by T the upper triangle with vertices J', J, J. Then $|{}_{J'}\diagup^{J}|$ is also the length of the top edge of T. Consequently $|{}_{J'}\diagup^{J}| \geqslant r(J)$.

Let E, F be two consecutive critical spots on row(J') of $\mathcal{T}(c)$ with F on the right of E. By Rule 2 the triangle T appears from column(E) with E as the lower vertex.

Assume at first $k = 1$, i.e. ${}_{[J'}\diagup^{J\,]}$ contains no critical spots. Applying Rule 1 one sees that the length of the horizontal segment E–F must be at least $|{}_{J'}\diagup^{J}|$. Therefore $|{}_{J'}\diagup^{J}| \leqslant r(J')$.

In the general case, let $(J' =)J^k, J^{k-1}, \ldots, J^1, J^0 (= J)$ be the consecutive critical spots on the closed diagonal ${}_{[J'}\diagup^{J\,]}$. Applying the above argument on each ${}_{[J^{l+1}}\diagup^{J^l\,[}$ we get

$$|{}_{J'}\diagup^{J}| \leqslant r(J^k) + r(J^{k-1}) + \cdots + r(J^1) \overset{\text{(i)}}{\leqslant} k \cdot r(J^k) = k \cdot r(J').$$

(iii) In $\mathcal{T}(c)$, let J—J be a horizontal segment bounded by two consecutive critical spots on row(J). Form an upper triangle T with this segment as the upper edge. Denote its lower vertex by W, and its length by l.

If W is critical, we use Rule 2 to compare T with the upper triangle in $\mathcal{T}(c)$ with left edge $\begin{array}{c} J \\ | \\ \mathcal{L}_c(J) \end{array}$ and (automatically) right vertex J, one concludes that $W = \mathcal{L}_c(J)$ and W is the first child of J. But there are at least two children, so $l < |{}_{\Gamma(J)}\diagup^{J}|$.

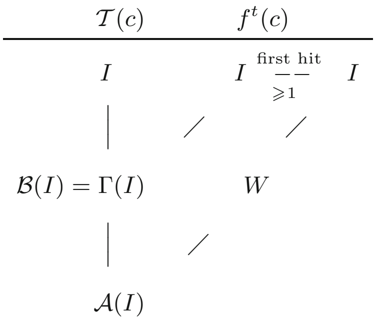
If W is not critical, follow its left-down diagonal in $\mathcal{T}(c)$ until the first critical vertex W' (such W' exists since the 0-th column vertex on that diagonal is critical). Then $l < |{}_{W'}\diagup^{J}| \leqslant |{}_{\Gamma(J)}\diagup^{J}|$.

This proves $R(J) < |{}_{\Gamma(J)}\diagup^{J}|$.

By the definition of a child, the half open diagonal ${}_{[\Gamma(J)}\diagup^{J\,[}$ contains only one critical spot: $\Gamma(J)$. So $|{}_{\Gamma(J)}\diagup^{J}| \leqslant r(\Gamma(J))$ by (ii). Denote by T_Γ the upper triangle with vertices $\Gamma(J), J, J$. As J has at least two children (Lemma 2.3) and $\Gamma(J)$ is the last child, the top edge of T_Γ contains at least three critical spots (counting the ends). Therefore $2 \cdot r(J) \leqslant |T_\Gamma| = |{}_{\Gamma(J)}\diagup^{J}|$.

2.3 The operators $\mathcal{A}$ and $\mathcal{B}$

Definition. *Given any critical puzzle piece I, define $\mathcal{B}(I)$ and $\mathcal{A}(I)$ as follows*:



(2.2)

Lemma 2.5. *For any critical puzzle piece I, the vertical segment* $\overset{\mathcal{B}(I)}{\underset{\mathcal{A}(I)}{|}}$ *is vacuous*; *and*

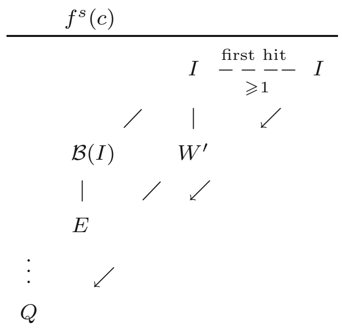
$$\begin{cases} \deg\left({}_{\mathcal{B}(I)}\diagup^{I}\right) = \delta =: C_{\mathcal{B}}; \\ \sharp\left(\{\textit{critical spots}\} \cap {}_{[\mathcal{B}(I)}\diagup^{I[}\right) = 1; \end{cases} \qquad \begin{cases} \deg\left({}_{\mathcal{A}(I)}\diagup^{I}\right) = \delta^2 =: C_{\mathcal{A}}; \\ \sharp\left(\{\textit{critical spots}\} \cap {}_{[\mathcal{A}(I)}\diagup^{I[}\right) = 2. \end{cases}$$

Proof. Let t be the length of $\overset{I}{\underset{\Gamma(I)}{|}}$. Let $W = \mathcal{L}_{f^t(c)}(I)$.

At first W has to be critical (for otherwise $\mathcal{A}(I)$ is a child of I deeper than $\Gamma(I)$).

Using then Rule 2 to compare the two triangles, one with vertices W, I, I the other with vertices $\mathcal{L}_c(I), I, I$, one sees that $W = \mathcal{L}_c(I)$.

Now assume by contradiction that $f^s(c) \in \mathcal{B}(I)\backslash\overline{\mathcal{A}(I)}$ for some $s > 0$. This is seen in $\mathcal{T}(c)$ as below, with E non critical:



$\mathrm{row}(E) = \mathrm{row}(\mathcal{A}(I)),\ \mathrm{row}(W') = \mathrm{row}(W) = \mathrm{row}(\mathcal{L}_c(I))$.

Consider the two parallelograms, the first with vertical edges $\overset{\mathcal{B}(I)}{\underset{\mathcal{A}(I)}{|}}$ and $\overset{I}{\underset{W}{|}}$, the second with vertical edges $\overset{\mathcal{B}(I)}{\underset{E}{|}}$ and $\overset{I}{\underset{W'}{|}}$. Applying Rule 3 to them one sees that W' is not critical. Now start from the upper-right vertex of the second parallelogram and march right until the first critical vertex, and from there march diagonally left-downwards until the first critical vertex Q. Now Q cannot be on column(W') (for otherwise Q has to be above W' and one gets a smaller triangle than the one with vertices $I, I, \mathcal{L}_c(I)$). Then Q must be a child of I deeper than $\mathcal{B}(I) = \Gamma(I)$. This is not possible.

Therefore $\overset{\mathcal{B}(I)}{\underset{\mathcal{A}(I)}{|}}$ is vacuous. The rest is trivial.

2.4 The KSS nest

We now construct inductively the KSS nest $(K'_n, K_n)_n$ from a critical piece K_0 by $K'_n = \mathcal{B}\Gamma(K_{n-1}) = \Gamma^2(K_{n-1})$ and $K_n = \mathcal{A}\Gamma(K_{n-1})$. See Figure 4.

By Corollary 2.4 (iii), we have $2\,r(J) \leqslant |{}_{\mathcal{B}(J)}\diagup^{J}| = |{}_{\Gamma(J)}\diagup^{J}| \leqslant r(\mathcal{B}(J))$, therefore $3\,r(J) \leqslant |{}_{\mathcal{A}(J)}\diagup^{J}| \leqslant 2\,r(\mathcal{A}(J))$, by the construction of $\mathcal{A}$ and Corollary 2.4.(ii).

