Local connectivity of the Julia set for geometrically finite rational maps

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with an appendix by Yin Yongcheng

Abstract

In this paper we prove that each (eventually) periodic connected component of the Julia set of a geometrically finite rational map is locally connected.

1 Introduction

We say that a rational map $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ with degree $d > 1$ is *geometrically finite* if every critical point is either eventually periodic or attracted after infinite iterates to an attracting or parabolic periodic cycle. Denote by J the Julia set of f. The *postcritical set* P_f is the union of forward orbits of critical points. Then f is geometrically finite iff $\overline{P}_f \cap J$ is a finite set.

The main results of this paper are the following:

Theorem A Let $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a geometrically finite rational map of degree $d > 1$. Assume that the Julia set J is connected. Then J is locally connected .

Corollary B If J is not connected, then each eventually periodic connected component of J is locally connected .

The similar statement of theorem A concerning geometrically finite Kleinian groups is known.

The family of geometrically finite rational maps contains all hyperbolic and sub-hyperbolic rational maps. The local connectivity of a connected Julia set is known, in the case that f is a geometrically finite polynomial (Douady-Hubbard, [DH], see also [CG]), or f is a hyperbolic rational map (Milnor, [Mi2]). There are also many results concerning geometrically infinite polynomials (Branner-Hubbard, Yoccoz and others), but the proofs are completely different, and much harder.

It is conjectured (cf. McMullen's section in [Bi]) that each Julia component of a geometrically finite rational map is locally connected . From our theorem it remains to show that each wandering component is locally connected . The question is still open.

The proof of Theorem A follows essentially the idea of Douady-Hubbard ([DH]). We use also some idea of Carleson-Gamelin ([CG]).

This paper is organized as follows:

In section 2 we show that each periodic Fatou component of a geometrically finite rational map has locally connected boundary. The proof consists of several parts, one for the construction of a suitable neighborhood of the Julia set (the neighborhood is degenerate at parabolic periodic points), and a suitable covering space of it; one for the construction of a metric so that the mapping is strictly expanding with respect to it; and one for the uniform convergence of sequences of curves to the boundary of periodic Fatou components.

Section 3 controls the diameter of Fatou components, with the help of the metric obtained in section 2, and ends the proof of Theorem A. As a corollary of theorem A and a result of McMullen, we get corollary B.

This paper contains also an appendix written by Yin Yongcheng. It consists of a useful technical result for general rational maps. It provides a second proof of what we have down in section 3. It also gives short proofs of some known results, for example, none rational map with no rotation domain can accept a Levy cycle.

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2 The boundary of periodic Fatou components

The aim of this section is to prove the following result.

Theorem 2.1 *Let $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a geometrically finite rational map with connected Julia set, $deg(f) = d > 1$. Then every periodic Fatou component has locally connected boundary.*

Definition:

$A = \{$(super)attracting periodic points of $f\}$;

$P = \{$parabolic periodic points of $f\}$;

$C = \{f^n(c) \mid n \geq 0, c$ critical point of $f\}$.

Step 1. The compact set Ω

Lemma 2.2 *Let V be a periodic parabolic component. There is a unique parabolic periodic point $a \in \partial V$ such that a is in the limit set of*

$\{f^n(z)\}$ for any $z \in V$. Assume the period of V is kq, with k the period of a. Then there is an open Jordan domain $D' \subset V$ with the following properties: a) $\overline{D'} \cap J = \{a\}$; b) $D' \supset (V \cap C)$; c) $f^{kq}(\overline{D'}) \subset D' \cup \{a\}$.

Proof. There is a surjective holomorphic mapping $\phi : V \to \mathbf{C}$ such that $\phi \circ f^{kq}|_V = T_1 \circ \phi$ (where T_1 is the translation $z \mapsto z + 1$). The critical points of ϕ are points z such that for some integer $n \geq 0$, the image $f^{nkq}(z)$ is a critical point of f^{kq}.

Let $x_0 \in \mathbf{R}_+$ large enough such that there is no critical value of ϕ in the half-plan $\{x+iy|x>x_0\}$. In $\phi^{-1}(\{x+iy|x>x_0\})$ there is a unique connected component D_0 which has a in its closure. One can choose the value of x_0 such that $\partial D_0 - \{a\}$ is disjoint from the postcritical set of f. By construction, we have $\overline{f^{kq}(D_0)} \subset D_0 \cup \{a\}$.

For each m, denote by D_m the unique connected component of $f^{-mkq}(D_0)$ containing D_0. Note that $f^{kq} : D_m \to D_{m-1}$ is a ramified covering and the set $\overline{D}_m \cap J$ consists of finitely many points b verifying $f^{mkq}(b) = a$. Moreover $\overline{D}_m \cap J \supset \overline{D}_{m-1} \cap J$ and $\overline{D}_{m-1} - J \subset D_m$. Let n be an integer such that $V \cap C \subset D_n$. We will modify $\overline{D}_n$ in the following way:

For $j = 1, \cdots, n$, set $S_j = \{x+iy|x \geq -y^2 + c_j\}$, where the constants c_j are real and are chosen such that $c_n > x_0$, there is no critical value of ϕ on S_n, and $c_{j-1} > c_j + 1$ for all j. Take $b \in \partial V \cap \overline{D}_n = J \cap \overline{D}_n$. Assume j is the minimal integer such that $f^{jkq}(b) = a$. Let $W(b)$ be the unique connected component of $\phi^{-1}(S_j)$ containing b in the closure. Since

$$\phi \circ f^{kq}(int(W(b))) = T_1 \circ \phi(int(W(b))) = T_1(int(S_j)) \supset S_{j-1} = \phi(W(f^{kq}(b))),$$

we have $int(f^{kq}(W(b))) \cup \{f^{kq}(b)\} \supset W(f^{kq}(b))$.

Then $D' = D_n - \bigcup_{b \in J \cap \overline{D}_n} (W(b) \cup \{b\})$ verifies the required properties. ■

Proposition 2.3 *Given any finite set P' containing P such that $f(P') \subset P'$, and $f^n(P') = P$ for some n, one can find a finitely connected and locally connected compact set Ω such that $(\partial\Omega) \cap J = P'$, $(C_A \cup A) \cap \Omega = \emptyset$; $J \subset int(\Omega) \cup P'$; $f^{-1}(\Omega) \subset int(\Omega) \cup P'$; $int(\Omega)$ is connected, and $\Omega \cap C_{P'} = \emptyset$ (where $C_{P'}$ is the subset of C contained in the union of eventually parabolic periodic components attached to points of P').*

Proof. For each $a \in A$, take an open disc neighborhood Δ_a of a such that $f(\overline{\Delta}_a) \subset \Delta_{f(a)}$. Set $C_A = \{f^n(c) \mid n \geq 0, c \text{ critical point of } f \text{ attracted by an attracting cycle}\}$; Let n_A be the minimal integer such that $B_A = f^{-n_A}(\bigcup_{a \in A} \Delta_a)$ contains C_A. The closure of B_A is contained in the Fatou set. By the assumption that the Julia set of f is connected, each connected component of $\overline{B}_A$ is a closed disc.

