Local properties of the Mandelbrot set at parabolic points

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Abstract

We formulate the technique of parabolic implosion into an easy-to-use result: Orbit correspondence, and apply it to show that for $c_0$ a primitive parabolic point, the Mandelbrot set $M$ outside the wake of $c_0$ is locally connected at $c_0$. This, combined with known results inside the wake, shows that $M$ is locally connected at $c_0$. The appendices contain sketches of relative results and their proofs.

1 Introduction

Denote by $Q_c$ the map $z \mapsto z^2 + c$. The Mandelbrot set $M$ is defined to be the set of $c$ such that $\lim_{n \to \infty} Q^n_c(0) \not\to \infty$.

We say that $c_0$ is a primitive (resp. non-primitive) parabolic point if $Q_{c_0}$ has a periodic point of multiplier 1 (resp. $e^{2\pi i \theta}$ with $\theta \in \mathbb{Q} \setminus \mathbb{Z}$).

For precise definition of the following notations, see §3.2. Denote by $\varphi_M : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus M$ the conformal isomorphism fixing $\infty$ and tangent to the identity there. If two $M$-rays $R_M(\theta^\pm)$ land at the same point $c_0$ we denote by $\text{wake}(\theta^\pm)$ the component of $\mathbb{C} \setminus R_M(\theta^+) \cup R_M(\theta^-) \cup \{c_0\}$ which does not contain 0. Denote by $R_{c_0}(\theta)$ the external ray of $Q_{c_0}$ of angle $\theta$.

Theorem 1.1 Let $c_0 \neq 1/4$ be a primitive parabolic point. More precisely $Q_{c_0}$ has a $k$-periodic point of multiplier 1, with $k > 1$.

a) (landing property) For the two angles $\theta^\pm$ for which $R_{c_0}(\theta^\pm)$ land at a common parabolic point $z_0$ and are adjacent to the Fatou component containing $c_0$, the $M$-rays $R_M(\theta^\pm)$ land at $c_0$.

b) (parametrisation) A germ of $R_M(\theta^\pm)$ can be written as $\bigcup_{n \geq n_0} c_{n^\pm}(0, 1)$ for some $n_0 > 0$ satisfying: $c_n$ injective, $c_{n+1}^\pm(0) = c_n^\pm(1)$, and, for $T(z) = z^2$ and some $C, C' > 0$,

$$\varphi_M \circ T^k \circ \varphi_M^{-1}(c_n^\pm(t)) = c_{n-1}^\pm(t) \quad \text{and} \quad \frac{C'}{n^2} \leq |c_n^\pm(t) - c_0| \leq \frac{C}{n^2}. \quad (1)$$

c) (transversality) The local solution of $Q_c^k(z) - z = 0$ can be written as $(c_0 + v^2, g_{\pm}(v))$ with $g_{\pm}$ holomorphic in $v$, $g_{\pm}(0) = z_0$, $g'_{\pm}(0) \neq 0$. Moreover the multiplier map $\lambda_{\pm}(v) = (Q_{c_0 + v^2}^k)'(g_{\pm}(v))$ satisfies $\lambda_{\pm}(0) = 1$ and $\lambda'_{\pm}(0) \neq 0$.

d) (local connectivity) The set $M \setminus \text{wake}(\theta^\pm)$ is locally connected at $c_0$. 

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e) (no other rays involved) For any other ray $R_M(\zeta)$ outside of the wake, the impression of its corresponding prime end does not contain $c_0$ (in particular the ray does not land at $c_0$) and $M \setminus (\text{wake}(\theta^\pm) \cup \{c_0\})$ is connected.

The last statement says that $M$ has no ghost umbilical cords at $c_0$, in the terminology of D. Sørensen. We will formulate techniques of parabolic implosion into a parabolic orbit correspondence result and then apply it to construct a sequence of puzzle pieces $P_n$ in $\mathbb{C} \setminus \text{wake}(\theta^\pm)$, with $c_0$ on the boundary, such that $\varphi_M \circ T^k \circ \varphi_M(\partial P_n) = \partial P_{n-1}$ and $\text{diam}(P_n) \approx 1/n^2$.

Together with known results inside the wake, see [Hu, Mi, Sø] and Appendix D, we can conclude that $M$ is locally connected at every primitive parabolic point.

A similar statement for non-primitive parabolic points is included in [DH2], [Mi] and [Sø]. In Appendix A and D we give a short account of these results, together with interesting consequences.

Our treatment follows very closely the original approach of Douady and Hubbard ([DH2]), who invented the theory of parabolic implosion while proving landing properties of $M$-rays.

J. Hubbard in [Hu, Theorem 14.6] (see also Appendix E in the present paper) sketched a proof of the local connectivity of $M$ at a primitive parabolic point, using the theory of Mandelbrot-like families together with the fact that $M$ is locally connected at the cusp $c = 1/4$. This proof is less elementary than the one in the present paper, but has an advantage that it turns the local connectivity property at a cusp into a semi-direct consequence of the existence of sub-Mandelbrot-like families (semi-direct means that extra combinatorial information is needed), and thus can be adopted to more general settings.

We show in Appendix B that the similar statement for repelling orbit correspondence can be applied to show that $M$ is locally connected at Misiurewicz points (i.e. $c$ values such that $Q_c(0)$ is strictly preperiodic).

For a more combinatorial approach of landing properties and local connectivity of $M$ at various points, see [Sc1] and [Sc2].

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2 Orbit correspondence in parabolic implosion

We will use the terminology and results in [S1].

**Definition.** By a *Fatou coordinate*, associated with a rational map $f$ on a region $U$, we will mean a univalent map $\Phi : U \to \mathbb{C}$ which satisfies the
following condition for every $z \in U$,

$$f(z) \in U \iff \Phi(z) + 1 \in \Phi(U) \iff \Phi(f(z)) = \Phi(z) + 1,$$

and moreover satisfies the convexity condition that whenever both $w$ and $w + n$ belong to $\Phi(U)$ the intermediate points $w + 1, w + 2, \ldots, w + (n - 1)$ must also belong to $\Phi(U)$. Note that if $\Phi$ exists $f$ is also univalent in $U$.

