CODY conference

Conformal methods in analysis and dynamics

Mapping properties of planar harmonic polynomials

joint work with J. J. Loeb and M. El Amrani

Tan Lei, Université d'Angers

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A planar harmonic polynomial takes the real form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}, \quad P, Q \text{ harmonic}$$

or the complex form

$$f(z) = p(z) - \overline{q(z)}, \quad p, q \text{ holomorphic} .$$

$$\deg f := \max\{\deg p, \deg q\} .$$

Questions:

- Number of roots of f ?
- $\#f^{-1}(c) =$?
- Fiber structure of $f : \mathbb{C} \to \mathbb{C}$?
- How does f map $\mathbb{C}$ into $\mathbb{C}$?
- Is f open? proper? surjective? Does $f(\mathbb{C})$ contain interior?
- dependence on parameter?

Why?

- real algebraic geometry
- global singularity theory of planar maps
- holomorphic dynamics
- gravitational lensing

The equivalence relation

$$f \sim \boxed{\underset{\text{bi-holomorphic}}{\xrightarrow{\psi}} \quad \xrightarrow{f} \quad \underset{az+b\overline{z}+c,\ |a|\neq|b|}{\xrightarrow{\Phi}}}$$

keeps the above properties invariant.

Lemma (degree 1 normal forms): $z, \quad z-\overline{z} = \begin{pmatrix} 0 \\ 2y \end{pmatrix}$.

Theorem 1 (quadratic normal forms + properties)

	open	proper	surjective	interior$(f(\mathbb{C})) \neq \emptyset$
holomorhpic z^2	yes	yes	yes	yes
$(z - \overline{z}) \circ (z^2)$	no	no	no	no
cone $\begin{pmatrix} x \\ xy \end{pmatrix}$	no	no	no	yes
fold $\approx \begin{pmatrix} x \\ y^2 \end{pmatrix}$	no	yes	no	yes
deltoid $z^2 - \overline{z}$	no	yes	yes	yes

	open	proper	surjective	$f(C) \ni$ interior
holomorhpic z^2	yes	yes	yes	yes
$(z-\overline{z}) \circ (z^2)$	no	no	no	no
cone $\begin{pmatrix} x \\ xy \end{pmatrix}$	no	no	no	yes
fold $\approx \begin{pmatrix} x+y \\ xy \end{pmatrix}$	no	yes	no	yes
deltoid $z^2 - \overline{z}$	no	yes	yes	yes

Theorem 2 Except $f \sim$ hol. or $(z-\overline{z})\circ$ hol.
(first and last columns true) f not open and $f(\mathbb{C}) \ni$ interior.

Theorem 3 The last line is valid for $f \sim p - \overline{q}$, $\deg(p) \neq \deg(q)$.

Lemma. proper $\not\Leftrightarrow$ surjective, proper $\Longrightarrow$ $int(f(\mathbb{C})) \neq \emptyset$.

Critical set for $f = p - \overline{q} \not\sim$ hol.

$\mathcal{C}_f = \text{cr. set} = \{z \mid Jac(f) = 0\} = \{z \mid |p'|^2 - |q'|^2 = 0\}$

$\mathcal{V}_f = \text{cr. value set} = f(\mathcal{C}_f)$.

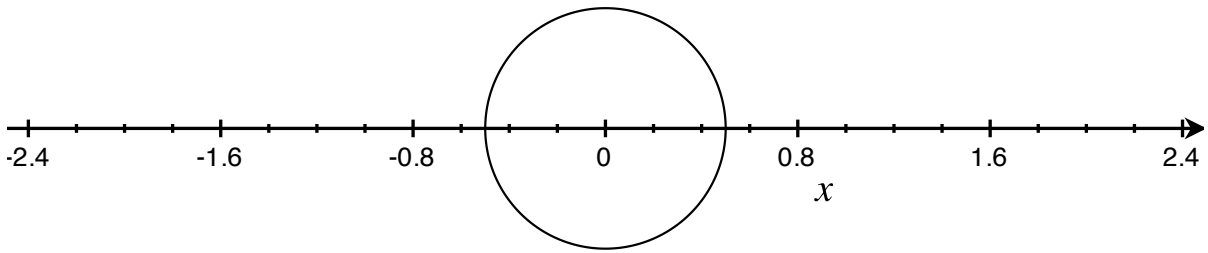
	$\mathcal{C}_f$	$\mathcal{V}_f$
z^2	$\{0\}$	$\{0\}$
$(z - \overline{z}) \circ z^2$	$\mathbb{C}$	$\mathbb{R}$
$\begin{pmatrix} x \\ xy \end{pmatrix}$	$i\mathbb{R}$	$\{0\}$
fold $\approx \begin{pmatrix} x \\ y^2 \end{pmatrix}$	a line	a parabola