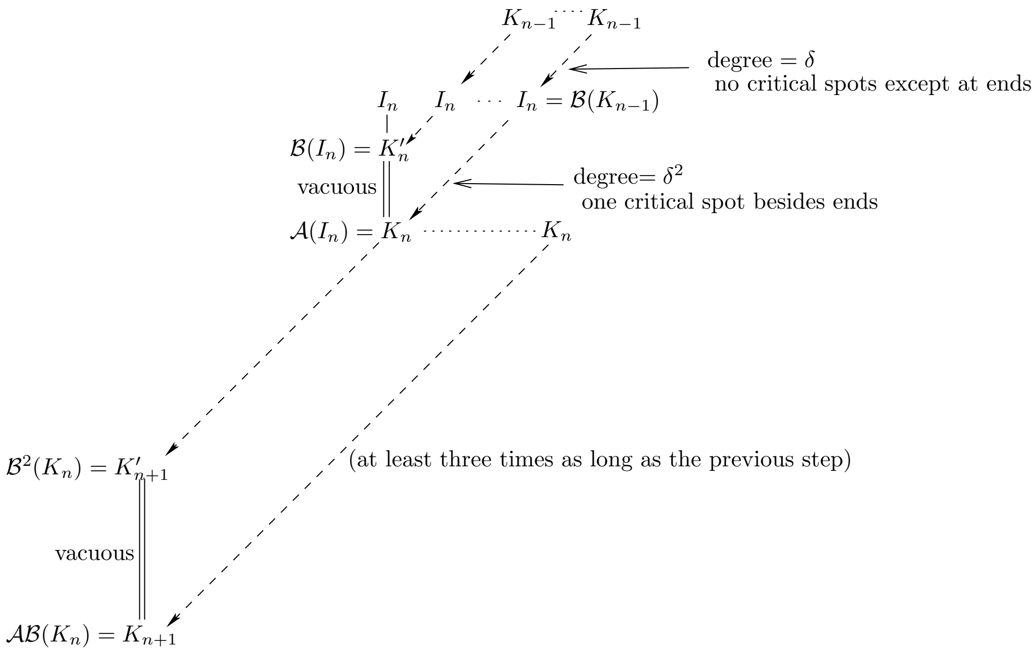


Figure 4 The operators $\mathcal{A}$, $\mathcal{B}$, Γ and the KSS nest

Set $I_n = \Gamma(K_{n-1})$, $u_n = |_{K_n=\mathcal{A}(I_n)}\diagup^{I_n}|$, $p_n = |_{K_n}\diagup^{K_{n-1}}|$, $q_n = |_{I_{n+1}=\Gamma(K_n)}\diagup^{K_n}|$.

We get successively the following estimates:

$$
\begin{aligned}
&3\,r(I_n) \leqslant u_n \leqslant 2\,r(K_n) \leqslant q_n \leqslant r(I_{n+1}); \qquad (2.3)\\
&r(I_n) \leqslant \frac{1}{3}u_n \leqslant \frac{2}{3}r(K_n), \quad p_n = q_{n-1} + u_n \leqslant r(I_n) + u_n \leqslant \frac{4}{3}u_n \leqslant \frac{8}{3}r(K_n);\\
&p_{n+1} = q_n + u_{n+1} \geqslant r(K_n) + 3\,r(I_{n+1}) \geqslant r(K_n) + 3\cdot 2\,r(K_n) = 8\,r(K_n) \geqslant 3\,p_n;\\
&|_{K_n}\diagup^{K_0}| = p_n + \cdots + p_1 < \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)p_n = \frac{3}{2}p_n.
\end{aligned}
$$

Proposition 2.6. *Starting from any critical piece K_0, the sequence of pairs of critical pieces $(K'_n, K_n)_{n\geqslant 1}$ satisfies*

$$
\begin{cases}
K_0 \supset\supset K'_1 \supset\supset K_1 \supset\supset K'_2 \supset\supset K_2 \supset\supset \cdots;\\
\forall\, n \geqslant 1, \quad \text{both } K'_n \text{ and } K_n \text{ are pullbacks of } K_{n-1};\\
\forall\, n \geqslant 1, \quad (K'_n \backslash \overline{K}_n) \cap \mathbf{P} = \emptyset.
\end{cases} \qquad (2.4)
$$

And, for any $m \geqslant 1$, for the maps g, f^ξ constructed below, and for $C_K = \delta^3, d = \delta^4, \beta = \delta^5$ such that:

(a) $\deg(_{K_m}\diagup^{K_{m-1}}) \leqslant C_K$, $\deg(_{K'_m}\diagup^{K_{m-1}}) \leqslant C_K$,

(b) $\deg(_{B'}\diagup^{K'_m}) = 1$,

(c) $\deg(_{A'}\diagup^{B'}) \leqslant d$,

(d) $\deg(_{K'_{m+2}}\diagup^{U}) = \deg(_{K'_{m+2}}\diagup^{K_m}) \leqslant \beta$,

(e) $f^\xi(K_{m+2}) \subset A$.

For any $m \geqslant 1$, we construct a proper map $g : (U, A', A) \to (V, B', B)$ as follows (see Figure 5). We will look at $\mathcal{T}(c)$, rows of K_0, K_m, K'_{m+2} and K_{m+2}.

Set $\xi = |_{K'_{m+2}}\diagup^{K_m}|$, $M = |_{K_m}\diagup^{K_0}|$, $\hat{x} = f^\xi(c)$ and $\hat{y} = f^M(\hat{x})$. Set the triple U, V, g as:

$$
\begin{array}{lcl}
V := K_0 & - & V\\
\quad\;\; | & \nearrow_{g:=f^M} & \\
U := K_m & &
\end{array}
$$

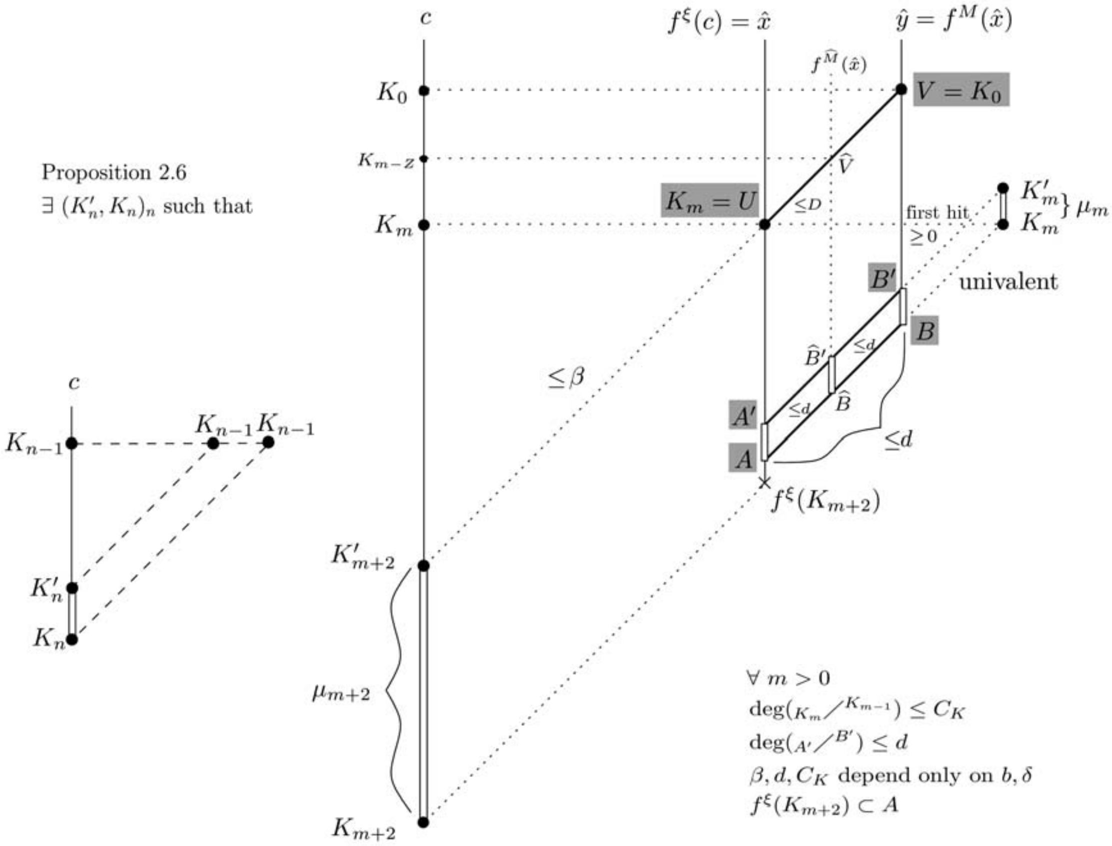


Figure 5 Construction of KL-maps from a KSS nest

By the hypothesis of recurrence, there is a minimal $l \geqslant 0$ such that from $\mathcal{T}(c)$ column of $\hat{y}$, row(K_m), marching horizontally to the right l steps one will meet again K_m. If $l = 0$, i.e. $\hat{y} \in K_m$, set $B = K_m$ and $B' = K'_m$. If $l > 0$, let $B' \supset\supset B$ be the pair of puzzle pieces on column$(\hat{y})$ forming a parallelogram with K'_m, K_m (i.e. $\hat{y} \in B$ and $f^l(B) = K_m$). In both cases let $A' \supset\supset A$ be the pair of puzzle pieces on column$(\hat{x})$ forming a parallelogram with B', B.