**The set up.** Fix $r > 0$ small. Let $\Delta_r \subset \mathbb{C}$ be a bounded connected open set with $0$ on the boundary, $U$ an open neighbourhood of $0$ and $f_s(z)$ be a family of holomorphic maps on $U$ satisfying:

1. $(s, z) \mapsto f_s(z)$ is well defined, continuous on $\overline{\Delta_r} \times U$ and holomorphic in $z$, with $f_s(z) = \lambda(s)z + O(z^2)$;
2. $(s, z) \mapsto f_s(z)$ is holomorphic on $\Delta_r \times U$;
3. $f_s''(0) \neq 0$;
4. Set $\sigma(s) = (\lambda(s) - 1)/(2\pi i)$. Then $\sigma$ is continuous on $\overline{\Delta_r}$ with $\sigma(0) = 0$, and $\sigma$ maps $\Delta_r$ univalently onto $\{z \mid |z| < r, \arg z < \pi/4\}$.

**Remark.** In practice $f_s(z)$ and $\sigma(s)$ are often defined in a larger sector neighbourhood of $s = 0$ than $\Delta_r$. But we are only interested in the subsector of parameters which is mapped by $\sigma(s)$ bijectively onto $\{z \mid |z| < r, \arg z < \pi/4\}$. Here the width $\pi/4$ is not really relevant, it can be any angle in the interval $(0, \pi)$.

**Theorem 2.1 (Douady-Lavaurs-Shishikura):** If $r$ is sufficiently small, then there exist the followings (see Figure 1):

For $s = 0$ there are open Jordan domains $\Omega_{-0}$ and $\Omega_{+0}$ which are respectively an attracting and a repelling petal for $f_0$, satisfying $\overline{\Omega_{-0}} \cap \Omega_{+0} = \{0\}$, with associated Fatou coordinates $\Phi_{\pm, 0} : \Omega_{\pm, 0} \to \mathbb{C}$.

For $s \in \overline{\Delta_r} \setminus \{0\}$ there is a Jordan domain $\Omega_s$ containing two fixed points for $f_s$ on its boundary and two Fatou coordinates $\Phi_{\pm, s} : \Omega_s \to \mathbb{C}$ with

$$\Phi_{-, s}(z) - \frac{1}{\sigma(s)} = \Phi_{+, s}(z). \tag{2}$$

Taking $\Omega_0$ to be $\Omega_{-0} \cup \Omega_{+0}$, both the closure $\overline{\Omega_s}$ and the complement $\mathbb{C} \setminus \Omega_s$ depend continuously on $s \in \overline{\Delta_r}$, using the Hausdorff metric for the space of compact subsets of the Riemann sphere, and the correspondence $(s, z) \mapsto \Phi_{\pm, s}(z)$ is continuous in both variables for either choice of sign, wherever it is defined. Moreover $\Phi_{\pm, s}(z)$ is holomorphic in $s$ for $s \in \Delta_r$.

**Proof.** Let $\alpha(s) \in \mathbb{C}$ be defined so that $\lambda(s) = e^{2\pi i \alpha(s)}$ and $|\Re(\alpha(s))| < 1/2$. This theorem was proved in [S1], Proposition 3.2.2 and Appendix, with $\frac{1}{\alpha(s)}$ in place of $\frac{1}{\sigma(s)}$, and with $|\arg(\alpha(s))| < \pi/4 + \varepsilon$ in place of $s \in \Delta_r$. For $s \in \Delta_r$,
we have \(|\arg \alpha(s)| \approx |\arg \sigma(s)| < \frac{\pi}{4}\). Using the Taylor series expansion of \(e^x\) one can see easily that the difference \(\frac{1}{\sigma(s)} - \frac{1}{\alpha(s)}\) is continuous on \(\Delta_r\). Since the Fatou coordinates are defined up to addition of constants (relative to \(z\)), one can add the above difference to \(\Phi_{-,s}\) in Shishikura’s normalisation without alerting the continuous dependence on \((s, z)\). Therefore (2) holds together with the rest part of the theorem.

Using this result, we can restate and prove the “Tour de Valse” theorem of Douady and Sentenac ([DH2], exposé n°XI) as follows.

**Figure 1** : Parabolic orbit correspondence

**Proposition 2.2 (Parabolic orbit correspondence).** Let \(X \subset \Omega_{-,0}\) and \(Y \subset \Omega_{+,0}\) be compact sets. If \(r\) is sufficiently small, then given continuous functions \(x: \Delta_r \times [0, 1] \to X\) and \(y: \Delta_r \times [0, 1] \to Y\), holomorphic in \(s\) for \(s \in \Delta_r\), the equations

\[
  f^n_s(x(s,t)) = y(s,t) \quad \text{and} \quad \Phi_{-,s}(x(s,t)) + n - \frac{1}{\sigma(s)} = \Phi_{+,s}(y(s,t)) \quad (3)
\]

have at least one common solution \(s = s_n(t)\) in \(\Delta_r\), which depends continuously on \(t\), provided that \(n\) is sufficiently large. Moreover

\[
  \lim_{n \to \infty} n - \frac{1}{\sigma(s_n(t))} = C_0(t) \quad \text{and} \quad \lim_{n \to \infty} \left| 1 - \frac{1}{n\sigma(s_n(t))} \right| = 0 \quad \text{as} \quad n \to \infty \quad (4)
\]

where \(C_0(t)\) is finite and continuous on \(t\), and the convergences are uniform on \(t\) (see Figure 1).

Before giving the proof we give a quick application showing that the Julia sets move discontinuously:

**Application I.** Let \(f_s(z) = (1+s)z + z^2\). For \(\Delta_r\) a sector neighbourhood of 0 containing the direction of \(i\mathbb{R}\) the family satisfies all the required conditions.