For each $a \in P$, by the same method as showed in the above lemma we can take a big flower F_a centered at a such that $f(F_a - \{a\}) \subset int(F_{f(a)})$ and $\bigcup_{a \in P} int(F_a) \supset C_P$. Now to each point $b \in P'$ we associate the minimal integer $j(b) \geq 0$ such that $f^{j(b)}(b) \in P$. Define a total order on P' such that if $b_1 < b_2$, then $j(b_1) \leq j(b_2)$. For $b \in P' - P$, we will construct disjoint flowers F_b by induction with respect to the order. There are finitely many Fatou components V attached to b such that $f^{j(b)}(V)$ is periodic. For V one of them which is disjoint from $\bigcup_{b' < b} F_{b'}$ we choose a closed petal (a closed Jordan domain) F_V such that $b \in F_V \subset V \cup \{b\}$, $F_V \supset (C \cap V)$ and $f(F_V - \{b\}) \subset int(F_{f(b)})$ (note that $f(b) \in P'$ and $f(b) < b$). This is possible since the Julia set is connected. We set F_b to be union of these petals. Set $B_{P'} = \bigcup_{b \in P'} int(F_b)$. Then $\overline{B}_{P'} = \bigcup_{b \in P'} F_b = B_{P'} \cup P'$, and $\overline{B}_{P'}$ is contained in the union of P' with the Fatou set.

We claim that the set $\Omega = \overline{\mathbf{C}} - (B_A \cup B_{P'})$ verifies the required properties. The only thing to check is that $int(\Omega)$ is connected. This is guaranteed by the ordering on P' and the choice of V to define F_V (in fact there is a unique point in $J \cap F_b$). ■

Step 2. The covering space of Ω

We will need the following generalization of Caratheodory's theorem:

Denote by D the open unit disc.

Let W be an open connected relatively compact subset of a Riemann surface. Assume that the universal cover of W is D. Let $\pi : D \to W$ be the projection. Assume also ∂W has finitely many connected components . Denote by $\hat{W}$ the ideal closure of W, i.e. $\hat{W} = W \cup \{\text{primes ends of } W\}$ (recall that a prime end of W is defined by a sequence of cross-cuts with distinct endpoints). Note that a puncture component of ∂W is not in the ideal closure). Each continuous curve $\gamma : [0,1] \to \overline{W}$ with $\gamma([0,1[) \subset W$, $\gamma(1) \in \partial W - \{\text{punctures}\}$ defines a unique prime end, and γ_0, γ_1 define the same prime end iff they define equivalent accesses, that is for any neighborhood Δ of $\gamma_0(1)$, there is a continuous arc $\alpha : [0,1] \to U \cap \Delta$ with $\alpha(0) \in \gamma_0$ and $\alpha(1) \in \gamma_1$. This argument was shown to Tan Lei by C. Petersen. For details, please refer [Go] or [Pe]. Denote by $\hat{W}^{acc}$ the union of W with its accessible primes ends (i.e. defined by some class of paths $[\gamma]$). Denote by Γ the finitely generated Fuchsian group $Aut_W(D)$, and Λ_Γ its limit set.

Theorem 2.4 *a) The mapping $\pi : D \to W$ extends to a topological covering from $\overline{D} - \Lambda_\Gamma$ to $\hat{W}$, and π restricted to a connected component of $\partial D - \Lambda_\Gamma$ is a covering over a connected component of $\hat{W} - W$.*

b) $\overline{W}$ is locally connected iff $\hat{W} = \hat{W}^{acc}$. In this case the identity mapping $Id : W \to W$ extends to a continuous mapping from $\hat{W}^{acc}$ to $\overline{W}$.

The proof of this theorem is easy, by applying locally the classical result of Caratheodory for simply connected domains. For more details, please refer [Go] (the terminology there is slitely different).

Returning to our compact set Ω . Denote by U the interior of Ω. By construction U is finitely connected and $\overline{U}$ is locally connected. For $b \in P'$, a suitable neighborhood of b would intersecte U in q connected components , called interpetals, with q the number of petals of the flower F_b. The ideal closure $\hat{U}$ of U can be identified to either

$$(\overline{U} - P') \cup \{(b, [\gamma]) \mid b \in P', [\gamma] \text{ an interpetal of } b\},$$

or $\{(x, [\gamma]) | x \in \overline{U}, [\gamma]$ is an equivalence class of paths landing on $x\}$, where a *path landing on x* is a continuous map $\gamma : [0,1] \to \overline{U}$, with $\gamma([0,1[) \subset U$, and $\gamma(1) = x$. Two pathes γ and γ' are *equivalent* if they define equivalent accesses of x in the above sense.

If Ω does not contain critical points, we take $p : D \to U$ a universal covering.

There are maybe three kind of critical points contained in Ω: those who are mapped eventually out of Ω, those who are eventually parabolic and those who are eventually repelling. By making a suitable choice of P' in the Proposition 2.3, one can get rid of the first kind of critical points in Ω . But this simplification is not useful if there are other critical points in Ω .

Anyway, in case that Ω contains eventually periodic critical points, we will have to take at first a ramified covering of Ω before taking universal covering. This is to help the construction of an inverse map of f in the sequel.

It is possible to define a function $v : \Omega \to \mathbf{N}$ such that $v|_{\Omega - C} = 1$, and $v(x)$ is a multiple of $v(y) \cdot \deg_y(f)$ for each $y \in f^{-1}(x)$ (cf. [DH]). Let v be the function with minimal possible values.

Let O be an open tubular neighborhood of Ω and extend v to $O - \Omega$ by $v = 1$. Let $p_* : O^* \to O$ be a finitely branched covering with O^* connected such that the degree of ramification equals to $v(x)$ for each point over x. Let U^* be a connected component of $p_*^{-1}(U)$. Then p_* extends to the ideal closure of U^* as a branched covering over $\hat{U}$. Since the universal cover of U^* is the unit disc, the boundary ∂U^* has finitely many connected components and the closure $p_*^{-1}(\Omega)$ in O^* is locally connected , one can apply theorem 2.4 to U^* and then get two continuous mappings $\pi : \overline{D} - \Lambda_\Gamma \to p_*^{-1}(\Omega)$ (it is also comformal in D) and $p_* \circ \pi : \overline{D} - \Lambda_\Gamma \to \Omega$ (where Λ_Γ is the limit set of $\Gamma = Aut_{U^*} D$). Set $p = p_* \circ \pi$. We denote also by p the mapping to the ideal closure of U.