Proof. Points (a) and (d) follow from

$$\deg\left({}_{K_n}\diagup^{K_{n-1}}\right) = C_{\mathcal{A}} \cdot C_{\mathcal{B}} = \delta^3 = C_K; \quad \deg\left({}_{K'_n}\diagup^{K_{n-1}}\right) = C_{\mathcal{B}}^2 < C_K; \tag{2.5}$$

$$(K'_n \backslash \overline{K}_n) \cap \mathbf{P} = \emptyset; \tag{2.6}$$

$$\deg({}_{K'_{n+1}}\diagup^{K_{n-1}}) = \deg({}_{K'_{n+1}}\diagup^{K_n}) \cdot \deg({}_{K_n}\diagup^{K_{n-1}}) = \delta^2 \cdot \delta^3 = \beta\ . \tag{2.7}$$

Point (b) is obvious.

Starting from $\mathcal{T}(\hat{x})$, we define inductively the sequence of column numbers κ_j as follows: set $\kappa_0 = 0$. From κ_j-th column, row(K_m), march at first u_m steps to the right, then march to the right until the first $\geqslant 0$ hit of a critical spot, and set κ_{j+1} to be the corresponding column number. Let now L be the minimal integer so that $\kappa_L > M := |{}_{K_m}\diagup^{K_0}|$.

Therefore $\kappa_{L-1} \leqslant M$. This implies that $(L-1)u_m \leqslant M < \frac{3}{2}p_m \leqslant \frac{3}{2} \cdot \frac{4}{3}u_m = 2\,u_m$. So $L \leqslant 2$.

Now $\deg\left({}_{A}\diagup^{B}\right) \leqslant (\deg\left({}_{K_m}\diagup^{I_m}\right))^L = (\deg\left({}_{\mathcal{A}(I_m)}\diagup^{I_m}\right))^L \leqslant (C_{\mathcal{A}})^2 = \delta^4 = d$.
But $\deg({}_{A}\diagup^{B}) = \deg({}_{A'}\diagup^{B'})$ (as $K'_m \backslash \overline{K}_m$ contains no postcritical points), this gives Point (c).

Point (e). For this we will estimate the number of critical spots on the top edge of the upper triangle T' (resp. T), defined so to have a left edge $\begin{smallmatrix} K_m \\ | \\ f^{\xi}(K_{m+2}) \end{smallmatrix}$ (resp. $\begin{smallmatrix} K_m \\ | \\ A \end{smallmatrix}$) on column$(\hat{x})$.

At first, recall that $B = \widehat{\mathcal{L}}_{\hat{y}}(K_m)$, so κ_L is actually greater or equal to the column number (in $\mathcal{T}(\hat{x})$) of the right vertex of T. Also, by Rule 1, the number of critical spots on row(K_m) from the κ_j-th column (excluded) to the κ_{j+1}-th column (included) is at most equal to the number of critical spots on $[{}_{K_m=\mathcal{A}(I_m)}\diagup^{I_m}]$, which is equal to 2 by Lemma 2.5. Therefore, denoting by top(T) the top edge of in T including the end vertices,

$$\begin{aligned}&\sharp\{\text{critical spots} \in \text{top}(T)\} - 1 \overset{\text{def. of L}}{\leqslant} L \cdot 2 \leqslant 4,\\ &|{}_{\Gamma(K_m)}\diagup^{K_m}| \cdot (\sharp\{\text{critical spots} \in \text{top}(T')\} - 1)\\ &\qquad \overset{\text{Cor.2.4 (iii)}}{>} R(K_m) \cdot (\sharp\{\text{critical spots} \in \mathit{top}(T')\} - 1)\\ &\qquad = R(K_m) \cdot (\sharp\{\text{segments on top}(T') \text{ linking two consecutive critical spots}\})\\ &\qquad \overset{*}{\geqslant} \text{sum of the lengths of these segments} = |T'|\\ &\qquad \overset{**}{\geqslant} r(I_{m+2}) = \frac{r(I_{m+2})}{r(I_{m+1})} r(I_{m+1}) \overset{(2.3)}{\geqslant} 3 \cdot |{}_{\Gamma(K_m)}\diagup^{K_m}|,\end{aligned}$$

the one marked by $*$ is due to the definition of $R(K_m)$ as the maximal possible length between two consecutive critical spots on row(K_m); the one marked by $**$ is due to the fact that the left edge of T' forms a parallelogram with $\begin{smallmatrix}K'_{m+2}\\ |\\ K_{m+2}\end{smallmatrix}$ (see Figure 2.4), therefore $|T'| = \left|\begin{smallmatrix}K'_{m+2}\\ |\\ K_{m+2}\end{smallmatrix}\right| = \left|\begin{smallmatrix}\mathcal{B}(I_{m+2})\\ |\\ \mathcal{A}(I_{m+2})\end{smallmatrix}\right|$, which in turn is greater than or equal to $r(I_{m+2})$ by (2.2).

It follows then $\sharp\{\text{critical spots} \in \text{top}(T')\} - 1 \geqslant 4 \geqslant \sharp\{\text{critical spots} \in \text{top}(T)\} - 1$. Point (e) follows.

2.5 Further estimates

For any $n \geqslant 1$, let K_n^- be the critical puzzle piece so that $\begin{smallmatrix}K_n\\ |\\ K_n^-\end{smallmatrix}$ and $\begin{smallmatrix}\mathcal{B}(K_{n-1})\\ |\\ \mathcal{A}(K_{n-1})\end{smallmatrix}$ form a parallelogram. See Figure 6.

Corollary 2.7. *For every $n \geqslant 1$, the segment $\begin{smallmatrix}K_n\\ |\\ K_n^-\end{smallmatrix}$ is vacuous,*

$$\left|\begin{matrix}K_{n+1}\\ |\\ K_{n+1}^-\end{matrix}\right| > \left|\begin{matrix}K_n\\ |\\ K_n^-\end{matrix}\right|; \quad \left|\begin{matrix}K'_{n+1}\\ |\\ K_{n+1}\end{matrix}\right| > \left|\begin{matrix}K'_n\\ |\\ K_n\end{matrix}\right|, \tag{2.8}$$

and

$$\text{mod}(K_{n+1}\backslash\overline{K_{n+1}^-}) > \frac{\text{mod}(K'_{n-1}\backslash\overline{K_{n-1}})}{C_{\mathcal{A}}^3 C_{\mathcal{B}}}. \tag{2.9}$$

Proof. The vacuousness follows from the property of $\mathcal{A}$ and $\mathcal{B}$ (Lemma 2.5). Now

$$r(I_n) = r(\Gamma(K_{n-1})) \overset{\text{Cor.2.4.(iii)}}{>} R(K_{n-1}) \geqslant \begin{cases}\left|\begin{matrix}\mathcal{B}(K_{n-1})\\ |\\ \mathcal{A}(K_{n-1})\end{matrix}\right| = \left|\begin{matrix}K_n\\ |\\ K_n^-\end{matrix}\right|\\ R(I_{n-1}) \geqslant \left|\begin{matrix}K'_{n-1}\\ |\\ K_{n-1}\end{matrix}\right|\end{cases}. \tag{2.10}$$

Hence

$$\left|\begin{matrix}K_{n+1}\\ |\\ K_{n+1}^-\end{matrix}\right| = \left|\begin{matrix}\mathcal{B}(K_n)\\ |\\ \mathcal{A}(K_n)\end{matrix}\right| \geqslant r(K_n) \geqslant r(I_n) \overset{(2.10)}{>} \begin{cases}\left|\begin{matrix}K_n\\ |\\ K_n^-\end{matrix}\right|\\ \left|\begin{matrix}K'_{n-1}\\ |\\ K_{n-1}\end{matrix}\right|\end{cases}, \quad \left|\begin{matrix}K'_{n+1}\\ |\\ K_{n+1}\end{matrix}\right| \geqslant r(I_{n+1}) \overset{(2.10)}{>} \left|\begin{matrix}K'_n\\ |\\ K_n\end{matrix}\right|.$$