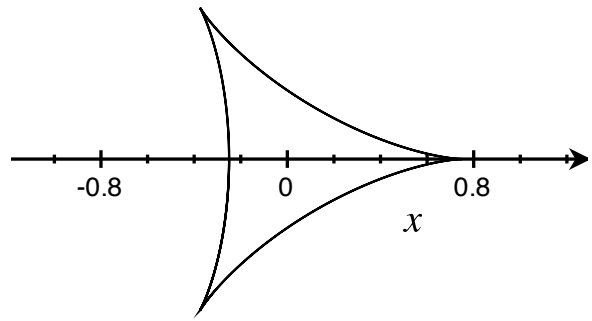
Lemma. $f \sim$ hol. or $(z - \overline{z}) \circ$ hol.

$\iff \mathcal{C}_f$ is finite or the entire plane $\mathbb{C}$

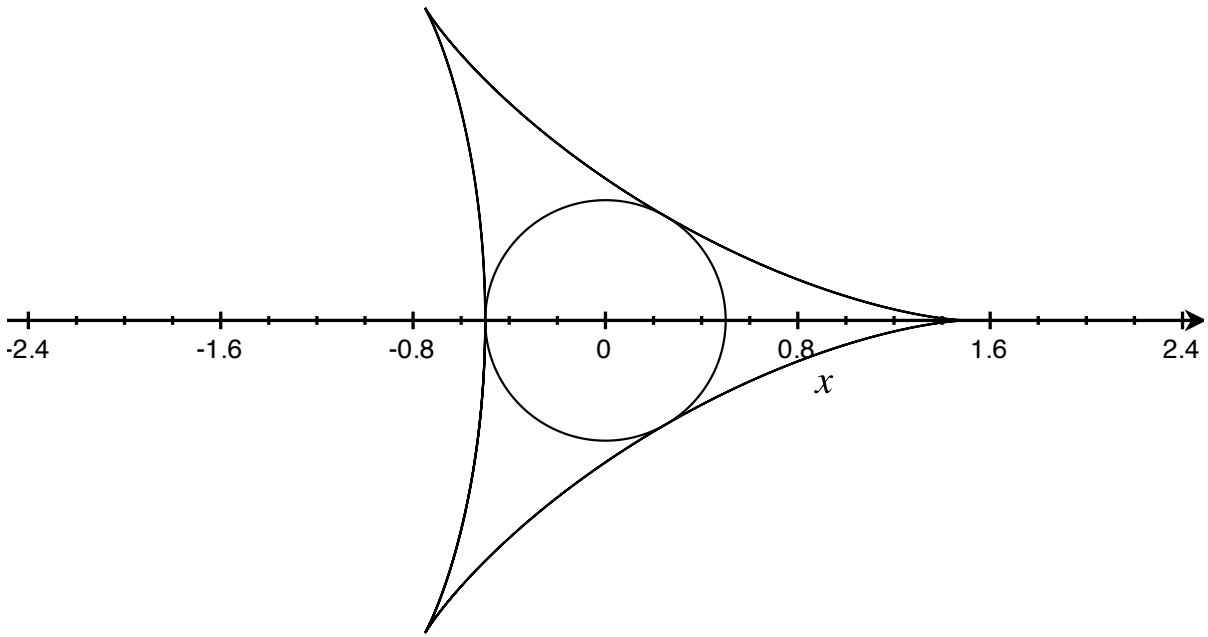
$\iff \dfrac{p'(z)}{q'(z)} \equiv$ constant.

