Local properties of the Mandelbrot set M
Similarities between M and Julia sets

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1 Introduction

Local properties of the Mandelbrot set around a point c_0, such as self similarity, Hausdorff dimension, local connectivity, are closely related to properties of the filled Julia set K_c for c in a neighborhood of c_0. Recall that $M = \{c \in \mathbb{C} \mid 0 \in K_c\}$. For each $N \geq 0$, let F_N denote the holomorphic mapping $c \mapsto Q_c^N(0)$, where Q_c denotes the polynomial $z \mapsto z^2 + c$. Since K_c is totally invariant, for any $N \in \mathbb{N}$, we have

$$M = \{c \in \mathbb{C} \mid F_N(c) \in K_c\} \ .$$

It is often convenient to go to the product space $(c, z) \in \mathbb{C} \times \mathbb{C}$ to study both M and K_c. We may regard $c \mapsto K_c$ as a map and $\mathcal{K} = \{(c, z) \mid c \in \mathbb{C},\ z \in K_c\}$ as its graph in $\mathbb{C} \times \mathbb{C}$. Then M can be interpreted as

$$Proj_c\left(graph(F_N) \bigcap \mathcal{K}\right) \tag{1}$$

where $Proj_c$ denotes the projection mapping to the first coordinate (the c coordinate).

Let c_0 be a point of M, so $F_N(c_0) \in K_{c_0}$. A typical way to relate K_{c_0} to the local structure of M around c_0 is to study the local structure of $\mathcal{K}$ around the point $(c_0, F_N(c_0))$ (which depends on the regularity of the mapping $c \mapsto K_c$ at c_0, see for example conditions * and *' below), and the slope of F_N at c_0 (i.e. the constant $F_N'(c_0)$). In this paper we illustrate this technique by showing two related results.

The first theorem states a similarity result between the dynamical plane and the parameter plane around Misiurewicz points. We know that the set of such points form a dense subset of ∂M, and for each Misiurewicz point c_0, we have $J_{c_0} = K_{c_0}$ (see B. Branner's paper [B1]).

Denote by D_H the Hausdorff distance on the space $Comp^*(\mathbb{C})$ of non-empty compact subsets of $\mathbb{C}$. Let τ_{-c} denote the translation $z \mapsto z - c$. For any closed set $A \subset \mathbb{C}$, define

$$A_r = (A \cap \overline{\Delta}(0, r)) \cup \partial\Delta(0, r)$$

where $\Delta(z, r)$ denotes the open disc centered at z and with radius r. For technical reasons it is important to include the circle $\partial\Delta(0, r)$ when measuring the Hausdorff distance of two closed sets within the disc $\overline{\Delta}(0, r)$.

Let ρ be a complex number with $|\rho| > 1$. A closed set $A \subset \mathbb{C}$ is said to be *ρ-self-similar* about x if $\rho\tau_{-x}(A) = \tau_{-x}(A)$; it is said to be *asymptotically ρ-self-similar* about x if there is a ρ-self-similar set L (about 0) so that the Hausdorff distance $D_H((\rho^n\tau_{-x}(A))_r, L_r)$ tends to zero as n tends to infinity for some $r > 0$ (hence every $r > 0$). For an example of a self-similar set, see the appendix.

Theorem 1.1 *([T1]): For every Misiurewicz point c_0, there are two constants $\rho \in \mathbb{C}, |\rho| > 1$ and $\mu \in \mathbb{C} - \{0\}$, and a closed ρ-self-similar set $L \subset \mathbb{C}$ such that for any $r > 0$*

a) $D_H((\rho^n \tau_{-c_0}(K_{c_0}))_r, L_r) \to 0$ *as* $n \to \infty$, *i.e.* K_{c_0} *is asymptotically* ρ*-self-similar about* c_0. *Moreover, for* c *in a neighborhood of* c_0, *we have* $D_H((\rho(c)^n \tau_{-\zeta(c)}(K_c))_r, L(c)_r) \to 0$, *where* $c \mapsto \rho(c)$ *and* $c \mapsto \zeta(c)$ *are holomorphic, and* $c \mapsto L(c)$ *is a map satisfying the condition* * *below.*

b) $D_H((\rho^n \tau_{-c_0}(M))_r, (\mu L)_r) \to 0$, *in other words* M *is also asymptotically* ρ*-self-similar about* c_0.

c) $\lim_{t\in\mathbb{C},|t|\to\infty} D_H((t\mu \cdot \tau_{-c_0}(K_{c_0}))_r, (t\tau_{-c_0}(M))_r) = 0$. *Hence up to multiplication by* μ *the sets* K_{c_0} *and* M *are asymptotically similar about* c_0.

(We will give the precise form of ρ, L *and* μ *later in the formulas (2) , (6)).*

Part c) is an easy consequence of a) and b). As an application of this result, one can show that M is locally connected at each Misiurewicz point (see the appendix).

The second result can be considered as a quantitative study of the similarity between Julia sets and the Mandelbrot set. It gives estimates for the Hausdorff dimension of these sets. The exact definition of the Hausdorff dimension is not so important for the purpose of this paper. It can be found in the appendix. We only state two basic properties of it: For any compact set K of $\mathbb{C}$, we have H-dim$(K) \in [0,2]$, and for any compact subset $K' \subset K$, we have H-dim$(K') \leq$ H-dim(K). One should consider that the Hausdorff dimension measures a kind of density or complexity of a set. So if $K \subset \mathbb{C}$ is a compact set without interior, but with H-dim$(K) = 2$, then K must be very complicated.