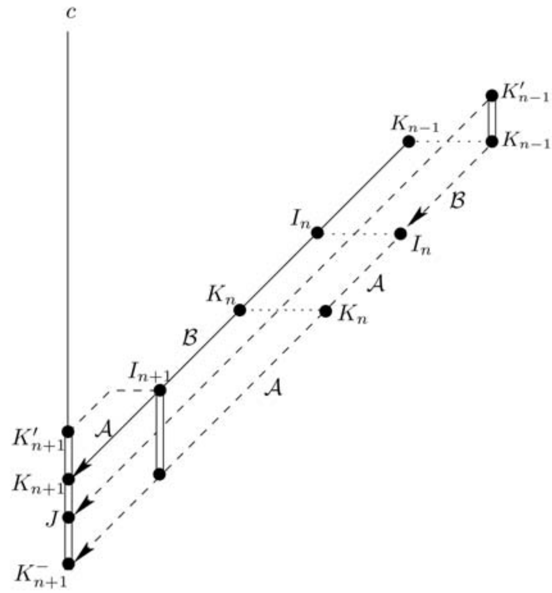


Figure 6 Control of moduli

Let J be the puzzle piece so that $\begin{array}{c} J \\ | \\ K^-_{n+1} \end{array}$ and $\begin{array}{c} K'_{n-1} \\ | \\ K_{n-1} \end{array}$ form a parallelogram. See Figure 6. As $\left|\begin{array}{c} J \\ | \\ K^-_{n+1} \end{array}\right| = \left|\begin{array}{c} K'_{n-1} \\ | \\ K_{n-1} \end{array}\right| < \left|\begin{array}{c} K_{n+1} \\ | \\ K^-_{n+1} \end{array}\right|$, we know that $J \subset K_{n+1}$. Therefore

$$\operatorname{mod}(K_{n+1}\backslash\overline{K^-_{n+1}}) \geqslant \operatorname{mod}(J\backslash\overline{K^-_{n+1}}) \overset{\text{vacuous}}{=} \frac{\operatorname{mod}(K'_{n-1}\backslash\overline{K_{n-1}})}{\deg({K^-_{n+1}}\diagup{K_{n-1}})} \geqslant \frac{\operatorname{mod}(K'_{n-1}\backslash\overline{K_{n-1}})}{C^3_{\mathcal{A}}C_{\mathcal{B}}}.$$

2.6 Proof of Theorem 2.1

In the proof, we will use the following lemma:

Lemma 2.8 (Kahn-Lyubich Covering Lemma [18]). *Fix $\eta > 0$ and $D \in \mathbb{N}$. There is $\varepsilon(\eta, D) > 0$ such that: given any $g : U \to V$,*

$$A \subset\subset A' \subset\subset U,\ B \subset\subset B' \subset\subset V\ (\textit{all discs}),$$
$$g : U \to V,\ A' \to B',\ A \to B \textit{ are proper holomorphic maps},$$

$$\begin{cases} \deg(g|_U) \leqslant D, \\ \deg(g|_{A'}) \leqslant d, \\ \operatorname{mod}(B'\backslash\overline{B}) \geqslant \eta \cdot \operatorname{mod}(U\backslash\overline{A}), \end{cases} \Longrightarrow \operatorname{mod}(U\backslash\overline{A}) \begin{array}{l} \overset{either}{>}\ \varepsilon(\eta, D) > 0; \\ \overset{or}{>}\ \dfrac{\eta}{2d^2}\operatorname{mod}(V\backslash\overline{B}). \end{array}$$

Now we prove Theorem 2.1.

The fact (i) on the uniform bound of degree from K_n to K_{n-1} is given by Proposition 2.6. The part in (ii) that $K_n\backslash\overline{K_n^-}$ is vacuous is proved in Corollary 2.7.

We will prove that

$$\exists\ C = C(\delta, \widehat{\mu}) > 0,\ \forall\, n, \quad \operatorname{mod}(K'_n\backslash\overline{K}_n) \geqslant C. \tag{2.11}$$

It follows then from Corollary 2.7 that $\operatorname{mod}(K_n\backslash\overline{K_n^-})$ for any n have also a positive lower bound, denoted by Δ, depending only on δ and $\widehat{\mu}$. This proves the remaining part of (ii).

Proof of (2.11). Set $Z = 2d^3\beta^2 + 1 = 2\delta^{22} + 1$. Set $\mu_n = \operatorname{mod}(K'_n\backslash\overline{K}_n)$, $n \geqslant 1$.

Case 1. Assume there is an increasing sequence $k_n \to \infty$ such that $\mu_{m'} \overset{\forall m'<k_n}{\geqslant} \mu_{k_n}$.

There is n_0 so that any $n \geqslant n_0$ satisfies $k_n - 2 \geqslant Z$.

Fix $n \geqslant n_0$ and set $m = k_n - 2$. Then $\mu_{m'} \geqslant \mu_{m+2}$ for any $m' \leqslant m+2$. For this m, define as in Proposition 2.6 the collection of objects

$$(\xi, M, U, V, g, B, B', A, A').$$

Set then $\widehat{V} = K_{m-Z}$; $\widehat{g} = f^{\widehat{M}} : U \to \widehat{V}$ where $\widehat{M}$ is chosen so that $f^{\widehat{M}}(U) = \widehat{V}$; $\widehat{B}' = \widehat{g}(A')$ and $\widehat{B} = \widehat{g}(A)$.

We then apply the estimates in Proposition 2.6. Note that, by (a) of Proposition 2.6, $\deg(\widehat{g}) \leqslant C_K^Z =: D$ with D independent of m. Note also

$$\deg({}_{A'}\!\diagup^{\widehat{B}'}), \deg({}_{\widehat{B}'}\!\diagup^{B'}) \leqslant \deg({}_{A'}\!\diagup^{B'}) \overset{(c)}{\leqslant} d, \tag{2.12}$$

$$\deg({}_{\widehat{B}'}\!\diagup^{K'_m}) = \deg({}_{\widehat{B}'}\!\diagup^{B'}) \deg({}_{B'}\!\diagup^{K'_m}) \overset{(b)}{\leqslant} d\ . \tag{2.13}$$

We have (see Figure 2.4)

$$\mu_{m+2} \geqslant \frac{\mathrm{mod}(U \backslash \overline{f^{\xi}(K_{m+2})})}{\deg({}_{K'_{m+2}}\!\diagup^{U})} \underset{(d)}{\overset{(e)}{\geqslant}} \frac{\mathrm{mod}(U\backslash\overline{A})}{\beta}; \tag{2.14}$$

$$\mathrm{mod}(\widehat{B}'\backslash\overline{\widehat{B}}) \overset{\text{no}\,\mathbf{P}}{=} \frac{\mu_m}{\deg\left({}_{\widehat{B}'}\!\diagup^{K'_m}\right)} \underset{(2.13)}{\overset{\text{choice of } k_n}{\geqslant}} \frac{\mu_{m+2}}{d} \overset{(2.14)}{\geqslant} \frac{\mathrm{mod}(U\backslash\overline{A})}{d\beta} =: \eta \cdot \mathrm{mod}(U\backslash\overline{A}),$$

with $\eta := \frac{1}{d\beta}$ independent of m.

Therefore for any $n \geqslant n_0$ and $m = k_n - 2$, we may then apply Kahn-Lyubich Covering Lemma to $\widehat{g} : (U, A', A) \to (\widehat{V}, \widehat{B}', \widehat{B})$, to conclude that,

$$\beta \cdot \mu_{m+2} \overset{(2.14)}{\geqslant} \mathrm{mod}(U\backslash\overline{A}) \begin{cases} \overset{\text{either}}{>} \varepsilon(\eta, D) > 0; \\ \overset{\text{or}}{>} \frac{\eta \cdot \mathrm{mod}(\widehat{V}\backslash\overline{\widehat{B}})}{2\deg\left({}_{A'}\!\diagup^{\widehat{B}'}\right)^2} \overset{(*)}{\geqslant} \frac{\eta}{2d^2}(\mu_m + \cdots + \mu_{m-Z+1}) \overset{\text{choice of } k_n}{\geqslant} \frac{\eta Z}{2d^2}\mu_{m+2}, \end{cases}$$

where the inequality $(*)$ is proved as follows: For $j = m, m-1, \ldots, m-Z+1$, and for $l_j \geqslant 0$ minimal so that $f^{l_j}(\widehat{g}(\hat{x})) \in K_j$, denote by B_j, resp. B'_j, the connected component of $f^{-l_j}(K_j)$, resp. $f^{-l_j}(K'_j)$, containing $\widehat{g}(\hat{x})$. Then $\{B'_j\backslash\overline{B_j}\}_j$ are pairwise disjoint essential annuli in $\widehat{V}\backslash\overline{\widehat{B}}$ with $\mathrm{mod}(B'_j\backslash\overline{B_j}) = \frac{\mu_j}{\deg(B_j \to K_j)} = \mu_j$ (due to $(K'_j\backslash\overline{K}_j) \cap \mathbf{P} = \emptyset$ and the fact that $\deg(B_j \to K_j) = 1$).