The Julia set \(J(f_0)\) of \(f_0\) is a cauliflower (see the left picture in Figure 1 of [S1], in this volume). Let \(x\) be any point in \(\Omega_{-,0}\).

So \(x \notin J(f_0)\). We will show that there is a sequence \(s_n \in \Delta_r\) converging to 0 such that \(x \in J(f_{s_n})\) for all \(n\). It follows that for any converging subsequence \(J(f_{s_{n_k}}), \lim J(f_{s_{n_k}}) \neq J(f_0)\).

In order to find \(s_n\), we define at first an appropriate function \(y(s)\). On \(J(f_0)\) there is a repelling periodic cycle of period two. Let \(y(0)\) be a preimage of this cycle such that \(y(0) \in \Omega_{+,0}\). For any \(s\) near 0, there is a unique
$y(s)$ which is holomorphic on $s$ and is mapped by some iterates of $f_s$ onto a repelling periodic cycle of period two. Clearly $y(s) \in J(f_s)$.

Now we can apply Proposition 2.2 for $x(s) \equiv x$ to conclude that there is a sequence $s_n \in \Delta_r$, $s_n \to 0$, such that $f^n_s(x) = y(s_n)$. As a consequence $x \in J(f_{s_n})$ for every large $n$.

**Proof of Proposition 2.2.** First note that we only need to find a solution for the right equation in (3), since combining it with (2) we get $\Phi_{+s}(x(s,t)) + n = \Phi_{+s}(y(s,t))$. By definition of Fatou coordinates we have $\Phi_{+s}(f^n_s(x(s,t))) = \Phi_{+s}(x(s,t)) + n = \Phi_{+s}(y(s,t))$. Since $\Phi_{+s}$ is univalent, we have $f^n_s(x(s,t)) = y(s,t)$. This is the left equation in (3).

To simplify the situation, assume at first $x(s,t) = x(s)$ and $y(s,t) = y(s)$ (i.e. they are independent of $t$). Set $\bar{x}(s) = \Phi_{-s}(x(s))$ and $\bar{y}(s) = \Phi_{+s}(y(s))$. Then $\bar{x}(s)$ and $\bar{y}(s)$ are continuous in $\Delta_r$. For $n$ large, we are seeking for solutions of the following:

$$\bar{x}(s) + n - \frac{1}{\sigma(s)} = \bar{y}(s). \quad (5)$$

Let $\bar{x}(s) - \bar{y}(s) = C_0 + H(s)$, with $C_0 = \bar{x}(0) - \bar{y}(0)$, $H(s)$ continuous in $\Delta_r$ and $H(0) = 0$. We may assume $r$ is small so that $|H(s)|_{\Delta_r} \leq 1/4$.

Set $F(s) = C_0 + n - \frac{1}{\sigma(s)} = \frac{1}{z_n} - \frac{1}{\sigma(s)}$ with $z_n = 1/(C_0 + n)$, and $G(s) = F(s) + H(s)$. Then (5) becomes $G(s) = 0$. We will at first find a solution for $F(s) = 0$ and then use it to approximate a solution of $G = 0$.

For $n$ large enough,

$$z_n \in \overline{D}(z_n, |z_n|^2) \subset \{ z \mid |z| < r, \ |\arg(z)| < \frac{\pi}{4} \}.$$ 

Let $\overline{D} = \sigma^{-1}(\overline{D}(z_n, |z_n|^2)) \subset \Delta_r$. By univalence of $\sigma$, the set $\overline{D}$ is a closed topological disc. Moreover there is a unique $s_n' \in D$ such that $\sigma(s_n') = z_n$, i.e. $F(s_n') = 0$. Now

$$|F(s)|_{\partial D} = \left| \frac{1}{z_n} - \frac{1}{\sigma(s)} \right|_{\partial D} = \left| \frac{\sigma(s) - z_n}{z_n \sigma(s)} \right|_{\partial D} \geq \frac{|z_n|^2}{2|z_n|^2} \geq \frac{1}{2} \quad \text{and}$$

$$|F(s) - G(s)|_{\partial D} = |H(s)|_{\partial D} \leq \frac{1}{4}.$$ 

Therefore $|F - G|_{\partial D} < |F|_{\partial D}$. By Rouché’s theorem we conclude that there is a unique $s_n \in D$ such that $G(s_n) = 0$. (Here is another argument showing the existence of $s_n$: We have $|\arg(F(s)) - \arg(G(s))|_{\partial D} < \pi/4$. Since $\arg(F)|_{\partial D}$ is of degree one so is $\arg(G)|_{\partial D}$. Now $G$ must have at least one zero in $D$ for otherwise $\arg(G(s))$ would extend to a continuous, homotopically non-trivial map from $\overline{D}$ to $\mathbb{R}/2\pi\mathbb{Z}$. This is impossible.)
Therefore $s_n$ is a common solution of the equations in (5) and (3). Furthermore, by definition of $C_0$ and (5),

$$\lim_{n \to \infty} n - \frac{1}{\sigma(s_n)} = C_0.$$  

This is part of (4), the other part of (4) easily follows.

Finally assume that $x(s,t)$ and $y(s,t)$ do depend on $t$. Set as above $\tilde{x}(s,t) = \Phi_{-s}(x(s,t))$ and $\tilde{y}(s,t) = \Phi_{+s}(y(s,t))$. Then $\tilde{x}(s,t) - \tilde{y}(s,t) = C_0(t) + H(s,t)$, with $C_0(t) = \tilde{x}(0,t) - \tilde{y}(0,t)$, $H(s,t)$ continuous on $\overline{\Delta_r} \times [0,1]$ and $H(0,t) = 0$. By uniform continuity of $H(s,t)$ we may choose $r$ small such that $|H(s,t)| \leq 1/4$ on $\overline{\Delta_r} \times [0,1]$. The same argument above would find a unique $s_n(t)$ as zero of $G(s,t)$. This implies that $s_n(t)$ is continuous in $t$. ♦

## 3 Proof of Theorem 1.1

In this section let $(c_0, z_0)$ be such that $z_0$ is a $k$-periodic point ($k > 1$) of $Q_{c_0}$ with multiplier 1 and it is on the boundary of the Fatou component of $Q_{c_0}$ containing $c_0$. We would like to find an appropriate family $f_s(z)$ and two appropriate functions $x(s,t)$ and $y(s,t)$ in order to apply Proposition 2.2.