Step 3. Lifting f^{-1} in the covering space

By construction, the set $f^{-1}(\Omega)$ is a proper subset of Ω and $f^{-1}(\Omega) \cap \partial\Omega = P'$. When $\Omega \cap C = \emptyset$, the mapping $f : f^{-1}(\Omega) \to \Omega$ is a non-ramified covering. The following proposition is easier in this case.

Proposition 2.5 *Let $(x_0, [\gamma]) \in \hat{U}$. Take a point $(y_0, [\gamma']) \in f^{-1}(x_0, [\gamma])$ (which means $y_0 \in f^{-1}(x_0)$ and γ' is a curve equivalent to one branch of $f^{-1} \circ \gamma$ landing at y_0). Let $\tilde{x}_0, \tilde{y}_0 \in \overline{D} - \Lambda_\Gamma$ such that $p(\tilde{x}_0) = (x_0, [\gamma])$ and $p(\tilde{y}_0) = (y_0, [\gamma'])$. Then there is a unique continuous mapping $g : \overline{D} - \Lambda_\Gamma \to \overline{D} - \Lambda_\Gamma$ such that $g(\tilde{x}_0) = \tilde{y}_0$, $f \circ p \circ g = p$. Moreover, $g|_D$ is holomorphic.*

Proof. We will define at first a finite set $\mathcal{E}$ of function elements in $p_*^{-1}(\Omega)$ such that any element of $\mathcal{E}$ admits an analytic continuation along a curve in $p_*^{-1}(\Omega)$, within $\mathcal{E}$. We will follow the terminology in [Ru] and the idea indicated in [CG].

For any $x \in \Omega - C$, let B_x be a small open circular disc centered at x such that p_* (resp. $f \circ p_*$) is a conformal homeomorphism restricted to every connected component G of $p_*^{-1}(B_x)$ (resp. to every connected component H of $(f \circ p_*)^{-1}(B_x)$). Define $g_{G,H} : G \to H$ to be $(f \circ p_*)^{-1} \circ p_*$.

For $x \in \Omega \cap C$, let B_x be a small open circular disc centered at x such that the local coordinate of p_* restricted to every connected component G of $p_*^{-1}(B_x)$ is $z^{v(x)}$, and that the local coordinate of $f \circ p_*$ restricted to a connected component H of $(f \circ p_*)^{-1}(B_x)$ is $z^{v(y) \cdot \deg_y(f)}$, where y is the unique element of $f^{-1}(x) \cap p_*(H)$. Since $v(x)$ is a multiple of $v(y) \cdot \deg_y(f)$, there is a unique holomorphic mapping $g_{G,H} : G \to H$ having local coordinate $z^{v(x)/(v(y) \cdot \deg_y(f))}$.

Now choose a finite collection of B_x's for $x \in \Omega$ such that they make an open cover of Ω . Pull-back the discs by p_*, we get an open cover $\mathcal{G}$ of $p_*^{-1}(\Omega)$. Denote by $\mathcal{E}$ the set of holomorphic function elements $(G, g_{G,H})$ for all $G \in \mathcal{G}$ and all (finitely many) possible H such that $g_{G,H}$ is defined. By construction, the set $G_1 \cap G_2$ is connected for any pair $G_1, G_2 \in \mathcal{G}$. This guarantees the existence of analytic continuation of any $(G, g_{G,H})$ along a curve η in $p_*^{-1}(\Omega)$, among elements of $\mathcal{E}$. Moreover, the image end point of η under the continuation depends only on the homotopy class of η.

Any $g_{G,H}$ maps $G \cap p_*^{-1}(\Omega)$ into $H \cap p_*^{-1}(\Omega)$, and maps equivalent paths in $G \cap p_*^{-1}(\Omega)$ to equivalent paths in $H \cap p_*^{-1}(\Omega)$. It maps then the ideal closure of $G \cap p_*^{-1}(\Omega)$ into the ideal closure of $p_*^{-1}(\Omega)$.

Return to the situation of our proposition. There are components G, H containing $\pi(\tilde{x}_0)$ and $\pi(\tilde{y}_0)$ respectively. At first define $g : \overline{D} - \Lambda_\Gamma \to \overline{D} - \Lambda_\Gamma$ in a neiborghhood of $\tilde{x}_0$ to be the unique lifting of $g_{G,H}$ restricted to $G \cap p_*^{-1}(\Omega)$ such that $g(\tilde{x}_0) = \tilde{y}_0$. For any $\tilde{x} \in \overline{D} - \Lambda_\Gamma$ we take a path $\tilde{\eta} : [0, 1] \to \overline{D} - \Lambda_\Gamma$ with $\tilde{\eta}(0) = \tilde{x}_0$ and $\tilde{\eta}(1) = \tilde{x}$. Now an analytic continuation of $g_{G,H}$ along $\pi \circ \tilde{\eta}$ maps $\tilde{\eta}$ to a path with initial point $\pi(\tilde{y}_0)$. Lift this path to $\overline{D} - \Lambda_\Gamma$ with initial point $\tilde{y}_0$ and define $g(x)$ to be its end point. It is then easy to check that $g(x)$ does not depend on either the choice of $\tilde{\eta}$ nor the analytic continuation. So g is a well defined continuous map. And $g|_D$ is automatically holomorphic.

Now suppose that $g_1 : \overline{D} - \Lambda_\Gamma \to \overline{D} - \Lambda_\Gamma$ is another continuous mapping such that $f \circ p \circ g_1 = p$. One can easily check that the set $\{x \in \overline{D} - \Lambda_\Gamma \mid g_1(x) = g(x)\}$ is both open and closed. So either the set is empty or $g_1 \equiv g$. This proves the uniqueness. ■

Step 4. Construction of a metric

In the following, we take $P' = P$. Let Ω be the compact set constructed in proposition 2.3 and $U = int(\Omega)$.

Denote by λ_D the Poincaré metric of D and $\lambda_1(z) = \rho_1(z)|dz|$ the metric of $U - C$ induced by λ_D.