By our choice of Z we have $Z > 2d^3\beta^2 = \frac{2d^2\beta}{\eta}$. Hence the second line above is impossible. So $\mu_{k_n} = \mu_{m+2} > \frac{1}{\beta} \cdot \varepsilon(\eta, D) > 0$. Therefore $\forall\, l \in \mathbb{N}$, $\mu_l \geqslant \lim_{n\to\infty} \mu_{k_n} \geqslant \frac{1}{\beta} \cdot \varepsilon(\eta, D)$.

Case 2. Assume there is k_0 such that $\mu_k \geqslant \mu_{k_0}$ for all $k \geqslant 1$.

Case 2.1. $k_0 - 2 \geqslant Z$. Then repeating the same argument as above we know that $\mu_{k_0} \geqslant \frac{1}{\beta} \cdot \varepsilon(\eta, D)$.

Case 2.2. $k_0 - 2 < Z$. Then $k_0 \leqslant Z + 1 = 2\delta^{22} + 2$. Notice that K'_{k_0} is mapped by some iterate of f onto $K_0 = P_0(c)$ with degree bounded by C_K^{Z+1}, due to (a) of Proposition 2.6. The same iterate of f maps K_{k_0} into a component W of $\mathbf{U}$ contained in $P_0(c)$. So $\mu_{k_0} \geqslant \frac{1}{C_K^{Z+1}} \mathrm{mod}(P_0(c)\backslash\overline{W}) \geqslant \frac{1}{C_K^{Z+1}}\widehat{\mu}$.

We have now proved (2.11) for $C = \min\{\frac{1}{\beta} \cdot \varepsilon(\eta, D), \frac{1}{C_K^{Z+1}}\widehat{\mu}\}$.

3 Proof of Theorem 1.2

Now we come to the proof of Theorem 1.2. Clearly if f and $\tilde{f}$ are qc-conjugate then they are c-equivalent and qc-conjugate off $\mathbf{K}_f$.

3.1 C-equivalence implies qc-conjugacy off $\mathbf{K}_f$

We just repeat the standard argument (see for example Appendix in [13]).

Assume that $f, \tilde{f}$ are c-equivalent. Set $\mathbf{U} = \mathbf{U}_1$, and $\mathbf{U}_n = f^{-n}(\mathbf{V})$. The same objects gain a tilde for $\tilde{f}$. For $t \in [0, 1]$, let $h_t : \overline{\mathbf{V}} \to \overline{\tilde{\mathbf{V}}}$ be an isotopy path linking h_0 to h_1.

Then there is a unique continuous extension $(t, z) \mapsto h(t, z), [0, \infty[\times\overline{\mathbf{V}} \to \overline{\tilde{\mathbf{V}}}$ such that

0) each $h_t : z \to h(t, z)$ is a homeomorphism;

1) $h_t|_{\partial \mathbf{V} \cup \mathbf{P}} = h_0|_{\partial \mathbf{V} \cup \mathbf{P}}, \forall t \in [0, +\infty[$;

2) for $n \geqslant 1$, $t > n$ and $x \in \mathbf{V} \backslash \mathbf{U}_n$ we have $h_t(x) = h_n(x)$;

3) for $t \in [0, 1]$ the following diagram commutes:

$$\begin{array}{ccc} \vdots & & \vdots \\ \mathbf{U}_2 & \xrightarrow{h_{t+2}} & \tilde{\mathbf{U}}_2 \\ f\downarrow & & \downarrow \tilde{f} \\ \mathbf{U}_1 & \xrightarrow{h_{t+1}} & \tilde{\mathbf{U}}_1 \\ f\downarrow & & \downarrow \tilde{f} \\ \mathbf{V} & \xrightarrow{h_t} & \tilde{\mathbf{V}}. \end{array}$$

Set then $\Omega = \bigcup_{n \geqslant 1} \mathbf{V} \backslash \mathbf{U}_n = \mathbf{V} \backslash \mathbf{K}_f$, and $\tilde{\Omega} = \tilde{\mathbf{V}} \backslash \mathbf{K}_{\tilde{f}}$. Then there is a qc map $H : \Omega \to \tilde{\Omega}$ such that $H(x) = h_n(x)$ for $n \geqslant 1$ and $x \in \mathbf{V} \backslash \mathbf{U}_n$ and that $H \circ f|_{\Omega \cap \mathbf{U}} = \tilde{f} \circ H|_{\tilde{\Omega} \cap \tilde{\mathbf{U}}}$, i.e. H realizes a qc-conjugacy from f to $\tilde{f}$ off $\mathbf{K}_f$. The qc constant of H is equal to C_0, the qc constant of h_1 on $\mathbf{V} \backslash \mathbf{U}$.

3.2 qc-conjugacy off $\mathbf{K}_f$ implies qc-conjugacy

Assume now that f and $\tilde{f}$ are qc-conjugate off $\mathbf{K}_f$, i.e. there is a C_0-qc map $H : \mathbf{V} \backslash \mathbf{K}_f \to \tilde{\mathbf{V}} \backslash \mathbf{K}_{\tilde{f}}$ conjugating f to $\tilde{f}$. We will show now that H admits a qc extension across $\mathbf{K}_f$ which is again a conjugacy. This will be done in three steps.

Uniform regularity of H on ∂K_n

Proposition 3.1. *Let $\{K_n,\ n \geqslant 1\}$ be the KSS nest for f given by Theorem* 2.1. *There is a constant L', such that for any $n \geqslant 1$, $H|_{\partial K_n}$ admits an L'-qc extension inside K_n.*

The basic step of the proof of Proposition 3.1 is the following lemma on covering maps of the unit disk.

Lemma 3.2. (*See Lemma* 3.2 *in* [1]): *For every integer $d \geqslant 2$ and every $0 < \rho < r < 1$ there exists $L_0 = L_0(\rho, r, d)$ with the following property. Let $g, \tilde{g} : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ be holomorphic proper maps of degree d, with critical values contained in $\mathbb{D}_\rho$. Let $\eta, \eta' : \mathbb{T} \to \mathbb{T}$ be two homeomorphisms satisfying $\tilde{g} \circ \eta' = \eta \circ g$. Assume that η admits an L-qc extension $\xi : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. Then η' admits an L'-qc extension $\xi' : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where $L' = \max\{L, L_0\}$.*

Proof of Proposition 3.1. We will just adapt the proof of Theorem 3.1 in [1] to our KSS nest.

For all $n \geqslant 1$, let $\tilde{K}_n$ be the puzzle piece bounded by $H(\partial K_n)$.

Notice that $H|_{\partial K_1}$ has a qc extension on a neighborhood of ∂K_1. It extends thus to an L_1-qc map $K_1 \to \tilde{K}_1$, $c \to \tilde{c}$ for some $L_1 \geqslant 1$ (see e.g. [13, Lemma C.1]).

For the sequence K_n^- given by Theorem 2.1, define $\tilde{K}_n^-$ accordingly. As the operators $\Gamma, \mathcal{A}, \mathcal{B}$ in the definition of the KSS nest can be read off from the dynamical degree on the boundary of the puzzle pieces, and H preserves this degree information, the sequence $\tilde{K}_n$ is precisely the KSS nest for $\tilde{f}$ starting from $\tilde{K}_0$. Therefore Theorem 2.1 is valid for this sequence as well, with the same constant $C_K = \delta^3$, and probably a different $\tilde{\Delta}$ as a lower bound for $\mathrm{mod}(\tilde{K}_n \backslash \overline{\tilde{K}_n^-})$.