### 3.1 The family $f_s$ relative to $\vec{w}$

We will at first define holomorphic functions $g_{\pm}(v), \sigma_{\pm}(v)$ for $v = \sqrt{c - c_0}$, define a holomorphic map $I(c)$, choose a vector $\vec{w}$ with certain properties and then define a family of maps $f_s$ relative to $\vec{w}$ satisfying the set up of §2.

**Lemma 3.1** Set $F(c, z) = Q^k_{c_0}(z) - z$. Then $(Q^k_{c_0})''(z_0) \neq 0$ and $F(c, z) = 0$ has two local solutions which can be written as $(c_0 + v^2, g_{\pm}(v))$ with $g_{\pm}(v)$ holomorphic.

**Proof.** By assumption $Q^k_{c_0}$ is the first return map about $z_0$ with derivative 1 at $z_0$. It follows that each parabolic basin attached to the point $z_0$ is fixed by $Q^k_{c_0}$. In other words, different parabolic basins attached to $z_0$ belong to different orbits under the iteration of $Q^k_{c_0}$. On the other hand, $Q^k_{c_0}$ has only one critical point so there can be at most one periodic cycle of parabolic basins. It follows that there is exactly one such basin attached to $z_0$. As a consequence $(Q^k_{c_0})''(z_0) \neq 0$ (by local theory of Fatou flowers).

Now consider $F(c, z)$. We have

$$F(c_0, z_0) = 0, \quad \frac{\partial F}{\partial z} \bigg|_{(c_0, z_0)} = 0, \quad \frac{\partial^2 F}{\partial^2 z} \bigg|_{(c_0, z_0)} = (Q^k_{c_0})''(z_0) \neq 0.$$
By Weierstrass preparation theorem the local solutions of \( F(c, z) = 0 \) coincide with the solution of
\[
(z - z_0)^2 + (c - c_0) \cdot A(c - c_0) \cdot (z - z_0) + (c - c_0) \cdot B(c - c_0) = 0 \tag{6}
\]
where \( A(*) , B(*) \) are holomorphic functions. Moreover the analytic set \( \{ c \mid F(c, z) \) has multiple solutions\} can not have dimension one, so it has dimension zero, hence is equal to the singleton \( \{ c_0 \} \) in a neighbourhood of \( c_0 \).

Now replace \( c - c_0 \) by \( v^2 \) in (6), we get
\[
(z - z_0)^2 + v^2 A(v^2) \cdot (z - z_0) + v^2 B(v^2) = 0 .
\]
As the discriminant is not constantly 0, it is in the form \( v^{2m} H(v^2) \) with \( H(0) \neq 0 \). Therefore \( \sqrt{H(v^2)} \) is a locally well defined holomorphic function. So the solutions of (6) (and of \( F(c, z) = 0 \)) are the two holomorphic functions
\[
g_\pm(v) = -v^2 A(v^2) \pm v^m \sqrt{H(v^2)}. \tag{We will show later that \( g'_\pm(0) \neq 0 \) therefore \( m = 1 \).}
\]
Set
\[
\sigma_\pm(v) = \frac{\left(Q^k_{c_0+v^2}\right)'(g_\pm(v)) - 1}{2\pi i}.
\]

**Lemma 3.2** For \( c = c_0 + v^2 \) set \( I(c) = \sigma_+(v)\sigma_-(v) \). The correspondence \( c \mapsto I(c) \) is holomorphic throughout a neighbourhood of \( c_0 \) with \( I(c_0) = 0 \).

Furthermore, given a direction \( \vec{w} \) such that \( I(c_0 + hw)/|I(c_0 + hw)| \to -1 \) as \( h \to 0 \), and a small number \( r > 0 \), there is an open connected set \( \Delta_r \) containing \( \{ hw, h > 0 \text{ small} \} \) and with 0 on the boundary, such that \( \sigma_+ \) maps a choice \( \Delta \) of \( \sqrt{\Delta} \) univalently onto \( \{ z \mid |z| < r, |\text{arg}(z)| < \pi/4 \} \), and maps \( \{ h\sqrt{\vec{w}}, h > 0 \text{ small} \} \) onto an arc tangent to \( \mathbb{R}^+ \) at 0 (see Figure 2).

**Proof.** Since \( z - Q^k_{c_0+v^2}(z) = 0 \) has only two solutions \( g_\pm(v) \), we have (by residue theorem)
\[
\int_{|z-z_0|=t} \frac{1}{Q^k_{c_0+v^2}(z) - z} dz = \frac{2\pi i}{(Q^k_{c_0+v^2})'(g_+(v)) - 1} + \frac{2\pi i}{(Q^k_{c_0+v^2})'(g_-(v)) - 1} = \frac{1}{\sigma_+(v)} + \frac{1}{\sigma_-(v)}.
\]
But the left hand side is holomorphic on \( v \) and has finite value at \( v = 0 \). So
\[
\lim_{v \to 0} \frac{1}{\sigma_+(v)} + \frac{1}{\sigma_-(v)} \text{ is finite.} \tag{7}
\]
Set \( \sigma_\pm(v) = r_\pm(v)e^{i\eta_\pm(v)} \). Then (7) gives \( \eta_-(v) - \eta_+(v) \to \pi \) as \( v \to 0 \) (exchange \( g_+ \) and \( g_- \) if necessary).
Now we turn to $I(c)$. As a symmetric function of the roots of (6), $I(c)$ is a function of the coefficients of (6). So $c \mapsto I(c)$ is holomorphic in a neighbourhood of $c_0$. Clearly $I(c_0) = 0$. It has then a power series expansion $I(c_0 + s) = as^\nu + \cdots$ with $\nu \geq 1$. Thus there are $\nu$ distinct directions $\vec{w}$ such that $I(c_0 + hw)/|I(c_0 + hw)| \to -1$ as $h \searrow 0$.