Theorem 1.2 *(Shishikura): For each* $\varepsilon > 0$, *there is a dense subset of* ∂M *satisfying that for every point* c_0 *in this set, there is a closed set* $X \subset \partial K_{c_0}$ *and a constant* $r_0 > 0$ *such that*

a') H-dim$(\partial K_{c_0}) \geq$ H-dim$(X) > 2 - \varepsilon$. *Moreover for* $c \in \Delta(c_0, r_0)$, *we have similarly* H-dim$(X(c)) > 2 - \varepsilon$, *where* $X(c)$ *is a subset of* ∂K_c, *and* $c \mapsto X(c)$ *satisfies the condition* *' *below.*

b') H-dim$(\partial M \cap \Delta(c_0, r_0)) > 2 - \varepsilon$.

Corollary. We have H-dim$(\partial M) = 2$.

The existence of c_0 and X satisfying a') involves deep analysis of parabolic perturbations and renormalizations. We will give some ideas of it in the appendix. The set X is in fact a hyperbolic set (see the appendix). It is this hyperbolicity that guarantees the stability property.

We will sketch the proofs of a),b),b') in the following sections.

Condition *. Consider a mapping $c \mapsto A(c)$ with $c \in \Delta(c_0, r_0)$, $A(c)$ a closed subset of $\mathbb{C}$, such that $c \mapsto A(c)_r$ is continuous at c_0, for every $r > 0$. The mapping *admits a dense set of continuous sections at* c_0, if there exists a dense subset $Z \subset A(c_0)$ and, for each $z \in Z$, a neighborhood $U_z \subset \Delta(c_0, r_0)$ of c_0, and a mapping

$$s : \{(c, z) \mid z \in Z,\ c \in U_z\} \to \mathbb{C}$$

such that $s(c_0, \cdot) = id$ and $s(\cdot, z) : U_z \to \mathbb{C}$ is continuous with $s(c, z) \in A(c)$.

Condition *'. Consider a mapping $c \mapsto X(c)$, with $c \in \Delta(c_0, r_0)$, $X(c)$ a subset of $\mathbb{C}$. The mapping admits a *holomorphic motion* if there is a mapping $i : \Delta(c_0, r_0) \times X \to \mathbb{C}$, where $X = X(c_0)$, such that $i(c_0, \cdot) = id$, $i(c, \cdot) : X \to \overline{\mathbb{C}}$ is injective with $i(c, X) = X(c)$ and $i(\cdot, z) : \Delta(c_0, r_0) \to \overline{\mathbb{C}}$ is holomorphic for each $z \in X$.

2 Dynamical planes, Proof of a)

First remark that asymptotic similarity is invariant under conformal transformations. More precisely:

Proposition 2.1 : *Let U and V be neighborhoods of 0 and $u : U \to V$ be an injective analytic map satisfying $u(0) = 0$ and $u'(0) \neq 0$. Suppose A is a closed subset of U containing 0, and suppose $u(A)$ is asymptotically ρ-self-similar for some $\rho \in \mathbb{C}$ with $|\rho| > 1$, i.e. for any $r > 0$,*

$$D_H((\rho^n u(A))_r, L_r) \to 0 \quad as \quad n \to \infty$$

where L is a closed ρ-self-similar set. Then A is asymptotically ρ-self-similar to $(1/u'(0))L$, i.e. for any $r > 0$,

$$D_H\left((\rho^n A)_r, \left(\frac{1}{u'(0)}L\right)_r\right) \to 0 \ .$$

Now assume that c_0 is a Misiurewicz point. If there is no ambiguity, we simplify the notation by setting $Q_{c_0} = Q$ and $K_{c_0} = K$. By definition, there is a smallest number k such that $\alpha = Q^k(c_0)$ is a periodic point. Let $\rho = (Q^p)'(\alpha)$ denote the multiplier.

It follows from classical results of Fatou and Julia that the point α is repelling, i.e. $|\rho| > 1$. Let $\varphi : U \to \Delta(0, r)$ denote linearizing coordinates in a neighborhood U of α, hence satisfying $\varphi(\alpha) = 0$, $\varphi'(\alpha) = 1$ and $\varphi \circ Q^p \circ \varphi^{-1}(z) = \rho z$ for all $z \in \Delta(0, r/|\rho|)$, for some $r > 0$.

Since K is totally invariant ($Q(K) = Q^{-1}(K) = K$) and φ is a linearizing coordinate we have

$$(\rho\varphi(K \cap U))_r = (\varphi(K \cap U))_r \ .$$

Applying proposition 2.1 to $(u, A) = (\varphi, K \cap \overline{U})$ we get

$$D_H((\rho^n \tau_{-\alpha}(K))_{r'}, (\varphi(K \cap \overline{U}))_{r'}) \to 0$$

for any $0 < r' < r$.

Since $(Q^k)'(c_0) \neq 0$, there exists a neighborhood V of c_0 and $r'' > 0$ such that $\varphi \circ Q^k : V \to \Delta(0, r'')$ is a homeomorphism. Applying proposition 2.1 to $(u, A) = (\varphi \circ Q^k, K \cap \overline{V})$ we obtain

$$D_H\left((\rho^n \tau_{-c_0}(K))_{r''}, \left(\frac{1}{(Q^k)'(c_0)}\varphi(K \cap \overline{U})\right)_{r''}\right) \to 0 \ .$$

This proves the first assertion of a). Note that ρ is the multiplier of the periodic point α and that the ρ-self-similar limit set L is determined (locally) by

$$\frac{1}{(Q^k)'(c_0)}\varphi \circ Q^k(K \cap \overline{V}) \tag{2}$$

where V is any neighborhood of c_0 which is mapped homeomorphically onto its image under $\varphi \circ Q^k$.