Recall that for each $i \geqslant 1$, p_i denotes the integer such that $f^{p_i}(K_i) = K_{i-1}$. We have $\tilde{f}^{p_i}(\tilde{K}_i) = \tilde{K}_{i-1}$, and $f^{p_i} : K_i \to K_{i-1}$ and $\tilde{f}^{p_i} : \tilde{K}_i \to \tilde{K}_{i-1}$ are proper holomorphic maps of degree δ^3.

Fix now $n \geqslant 1$. We will show that $H|_{\partial K_n}$ has an L'-qc extension inside K_n with L' independent of n. This will not be an induction argument on n. In other words, this extension will not be a pullback of a previously defined extension of H inside K_{n-1}.

Set $v_n = c$, and then, for $i = n-1, n-2, \ldots, 1$, set consecutively $v_i = f^{p_{i+1}+\cdots+p_n}(c)$.

Since $K_i \backslash \overline{K_i^-}$ is vacuous, the critical values of $f^{p_{i+1}}|_{K_{i+1}}$, as well as v_i, are contained in K_i^-, $1 \leqslant i \leqslant n-1$.

Let $\psi_i : (K_i, v_i) \to (\mathbb{D}, 0)$ be a bi-holomorphic uniformization, $i = 1, \ldots, n$. For $i = 2, \ldots, n$, let $g_i = \psi_{i-1} \circ f^{p_i} \circ \psi_i^{-1}$. These maps fix the point 0, are proper holomorphic maps of degree δ^3, with the critical values contained in $\psi_{i-1}(K_{i-1}^-)$.

Let $\psi_i(K_i^-) = \Omega_i$. Since $\mathrm{mod}(\mathbb{D} \backslash \overline{\Omega_i}) = \mathrm{mod}(K_i \backslash \overline{K_i^-}) \geqslant \Delta > 0$ and $\Omega_i \ni \psi_i(v_i) = 0$, $1 \leqslant i \leqslant n$, these domains are contained in some disk $\mathbb{D}_s$ with $s = s(\Delta) < 1$. So the critical values of g_i are contained in $\Omega_{i-1} \subset \mathbb{D}_s$, $2 \leqslant i \leqslant n$.

The corresponding objects for $\tilde{f}$ will be marked with a tilde. The same assertions hold for $\tilde{g}_i$. Then all the maps g_i and $\tilde{g}_i$ satisfy the assumptions of Lemma 3.2, with $d = \delta^3$, and $\rho = \max\{s, \tilde{s}\}$.

$$\begin{array}{ccccccc}
(\mathbb{D},0) & \xleftarrow{\psi_1} & (K_1, v_1) & & (\tilde{K}_1, \tilde{v}_1) & \xrightarrow{\tilde{\psi}_1} & (\mathbb{D},0) \\
g_2 \uparrow & & \uparrow f^{p_2} & & \tilde{f}^{p_2} \uparrow & & \uparrow \tilde{g}_2 \\
(\mathbb{D},0) & \xleftarrow{\psi_2} & (K_2, v_2) & & (\tilde{K}_2, \tilde{v}_2) & \xrightarrow{\tilde{\psi}_2} & (\mathbb{D},0) \\
g_3 \uparrow & & \uparrow f^{p_3} & & \tilde{f}^{p_3} \uparrow & & \uparrow \tilde{g}_3 \\
\vdots & & \vdots & & \vdots & & \vdots \\
(\mathbb{D},0) & \xleftarrow{\psi_{n-1}} & (K_{n-1}, v_{n-1}) & & (\tilde{K}_{n-1}, \tilde{v}_{n-1}) & \xrightarrow{\tilde{\psi}_{n-1}} & (\mathbb{D},0) \\
g_n \uparrow & & \uparrow f^{p_n} & & \tilde{f}^{p_n} \uparrow & & \uparrow \tilde{g}_n \\
(\mathbb{D},0) & \xleftarrow{\psi_n} & (K_n, v_n), & & (\tilde{K}_n, \tilde{v}_n) & \xrightarrow{\tilde{\psi}_n} & (\mathbb{D},0).
\end{array}$$

Note that each of $\psi_i, \tilde{\psi}_i$ extends to a homeomorphism from the closure of the puzzle piece to $\overline{\mathbb{D}}$.

Let us consider homeomorphisms $\eta_i : \mathbb{T} \to \mathbb{T}$ given by $\eta_i = \tilde{\psi}_i \circ H|_{\partial K_i} \circ \psi_i^{-1}$. They are equivariant with respect to the g-actions, i.e., $\eta_{i-1} \circ g_i = \tilde{g}_i \circ \eta_i$.

Due to the qc extension of $H|_{\partial K_1}$, we know that η_1 extends to an L_1-qc map $(\mathbb{D}, \psi_1(c)) \to (\mathbb{D}, \tilde{\psi}_1(\tilde{c}))$. Fix some r with $\rho < r < 1$. Since $c \in K_1^-$, $\tilde{c} \in \tilde{K}_1^-$, we have $\psi_1(c), \tilde{\psi}_1(\tilde{c}) \in \mathbb{D}_\rho$. We conclude that η_1 extends to an L-qc map $\xi_1 : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where L depends on L_1, ρ and r.

Let $L_0 = L_0(\rho, r, \delta^3)$ be as in Lemma 3.2, and let $L' = \max\{L, L_0\}$. For $i = 2, 3, \ldots, n-1$, apply consecutively Lemma 3.2 to the following left diagram (from top to bottom):

$$\begin{array}{ccccccc}
\mathbb{T} & \xrightarrow{\eta_1} & \mathbb{T} & & (\mathbb{D},0) & \xrightarrow{\xi_1} & (\mathbb{D},0) \\
g_2 \uparrow & & \uparrow \tilde{g}_2 & & g_2 \uparrow & & \uparrow \tilde{g}_2 \\
\mathbb{T} & \xrightarrow{\eta_2} & \mathbb{T} & & (\mathbb{D},0) & \xrightarrow{\xi_2} & (\mathbb{D},0) \\
g_3 \uparrow & & \uparrow \tilde{g}_3 & \text{we get} & g_3 \uparrow & & \uparrow \tilde{g}_3 \\
\vdots & & \vdots & & \vdots & & \vdots \\
\mathbb{T} & \xrightarrow{\eta_{n-1}} & \mathbb{T} & & (\mathbb{D},0) & \xrightarrow{\xi_{n-1}} & (\mathbb{D},0) \\
g_n \uparrow & & \uparrow \tilde{g}_n & & g_n \uparrow & & \uparrow \tilde{g}_n \\
\mathbb{T} & \xrightarrow{\eta_n} & \mathbb{T}, & & (\mathbb{D},0) & \xrightarrow{\xi_n} & (\mathbb{D},0),
\end{array}$$

so that for $i = 2, \ldots, n$, the map η_i admits an L'-qc extension $\xi_i : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. The desired extension of $H|_{\partial K_n}$ inside K_n is now obtained by taking $\tilde{\psi}_n^{-1} \circ \xi_n \circ \psi_n$.

Spreading principle

Fix now $n \geqslant 1$. Recall that K_n is the unique critical puzzle piece of depth M_n. Set $W_0 = \mathbf{U}_{M_n} \backslash K_n$ and $X_0 = K_n$. For $j \geqslant 0$, define a sequence $H^{(j)}$ of qc maps as follows:

$$H^{(0)} = \begin{cases} H, & \text{on } \mathbf{V} \backslash (X_0 \cup W_0) = \mathbf{V} \backslash \mathbf{U}_{M_n}, \\ \text{the } L'\text{-qc map given by Proposition 3.1}, & \text{on } X_0 = K_n, \\ \text{a } C_n'\text{-qc extension for some } C_n' \geqslant 1, & \text{on } W_0 = \mathbf{U}_{M_n} \backslash K_n. \end{cases}$$

The constant C_n' may depend on n.