In a small neighborhood $N(a)$ of $a \in C$, $m_a \le \rho_1(z) \le \frac{c_a}{|z-a|^{\beta_a}}$, where $m_a > 0, 0 < c_a < \infty, \beta_a = \frac{v(a)-1}{v(a)}$. Then $m_a|z - a| \le d(z, a) \le \frac{c_a}{1-\beta_a}|z - a|^{1-\beta_a}$ for $z \in N(a)$. Hence for any rectifiable curve $\gamma : [0, 1] \to \Omega'$ such that $\gamma(]0, 1[) \subset U$, $length_\lambda(\gamma) < \infty$

For each $a \in P$, one can find a disc-like neighborhood Δ_a and a homeomorphism $\zeta_a : \Delta_a \to D_{r_a}$ such that $f : \Delta_a \to \Delta_{f(a)}$ in the new coordinates becomes:

$$\zeta \mapsto \lambda(\zeta + b_a\zeta^{q_a+1} + \cdots),$$

with $b_a > 0$, q_a equals to the number of petals attached to a, and $\lambda^{q_a} = 1$.

One can choose Δ_a small enough such that $U \cap \Delta_a$ is contained in the union of interpetals, and $|f'_{U\cap\Delta_a}| > 1$. Moreover, $f(U \cap \Delta_a) \supset U \cap \Delta_{f(a)}$, $f(\overline{\Delta}_a) \cap \overline{\Delta}_b = \emptyset$ for $b \ne f(a)$.

For $a \in P - f(C)$, denote by $\lambda_a(z) = \rho_a(z)|dz|$ the metric $|d\zeta_a|$ on Δ_a.

For $a \in P \cap f(C)$, denote by $\lambda_a(z) = \rho_a(z)|dz| = \frac{|dz|}{|z-a|^{\beta_a}}$ the metric on Δ_a, where $\beta_a = \frac{v(a)-1}{v(a)}$.

For $M \in \mathbf{R}_+$ large define a riemannien metric $\lambda(z) = \rho(z)|dz|$ on $U \cup \bigcup_{a\in P} \Delta_a - f(C)$, where $\rho(z) = inf(\rho_1(z), M \cdot \rho_a(z))$. The function ρ restricted to $U - (\bigcup_{a\in P} \partial\Delta_a \cup f(C))$ is continuous.

Proposition 2.6 *Take the compact set $\Omega' = f^{-1}(\Omega) \subset U \cup P$. When M is large enough, f is expanding for λ on Ω' in the sense that $length(f \circ \gamma) > length(\gamma)$ for all C^1-smooth curve $\gamma \subset \Omega'$ (this makes sense eventhough $f \circ \gamma([0, 1]) \cap \partial\Omega \ne \emptyset$).*

Proof. For $a \in P$, $f^{-1}(\Omega \cap \Delta_a)$ has the form $L'_a \cup L''_a$, where $L'_a \subset \Omega' \cap (\bigcup_{a' \in f^{-1}(a) \cap P} \Delta_{a'})$ and L''_a is a relatively compact subset of U.

Put $m_a = inf\{\frac{\rho_a(f(z))|f'(z)|}{\rho_1(z)} \mid z \in L''_a - C\}$, it is easy to check that $m_a > 0$ by calculating the order of singularity at a for each involved function. Choose M such that

$$M \cdot inf\{m_a \mid a \in P\} > 1 \ .$$

It is sufficient to prove:

$$\rho(f(z))|f'(z)| > \rho(z) \tag{1}$$

for $z \in \Omega' - (P \cup f^{-1}(f(C)))$.

There are four possibilities:

a) $\rho(z) = \rho_1(z)$ and $\rho(f(z)) = \rho_1(f(z))$. Since the mapping $f : f^{-1}(U) - f^{-1}(f(C)) \to U - f(C)$ is strictly expanding for λ_1, (1) holds.

b) $\rho(z) = M \cdot \rho_a(z) \leq \rho_1(z)$ and $\rho(f(z)) = \rho_1(f(z))$. By a), (1) holds.

c) $\rho(z) = M \cdot \rho_a(z)$ and $\rho(f(z)) = M \cdot \rho_{f(a)}(f(z))$. From the definition of λ_a, (1) holds.

d) $\rho(z) = \rho_1(z)$ and $\rho(f(z)) = M \cdot \rho_{f(a)}(f(z))$. If $z \in L'_{f(a)} - C$, we have $\rho_1(z) \leq M \cdot \rho_a(z)$. From c), (1) holds. If $z \in L''_{f(a)} - f^{-1}(f(C))$, (1) holds from the condition $M \cdot m_{f(a)} > 1$. ∎

Step 5. A continuous modulus for the liftings

Let $\tilde{\lambda}$ be the lifting of λ on $p^{-1}(U \cup P - f(C))$, then $\tilde{\lambda}(z) = \lambda_D(z)$ for $z \in p^{-1}(U - (f(C) \cup \bigcup_{a \in P} \Delta_a))$. Hence for any rectifiable curve $\tilde{\gamma} : [0,1] \to p^{-1}(\Omega')$ such that $\tilde{\gamma}(]0,1[) \subset D$, $length_{\tilde{\lambda}}(\tilde{\gamma}) < \infty$.

For any points $x, y \in p^{-1}(\Omega')$, define the distance

$$\tilde{d}(x,y) = inf\{length_{\tilde{\lambda}}(\tilde{\gamma}) \mid \tilde{\gamma} : [0,1] \to p^{-1}(\Omega') \text{ is a rectifiable curve, } \tilde{\gamma}(0) = x,\ \tilde{\gamma}(1) = y\},$$

then $(p^{-1}(\Omega'), \tilde{d})$ is a complete metric space.

Denote $\Gamma = Aut_{U^*}(D)$. It acts as isometry for $\tilde{d}$ on $p^{-1}(\Omega')$.

Lemma 2.7 *If $g \circ \sigma(x) = \delta \circ g(x)$ for some non-ramified point $x \in p^{-1}(\Omega')$ and $\sigma, \delta \in \Gamma$, then $g \circ \sigma = \delta \circ g$.*

Proof. For any non-ramified point $y \in p^{-1}(\Omega')$, there is a path $\gamma : [0,1] \to p^{-1}(\Omega')$ connecting x and y such that $p \circ \gamma(t) = p \circ \sigma \circ \gamma(t) \in \Omega' - C$ for $t \in [0,1]$.