Example. Let us take the example of $c_0 = i$. For $Q_i : z \mapsto z^2 + i$, the orbit of i is : $i \mapsto i - 1 \mapsto -i \mapsto i - 1$. In our notation, $k = 1, p = 2, \alpha = i - 1$ and $\rho = (Q_i^2)'(i - 1) = 4(1 + i) = 4\sqrt{2}e^{\pi i/4}$.

Remark. Note that a) could have been stated in greater generality. The statement is true for any repelling periodic point α with multiplier ρ and linearizing coordinates φ, and similarly for any pre-periodic repelling point. We have only chosen the special pre-periodic point c_0 in order to be able to compare with the parameter plane. These are the only properties we have used, together with the invariance of K.

The second assertion of a) is a consequence of a stability result. As before let c_0 be a Misiurewicz point and let k, p, α, ρ be as above for the map Q_{c_0}.

As an application of the implicit function theorem, (pre-)repelling periodic points are "stable" with respect to the parameter. That is for any c in a neighborhood W of c_0, the polynomial

$Q_c : z \mapsto z^2 + c$ has a p-periodic point $\alpha(c)$, depending analytically on c, with multiplier $\rho(c)$, and with a k-th pre-image $\zeta(c)$ of $\alpha(c)$, both depending analytically on c, satisfying

$$\zeta(c_0) = c_0 \ , \ \alpha(c_0) = \alpha \ , \ \rho(c_0) = \rho \ \text{ and } (Q_c^k)'(\zeta(c)) \neq 0 \ .$$

Let φ_c denote the linearization coordinate around $\alpha(c)$. The same proof as above shows that there is a neighborhood $V_{\zeta(c)}$ of $\zeta(c)$ which is mapped by $\varphi_c \circ Q_c^k$ comformally onto its image, and that K_c is asymptotically $\rho(c)$-self-similar about $\zeta(c)$ to the limit set (locally)

$$L(c) = \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(K_c \cap \overline{V}_{\zeta(c)}) \ . \tag{3}$$

As for the condition *: the mapping $c \mapsto L(c)$ is continuous at c_0 because $c \mapsto K_c$ is continuous at c_0 (Douady-Hubbard, see B. Branner's paper [B1]); each repelling periodic point has a continuous section, and the set of repelling points is dense in $J_{c_0} = K_{c_0}$.

3 Parameter plane, Proofs of b),b')

The proof of b) is done in two steps, one consists of a general result, one is the adaptation.

Proposition 3.1 *Suppose Λ is a neighborhood of λ_0 in $\mathbb{C}$. Assume we have a mapping $\lambda \mapsto A(\lambda)$ satisfying the condition * at λ_0, and that $A(\lambda)$ is $\rho(\lambda)$-self-similar about 0, where $\lambda \mapsto \rho(\lambda)$ is holomorphic with $|\rho(\lambda_0)| > 1$. Assume $u : \Lambda \to \mathbb{C}$ is a holomorphic mapping, with $u(\lambda_0) = 0$, and $u'(\lambda_0) \neq 0$ (transversality). Set*

$$M_u = \{\lambda \in \Lambda | u(\lambda) \in A(\lambda)\} \tag{4}$$

Then M_u is asymptotically $\rho(\lambda_0)$-self-similar about λ_0 to the $\rho(\lambda_0)$- self-similar set $A(\lambda_0)/u'(\lambda_0)$.

Proof. (sketch) Assume $\lambda_0 = 0$. For $z \in \mathbb{C}$ a point and $K \subset \mathbb{C}$ a compact set, we use $d(z, K)$ to denote the euclidean distance from z to K, i.e. $d(z, K) = \min_{z' \in K} |z - z'|$.

To fix our ideas we treat first two simple cases. Assume that $\lambda \mapsto A(\lambda)$ is a constant map (i.e. $A(\lambda) \equiv A(\lambda_0)$) and $u(z) = u'(0)z$ is linear. Then obviously M_u is ρ_0-self-similar about 0, to the set $A(0)/u'(0)$. Assume now that $\lambda \mapsto A(\lambda)$ is still constant but $u(z)$ is no longer linear. Then M_u coincides with $u^{-1}(A(0))$ and the conclusion holds by Proposition 2.1.

Now let us come back to the setting of our proposition. Set $\rho(0) = \rho$, $u'(0) = u'$ and $A(0) = A$. Choose $r > 0$ sufficiently small. We must prove that $D_H((\rho^n u' M_u)_r, A_r) \to 0$ as $n \to \infty$. Recall that by definition $D_H(A, B) = \max(\delta(A, B), \delta(B, A))$ (see B. Branner's paper [B1]).