For $j \geqslant 1$, set

$$X_j = \{z \in \mathbf{U}_{M_n} \mid f^l(z) \in K_n \text{ for some } 0 \leqslant l \leqslant j\}$$

and

$$W_j = \{z \in \mathbf{U} \mid z, f(z), f^2(z), \dots, f^j(z) \in W_0\} = U_{M_n+j} \backslash X_j.$$

Notice that for any $z \in X_j$, $\widehat{\mathcal{L}}_z(K_n)$ is a connected component of X_j, and there is $j \geqslant l(z) \geqslant 0$ so that $f^{l(z)}$ maps $\widehat{\mathcal{L}}_z(K_n)$ univalently onto K_n. On the other hand, any component S of W_j is a puzzle piece of depth $M_n + j$ and f^j maps S univalently onto a component of W_0. Set then

$$H^{(j)} = \begin{cases} H, & \text{on } \mathbf{V} \backslash (X_j \cup W_j), \\ \text{the } L'\text{-qc map given by univalent pullback of } H^{(0)}|_{X_0}, & \text{on } X_j, \\ \text{the } C_n'\text{-qc map given by univalent pullback of } H^{(0)}|_{W_0}, & \text{on } W_j. \end{cases}$$

Set $C_n = \max\{C_0, L', C_n'\}$. The $\{H^{(j)}\}_{j \geqslant 0}$ is a sequence of C_n-qc maps. Hence it is precompact in the uniform topology.

Notice that $H^{(j)} = H^{(j-1)}$ outside W_{j-1}. Thus, the sequence $\{H^{(j)}\}$ converges pointwise outside

$$\bigcap_j W_j = \{x \in \mathbf{K}_f \mid f^k(x) \notin K_n, k \geqslant 0\}.$$

This set is a hyperbolic subset, on which f is uniformly expanding, and hence has zero Lebesgue measure, in particular no interior. So any two limit maps of the sequence $\{H^{(j)}\}_{j \geqslant 0}$ coincide on a dense open set of $\mathbf{V}$, therefore coincides on $\mathbf{V}$ to a unique limit map. Denote this map by H_n. It is C_n-qc.

By construction, H_n coincides with H on $\mathbf{V} \backslash ((\bigcup_j X_j) \cup (\bigcap_j W_j))$ is therefore C_0-qc there; and is L'-qc on $\bigcup_j X_j$. It follows that the dilatation of H_n is bounded by $\max\{C_0, L'\}$ except possibly on the set $\bigcap_j W_j$. But this set has zero Lebesgue measure. It follows that the dilatation of H_n is $\max\{C_0, L'\}$, which is independent of n.

Conclusion

The sequence $H_n : \mathbf{V} \to \tilde{\mathbf{V}}$ has a subsequence converging uniformly to a limit qc map $H' : \mathbf{V} \to \tilde{\mathbf{V}}$, with $H'|_{\mathbf{V} \backslash \mathbf{K}_f} = H$. Therefore H' is a qc extension of H. On the other hand, $H \circ f = \tilde{f} \circ H$ on $\mathbf{U} \backslash \mathbf{K}_f$, and $\mathbf{K}_f$ has empty interior by assumption. So $H' \circ f = \tilde{f} \circ H'$ holds on $\mathbf{U}$ by continuity. Therefore H' is a qc-conjugacy from f to $\tilde{f}$. This ends the proof of Theorem 1.2.

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Appendix A: Proof of Theorem 1.1

For any $x \in \mathbf{K}_f$, we say that the forward orbit of x *combinatorially accumulates* to c, written as $x \to c$, if a critical vertex appears in each row of $\mathcal{T}(x)\backslash\{$0-th column$\}$.

For the unique critical point c, one of the following cases will occur.

Case a. $c \not\to c$.

Case b. $c \to c$ and $\mathcal{T}(c)$ is not persistently recurrent (this is commonly called *reluctantly recurrent*).

Case c. $c \to c$ and $\mathcal{T}(c)$ is persistently recurrent.

Lemma A.1. *In Case* a *or Case* b*, there are a puzzle piece $P_{n_0}(x)$ of depth n_0 containing $x \in \mathbf{V}$ and infinitely many integers i_n such that*

$$f^{i_n}(P_{n_0+i_n}(c)) = P_{n_0}(x), \quad f^{i_n}(P_{n_0+i_n+1}(c)) = P_{n_0+1}(x)$$

and $\deg({}_{P_{n_0+i_n}(c)}\diagup^{P_{n_0}(x)}) = \delta$.

Proof. In Case a, there is an integer $n_0 \geqslant 0$ such that each vertex of at least n_0-th row in $\mathcal{T}(c)\backslash\{$0-th column$\}$ is non-critical. So for each $n \geqslant 1$,

$$\deg({}_{P_{n_0+n}(c)}\diagup^{P_{n_0}(f^n(c))}) = \delta.$$

Since there are finitely many puzzle pieces in every depth, we can take a subsequence i_n such that

$$f^{i_n}(P_{n_0+i_n}(c)) = P_{n_0}(x), \quad f^{i_n}(P_{n_0+i_n+1}(c)) = P_{n_0+1}(x)$$

for some fixed puzzle piece $P_{n_0}(x)$.

In Case b, since $\mathcal{T}(c)$ is reluctantly recurrent, there exist an integer $n_0 \geqslant 0$ and infinitely many integers $m_k \geqslant 1$ such that $\{P_{n_0+m_k}(c)\}_{k\geqslant 1}$ are children of $P_{n_0}(c)$ and then $\deg({}_{P_{n_0+m_k}(c)}\diagup^{P_{n_0}(c)}) = \delta$. Now we take a subsequence i_n of m_k such that

$$f^{i_n}(P_{n_0+i_n}(c)) = P_{n_0}(c) = P_{n_0}(x), \quad f^{i_n}(P_{n_0+i_n+1}(c)) = P_{n_0+1}(x)$$

for some fixed puzzle piece $P_{n_0+1}(x)$.

Now we prove $\mathbf{K}_f$ is a Cantor set. It is equivalent to show that for each $z \in \mathbf{K}_f$, the connected component of $\mathbf{K}_f$ containing z, denoted by $\mathbf{K}_f(z)$, is a single point. The argument is similar as the proof of Lemma 7 in [18].

For $z \in \mathbf{K}_f$, there are four possibilities as follows:

Case I. $z \not\to c$.

Case II. $z \to c$ and $c \not\to c$.

Case III. $z \to c$, and $\mathcal{T}(c)$ is reluctantly recurrent.

Case IV. $z \to c$ and $\mathcal{T}(c)$ is persistently recurrent.

In Case I, there is an integer $n_1 \geqslant 0$ such that each vertex of at least n_1-th row in $\mathcal{T}(z)\backslash\{$0-th column$\}$ is non-critical. So for each $n \geqslant 1$,

$$\deg({}_{P_{n_1+n}(z)}\diagup^{P_{n_1}(f^n(z))}) \leqslant \delta.$$

Take a subsequence j_n of n_1+n with $P_{n_1}(f^{j_n}(z)) = P_{n_1}$ for some fixed puzzle piece P_{n_1}. Let

$$\nu_{n_1} = \min\{\mathrm{mod}(P_{n_1}\backslash\overline{W}) \mid W \text{ a piece of depth } n_1+1 \text{ contained in } P_{n_1}\}\ .$$

Then

$$\mathrm{mod}(P_{j_n}(z)\backslash\overline{P_{j_n+1}(z)}) \geqslant \frac{\nu_{n_1}}{\delta}.$$

Thus $\sum_{n=1}^{\infty}\mathrm{mod}(P_{j_n}(z)\backslash\overline{P_{j_n+1}(z)}) = +\infty$. Applying then Grötzsch's inequality one could conclude immediately that $\mathrm{mod}(P_0(z)\backslash\mathbf{K}_f(z)) = +\infty$ and $\mathbf{K}_f(z)$ is a point.