\begin{equation}
\begin{aligned}
\text{Figure 2 : Finding $\vec{w}$, $\Delta$ and $\Delta_r$, in the case $\sigma'_+(0) \neq 0$}
\end{aligned}
\end{equation}

Fix a choice of $w$ and a choice of $\sqrt{w}$. Let $v = v(h) = h \cdot \sqrt{w}$. Then $I(c_0 + v^2) = \sigma_+(v)\sigma_-(v)$ and $\frac{|v|}{|v|} = e^{i(\eta_+(v) + \eta_-(v))}$. So $\eta_+(v) + \eta_-(v) \to \pi$ as $h \searrow 0$.

Combining the two equations on $\eta_\pm(v)$ we conclude that, as $v = h\sqrt{w}$, $\lim_{h \searrow 0} \eta_+(v)$ exists and is equal to 0. In other words, $\sigma_+(v)$ maps $\{h\sqrt{w}, h > 0 \text{ small}\}$ onto an arc tangent to $\mathbb{R}^+$ at 0. (Similarly $\sigma_-(v)$ maps it onto an arc tangent to $\mathbb{R}^-$ at 0.)

As a holomorphic function fixing 0, $\sigma_+(v) = b(v)^p$ in a neighbourhood of 0, with $b$ univalent, $b(0) = 0$ and $p > 0$ some integer. There is therefore a (almost) sector neighbourhood $\Delta$ of 0 containing the direction of $\sqrt{w}$ which is mapped by $\sigma_+(v)$ univalently onto $\{z \mid |z| < r, |\arg(z)| < \pi/4\}$. Since $\Delta$ has an opening angle close to $\pi/(4p) \leq \pi/4$, the map $v^2$ is univalent on $\Delta$, whose image will be the required set $\Delta_r$.

**Definition of $f_s$ relative to $\vec{w}$.** Fix a choice of $\vec{w}$ such that $I(c_0 + hw)/|I(c_0 + hw)| \to -1$ as $h \searrow 0$. Lemma 3.2 provides two sets $\overline{\Delta}$ and $\overline{\Delta}_r$. For $v \in \overline{\Delta}$ and $s = v^2 \in \overline{\Delta}_r$, set $g(s) = g_+(v)$ and define $f_s(z)$ to be $Q^k_{c_0+s}(z + g(s)) - g(s)$.

**Corollary 3.3** The family $f_s$ satisfies the hypothesis in the set up of §2 with $(s, \sigma(s))$ parametrised by $(v^2, \sigma_+(v))$.

**Proof.** The conditions 1 and 2 about regularities of $f_s(z)$ are easily verified, with $\lambda(s) = f_s'(0) = (Q^k_{c_0+s})'(g(s)) = (Q^k_{c_0+s})'(g_+(v))$. So $\sigma(s) := \frac{\lambda(s) - 1}{2\pi i} =$
σν(v). For condition 3, we have \( f''_0(0) = (Q^{k}_{c_0})''(z_0) \neq 0 \) (Lemma 3.1). Condition 4 follows from Lemma 3.2.

### 3.2 The angles \( \theta^\pm, \eta^\pm \) and the functions \( x(s,t), y(s,t) \)

Here we need to understand more about the dynamics of \( Q_{c_0} \) and of \( Q_c \) for nearby \( c \). We will at first recall the classic theory of Riemann representations, external rays, state two results about the dynamics of \( Q_{c_0} \), and then use perturbations to construct \( x(s,t) \) and \( y(s,t) \).

We consider a Model plane as a complex plane together with the action of \( q(= Q_0) : z \mapsto z^2 \). This plane is used to give (external) coordinates in the dynamical plane of \( Q_c \) as well as in the parameter plane in the following sense:

For \( c \in \mathbb{C} \) define \( \varphi_c \) from a subset of the model plane onto a subset of the dynamical plane of \( Q_c \) such that \( \varphi_c \) conjugates \( q \) to \( Q_c \), fixes \( \infty \) and is tangent to the identity there (this is often called the Böttcher coordinates). It is known that the maximal domain of definition of \( \varphi_c \) is \( \mathbb{C} \setminus X_c \), with \( X_c \) a bounded star-like compact set containing the closed unit disc. Moreover, if \( c \in M \), the set \( X_c \) coincides with the closed unit disc, and if \( c \notin M \), there is \( z_c \in \mathbb{C} \setminus X_c \) such that \( \varphi_c(z_c) = c \).

Define \( \varphi_M \) from \( \{ |z| > 1 \} \) (a subset of the model plane) onto \( \mathbb{C} \setminus M \) (a subset of the parameter plane) by \( \varphi_M(z_c) = c \). Due to a result of Douady-Hubbard, and independently Sibony, we know that \( \varphi_M \) is a well defined conformal isomorphism, fixing \( \infty \) and tangent to the identity there.

The rays \( R_M(\theta), R_c(\theta) \) are defined to be \( \varphi_M(\{ e^{\mu+2\pi i\theta} \mid 0 < \mu \}) \) and \( \varphi_c(\{ e^{\mu+2\pi i\theta} \mid 0 < \mu \}) \) respectively. Such a ray is said to land at a point \( z \) if the limit, as \( \mu \searrow 0 \), exists and is equal to \( z \). We say also that \( z \) has \( \theta \) as external angle.

The great importance of these rays are:
1. they are preserved by the dynamics;
2. when two or more rays land at the same point, they provide a clean cut of the plane, and, together with equipotentials, provide puzzle pieces which are candidates of local connected neighbourhoods;
3. they often depend analytically on the parameter.