First we prove $\delta((\rho^n u' M_u)_r, A_r) \to 0$. Let

$$\lambda \in M_u \cap \overline{\Delta}(0, \frac{r}{\rho^n u'})$$

so that $\rho^n u' \lambda \in \overline{\Delta}(0, r)$ and $|\lambda| \sim |\rho|^{-n}$.

$$d(\rho^n u' \lambda, A_r) \leq |\rho^n u' \lambda - \rho^n u(\lambda)| + |\rho^n u(\lambda) - \rho(\lambda)^n u(\lambda)| + d(\rho(\lambda)^n u(\lambda), A_r) = I_1 + I_2 + I_3 \ .$$

We have $I_1 \sim |\rho|^n |\lambda|^2 \sim |\rho|^n |\rho|^{-2n} \to 0$, and

$$I_2 = |\rho^n - \rho(\lambda)^n| \cdot |u(\lambda)| \leq \sum_{i=0}^{n-1} |\rho|^{n-i-1} |\rho(\lambda)|^i |\rho - \rho(\lambda)| \cdot |u(\lambda)| \sim n|\rho|^{n-1}|\lambda|^2 \to 0 \ .$$

$I_3 \to 0$ because $\lambda \in M_u$ so $\rho(\lambda)^n u(\lambda) \in A(\lambda)$ and $|\rho(\lambda)^n u(\lambda)| \sim |\rho^n u'\lambda| \leq r$, moreover $\delta(A(\lambda)_r, A_r) \to 0$ as $\lambda \to 0$.

To prove $\delta(A_r, (\rho^n u' M_u)_r) \to 0$, we only need to show $d(z, (\rho^n u' M_u)_r) \to 0$ for each $z \in Z$, where Z is the dense set in A_r where we have continuous sections $s(\cdot, z)$ (see condition *****). Let $z \in Z$. By Rouché's theorem, for n large, there is a $\lambda_n \in U_z$ as a solution of the equation $s(\lambda, z) = \rho(\lambda)^n u(\lambda)$. Hence $\lambda_n \in M_u$ and $|\lambda_n| \leq C|u(\lambda_n)| \sim |\rho|^{-n}$. Moreover

$$|\rho(\lambda_n)^n u(\lambda_n)| = |s(\lambda_n, z)| \sim |s(0, z)| = |z| \leq r \ .$$

So $d(z, (\rho^n u' M_u)_r) \leq |z - \rho^n u' \lambda_n| \leq |z - s(\lambda_n, z)| + |\rho(\lambda_n)^n u(\lambda_n) - \rho^n u' \lambda_n| = I_4 + I_5$.
We have $I_5 \to 0$ as above, and $I_4 \to 0$ by the continuity of $s(\cdot, z)$, and the fact $s(0, z) = z$. ■

Remark. This proposition is also true in a higher dimensional setting, and under a weaker hypothesis of differentiability for ρ and u. See [T1] for details.

Let c_0 be a Misiurewicz point. We now will adapt the situation just considered. Recall from the proof of the second part of a) that there exist a neighborhood W of c_0 and a holomorphic mapping $\zeta(c)$ such that K_c is asymptotically $\rho(c)$-self-similar about the point $\zeta(c)$, to the $\rho(c)$-self-similar limit set $L(c)$ (see (3)). Shrinking W if necessary we have $c \in V_{\zeta(c)}$ whenever $c \in W$. Hence the Mandelbrot set in the region W can be interpreted as

$$M \cap W = \{c \in W \mid c \in K_c\} = \left\{ c \in W \;\middle|\; \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(c) \in L(c) \right\} . \tag{5}$$

To apply the above proposition, set $\Lambda = W$, $\lambda_0 = c_0$, $A(c) = L(c)$ and

$$u(c) = \frac{1}{(Q_c^k)'(z)|_{z=\zeta(c)}} \varphi_c \circ Q_c^k(c) \ .$$

We have proved in part a) that $c \mapsto L(c)$ satisfies the condition ***** at c_0, and $L(c)$ is $\rho(c)$-self-similar about 0. Moreover $c \mapsto \rho(c)$ is holomorphic, with $|\rho(c_0)| > 1$. It is clear that $u(c)$ is also holomorphic. As a consequence of the connectedness of M, Douady and Hubbard showed that $u'(c_0) \neq 0$. Set $L(c_0) = L$. Now since $M \cap W = M_u$, Proposition 3.1 shows that, for any $r > 0$,

$$D_H\left((\rho^n \tau_{-c_0} M)_r, (\frac{1}{u'(c_0)} L)_r \right) \to 0 \ .$$

Moreover, an elementary calculation (see [T1] for details) shows that the required μ is

$$\frac{1}{u'(c_0)} = \frac{(Q_{c_0}^k)'(z)|_{z=c_0}}{\frac{d}{dc}(Q_c^k(c))|_{c=c_0} - \frac{d}{dc}(\alpha(c))|_{c=c_0}} \ . \tag{6}$$

This ends the proof of b).

Example. Let us take again the example of $c_0 = i$. To find the value of μ , we first obtain $(Q_{c_0}^k)'(z)|_{z=c_0} = (Q_i)'(z)|_{z=i} = 2i$,

$$\frac{d}{dc}(Q_c^k(c))|_{c=i} = \frac{d}{dc}(c^2 + c)|_{c=i} = (2c+1)|_{c=i} = 2i + 1 \ .$$

The function $\alpha(c)$ is the solution near $i - 1$ of the implicit equation $Q_c^2(z) = z$, i.e. $(z^2 + c)^2 + c - z = 0$ for c close to i . Hence

$$\frac{d}{dc}(\alpha(c))|_{c=i} = -\frac{2((i-1)^2 + i) + 1}{2((i-1)^2 + i) \cdot 2(i-1) - 1} = \frac{2i - 1}{4i + 3} \quad \text{and} \quad \mu = \frac{2i}{2i + 1 - \dfrac{2i-1}{4i+3}} = 1 + \frac{1}{2}i \ .$$

Computer experiments confirm the similarity very impressively.