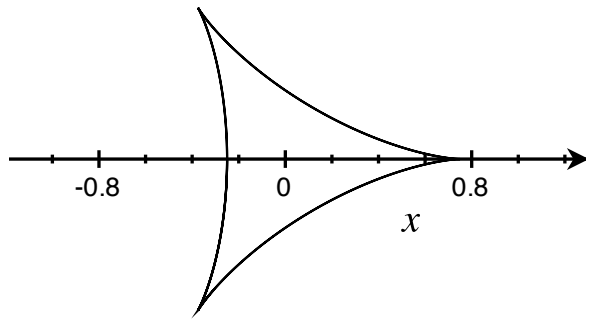
Case $z^2 - \overline{z}$, cr. set and cr. value set





Case $z^2 - \overline{z}$, co-critical set





Critical set for $f = p - \overline{q} \not\sim$ hol. , $\not\sim (z - \overline{z}) \circ$ hol.

Theorem 4

1. (topology) $\mathcal{C}_f = \textit{graph}\ h^{-1}(S^1)$ for $h = p'/q'$. Conversely any graph of this type is realizable by some f.

2. • $z \in (\mathcal{C}_f)^c$, f is a local homeo (may reverse the orientation);
   • for every $z \in \mathcal{C}_f$ (except a discrete subset)
   either $f|_z$ has a Whitney's fold singularity
   or $f|_{\mathcal{C}_f}$ is locally constant.

3. At every $z \in \mathcal{C}_f$ (except a finite subset), f takes the local form $z + O(z^2) - \overline{z} = \eta(z) - \overline{z}$ with η a parabolic germ. Setting $Q(z) = \overline{\eta} \circ \eta(z)$. Then

$$\begin{aligned} f|_{\mathcal{C}_f} =_{\text{locally}} \text{constant} &\iff Q(z) \equiv z \\ \text{fold at } 0 &\iff Q(z) = z + az^2 + \ldots,\ a \neq 0 \\ \text{cusp at } 0 &\iff Q(z) = z + az^3 + \ldots,\ a \neq 0 \end{aligned}$$

$\Longrightarrow$ **Theorem 2**. f is not open.

Upper bound for #roots for $f = p - \overline{q}$, $d = \deg(f)$

Assume $\deg(p) \neq \deg(q)$ ($\Longleftrightarrow \mathcal{C}_f$ bounded).

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$.

In case $p(z) - \overline{z}$, $d > 1$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$.

Bshouty-Lyzzaik (2004) The bound $3d - 2$ is sharp for $d = 4, 5, 6, 8$.

Geyer (2008) sharpness for $\forall\, d$ (see also Buff-Epstein-Koch).

Counting roots for $f = p - \overline{q}$, $d = \deg(f)$

Refined Theorem 3 (lower bound and surjectivity).

$$\deg p \neq \deg q \Longrightarrow \forall\, c \in \mathbb{C}, \quad \#f^{-1}(c) \geq d \ .$$

Geyer (again) $\forall\, d \geq 2$, $\exists\ p(z) \in \mathbb{R}[z]$ with ≤ 1 real critical point s.t.

$$\exists\, c \in \mathbb{C} \quad \#\{p(z) - \overline{z} = c\} = 3d - 2 \ .$$

Theorem 5. Let $p(z) \in \mathbb{R}[z]$, $\deg(p) = d > 1$ and all critical points real. Then

$$\forall c \in \mathbb{R}, \quad \#\{p(z) \pm \overline{z} = c\} \leq \begin{cases} 2d & \text{if } d \text{ is even} \\ 2d+1 & \text{if } d \text{ is odd} \end{cases} \ .$$

And the bound is sharp for every d.

Theorem 6 (cubic polynomials).

- $\boxed{f(z) = q(z) - \overline{z},\ \deg q = 3} \sim \boxed{\frac{z^3}{3} + pz - \overline{z}}$.