Concerning our polynomial \( Q_{c_0} \), we have the following known result:

**Lemma 3.4 (Douady-Hubbard)** The Julia set of \( Q_{c_0} \) is locally connected. The convex hull of the critical orbit forms a topologically finite tree (Hubbard tree) with the critical value \( c_0 \) as an extremity. The point \( z_0 \), as the unique parabolic point on the boundary of the Fatou component \( U \) containing \( c_0 \), has two external angles adjacent to \( U \). These angles are denoted by \( \theta^\pm \) (with \( \theta^+ \in (0,1) \) and \( \theta^- < \theta^+ \)). The period of them by angle doubling is equal to \( \text{per}(z_0) \).
For a proof, see [DH2], pages 51-52. From this one derives easily

**Lemma 3.5** In the same setting, there is a sequence of prerepelling (i.e. eventually periodic repelling) points \( z_n \) in the Hubbard tree such that \( z_n \to z_0 \), \( z_n \) has two external angles \( \eta_n^+ \) with \( \eta_n \not\to \theta^- \), \( \eta_n \not\to \theta^+ \), as \( n \to \infty \), and, for \( k \) the period of \( z_0 \), \( Q_{c_0}^k(z_n) = z_{n-1} \), \( 2^k \theta^+ = \theta^+ \) (mod 1) and \( 2^k \theta^- = \theta^- \) (mod 1).

**Proof.** In the segment \([\alpha, c_0] \) of the Hubbard tree (where \( \alpha \) is the fixed point of \( Q_{c_0} \) with a non-zero rotation number), choose a point \( x \) so that in \([x, z_0] \setminus \{z_0\} \) there is no branching points of the tree (this is possible since there are only finitely many branching points), and no points in the basin of attraction of the parabolic orbit (this is possible as we have assumed that \( c_0 \) is primitive). By expansion property on the Julia set, there is a minimal \( N \) such that \( Q_{c_0}^N \) restricted to \([x, z_0] \) is no more injective. This implies that there is \( z' \in [x, z_0] \) such that \( Q_{c_0}^{N-1}(z') = 0 \) and \( Q_{c_0}^N(z') = c_0 \). By minimality of \( N \), \( Q_{c_0}^N \) restricted to \([z', z_0] \) is a homeomorphism onto its image. As \( Q_{c_0}^N(z_0) \) is a point in the parabolic orbit, distinct from \( z_0 \), and is on the Hubbard tree, by the choice of \( x \) we have \([Q_{c_0}^N(z_0), z_0] \supset [x, z_0] \supset [z', z_0] \). Therefore

\[
Q_{c_0}^N([z_0, z']) = [Q_{c_0}^N(z_0), Q_{c_0}^N(z')] = [Q_{c_0}^N(z_0), c_0] \supset [z', z_0].
\]

As \( Q_{c_0}^N \) reverses the orientation, there is a point \( z_1 \in [z', z_0] \) fixed by \( Q_{c_0}^N \), i.e. \( z_1 \) is a periodic point. It must be repelling as it is not in the parabolic orbit. It must have exactly two external rays, as it is in the Hubbard tree, and is not a branching or extremal point. By expansion again the local inverse branch \( g \) of \( Q_{c_0}^N \) mapping \( z_0 \) to \( z_0 \) maps \([z_1, z_0] \) into itself. Denote by \( z_n \) the orbit of \( z_1 \) under \( g \). They satisfy the desired property of the Lemma. See Figure 3. □

**Figure 3** : The segment \([Q_{c_0}^N(z_0), c_0] \) in the Hubbard tree of \( Q_{c_0} \)

A proof of a similar statement in the case that 0 is preperiodic for \( Q_c \) can be found in [T], Lemma 2.3(2), and in the case 0 is periodic for \( Q_c \) can be found in [Sc2], Lemma 5.1.

In order to define an appropriate \( y(s, t) \), we define at first \( \Gamma_0(t) \) for \( t \in [0, 1] \) in the log-model plane as follows (we have chosen deliberately \( \log e^{2\pi i \theta^+} \) to be \( 2\pi i \theta^+ - 2\pi i \) for the purpose to give a similar picture as in the dynamical plane and in the parameter plane, Figures 4, 5 and 6): Fix \( \tilde{\mu} > 0 \) small and
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When \( n_1 \) large, the map \( t \mapsto \Gamma_0(t) \) should be piecewise linear and continuous on \([0, 1/2]\) and \((1/2, 1]\), with

\[
\begin{align*}
\Gamma_0(0) &= \hat{\mu}/2^k + 2\pi i\theta^+ - 2\pi i, \quad \Gamma_0(1/6) = \hat{\mu} + 2\pi i\theta^+ - 2\pi i, \\
\Gamma_0(1/3) &= \hat{\mu} + 2\pi i\eta_{n_1}^+ - 2\pi i, \quad \Gamma_0(1/2) = 2\pi i\eta_{n_1}^- - 2\pi i, \\
\Gamma_0(1/2^+) &= 2\pi i\eta_{n_1}^-, \quad \Gamma_0(2/3) = \hat{\mu} + 2\pi i\eta_{n_1}^-, \\
\Gamma_0(5/6) &= \hat{\mu} + 2\pi i\theta^-, \quad \Gamma_0(1) = \hat{\mu}/2^k + 2\pi i\theta^-.
\end{align*}
\]

**Figure 4**: The curve \( \Gamma_0(t) \) and its direction of parametrisation

We then define \( y(0, t) = \varphi_{c_0}(e^{\Gamma_0(t)}) \). Although \( \Gamma_0 \) has a discontinuity at \( t = 1/2 \), by the choice of \( \eta_{n_1}^\pm \), the map \( y(0, t) \) is continuous everywhere and parametrises a Jordan arc.