The two paths $p \circ g \circ \gamma$ and $p \circ g \circ \sigma \circ \gamma$ have the same initial point $p \circ g(x)$, $f \circ p \circ g \circ \gamma(t) = f \circ p \circ g \circ \sigma \circ \gamma(t) = p \circ \gamma(t)$. Define $T = \{t \in [0,1] \mid g \circ \sigma \circ \gamma(t) = \delta \circ g \circ \gamma(t)\}$, T is a closed set and $0 \in T$. If $t_0 \in T$, since f is local injective, there is a neighborhood V_0 of t_0 such that $p \circ g \circ \sigma \circ \gamma(t) = p \circ \delta \circ g \circ \gamma(t)$ for $t \in V_0$, $g \circ \sigma \circ \gamma(t) = \eta_t \circ \delta \circ g \circ \gamma(t)$, where $\eta_t \in \Gamma$. Since Γ is a discrete group, hence $\eta_t = \eta_{t_0} = id.$ for $t \in V_0$. This proves T is an open set, then $T = [0,1]$ and $g \circ \sigma(y) = \delta \circ g(y)$. Since $g \circ \sigma$ and $\delta \circ g$ are continuous mappings, hence $g \circ \sigma = \delta \circ g$. ∎

Proposition 2.8 *There is an increasing right semi-continuous mapping* $h : \mathbf{R}_+ \to \mathbf{R}_+$ *with the properties* $h(s) < s$ *for* $s > 0$ *and* $s - h(s) \to \infty$*, such that*

$$\tilde{d}(g(x), g(y)) \leq h(\tilde{d}(x, y)) ,$$

for all $(x, y) \in p^{-1}(\Omega') \times p^{-1}(\Omega')$*, and any lifting* g *of* f^{-1} *defined by proposition 2.5.*

Proof. Fix a non-ramified point $x_0 \in p^{-1}(\Omega'), f^{-1}(p(x_0))$ has d distinct points. We take $\{x_1, x_2, ..., x_d\} \subset p^{-1}(\Omega')$ such that $\{p \circ g(x_i) \mid 1 \leq i \leq d\} = f^{-1}(p(x_0))$, then $f \circ p \circ g(x_i) = p(x_i) = p(x_0)$. There is $\sigma_i \in \Gamma$ such that $x_i = \sigma_i(x_0), 1 \leq i \leq d$.

Suppose $F \subset p^{-1}(\Omega')$ is a compact subset such that $p^{-1}(\Omega') = \cup_{\sigma \in \Gamma} \sigma(F)$. Put $F_1 = \cup\{\sigma_i(F) \mid 1 \leq i \leq d\}$, it is a compact subset of $p^{-1}(\Omega')$. For $s > 0$, denote $B(F_1, s) = \{x \in p^{-1}(\Omega') \mid \tilde{d}(x, F_1) \leq s\}$, it is again a compact subset of $p^{-1}(\Omega')$.

Define $h_g(s)$ as following

$$h_g(s) = sup\{\tilde{d}(g(x), g(y)) \mid (x, y) \in p^{-1}(\Omega') \times p^{-1}(\Omega'), \tilde{d}(x, y) \leq s\} .$$

For a non-ramified point $x \in F$, $\{p \circ g \circ \delta(x) \mid \delta \in \Gamma\} = f^{-1}(p(x))$. There is $\sigma_{i(\delta)}$ such that $p \circ g \circ \delta(x) = p \circ g \circ \sigma_{i(\delta)}(x)$. Then $g \circ \delta(x) = \eta(\delta) \circ g \circ \sigma_{i(\delta)}(x)$ for some $\eta(\delta) \in \Gamma$. From lemma 2.7, $g \circ \delta = \eta(\delta) \circ g \circ \sigma_{i(\delta)}$. Hence $\tilde{d}(g(\delta(x)), g(\delta(y))) = \tilde{d}(g \circ \sigma_{i(\delta)}(x), g \circ \sigma_{i(\delta)}(y))$. So

$$h_g(s) = sup\{\tilde{d}(g(x), g(y)) \mid x \in F_1, y \in B(F_1, s), \tilde{d}(x, y) \leq s\} .$$

According to proposition 2.6, a lifting g is strictly contracting with respect to $\tilde{d}$, that is $\tilde{d}(g(x), g(y)) < \tilde{d}(x, y)$. Since g is a continuous mapping, we have $h_g(s) < s$.

It is easy to check that $h_g(s)$ is right semi-continuous, that is $lim_{\varepsilon \to 0^+} h_g(s+\varepsilon) = h_g(s)$. Let s_1 and s_2 be any two positive numbers. For any points $x, y \in p^{-1}(\Omega')$ satisfying $\tilde{d}(x, y) \leq s_1 + s_2$ and any positive ε, there exists a path $\gamma_\varepsilon \subset p^{-1}(\Omega')$ connecting x and y such that $length(\gamma_\varepsilon) \leq s_1 + s_2 + \varepsilon$. Take a point $x' \in \gamma_\varepsilon$ such that $\tilde{d}(x, x') \leq s_1 + \varepsilon/2, \tilde{d}(x', y) \leq s_2 + \varepsilon/2$. Then $\tilde{d}(g(x), g(y)) \leq \tilde{d}(g(x), g(x')) + \tilde{d}(g(x'), g(y)) \leq h_g(s_1 + \varepsilon/2) + h_g(s_2 + \varepsilon/2)$, that is $h_g(s_1 + s_2) \leq h_g(s_1 + \varepsilon/2) + h_g(s_2 + \varepsilon/2)$. Hence $h_g(s_1 + s_2) \leq h_g(s_1) + h_g(s_2)$. This implies that $s - h_g(s)$ is increasing and $h_g(ks_0) \leq k \cdot h_g(s_0)$ for any integer $k > 0$ and a fixed positive value s_0. This results that $s - h_g(s) \to \infty$ as $s \to \infty$.

Fixed a non-ramified point $y_0 \in p^{-1}(\Omega'), f^{-1}(p(y_0))$ has d distinct points. We take $\{y_i | 1 \leq i \leq d\} \subset p^{-1}(\Omega')$ such that $\{p(y_i) | 1 \leq i \leq d\} = f^{-1}(p(y_0))$. Let g_i be the unique lifting of f^{-1} such that $g_i(y_0) = y_i$ for $1 \leq i \leq d$. Take any lifting g of f^{-1}, we have $p \circ g(y_0) \in \{p(y_i) | 1 \leq i \leq d\}$. There exists $\delta_g \in \Gamma$ such that $g(y_0) = \delta_g(y_{i(g)}) = \delta_g \circ g_{i(g)}(y_0)$. So $g = \delta_g \circ g_{i(g)}$.

Now define

$$h(s) = sup\{\tilde{d}(g(x), g(y)) \mid (x, y) \in p^{-1}(\Omega') \times p^{-1}(\Omega'),\ \tilde{d}(x, y) \leq s,\ g \text{ a lifting of } f^{-1}\}$$

Then $h(s) = sup\{h_g(s) | g \text{ a lifting of } f^{-1}\} = sup\{h_{g_i}(s) | 1 \leq i \leq d\}$ is the needed mapping. ∎

Step 6. Convergence

From Sullivan's classification theorem, each periodic Fatou component of f is attracting or parabolic.

