Constructing rational dynamics using Thurston's theorem

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Theorem A. Let $G : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be

- a geometrically finite semi-rational map, i.e.
  - $P'_G := \text{cluster}(P_G) := \text{cluster}(\textit{critical orbits})$ is $\boxed{\text{finite}}$ and $\neq \emptyset$
  - G is hol. on neighborhood(P'_G), per. pts = attrac. / parabolic
- **with no obstructions** (generalized Thurston obstructions)

$\Longrightarrow \exists!$ R rational map $\sim_{\text{c-equivalent}} G$, i.e. for

$U = \sqcup(\text{round attract. discs}) \sqcup (\text{attract. petals}) = \textit{basic set}$

$$\begin{array}{ccccccc} U, \overline{\mathbf{C}} & \xrightarrow[\text{homeo}]{\theta} & U, \overline{\mathbf{C}} & \xrightarrow[\text{homeo}]{\phi} & V, \overline{\mathbf{C}} & & \theta|_U = id \\ G \downarrow & & & & \downarrow R & & \theta \sim_{P_G} id \\ U, \overline{\mathbf{C}} & \xrightarrow{id} & U, \overline{\mathbf{C}} & \xrightarrow{\phi} & V, \overline{\mathbf{C}} & & \phi|_U \text{ hol.} \end{array}$$

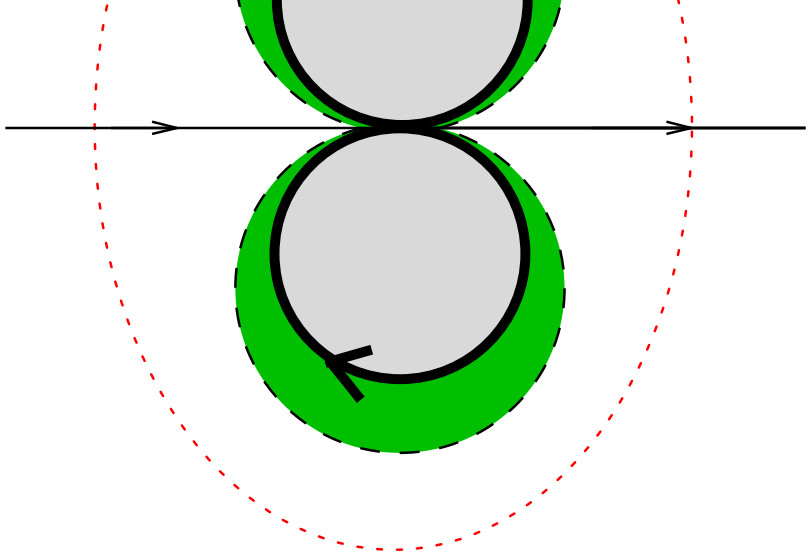
Moreover $R = \lim_{n\to\infty} R_n$ *(Thurston algorithm of successive liftings).*

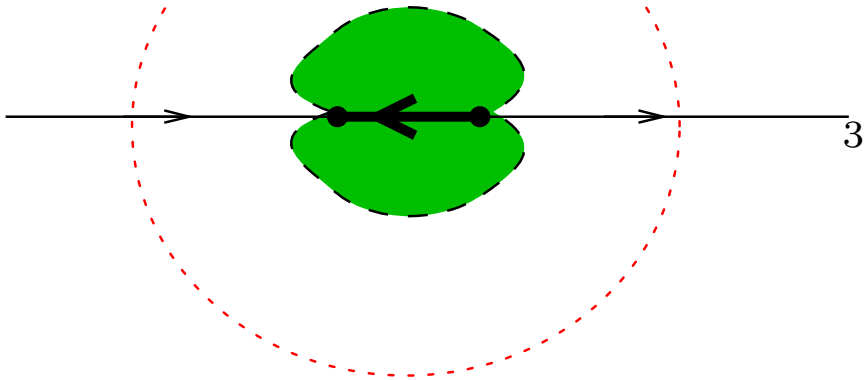
Theorem B (uniform control of distortions). Let g be a geometrically finite rational map ($J_g \neq \overline{\mathbf{C}}$), $X \subset J_g$ a periodic cycle, $V_x =$ *disc* around $x \in X$, $\boxed{V_x(r)\text{: bounded by equi-potential } r.}$
Then $\exists\ C(r) \searrow 0$ as $r \searrow 0$ such that