The proof of b') is also done in two steps:

Proposition 3.2 *Assume that we have a holomorphic motion* $i : \Delta \times X \to \overline{\mathbb{C}}$ *(where* Δ *denotes the unit disc) and an analytic mapping* $v : \Delta \to \overline{\mathbb{C}}$*, with* $v(0) = z_0 \in X$*,* $v(\lambda) \not\equiv i(\lambda, z_0)$ *(weak transversality). Set* $M_v = \{\lambda \in \Delta \mid v(\lambda) \in X(\lambda)\}$*. Then*

$$\text{H-dim}(M_v) \geq \lim_{r\to 0} \text{H-dim}(X \cap \Delta(z_0, r)) \ .$$

Proof. (sketch) For simplicity, we assume $v'(0) \neq 0$. In the simple case that $X(\lambda) \equiv X(0)$ for $\lambda \in \Delta$ we have $M_v = v^{-1}(X(0))$. Since v is bi-Lipschitz near 0, and hence preserves the Hausdorff dimension, we have $\text{H-dim}(M_v) \geq \text{H-dim}(X \cap \Delta(z_0, r))$ for some $r > 0$. As a consequence $\text{H-dim}(M_v) \geq \lim_{r\to 0} \text{H-dim}(X \cap \Delta(z_0, r))$.

Now let us come back to the setting of our proposition. We will apply Rouché's theorem to prove that, for $r \in]0, r_0[$ (where r_0 is a small constant) and $Y^r = v^{-1}(X \cap \Delta(z_0, r))$, there is $R_r > 1$ (with $R_r \to \infty$ as $r \to 0$) and a holomorphic motion: $j^r : \{|\mu| < R_r\} \times Y^r \to \overline{\mathbb{C}}$ such that $j^r(1, Y^r) \subset M_v$. Therefore

$$\text{H-dim}(M_v) \geq \text{H-dim}(j^r(1, Y^r)) \geq C(1/R_r)\text{H-dim}(Y^r) = C(1/R_r)\text{H-dim}(X \cap \Delta(z_0, r)) \ ,$$

where the existence of $C(1/R_r)$ in the second inequality is due to a non trivial property of holomorphic motions (see below). Furthermore $C(1/R_r) \to 1$ as $r \to 0$.

(More precisely by Słodkowski's theorem (see for example [D1]), the mapping $j^r(1, \cdot)$ extends to a $K(1/R_r)$-quasi-conformal mapping and $K(1/R_r) \to 1$ as $1/R_r \to 0$. On the other hand, by Mori's inequality K-quasi-conformal maps are $1/K$-bi-Hölder continuous. Furthermore a simple calculation shows that for any $1/K$-bi-Hölder continuous map j and any set Y, we have

$$(1/K) \cdot \text{H-dim}(Y) \leq \text{H-dim}(j(Y)) \leq K \cdot \text{H-dim}(Y) \ .$$

Setting $C(1/R_r) = 1/K(1/R_r)$, we get the desired inequality).

To define the constant R_r and the holomorphic motion j^r, we proceed as follows: Assume $z_0 = 0$ and $i(\lambda, 0) \equiv 0$. Since the family of holomorphic maps $\{i(\cdot, z)\}_{z\in X}$ is normal, and $i(\lambda, z) \neq 0$ for any $z \neq 0$ and any λ (by injectivity), any limit function of the family corresponding to a sequence $z_n \to 0$ must be the constant function 0. Fix $s < 1$ and $a > 0$ such that in $\Delta(0, s)$, $v(\lambda)$ is injective and $a|\lambda| \leq |v(\lambda)|$. Then for $b_r = \sup\{|i(\lambda, z)| \mid z \in X \cap \Delta(0, r),\ |\lambda| \leq s\}$, we have $b_r \to 0$ as $r \to 0$.

Fix r_0 such that $as > b_r$ for $0 < r < r_0$. Take $r \in]0, r_0[$, set $R_r = as/b_r$. Take $\mu \in \Delta(0, R_r)$ and $z \in X \cap \Delta(0, r)$. The equation $v(\lambda) - i_{\mu\lambda}(z) = 0$ has a unique solution $\lambda(r, \mu, z)$ in the disc $\Delta(0, \min\{s, s/|\mu|\})$ (we just apply Rouché's theorem here).

Now set $Y^r = v^{-1}(X \cap \Delta(0, r))$ and define $j^r : \{|\mu| < R_r\} \times Y^r \to \overline{\mathbb{C}}$ by $j^r(\mu, y) = \lambda(r, \mu, v(y))$. It is then easy to check that j^r is a holomorphic motion and $j^r(1, Y^r) \subset M_v$. ■

Now we should adapt the above result to the situation of the boundary ∂M of the Mandelbrot set. It is much less trivial than the similarity case because we want M_v to be a subset of ∂M.

Lemma 3.3 *For the holomorphic motion in part a'), there are* $z_0 \in X$*,* $c' \in \Delta' \subset \Delta$*, with* Δ' *a neighborhood of* c'*, and* $v(c) = Q_c^N(0)$ *for some* $N > 0$ *with* $v(c') = i_{c'}(z_0)$*,* $v(c) \not\equiv i_c(z_0)$ *such that* $\{c \in \Delta' \mid v(c) \in X(c)\} \subset \partial M \cap \Delta'$*, and* $\lim_{r\to 0} \text{H-dim}\,(X_{c'} \cap \Delta(i_{c'}(z_0), r)) > 2 - \varepsilon$.