Let V be a parabolic periodic component with the parabolic periodic point a on its boundary, h be the increasing mapping constructed in Step 5 and k be the period of V.

Note that $\Omega \cap V$ contains no point of the critical set C. Let $\gamma_0 : [0,1] \to (\partial\Omega \cap V) \cup \{a\}$ be a C^1-smooth parametrization such that $\gamma_0(0) = \gamma_0(1) = a$. Let $\tilde{\gamma}_0$ be a lifting of γ_0 in $\overline{D} - \Lambda_\Gamma$. Set $\tilde{a} = \tilde{\gamma}_0(0)$, $\tilde{b} = \tilde{\gamma}_0(1)$. We have $\tilde{a}, \tilde{b} \in p^{-1}(a)$.

Define a sequence of paths $\{\gamma_n(t) : [0,1] \to \Omega' \cap \overline{V}\}$ satisfying $f^k(\gamma_{n+1}(t)) = \gamma_n(d_1 t)$, where $d_1 = deg(f^k\mid_V)$. These conditions determine uniquely γ_n. We want to lift γ_n by p to some path homotopic to $\tilde{\gamma}_0$. There might be some ambiguity at points in $\gamma_n(]0,1[) \cap f(C)$, which could be taken away as follows: each γ_n is in the closure $\overline{\mathcal{B}}$ (with the topology of uniform convergence) of $\mathcal{B}$, which is by definition the set of curves: $\beta : [0,1] \to (\Omega \cap V) \cup \{a\}$ continuous, $\beta(0) = \beta(1) = a$, β is homotopic to γ_0 in $(\Omega \cap V) \cup \{a\}$. Since $f(C)$ is disjoint from $\Omega \cap V$, each curve $\beta \in \mathcal{B}$ has a unique lifting $\tilde{\beta}$ with initial point $\tilde{a}$, with end point $\tilde{b}$. Now for $\gamma \in \overline{\mathcal{B}}$, we define $\tilde{\gamma}$ as the uniform limit of liftings of curves in $\mathcal{B}$ converging to γ. By continuity, $\tilde{\gamma}(0) = \tilde{a}$, $\tilde{\gamma}(1) = \tilde{b}$, and $p(\tilde{\gamma}(t)) = \gamma(t)$.

Let G be the composition of k liftings of f^{-1} such that $f^k \circ p \circ G = p$ and $G(\tilde{a}) = \tilde{a}$. For $t = \frac{i+s}{d_1}$, $s \in [0,1]$, $0 \le i < d_1 - 1$, we have $f^k \circ \gamma_{n+1}(t) = \gamma_n(s)$. So $G(\tilde{\gamma}_n(s)) = \tilde{\gamma}_{n+1}(s/d_1)$. By induction on i, there is $\delta_i \in \Gamma$ independent of n and s such that $G(\delta_i \circ \tilde{\gamma}_n(s)) = \tilde{\gamma}_{n+1}((i+s)/d_1)$. Now define a distance in $\overline{\mathcal{B}}$ as well as in its lifting space by:

$$d(\gamma, \gamma') = d(\tilde{\gamma}, \tilde{\gamma}') = sup\{\tilde{d}(\tilde{\gamma}(t), \tilde{\gamma}'(t)) \mid t \in [0,1]\} \ .$$

Then $d(\gamma_{m+1}, \gamma_{n+1}) \le h^k(d(\gamma_m, \gamma_n))$.

There is a positive number L such that $L - h(L) \ge d(\gamma_0, \gamma_1)$. Since $d(\gamma_0, \gamma_n) \le d(\gamma_0, \gamma_1) + h(d(\gamma_0, \gamma_{n-1}))$, by induction $d(\gamma_0, \gamma_n) \le L$ for any $n \ge 1$.

For any l and m, we have $d(\gamma_l, \gamma_{l+m}) \le h^{kl}(d(\gamma_0, \gamma_m)) \le h^{kl}(L)$. Since $lim_{l\to\infty} h^{kl}(L) = 0$, $\tilde{\gamma}_n(t)$ converges uniformly to $\tilde{\gamma}_\infty(t)$ for the metric $\tilde{d}$. Denote $\gamma_\infty(t) = p \circ \tilde{\gamma}_\infty(t)$, then $\gamma_n(t)$ converges uniformly to $\gamma_\infty(t)$ for λ, hence also for the euclidean metric, because the two metrics define the same topology, and Ω' is compact. The image $\gamma_\infty([0,1])$ is the boundary of V. This proves that the boundary of V is locally connected.

This ends the proof that each parabolic periodic component has locally connected boundary. It remains to show the same for each attracting periodic component V. The method is analogue. We can either copy word by word Douady-Hubbard's proof in the polynomial case ([DH]), or proceed as follows: take an access γ from V to a fixed (or periodic) point a on the boundary. Choose an inverse mapping g such that it preserves a and the access. Take an equipotential Q of V close to the boundary. Reparametrize the set $Q \cup \gamma$ so that it is the image of some continuous mapping $\gamma_0 : [0,1] \to U$, with $\gamma_0(0) = \gamma_0(1) = a$. Then a sequence of liftings of γ_0 by f will converge uniformly to ∂V. We omit the details here.

This ends the proof of Theorem 2.1.

3 Proof of theorem A and corollary B

For geometrically finite rational map with $J \neq \overline{\mathbf{C}}$, we use the expanding metric λ to show that each Fatou component eventually land on a parabolic or attracting basin, and the diameter tends to zero.

For any Fatou component $V_0 \subset \Omega, V_n = f^n(V_0)$. Assume that $V_n \subset \Omega$ for all $n \geq 0$. Let $\tilde{V_n}$ be the lifting of V_n contained in a compact subset of $p^{-1}(\Omega')$, then $diam_{\tilde{d}}(\tilde{V_0}) \leq h^n(diam_{\tilde{d}}(\tilde{V_n}))$. Since $diam_{\tilde{d}}(\tilde{V_n}) \leq M$ for some positive M, then $diam_{\tilde{d}}(\tilde{V_0}) = 0$. It is impossible. Then each Fatou component is preperiodic and there are only finitely many periodic Fatou components.