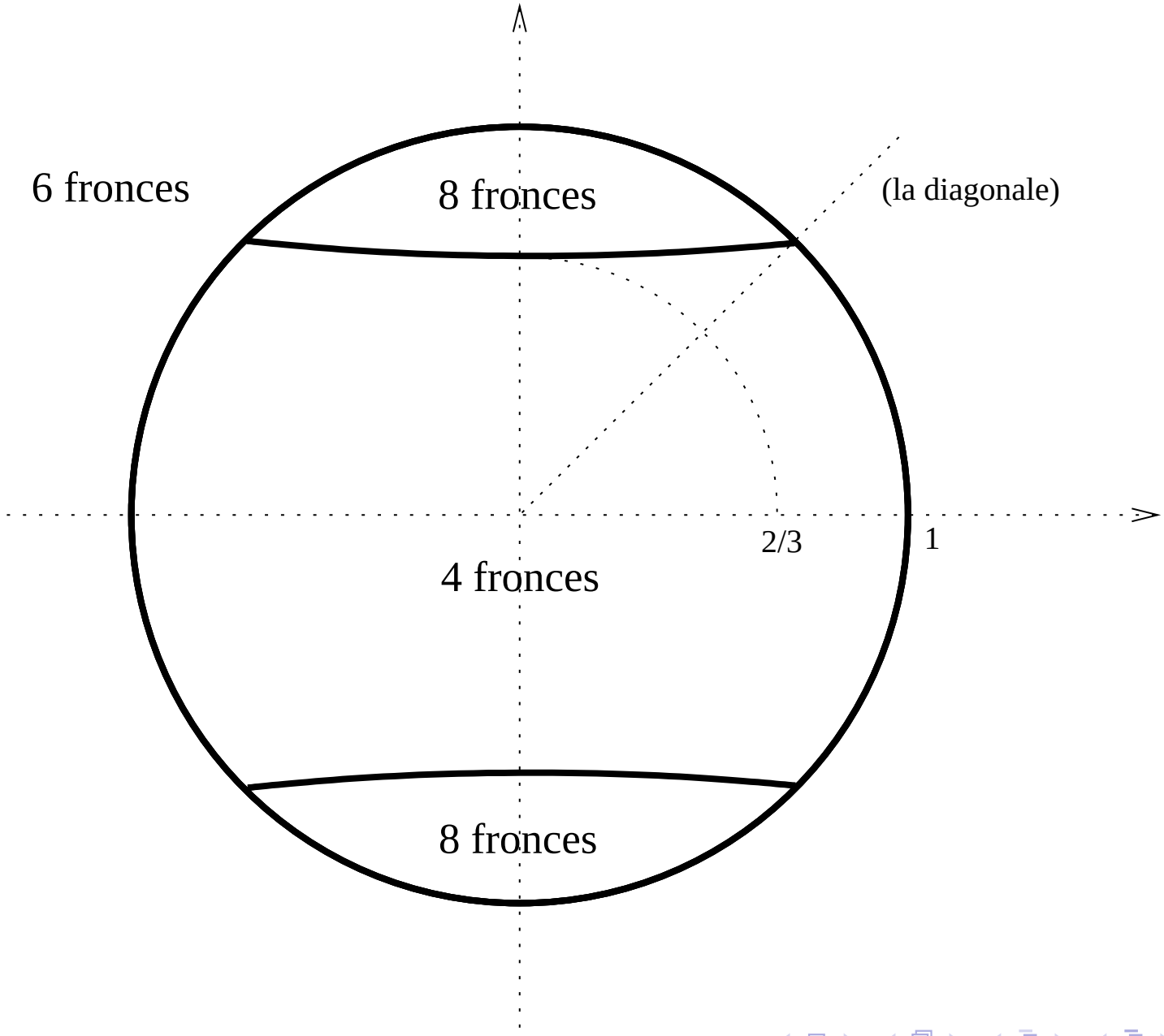
  - $\mathcal{C}_f = \begin{cases} \text{a Jordan curve} & \text{if } |p| < 1 \\ \text{a lemniscate} & \text{if } |p| = 1 \\ \partial(\Omega^1 \sqcup \Omega^2 \text{Jordan domains}) & \text{if } |p| > 1 \end{cases}$

- $|p| > 1 \Longrightarrow f|_{\overline{\Omega^k}}$ injective (behaves like $z^2 - \overline{z}$), and

  - $\#f^{-1}(c) = \begin{cases} 7 & | \ c \in f(\Omega^1) \cap f(\Omega^2) \\ 5 & | \ c \in f(\Omega^1) \triangle f(\Omega^2) \\ 3 & | \ c \in \overline{\mathbb{C}} \setminus f(\Omega^1) \cup f(\Omega^2) \end{cases}$

- $\#\{$Whitney's cusp singularities$\}$=8,4,6

p=plane, Location of swallowtail, butterfly bifurcations ...



Le plan de p est decoupe en 4 zones, dont 2 avec 8 fronces

Theorem 7.

$\mathcal{C}_f$ bounded	$\Longrightarrow$	f proper
$f\big\vert_{\mathcal{C}_f}$ nowhere loc. constant		
$\Updownarrow$	$\Longleftarrow$	f proper
every fiber $f^{-1}(c)$ is finite.		

Theorem 8. Convexity...