Proof. (sketch). By compactness of X, there exists a point $z_0 \in X$ such that H-dim$(X) = \lim_{r\to 0}$ H-dim$(X \cap \Delta(0,r)) > 2-\varepsilon$. Let $i : \Delta(c_0, r_0) \times X \to \mathbb{C}$ be the holomorphic motion given in part a'). Since for c close to c_0 the mapping $i(c,\cdot)$ does not change too much the Hausdorff dimension (see the proof of the above proposition), there is a small neighborhood $\Delta' \subset \Delta(c_0, r_0)$ of c_0 such that $\lim_{r\to 0}$ H-dim $(X(c) \cap \Delta(i(c,z_0),r)) > 2-\varepsilon$ for $c \in \Delta'$.

Recall that F_n denotes the map $c \mapsto Q_c^n(0)$. The boundary of M coincides with the set

$$\{\ c \in \mathbb{C} \mid \ \text{the family } \mathcal{F} = \{F_n\ ,\ n \in \mathbb{N}\ \} \text{ is not normal at } c\ \}$$

(see B. Branner's paper [B1]). By part a'), $c_0 \in \partial M$. We claim that there is $c' \in \Delta'$, an integer $N > 0$, such that $F_N(c') = i(c', z_0)$. For otherwise the family $\mathcal{F}$ would satisfy Montel's normality criterion at c_0 with respect to the two analytic functions $c \mapsto i(c, z_0)$ and one branch of $c \mapsto Q_c^{-1}(i(c,z_0))$, which contradicts the fact that $\mathcal{F}$ is not normal at c_0.

Set $v(c) = F_N(c)$. In order to apply the above proposition, we need to know that $v(c) \not\equiv i(c, z_0)$ and the set $M_v = \{c \in \Delta' \mid v(c) \in X(c)\}$ is a subset of ∂M.

One thing that was not explicitly stated in part a') is that, besides the other properties in a'), we have also $Q_{c_0}(X) = X$, moreover the mapping $i(c,\cdot)$ conjugates the dynamics, i.e.

$$i(c, Q_{c_0}(z)) = Q_c(i(c,z))\ . \tag{7}$$

As a consequence $Q_c(X(c)) = X(c)$ and $X(c) \subset J_c$.

There are (at least) two ways to see that $v(c) \not\equiv i(c, z_0)$: I. Since $c_0 \in \partial M$, there is $c'' \in \Delta' - M$. So $F_N(c'') = Q_{c''}^N(0) \notin J_{c''}$. But $i(c'', z_0) \in X(c'') \subset J_{c''}$. II. We may also use the normality argument. Assume $F_N(c) \equiv i(c, z_0)$ in Δ'. Then, for any integer $k > 0$,

$$F_{N+k}(c) = Q_c^k(F_N(c)) \equiv Q_c^k(i(c,z_0)) = i(c, Q_{c_0}^k(z_0))\ ,$$

where the last equality is due to the formula (7) . So the family $\mathcal{F}$ is uniformly bounded in Δ', hence normal. This contradicts the fact that $\mathcal{F}$ is not normal at c_0.

To prove $M_v \subset \partial M$ we need Mañé-Sad-Sullivan's characterization of ∂M: We say that Q_c is J-stable at c_1 if there is a continuous map $h : \Delta(c_1, r) \times J_{c_1} \to \overline{\mathbb{C}}$ such that $h_c \equiv h(c,\cdot)$ is a conjugacy from (J_{c_1}, Q_{c_1}) to (J_c, Q_c) and $h_{c_1} = Id$. Then

$$\partial M = \{c_1 \in \mathbb{C} \mid Q_c \text{ is NOT } J\text{-stable at } c_1\ \}\ .$$

(This formula can be used to get another way to prove $v(c) \not\equiv i(c, z_0)$, for otherwise one can pull back the formula (7) to get a holomorphic motion of $\bigcup_n Q_{c_0}^{-n}(v(c_0))$. Since this is a dense subset of J_{c_0}, we can apply the λ-lemma of Mañé-Sad-Sullivan to show that Q_c is J-stable at c_0, thus a contradiction.)

Now assume that c_1 is a point in $M_v - \partial M$. So Q_c is J-stable at c_1, and admits a conjugacy map $h : \Delta(c_1, r) \times J_{c_1} \to \overline{\mathbb{C}}$. Decreasing r if necessary, we may assume $\Delta(c_1, r) \subset \Delta'$. Set $X_1 = i(c_1, X)$. We claim that $h|_{\Delta(c_1,r)\times X_1}$ must coincide with the holomorphic motion i. The reason is that both maps are continuous and $h(c,z) = i(c,z)$ for z any repelling periodic point, and repelling periodic points are dense in X_1 (this is because X_1 is a hyperbolic subset). On the other hand, h must preserve the critical point, i.e. $h(c,0) = 0$. So $h(c, v(c_1)) = v(c)$. This gives rise to a contradiction since $v(c_1) = i(c_1, z_0)$ but $v(c) \not\equiv i(c, z_0)$.

Hence all the conditions required by the above proposition are satisfied. So

$$\text{H-dim}(\partial M) \geq \text{H-dim}(M_v) \geq \lim_{r\to 0} \text{H-dim}\left(X(c') \cap \Delta(i(c', z_0), r)\right) > 2-\varepsilon\ .$$

This completes the proof of b'). ■

4 Appendix

1. An example of a self-similar set. Let $A \subset [0,1]$ denote the standard middle third Cantor set. Take all logarithmic spirals through points in A which in logarithmic coordinates are straight lines parallel to the vector $\log 3 + 2\pi i$. This set is ρ-self-similar for $\rho = 3^t e^{2\pi i t}$ for any $t \in \mathbb{R}_+$.