Let V_0 be a Fatou component and V_n be a connected component of $f^{-n}(V_0)$. Denote $\tilde{V_n}$ the lifting of $V_n (n \geq 0)$ contained in a compact subset of $p^{-1}(\Omega')$. By proposition 2.8, $diam_{\tilde{d}}(\tilde{V_n}) \leq h^n(diam_{\tilde{d}}(\tilde{V_0})) \to 0$ as $n \to \infty$. Hence $diam_\lambda(V_n) = diam_\lambda(p(\tilde{V_n})) \to 0$ and $diam(V_n) \to 0$ in the standard metric. This proves that the diameter of Fatou components tends to zero. By theorem 2.1 and Torhost's theorem ([Wh]), the Julia set J is locally connected. We complete the proof of theorem A.

Lemma 3.1 *Let $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a rational map with non-connected Julia set J and K be a connected component of J. Then $f(K)$ is a connected component of J.*

Proof. The set $f(K)$ is connected and compact, $f(K) \subset J$. Suppose E is a connected component of J containing $f(K)$. By a lemma in [Be], $f^{-1}(E)$ consists of finitely many connected components and each is mapped by f onto E. Let E' be a connected component of $f^{-1}(E)$ containing K, then $E' = K$. Hence $f(K) = E$ is a connected component of J. ∎

Let $f(z)$, $g(z)$ be two rational maps, not necessarily of the same degree. Let F, G be closed subsets of $\overline{\mathbf{C}}$ such that $f(F) = F$, $g(G) = G$. A map $\phi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ determines a stable conjugacy $(F, f) \sim (G, g)$ if

(1) ϕ is quasiconformal on $\overline{\mathbf{C}}$;

(2) $\phi(F) = G$; and

(3) $\phi \circ f(z) = g \circ \phi(z)$ for all $z \in F$.

Let K be a connected component of J and $f(K) = K$.

Proposition 3.2 *([Mc1]) There is a stable conjugacy $(K, f) \sim (J(g), g)$, where $g(z)$ is a rational map with connected Julia set $J(g)$.*

For a geometrically finite rational map $f(z)$, K is a connected component of J and $f(K) = K$, there is a stable conjugacy $\phi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that $\phi(K) = J(g)$ for a rational map $g(z)$ with connected Julia set $J(g)$. If $z_0 \in J(g)$ is a critical point of $g(z)$, then $\phi^{-1}(z_0) \in K$ is a critical point of $f(z)$. Hence $g(z)$ is a geometrically finite rational map. By theorem A, $J(g)$ is locally connected, hence K is locally connected. From Lemma 3.1, each eventually periodic connected component of J is locally connected. This ends the proof of corollary B.

Proposition 3.3 *If V is a simply connected invariant Fatou component with locally connected boundary, then every periodic point on its boundary has at most finitely many accesses from V.*

Proof. If V is a Siegel disk, the boundary of V has no periodic point.

If V is a parabolic basin with a parabolic fixed point a on the boundary of V, take a simply connected domain $W \subset V$ such that $\overline{W} \cap \partial V = \{a\}, f^{-1}(W) \cup \{a\} \supset \overline{W}$ and $V - \overline{W}$ contains no critical point of f, then there is a quasiconformal mapping $\varphi : V - \overline{W} \to D_1$ such that $\varphi \circ f(z) = P_k \circ \varphi(z)$, where $P_k(z) = \frac{z^k+c}{1+cz^k}, k = deg(f\mid_V) > 1, c = \frac{k-1}{k+1}$, [Mc]. The mapping $\varphi^{-1} = \psi$ can be extended continuously to $\partial D_1 = S^1$. The mapping $P_k : S^1 \to S^1$ is a covering with degree k. By Shub's theorem [Sh], there exists a monotone and surjective mapping $R : S^1 \to S^1$ such that $R \circ P_k(z) = (R(z))^k$. The Julia set $J(P_k) = S^1$ implies that R is a homeomorphism. For any fixed point $\alpha \in \partial V$, f is locally injective in a neighborhood of α. Hence $P_k : \psi^{-1}(\alpha) \to \psi^{-1}(\alpha)$ is injective and $\psi^{-1}(\alpha) \subset S^1$ is a compact set. So $g_0 : R(\psi^{-1}(\alpha)) \to R(\psi^{-1}(\alpha))$ is injective and $R(\psi^{-1}(\alpha)) \subset S^1$ is a compact set, where $g_0(z) = z^d : S^1 \to S^1$. By a lemma of Douady, $R(\psi^{-1}(\alpha))$ is a finite set. Hence $\psi^{-1}(\alpha)$ is also a finite set. There are at most finitely many accesses from V landing at α. Therefore, every periodic point on ∂V has at most finitely many accesses from V.

Carrying out the same discussion if V is an attracting(superattracting) basin. We complete the proof of this proposition. ■

Remark. The above result can be also obtained from works of Ch. Pommerenke ([Po]) and C. Petersen ([Pe]).

Corollary 3.4 *Let f be a rational mapping with connected Julia set and every Fatou component has locally connected boundary. Then there are at most finitely many Fatou components attaching at any preperiodic point of f.*

Proof. It is straightly from proposition 3.3 and the finiteness of periodic Fatou components. ■

Combine corollary 3.4 and proposition A.3 in appendix, we have another proof of theorem A.

A Appendix (written by Yin Yongcheng)

Lemma A.1 *Let B be an open topological disk and $B \cap cl(P_f) = \emptyset$, $\{f_i^{-n} \mid 1 \le i \le d^n\}$ be the set of branches of f^{-n} on B. Then $\mathcal{F} = \{f_i^{-n} : B \to \overline{\mathbf{C}} \mid 1 \le i \le d^n, n \ge 1\}$ is a normal family.*

Proof. For any n and i, $f_i^{-n}(B) \cap cl(P_f) = \emptyset$.

If $\#(cl(P_f)) \geq 3$, Montel's theorem implies that $\mathcal{F}$ is a normal family.