$$\forall\ N, \forall\ \phi : \overline{\mathbf{C}} \smallsetminus \bigcup_{x \in g^{-N}(X)} \overline{V_x(r)} \to \overline{\mathbf{C}} \qquad \text{univalent}$$

$$\boxed{\text{distortion}(\phi) \leq C(r) \searrow 0.}$$

Applications of Theorem A + Theorem B + *easy surgeries* :
I. (Douady-Hubbard) convergence of preperiodic periodic rays of M.
II. Parabolic maps are accessible from various hyperbolic components.
III. Pinching deformations converge.
IV. Hubbard trees induce hyperbolic maps with disconnected Julia set.





Existence part of Theorem A with parabolics $\Longleftarrow$ { **Hyperbolic version** ($\Longleftarrow$ Prop. C); **Control of distortions *** }

$$
\begin{array}{ccc}
F & \overset{\text{Hyperbolic version}}{\sim} & f \\
\text{star-like plumb. surgery}\ \uparrow & & \searrow\ \text{star-like pinching} \\
G & \overset{\text{conclusion}}{\sim} & g \overset{*}{\Longleftarrow} f_t
\end{array}
$$

Proof of Hyperbolic version: **1.** Inducing a new map F:

(G, U) no parabolics and no obstructions

$\downarrow$ isotopic

G quasi-regular and no obstructions

$\downarrow$ complement and pull-back

$$L_0 = \overline{\mathbf{C}} \setminus U$$
$$\cdots \xleftarrow{G^{-1}} L_{n+1} \xleftarrow{G^{-1}} L_n \xleftarrow{G^{-1}} \cdots \xleftarrow{G^{-1}} L_1 \xleftarrow{G^{-1}} L_0$$

$\downarrow$ finite pull-back and fill

$$\widetilde{L} = fill(L_N),\ \widetilde{E} = G^{-1}(\widetilde{L}),\ \partial\widetilde{E} \sim_{\mathcal{P}_G} \partial\widetilde{L},$$
$$F := G|_{\widetilde{E}} : \widetilde{E} \to \widetilde{L} \text{ with } J_F = J_G \subset \widetilde{E},$$
$$\lambda_F(\partial\widetilde{L}) < 1 \Longleftarrow \text{no obstruction.}$$

F is of **constant type**.

2. Proposition C. For the induced map F :

$\exists$ complex structure σ on $\widetilde{L}$ and $\theta = \begin{cases} id & \text{outside } \widetilde{L} \\ \widetilde{L} \to \widetilde{L} \text{ q.c. } \sim_{\partial \widetilde{L}} id \end{cases}$ such that

$$\begin{array}{ccc} \widetilde{E} \subset \widetilde{L} & \xrightarrow{\theta} & \theta(\widetilde{E}) \subset \widetilde{L} \\ \downarrow F & & \downarrow \ \sigma\text{-hol.} \\ \widetilde{L} & \xrightarrow{id} & \widetilde{L},\ \sigma \end{array} \qquad \text{So for} \quad \begin{array}{ccc} \overline{\mathbf{C}} & \xrightarrow{\theta} & \overline{\mathbf{C}} \\ G \downarrow & & \downarrow G_1 \\ \overline{\mathbf{C}} & \xrightarrow{id} & \overline{\mathbf{C}} \end{array}$$

$G_1 = G \circ \theta^{-1}$ is q.-regular and is $\begin{cases} \sigma_0\text{-hol. on } U & \text{nbhd of } \mathcal{P}'_{G_1} \\ \sigma\text{-hol. on } \theta(\widetilde{E}) & \text{nbhd of } J(G_1) \end{cases}$

Proposition C $\Longrightarrow$ $G_1 \sim_\phi R$ (a rat. map), ϕ q.c. and $\phi|_U$ hol.