2. Local connectivity. For c_0 a Misiurewicz point, [T2] constructed a sequence of Jordan curves Γ_n in the dynamical plane such that $A = \{c_0\} \cup \bigcup_n \Gamma_n$ is asymptotically ρ-self-similar and the set of bounded components U_n of $\overline{\mathbb{C}} - \Gamma_n$ forms a basis of nested neighborhoods of c_0 with $\overline{U}_n \cap K_{c_0}$ connected. Moreover there is a holomorphic motion $i : \Delta(c_0, r_0) \times A \to \overline{\mathbb{C}}$ with $A(c)$ asymptotically $\rho(c)$-self-similar. As $n \to \infty$, the sequence of subsets $\mathrm{P}\Gamma_n = \{c \in \Delta(c_0, r_0) \mid c \in \Gamma_n(c)\}$ bounds a nested sequence of neighborhoods W_n of c_0 in the parameter plane, with $\overline{W}_n \cap M$ connected. Applying Proposition 3.1 to $\mathrm{P}\Gamma_n$, we see that it is also asymptotically ρ-self-similar. In particular the diameters of W_n shrink to zero exponentially fast. This is a stronger statement than saying that M is locally connected at c_0.

3. Hausdorff dimension: Let (E, d) be a metric space. For $A \subset E$, denote by $|A|$ its diameter. For $X \subset E$, $t > 0$, $\varepsilon > 0$, we define

$$m_t^\varepsilon(X) = \inf_{\{A_j\}} \left\{ \sum_{j \in \mathbb{N}} |A_j|^t \;\middle|\; 0 < |A_j| \le \varepsilon,\ X \subset \bigcup_{j \in \mathbb{N}} A_j \right\} .$$

Fixing t, $m_t^\varepsilon(X)$ increases as ε decreases. We can then define $m_t(X) = \lim_{\varepsilon \to 0} m_t^\varepsilon(X) = \sup_{\varepsilon > 0} m_t^\varepsilon(X)$. Note that $m_t(X)$ can be ∞. An easy calculation shows that if for some t, $m_t(X) < \infty$, then $m_T(X) = 0$ for any $T > t$. As a consequence, there is a unique number $\delta \in [0, \infty]$ such that $m_t(X) = \infty$ for $t < \delta$ and $m_t(X) = 0$ for $t > \delta$. This δ is called the Hausdorff dimension of X.

It is not easy to calculate the Hausdorff dimension in general. However, in the following situation there is an easy lower bound: Let $G : V \to U$ be an analytic covering, with U an open disc, V a finite union of open discs with disjoint closures and $\overline{V} \subset U$. Then for the non-escaping set $X = \{z \in V \mid G^m(z) \in V \text{ for all } m > 0\}$, one has:

$$\text{H-dim}(X) \ge \frac{\log(\text{number of components of } V)}{\log(\max |G'(z)|)} .$$

The set X is a special example of *hyperbolic sets* and is automatically stable under perturbation.

4. Proof of theorem 1.2.a'). As an example, we give the main steps to show that there are $c_n \to c = 1/4$, $c_n \in \partial M$ and hyperbolic subsets $X_n \subset \partial K_{c_n}$ such that $\text{H-dim}(X_n) \to 2$ as $n \to \infty$. The technique is called geometric limits of a parabolic map, and parabolic implosion. A similar study can be done for each c in a dense subset of ∂M (namely the set of roots of primitive hyperbolic components).

Denote by f the map $z \mapsto z^2 + 1/4$. There will be two holomorphic functions g and h (the first and a second geometric limit of f) generating the family of maps

$$\mathcal{L} = \{f^k g^l h^m \mid k, l, m \in \mathbb{Z} ,\ m > 0\}$$

(with a certain convention on f^{-1} and g^{-1}) satisfying the following two properties:

4.1. There exists a sequence $c_n \to 1/4$ (with $c_n \in \partial M$, but one can also choose c_n in the main cardioid or $c_n \in \mathbb{C} - M$) such that for each $G_i \in \mathcal{L}$, there are integers $j(n,i) \to +\infty$ such that $Q_{c_n}^{j(n,i)}$ converges to G_i uniformly on compact sets in the domain of definition of G_i.

4.2. There exists a small neighborhood U of $1/2$ and constants $a, C, C' > 0$ such that for large $\eta > 0$, there are open sets $U_1, \cdots, U_N$, with $N > a\eta^2$, $\overline{U}_i \cap \overline{U}_j = \emptyset$, $\overline{U_i} \subset U$, and $G_i \in \mathcal{L}$ such that $G_i|_{\overline{U}_i} : \overline{U}_i \to \overline{U}$ is bijective and $C\eta(\log\eta)^2 < |G_i'|_{\overline{U}_i}| < C'\eta(\log\eta)^2$.