If $\#(cl(P_f)) = 2$, there is an open neighborhood B_1 of $cl(P_f)$ such that $B_1 \cap B = \emptyset$ and $f(B_1) \subset B_1$. Then $f_i^{-n}(B) \cap B_1 = \emptyset$ for any n and i. The family $\mathcal{F}$ is normal. ■

Lemma A.2 *Let V be an open set intersecting with J and K be a compact set containing no exceptional point. Then there is an integer N such that $f^n(V) \supset K$ for $n \geq N$.*

Proof. Let $z_0 \in V \cap J$ be a repelling periodic point with period k. Choose $V_1 \subset V$ be an open neighborhood of z_0 such that $f^k(V_1) \supset V_1$, then $f^{ik}(V_1) \supset f^{(i-1)k}(V_1)$ for any $i \geq 1$. Denote E the exceptional set of f, $\#(E) \leq 2$. Let $V(E)$ be an open neighborhood of E satisfying $f(V(E)) \subset V(E)$ and $V(E) \cap K = \emptyset$, $V(E) = \emptyset$ if $E = \emptyset$, then $\cup_{i\geq 0} f^{ik}(V_1) \supset \overline{\mathbf{C}} - E \supset \overline{\mathbf{C}} - V(E)$. There is i_0 such that $f^{i_0 k}(V_1) \supset \overline{\mathbf{C}} - V(E)$. Take $N = i_0 k$, then $f^n(V_1) \supset \overline{\mathbf{C}} - V(E)$ for any $n \geq N$. Hence $f^n(V) \supset K$ for any $n \geq N$. ■

Proposition A.3 *Let $\overline{D}$ be a topological disk satisfying $\overline{D} \cap cl(P_f) = \emptyset$. Then either (a) For any $\varepsilon > 0$, there exists N such that $diam(K_n) < \varepsilon$ for any connected component K_n of $f^{-n}(\overline{D})$ and $n \geq N$; or (b) $\overline{D}$ is contained in a rotation domain.*

Proof. Choose a topological disk $\overline{B}$ such that $\overline{D} \subset B$ and $\overline{B} \cap cl(P_f) = \emptyset$. Let $\{f_i^{-n} : B \to \overline{\mathbf{C}}\}$ be the set branchs of f^{-n} on B.

If (a) is false, then there are $\{n_k\}_{k\geq 1}$ and $i(n_k)$ such that $diam(f_{i(n_k)}^{-n_k}(\overline{D})) \geq \varepsilon_0$ for some positive ε_0.

The mapping $g_k = f_{i(n_k)}^{-n_k} : B \to \overline{\mathbf{C}}$ is conformal. By lemma A.1, there is a subsequence $\{k_j\}_{j\geq 1}$ such that $g_{k_j} \to g$ uniformly in any compact subset of B. Since $diam(f_{i(n_k)}^{-n_k}(\overline{D})) \geq \varepsilon_0$, $g : B \to \overline{\mathbf{C}}$ is a conformal mapping. Take $x_0 \in D, y_0 = g(x_0)$, there exists $\delta_0 > 0$ such that $D(y_0, \delta_0) = \{z \mid d(z, y_0) < \delta_0\} \subset g_{k_j}(D)$ for large j. Then $f^{n_{k_j}}(D(y_0, \delta_0)) \subset D$. From lemma A.2, $D(y_0, \delta_0) \cap J = \emptyset$. There is a Fatou component W containing $D(y_0, \delta_0)$. Suppose $f^q(W)$ is periodic for some $q \geq 0$. There exists a subsequence $\{m_i\}$ of $\{n_{k_j}\}$ such that $f^{m_i}(D(y_0, \delta_0)) \subset f^{q+l}(W)$ for some $l \geq 0$, $f^{q+l}(W)$ is a periodic Fatou component. If $f^{q+l}(W)$ is attracting or parabolic, then $f^{m_i}(z) \to z_0 \in \overline{f^{q+l}(W)}$ for any $z \in D(y_0, \delta_0)$, where z_0 is the attracting or parabolic periodic point, $z_0 \in cl(P_f)$. From $f^{m_i}(D(y_0, \delta_0)) \subset D, z_0 \in \overline{D}$. This contradicts with $\overline{D} \cap cl(P_f) = \emptyset$. Hence $f^{q+l}(W)$ is a rotation domain and $f^{q+l}(W) \cap D \neq \emptyset$. Since $\partial f^{q+l}(W) \subset cl(P_f)$ and $\overline{D} \cap cl(P_f) = \emptyset$, therefore $\overline{D}$ is contained in the periodic rotation domain $f^{q+l}(W)$. ■

Corollary A.4 *Let K be a connected compact set and $K \cap cl(P_f) = \emptyset$. If f^n maps K_n onto K by homeomorphism for $n \geq 1$, then either (a) $diam(K_n) \to 0$ as $n \to \infty$; or (b) K is contained in a periodic rotation domain.*

Proof. Choose finitely many open topological disks $D_1, ..., D_L$ such that $\{D_i | 1 \leq i \leq L\}$ cover K and $\overline{D_i} \cap cl(P_f) = \emptyset$ for $1 \leq i \leq L$. From above proposition, we get the corollary. ■

The following two corollaries are suggested by C. Petersen and Tan Lei.

A simple closed curve in $\overline{\mathbf{C}} - cl(P_f)$ is *peripheral* if it bounds a disk containing at most one point of $cl(P_f)$.

A system of non-peripheral simple closed curves $\{\gamma_1, ..., \gamma_n\}$ is called a Levy cycle if there exists a component γ_i' of $f^{-1}(\gamma_{i+1})(1 \leq i \leq n)$ such that γ_i and γ_i' are isotopic in $\overline{\mathbf{C}} - cl(P_f)$ and $f|_{\gamma_i'}$ is a homeomorphism, $\gamma_{n+1} = \gamma_1$.

Corollary A.5 *If a rational map has a Levy cycle, then this cycle is contained in a periodic rotation domain.*

Proof. It is straightly from corollary A.4. ■

Remark: C.McMullen proved this by means of Strebel's result about the problem of moduli ([Mc2]).

Corollary A.6 *Let $\gamma : [0, \infty) \to \overline{\mathbf{C}}$ be a path with $f(\gamma(t)) = \gamma(t-1)$ for $t \geq 1$. If $K \cap (cl(P_f) \cup R) = \emptyset$, where $K = \gamma([0,1])$ and R is the union of periodic rotation domains, then $\gamma(t)$ converges to a fixed point of f with the multiplier λ satisfying $|\lambda| > 1$ or $\lambda = 1$.*

Proof. Denote $K_n = \gamma([n, n+1])$, then f^n maps K_n onto K with degree one. By corollary A.4, $diam(K_n) \to 0$. Since $K_n \cap K_{n+1} \neq \emptyset$, we have $diam(K_n \cup K_{n+1}) \to 0$ as well. Let $x_{n_k} \in K_{n_k}$ be a sequence of points converging to a point x. By continuity $f(x_{n_k}) \to f(x)$. On the other hand, $d(x_{n_k}, f(x_{n_k})) \leq diam(K_{n_k} \cup K_{n_k+1}) \to 0$ as $k \to \infty$. So $f(x) = x$. This means that any limit point of the sequence of sets K_n is a fixed point of f. Since the set of all limit points is connected and f has only finitely many fixed points, this proves that $\gamma(t)$ converges to a fixed point z_0 of f. By snail lemma, $|\lambda| > 1$ or $\lambda = 1$, where $\lambda = f'(z_0)$ ([Su], see also [Mi1]). ■

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