The function \( y(0, t) \) is stable in a neighbourhood of \( c_0 \) (cf. [DH1], last chapter), in other words, \( y(s, t) = \varphi_{c_0+s}\varphi_{c_0}^{-1}(y(0, t)) = \varphi_{c_0+s}(e^{\Gamma_0(t)}) \) is well defined and is an analytic continuation of \( y(0, t) \). One may choose \( \hat{\mu} \) small enough and \( n_1 \) large enough so that \( y(s, t) \) belongs to a compact set \( Y \) in \( \Omega_{+,0} \) for all \((s, t) \in \Delta \times [0, 1]\).

Define \( x(s, t) = x(s) \) to be simply \( Q_{c_0+s}^{kl}(c_0 + s) \), where \( l \) is the minimal integer such that \( x(0) \in \Omega_{-,0} \). We may choose a small enough neighbourhood of \( c_0 \) so that \( x(s) \) belongs to a compact set \( X \) in \( \Omega_{-,0} \).

We now define \( \Gamma_n(t) \) for \( n \geq 0 \) inductively as follows: choose \( \Theta \in (\theta^-, \theta^+) \). Let \( \tau \) be the map from \( [\Theta - 2\pi i, \Theta] \) to its self mapping \( z \to 2z \) (mod \( 2\pi i\mathbb{Z} \)). Then \( 2\pi i\theta^- \) and \( 2\pi i\theta^+ - 2\pi i \) are fixed points of \( \tau^k \). Set

\[
\begin{align*}
\Gamma_n(0) &= \frac{\hat{\mu}}{2^n(k+1)} + 2\pi i\theta^+ - 2\pi i, \quad \Gamma_n(1) = \frac{\hat{\mu}}{2^n(k+1)} + 2\pi i\theta^-,
\end{align*}
\]
and define \( \Gamma_n([0,1/2]) \) (resp. \( \Gamma_n((1/2,1]) \)) to be the lift by \( \tau^{-k} \) of the arc \( \Gamma_{n-1}([0,1/2]) \) (resp. \( \Gamma_{n-1}((1/2,1]) \)) with the above initial condition. Note that
\[
\Gamma_n(1/6) = \Gamma_{n-1}(0), \quad \Gamma_n(5/6) = \Gamma_{n-1}(1),
\]
\[
\lim_{t \to \frac{1}{2}^- \Gamma_n(t) = 2\pi i \eta_{n_1+n}^+ - 2\pi i \eta_{n_1+n}^-
\]

Moreover
\[
\bigcup_{n \geq 0} \Gamma_n([0, \frac{1}{6}]) = \{ \mu + 2\pi i \theta^+ - 2\pi i \mid 0 < \mu \leq \hat{\mu} \}
\]
\[
\bigcup_{n \geq 0} \Gamma_n([\frac{5}{6}, 1]) = \{ \mu + 2\pi i \theta^- \mid 0 < \mu \leq \hat{\mu} \}.
\]

Note also that for \( q : z \mapsto z^2 \) in the model plane we have \( q^k(e^{\Gamma_n(t)}) = e^{\Gamma_{n-1}(t)} \).

\[\text{Figure 5 : The dynamical plane of } Q_{c_0+s}\]

### 3.3 Application II of Proposition 2.2 for a fixed \( \vec{w} \)

Fix a choice of \( c_0 + \vec{w} \) along which \( I(c)/|I(c)| \to -1 \), as in Lemma 3.2.

We can now apply Proposition 2.2 to the family \( f_s \) relative to \( \vec{w} \) and the functions \( x(s) - g(s) \) and \( y(s,t) - g(s) \) defined in the previous sub-sections. There exists then \( n_0 > 0 \) and, for every \( n \geq n_0 \), \( t \in [0,1] \), a point \( s_n(t) \in \Delta_r \) such that
\[
f^n_{s_n(t)}(x(s_n(t)) - g(s_n(t))) = y(s_n(t),t) - g(s_n(t)).
\]

In other words,
\[
Q_{c_n+s_n(t)}^{kn}(x(s_n(t))) = y(s_n(t),t).
\]

**Lemma 3.6** We have \( c_n(t) := c_0 + s_n(t) = \varphi_M(e^{\Gamma_{k+s_n(t)}}) \) for \( n \geq n_0 \) and \( t \in [0,1] \).
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3.4 Conclusions

**Lemma 3.7** (= Theorem 1.1.a) The M-ray of angle \( \theta^\pm \) lands at \( c_0 \) and is tangent to \( c_0 + \tilde{w} \) at \( c_0 \), for each direction \( c_0 + \tilde{w} \) along which \( \frac{I(c)}{|I(c)|} \) converges to \(-1\).

**Proof.** Note that for \( c_n(t) \) defined in Lemma 3.6,

\[
\gamma^+ := \bigcup_{n \geq n_0} c_n([0, 1/6]) = \varphi_M(\exp(\{\mu + 2\pi i\theta^+ \mid 0 < \mu \leq \hat{\mu}\}))
\]

is a germ of \( R_M(\theta^+) \). By Lemma 3.2 and Corollary 3.3, the arc \( \sigma(\{hw, \ h > 0 \ \text{small}\}) \) is tangent to \( \mathbb{R}^+ \) at 0. On the other hand, since \( \frac{1}{n} / \sigma(s_n(t)) \to 1 \) uniformly (by (4) of Proposition 2.2), the arc \( \sigma(T_{-c_0} \gamma^+) = \sigma(\bigcup_{n \geq 0} s_n([0, 1/6])) \)
is also tangent to \( \mathbb{R}^+ \) at 0 (where \( T_{-c_0} \) denotes the translation \( z \mapsto z - c_0 \)). Therefore \( \sigma(T_{-c_0} \gamma^+) \) and \( \sigma(\{hw, \ h > 0 \ \text{small}\}) \) are tangent. So \( R_M(\theta^+) \) lands at \( c_0 \) and is tangent to \( c_0 + \vec{w} \) at \( c_0 \).

Similarly \( R_M(\theta^-) \) lands at \( c_0 \) and is tangent to \( c_0 + \vec{w} \).