As a consequnce, for n large, $i = 1, \cdots, N$, there is $j(n,i)$ large and $U(n,i)$ close to U_i such that $Q_{c_n}^{j(n,i)}$ maps $U(n,i)$ bijectively onto U with derivative close that of G_i. For $X(n)$ the non-escaping set, the formula in Appendix 3 gives us:

$$\text{H-dim}(X(n)) \geq \frac{2\log\eta + \log a}{\log\eta + 2\log\log\eta + \text{constant}} > 2 - \varepsilon\ .$$

We will skip the proof of 4.1 (which can be found in the papers of Shishikura and in [D2]) and give a sketch of the construction of g, h, U_i, U, G_i and the estimate of $|G_i'|$.

a) Denote by B the basin of the parabolic point $1/2$ for f. By classical results there are holomorphic surjective maps $\varphi_1 : \mathbb{C} \to \mathbb{C}$, $w \mapsto z$ and $\Phi_1 : B \to \mathbb{C}$, $z \mapsto w$ such that $f \circ \varphi_1 = \varphi_1 \circ T$ and $\Phi_1 \circ f = T \circ \Phi_1$, where T denotes the translation $w \mapsto w + 1$ (the mappings φ_1, Φ_1 are called *Fatou coordinate changes* of f). Set $g = g_1 = \varphi_1 \circ \Phi_1 : B \to \mathbb{C}$; $\tilde{g}_1 = \Phi_1 \circ \varphi_1 : \varphi_1^{-1}B \to \mathbb{C}$ and $\overline{g}_1 = \pi \circ \tilde{g}_1 \circ \pi^{-1} : \pi(\varphi_1^{-1}(B)) \to \mathbb{C}^*$ (where $\pi(w) = e^{2\pi i w}$). Then there exist (unique) choices of φ_1, Φ_1 such that $\tilde{g}_1(w) = w + o(1)$ as $Im(w) \to +\infty$ and $\overline{g}_1(0) = 0$, $\overline{g}_1'(0) = \overline{g}_1''(0) = 1$. For this $\overline{g}_1$, we have $\overline{g}_1(\infty) = \infty$ and $|\overline{g}_1'(\infty)| > 1$.

b) Because B is simply connected and contains only one critical value, the immediate basin B' of 0 for the map $\overline{g}_1$ has the same property. Similarly one can find φ_2, Φ_2 (but with $\varphi_2(\mathbb{C}) = \mathbb{C}^*$ instead) and define $g_2, \tilde{g}_2, \overline{g}_2$ with the same asymptotic behavior at 0 and ∞. There are lifts of g_2 by π^{-1} and $\varphi_1 \circ \pi^{-1}$ successively. We call them $\tilde{h}$ and h.

c) Here is the diagram of our construction:

$$\begin{array}{ccccccc} U_i & \overset{\varphi_1}{\longleftarrow} & & \overset{\pi_0^{-1}}{\longleftarrow} & & \overset{\varphi_2}{\longleftarrow} & W_i \\ G_i \downarrow & & \tilde{s} \downarrow & & s \downarrow & & \downarrow T^{l_i}\tilde{g}_2^{m_i} \\ U & \overset{\varphi_1}{\longleftarrow} & & \overset{\pi^{-1}}{\longleftarrow} & & \overset{\varphi_2}{\longleftarrow} & W \end{array}$$

where $\tilde{s} = T^k \tilde{g}_1^{l_i} \tilde{h}^{m_i}$ and $s = \overline{g}_1^{l_i} g_2^{m_i}$. The other terms are going to be defined below.

Fix U a small disc neighborhood of $1/2$. There is a disc W and a choice of π^{-1} such that $\varphi_1 \circ \pi^{-1} \circ \varphi_2$ maps $\overline{W}$ onto $\overline{U}$ bijectively with bounded derivative.

Since ∞ is repelling for $\overline{g}_2$, the map $\tilde{g}_2$ behaves like a translation $w \mapsto w + \mu$ with $Im(\mu) > 0$ as $Im(w) \to -\infty$. So for $m \in \mathbb{N}$ there are W_m' disjoint discs such that $\tilde{g}_2 W_m' = W_{m-1}'$, $Im(W_m') \to -\infty$ as $m \to \infty$ and $\tilde{g}_2^m : \overline{W'}_m \to \overline{W}$ is a bijection with bounded derivative (independent of m).

Fix $\eta > 0$ large. In the rectangle $R(\eta) = \{w,\ |\Re w| < \eta\ ,\ Im(w) \in [-\eta, -2\eta]\}$, there are N disjoint discs $W_1, \cdots, W_N$, with $N > a\eta^2$, and $l_i, m_i \in \mathbb{Z}$, $m_i > 0$ such that $T^{l_i}\overline{W}_i = \overline{W'}_{m_i}$. Therefore for each i the map $T^{l_i}\tilde{g}_2^{m_i} : \overline{W}_i \to \overline{W}$ is a bijection with bounded derivative (independent of i and η).

Let $\pi_0^{-1} : \mathbb{C}^* - \mathbb{R}^+ \to \{w | 0 < \Re w < 1\}$ be the special branch of π^{-1}. Set $U_i = \varphi_1 \circ \pi_0^{-1} \circ \varphi_2(W_i)$.

One can easily check that for η large and a good choice of $\tilde{h}$, we have $\overline{U}_i \subset U$ and there is $k \in \mathbb{Z}$ (independent of i and η) such that $f^k g^{l_i} h^{m_i} \overline{U}_i = \overline{U}$. Set $G_i = f^k g^{l_i} h^{m_i}|_{U_i}$.

d) The derivative $|G_i'|$ is controlled by $1/|(\varphi_1 \circ \pi_0^{-1} \circ \varphi_2)'(w)|$, $w \in \overline{W}_i \subset R(\eta)$, the rest has bounded effects. On the other hand, when η is large, both φ_1 and φ_2 can be approximated by $I(w) = -1/w$. A simple calculation shows that $|G_i'| \sim \eta(\log\eta)^2$.

Some key words: dynamical spaces, parameter space, product space, implicit function theorem, Rouché's theorem, holomorphic motion.

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