(by Measurable Riemann Mapping Thm)

$\Longrightarrow$ Hyperbolic version of the existence part of Theorem A.

Proof of Proposition C

needs $\lambda_F(\partial\widetilde{L}) < 1$ *and Grötzsch inequality*

Decomposition: ↙ ↓

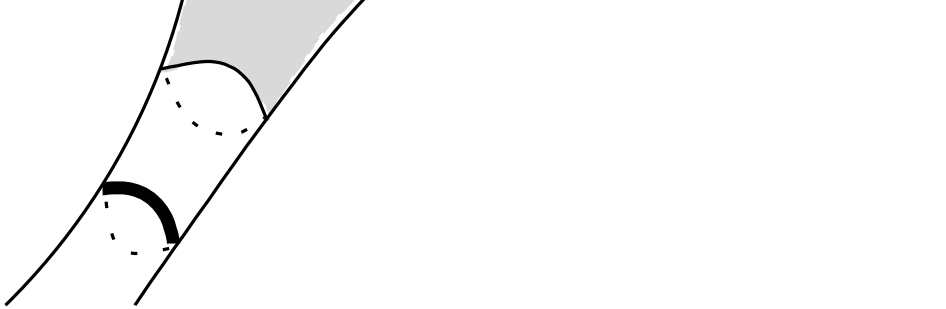
$\widetilde{L} =$ [pieces with complicated top.] $\sqcup$ [annular pieces] $\sqcup$ [disc-like pieces]

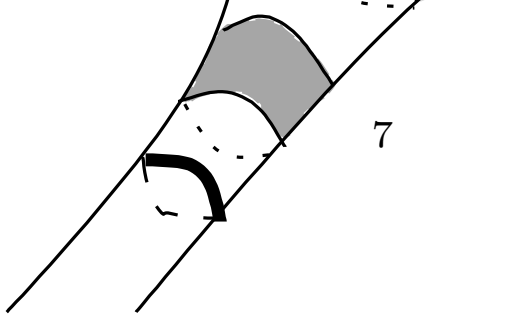
$\widetilde{E} =$ [complicated type] $\sqcup$ [annular type] $\sqcup$ [disc-type]

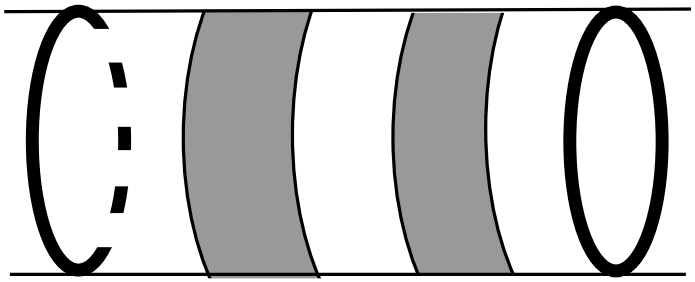
↑

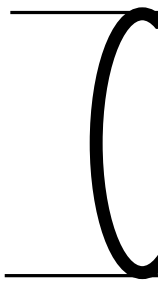
needs Thurston's original Thm.

$\left.\begin{array}{l} \widetilde{E}\text{-piece} \subset \widetilde{L}\text{-piece} \\ \widetilde{E}\text{-piece} \xrightarrow{F} \widetilde{L}\text{-piece} \end{array}\right\}$ both increase (weakly) the complexity









The pieces in $\widetilde{L}$ (bounded by dark curves) and in $\widetilde{E}$.
Find at first σ on each piece of $\widetilde{L}$ following Thurston's theorem (if applicable) and an eigenvector, and then adjust θ.