In Case II or Case III, by Lemma A.1, there are a puzzle piece $P_{n_0}(x)$ and infinitely many integers i_n such that $f^{i_n}(P_{n_0+i_n}(c)) = P_{n_0}(x)$. As in Subsection 2.1, we construct a puzzle piece $\mathcal{L}_z(P_{n_0+i_n}(c))$ for $n \geqslant 1$ and by Lemma 2.4,

$$\deg({}_{\mathcal{L}_z(P_{n_0+i_n}(c))}\diagup^{P_{n_0+i_n}(c)}) \leqslant \delta$$

and then

$$\deg({}_{\mathcal{L}_z(P_{n_0+i_n}(c))}\diagup^{P_{n_0}(x)}) = \deg({}_{\mathcal{L}_z(P_{n_0+i_n}(c))}\diagup^{P_{n_0+i_n}(c)})\cdot\deg({}_{P_{n_0+i_n}(c)}\diagup^{P_{n_0}(x)}) \leqslant \delta^2.$$

Let $l_n = |{}_{\mathcal{L}_z(P_{n_0+i_n}(c))}\diagup^{P_{n_0+i_n}(c)}|$. Set

$$\nu_{n_0} = \min\{\mathrm{mod}(P_{n_0}(x)\backslash\overline{W}) \mid W \text{ a piece of depth } n_0+1 \text{ contained in } P_{n_0}(x)\}.$$

Then

$$\mathrm{mod}(P_{n_0+i_n+l_n}(z)\backslash\overline{P_{n_0+i_n+l_n+1}(z)}) \geqslant \frac{\nu_{n_0}}{\delta^2}.$$

Thus $\sum_{n=1}^{\infty}\mathrm{mod}(P_{n_0+i_n+l_n}(z)\backslash\overline{P_{n_0+i_n+l_n+1}(z)}) = +\infty$ and $\mathbf{K}_f(z)$ is a point.

In Case IV, we use the estimate (2.11) to obtain that $\mathbf{K}_f(z)$ is a point. Let $T_n(z) = \mathcal{L}_z(K_n)$ and $T'_n(z)$ be the puzzle piece such that $\begin{smallmatrix} T'_n(z) \\ | \\ T_n(z) \end{smallmatrix}$ and $\begin{smallmatrix} K'_n \\ | \\ K_n \end{smallmatrix}$ form a parallelogram. Since $\begin{smallmatrix} K'_n \\ | \\ K_n \end{smallmatrix}$ is vacuous, applying tableau rule3b (see also Figure 1.2), we know that $\deg({}_{T'_n(z)}\diagup^{K'_n}) = \deg({}_{T_n(z)}\diagup^{K_n}) \leqslant \delta$. Then

$$\mathrm{mod}(T'_n(z)\backslash\overline{T_n(z)}) \geqslant \frac{\mathrm{mod}(K'_n\backslash\overline{K_n})}{\delta} \geqslant \frac{C}{\delta} > 0.$$

Until now, we come to the conclusion that $\mathbf{K}_f$ is a Cantor set.

We may assume that $\{P_{n_0+i_n}(c)\}_{n\geqslant 1}$ in Lemma A.1 is a nested sequence of critical puzzle pieces. Set $Q_n = P_{n_0+i_n}(c)$ and $Q_n^- = P_{n_0+i_n+1}(c)$. In the following, we will combine Lemma 3.2 and Lemma A.1 to prove

Proposition A.2. *For every $n \geqslant 1$, $H|_{\partial Q_n}$ admits a K'-qc extension inside Q_n, where K' is a constant independent of n.*

Proof. Fix $n \geqslant 1$.

Let $\tilde{Q}_n$, $\tilde{Q}_n^-$ be the puzzle pieces bounded by $H(\partial Q_n)$ and $H(\partial Q_n^-)$ respectively. Since H preserves the degree information, Lemma A.1 is valid for $\tilde{Q}_n$ with the same δ, more precisely, for the map $\tilde{f}$, there exists a piece $\tilde{P}_{n_0}(\tilde{x})$ of depth n_0 containing $\tilde{x} \in \tilde{\mathbf{V}}$ such that

$$\tilde{f}^{i_n}(\tilde{P}_{n_0+i_n}(\tilde{c})) = \tilde{P}_{n_0}(\tilde{x}), \quad \tilde{f}^{i_n}(\tilde{P}_{n_0+i_n+1}(\tilde{c})) = \tilde{P}_{n_0+1}(\tilde{x})$$

and

$$\deg({}_{\tilde{P}_{n_0+i_n}(\tilde{c})} \diagup {}^{\tilde{P}_{n_0}(\tilde{x})}) = \delta.$$

Let $\text{mod}(P_{n_0}(x)\backslash\overline{P_{n_0+1}(x)}) = \Delta$ and $\text{mod}(\tilde{P}_{n_0}(\tilde{x})\backslash\overline{\tilde{P}_{n_0+1}(\tilde{x})}) = \tilde{\Delta}$.

Let $\phi_0 : (P_{n_0}(x), f^{i_n}(c)) \to (\mathbb{D}, 0)$ and $\phi_n : (Q_n, c) \to (\mathbb{D}, 0)$ be bi-holomorphic uniformizations. Let $h_n = \phi_0 \circ f^{i_n} \circ \phi_n^{-1}$. These maps fix the point 0, are proper holomorphic maps of degree δ. The critical values of them are contained in $\phi_0(P_{n_0+1}(x))$ because c is the unique critical point in Q_n of the map f^{i_n} restricted in Q_n.

Since $\text{mod}(\mathbb{D} \setminus \overline{\phi_0(P_{n_0+1}(x))}) = \text{mod}(P_{n_0}(x)\backslash\overline{P_{n_0+1}(x)}) = \Delta > 0$ and $\phi_0(P_{n_0+1}(x)) \ni \phi_0(f^{i_n}(c)) = 0$, we have $\phi_0(P_{n_0+1}(x)) \subset \mathbb{D}_t$ with $t = t(\Delta) < 1$. So the critical values of h_n are contained in $\phi_0(P_{n_0+1}(x)) \subset \mathbb{D}_t$.

The corresponding objects for $\tilde{f}$ will be marked with a tilde. The same assertions hold for $\tilde{h}_n$. Then all the maps h_n and $\tilde{h}_n$ satisfy the assumptions of Lemma 3.2, with $d = \delta$, and $\rho = \max\{t, \tilde{t}\}$.

Note that each of $\phi_n, \tilde{\phi}_n$ extends to a homeomorphism from the closure of the puzzle piece to $\overline{\mathbb{D}}$.

Let us consider homeomorphisms $\sigma_0 : \mathbb{T} \to \mathbb{T}$ and $\sigma_n : \mathbb{T} \to \mathbb{T}$ given by $\sigma_0 = \tilde{\phi}_0 \circ H|_{\partial P_{n_0}(x)} \circ \phi_0^{-1}$ and $\sigma_n = \tilde{\phi}_n \circ H|_{\partial Q_n} \circ \phi_n^{-1}$ respectively. Then $\sigma_0 \circ h_n = \tilde{h}_n \circ \sigma_n$.

Notice that $H|_{\partial P_{n_0}(x)}$ has a K_1-qc extension on a neighborhood of $\partial P_{n_0}(x)$ for some $K_1 \geqslant 1$. Fix some r with $\rho < r < 1$. We conclude that σ_0 extends to a K-qc map $\zeta_0 : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$, where K depends on K_1, ρ and r.

Let $K_0 = K_0(\rho, r, \delta)$ be as in Lemma 3.2, and let $K' = \max\{K, K_0\}$. Apply Lemma 3.2 to the following left diagram :

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\sigma_n} & \mathbb{T} \\ h_n \downarrow & & \downarrow \tilde{h}_n \\ \mathbb{T} & \xrightarrow{\sigma_0} & \mathbb{T}, \end{array} \qquad \text{we get} \qquad \begin{array}{ccc} (\mathbb{D},0) & \xrightarrow{\zeta_n} & (\mathbb{D},0) \\ h_n \downarrow & & \downarrow \tilde{h}_n \\ (\mathbb{D},0) & \xrightarrow{\zeta_0} & (\mathbb{D},0), \end{array}$$

so that the map σ_n admits a K'-qc extension $\zeta_n : \mathbb{D} \to \mathbb{D}$ which is the identity on $\mathbb{D}_r$. The desired extension of $H|_{\partial Q_n}$ inside Q_n is now obtained by taking $\tilde{\phi}_n^{-1} \circ \zeta_n \circ \phi_n$. This completes the proof of this proposition.

Now we can use the spreading principle to the sequence $\{Q_n\}_{n\geqslant 1}$ as in Subsection 3.2 and conclude that under the assumptions that $c \not\to c$ or $\mathcal{T}(c)$ is reluctantly recurrent, the statements B and C in Theorem 1.1 are equivalent. Under these assumptions, it is easy to verify that the proof in Subsection 3.1 is also valid for the equivalence of statements A and B in Theorem 1.1. Thus we complete the proof of Theorem 1.1.