Convergence of preperiodic external rays of M (Douady-Hubbard)

$Q_c : z \mapsto z + c$, $\theta = \frac{p}{2q}$, $c(t)$ such that $\varphi_{c(t)}(e^{t+2\pi i\theta}) = c(t)$ $(t > 0)$.
Then $c(0)$ exists and $c(t) \to c(0)$ as $t \to 0$.

Existence of $c(0)$: $\boxed{Q_{c(t_0)} \overset{\text{cut-glue}}{\longrightarrow} F \overset{\text{Thurston}}{\longrightarrow} Q_{c(0)}}$

Perturbation of $c(0)$: $\forall 1 >> t > 0$,

$$
\begin{array}{ccccc}
 & & Q_{c(t)} & & \\
Q_{c(0)} & \overset{\text{cut-glue}}{\longrightarrow} & F_t & \overset{\text{Thm A}}{\longrightarrow} & \uparrow_{n\to\infty} \\
 & & Q_{c(t,n)} & &
\end{array}
\qquad
\begin{array}{ccc}
\vdots & \overset{\phi_{t,2}}{\longrightarrow} & \vdots \\
F_t \downarrow & & \downarrow Q_{c(t,2)} \\
 & \overset{\phi_{t,1}}{\longrightarrow} & \\
F_t \downarrow & & \downarrow Q_{c(t,1)} \\
 & \overset{\phi_{t,0}=id}{\longrightarrow} &
\end{array}
$$

$|c(0) - c(t)| = \lim_{n\to\infty} |c(0) - c(t,n)|$ and

$|c(0) - c(t,n)| = \text{distortion}(\phi_{t,n}) \underset{\text{indep. of } n}{\overset{\text{Thm. B}}{\leq}} \varepsilon(t) \overset{t\to 0}{\longrightarrow} 0.$

Parabolic perturbation:
given g, $\exists$ f_t with
prescribed combinatorics
such that $f_t \to g$

$$\begin{array}{ccc} F_t & \overset{\text{Theorem A}}{\sim} & f_t \\ {\scriptstyle\text{plumbing}}\uparrow & & \uparrow_{n\to\infty} \\ g & \underset{t\to\infty}{\overset{\text{Thm. B}}{\Leftarrow}} & f_{t,n} \end{array}$$

$$\begin{array}{ccccc} \vdots & \xleftarrow{\widehat{q}_2} & \vdots & \xleftarrow{\eta_2} & \vdots \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,2} \\ & \xleftarrow{\widehat{q}_1} & & \xleftarrow{\eta_1} & \\ g\downarrow & & F_t\downarrow & & \downarrow f_{t,1} \\ & \xleftarrow{\widehat{q}_0} & & \xleftarrow{\eta_0=id} & \end{array}$$

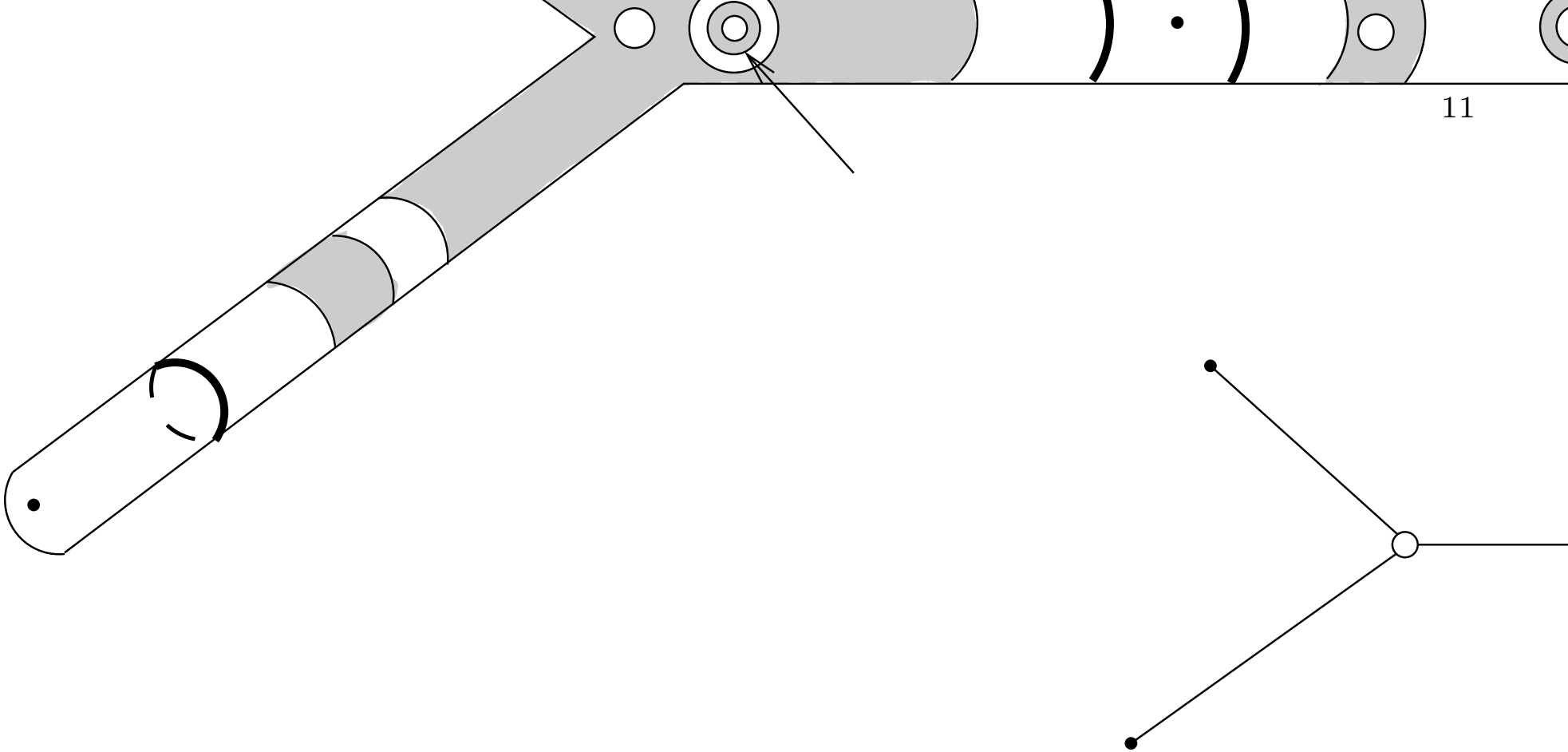
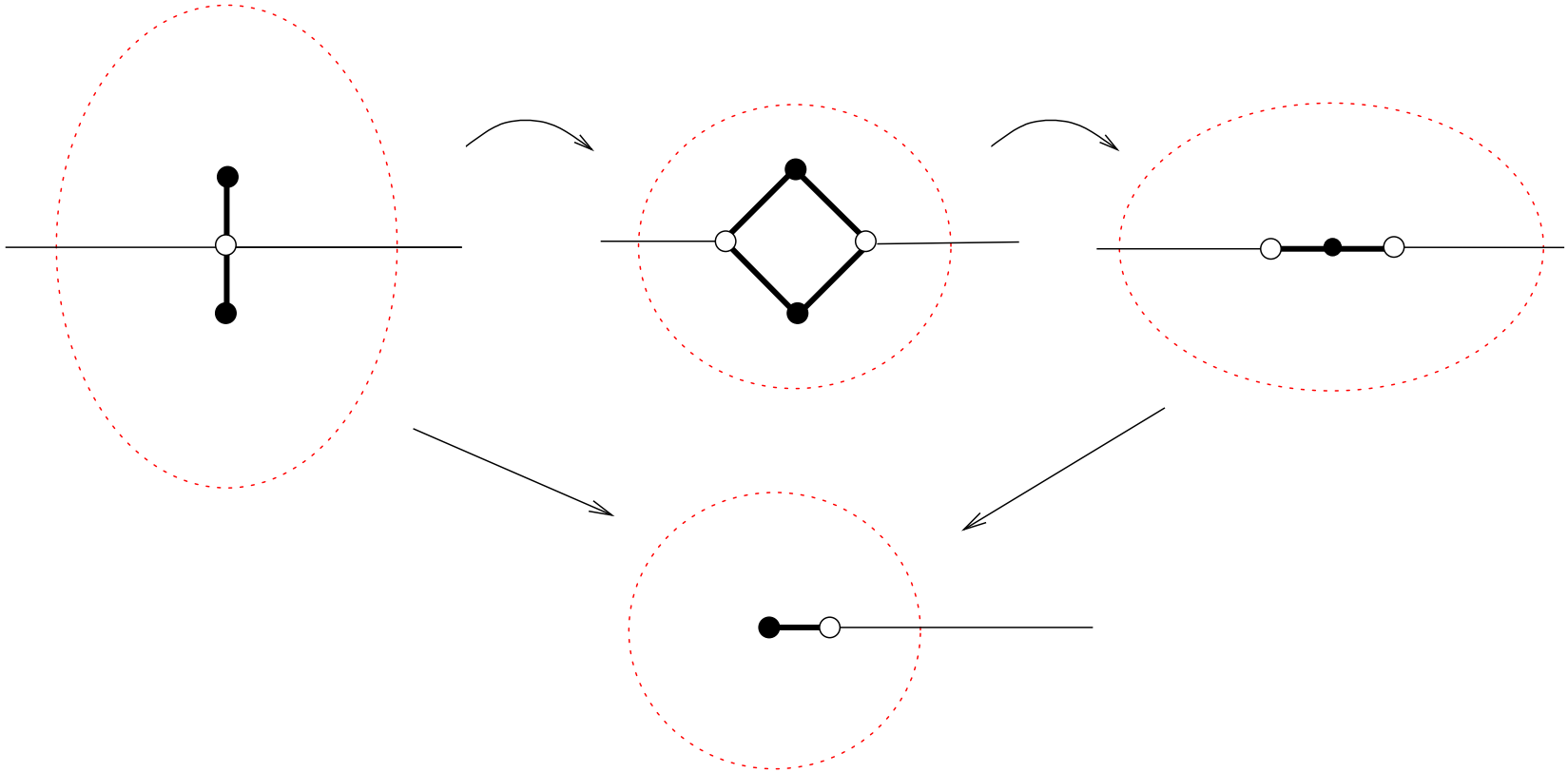
which is

$$\begin{array}{ccc} \vdots & \xrightarrow{\phi_{t,2}} & \vdots \\ g\downarrow & & \downarrow f_{t,2} \\ & \xrightarrow{\phi_{t,1}} & \\ g\downarrow & & \downarrow f_{t,1} \\ & \xrightarrow{\phi_{t,0}} & \end{array}$$

Pinching deformations of f

Let Γ be a multicurve in $\mathcal{R}_f$ whose lifts are Julia non-separating. μ_t : pinching structure supported on collar(Γ). Then $(f, \widetilde{\mu}_t)$ creates f_t and $f_t \underset{t\to\infty}{\to}$ some g.

$$\begin{array}{ccccc} f & \overset{\text{pinch } \Gamma,\mu_t}{\longrightarrow} & f_t & \overset{\text{unicity}}{=} & \hat{f}_t \\ & & & \underset{conclusion}{} & \\ \text{surgery} \downarrow & & & & \uparrow\downarrow\downarrow \text{ Parabolic perturbation} \\ G & & \overset{\text{Theorem A}}{\sim} & & g \end{array}$$



Maps with disconnected Julia sets, induced by Hubbard tree dynamics