**Corollary 3.8** (= Theorem 1.1.c)) The functions \( g_\pm(v) \) defined in Lemma 3.1 and their multiplier functions \( \lambda_\pm(v) \) satisfy \( (g_\pm)'(0) \neq 0 \) and \( \lambda'_\pm(0) \neq 0 \).

**Proof.** Relying on the fact that \( \varphi_M \) is a conformal isomorphism we know that there is only one \( M \)-ray of angle \( \theta^+ \). Combining with Lemma 3.7 we conclude that there is a unique \( \vec{w} \) such that \( I(c)/|I(c)| \) converges to \(-1\) along \( c_0 + \vec{w} \). It follows that \( I(c) \) is univalent.

If \( \sigma_\pm(v) \) were holomorphic on \( c \), the function \( I(c) = \sigma_+(v)\sigma_-(v) \) could not be univalent (since \( \sigma_\pm(0) = 0 \)). Therefore none of \( \sigma_\pm(v) \) is holomorphic on \( c \). It follows that none of \( g_\pm(v) \) can be holomorphic on \( c \).

Since \( I(c) \) is univalent, \( I(c_0 + v^2) \) as a function of \( v \) has a double zero at 0. It follows that each \( \sigma_\pm(v) \) has a simple zero at 0, i.e. \( \sigma'_\pm(0) \neq 0 \). It follows easily that \( \lambda'_\pm(0) \neq 0 \), as \( \lambda_\pm(v) = 1 + 2\pi i \sigma_\pm(v) \).

Finally

\[
\lambda_\pm(0) = \frac{d}{dv}(Q_{c_0+v^2}'(g_\pm(v))) \Big|_{v=0} = 0
\]

\[
\frac{\partial}{\partial c}(Q_{c_0+v^2}') \Big|_{(c_0,v^2)} \cdot \frac{d}{dv}(c_0 + v^2) \Big|_{v=0} + \frac{\partial}{\partial z}(Q_{c_0}') \Big|_{(c_0,v^2)} \cdot \frac{d}{dv}g_\pm(v) \Big|_{v=0} = (Q_{c_0}')^{\prime\prime}(z_0) \cdot g'_\pm(0)
\]

so \( g'_\pm(0) \neq 0 \).

**Corollary 3.9** There exist constants \( C,C' > 0 \) independent of \( n,t \) such that \( C'/n^2 \leq |c_n(t) - c_0| \leq C/n^2 \), for all \( t \in [0,1] \) and all \( n \geq n_0 \).

**Proof.** The curve \( (s,\sigma(s)) \) is parametrised by \( (v^2,\sigma_+(v)) \). By the left limit in (4) of Proposition 2.2 and the fact that \( C_0(t) \) is continuous on \( t \), we know that \( \sigma(s_n(t))/\frac{1}{n} \to 1 \) uniformly. Now \( \sigma(s_n(t)) = \sigma_+(v_n(t)) \), where \( v_n(t) = \sqrt{s_n(t)} \in \Delta \), and \( \sigma_+ \) is univalent by the above corollary, so \( \sigma_+(v_n(t))/(b \cdot v_n(t)) \to 1 \) (for some \( b \neq 0 \) uniformly). Therefore \( b \cdot v_n(t)/\frac{1}{n} \to 1 \) uniformly. Hence

\[
\frac{c_n(t) - c_0}{1/(bn)^2} = \frac{v_n(t)^2}{1/(bn)^2} \to 1
\]

uniformly. In particular the constants \( C,C' \) required in the Corollary exist.

**Proof of Theorem 1.1.b), d) and e).**
b) To get $c_n^+([0, 1])$ and $c_n^-([0, 1])$ for $n \geq n_0$ we just need to reparametrise $c_n([0, 1/6])$ and $c_n([5/6, 1])$ respectively. The rest follows from Lemma 3.6 and Corollary 3.9.

d) By construction and the fact that $R_M(\theta^\pm)$ land at $c_0$, for $n \geq n_0$, the set

$$c_n([1/6, 5/6]) \cup \bigcup_{m \geq n} c_m([0, 1/6]) \cup \bigcup_{m \geq n} c_m([5/6, 1]) \cup \{c_0\}$$

bounds a Jordan domain (called a puzzle piece, Figure 6). Let us denote it by $P_n$. We have $\text{diam}(P_n) \leq C/n^2$ due to the above corollary. Moreover $\partial P_n \cap M$ has only two points $c_0$ and $c_n(1/2)$. So $M \cap P_n$ is connected (this is a general fact, cf. for example Lemma 2.1 of [T]). Therefore $M \setminus \text{wake}(\theta^\pm)$ is locally connected at $c_0$ and $M \cap \text{wake}(\theta^\pm) \cup \{c_0\}$ is connected.

e) Given any $R_M(\zeta)$ outside the wake, there is $n$ such that $\zeta > \eta^+_n$ or $\zeta < \eta^-_n$. So the corresponding impression is contained in $(M \setminus \text{wake}(\eta^+_n)) \cup \{c_n(1/2)\}$, hence is disjoint from $c_0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{puzzle_piece.png}
\caption{The puzzle piece $P_n$}
\end{figure}

4 Application III of Proposition 2.2

Here is a qualitative version and the most common form of parabolic implosion:

\textbf{Corollary 4.1} In the set up of \S 2, for any $x \in \Omega_{-0}$, $y \in \Omega_{+0}$, there are $s_n \to 0$ and $k_n \to \infty$ such that $f_{k_n}^{s_n}$ converges uniformly on compact sets of $\Omega_{-0}$ to a holomorphic map $g$ with $g(x) = y$.

\textit{Proof.} Apply Proposition 2.2 to $x(s, t) \equiv x$ and $y(s, t) \equiv y$, we find a sequence $s_n \to 0$ such that $f_{s_n}^n(x) = y$. We will see that $f_{s_n}^n|_{\Omega_{-0}}$ converges uniformly on compact sets to a function $g$ satisfying $g(x) = y$ and $g \circ f_0 = f_0 \circ g$. 
Recall from the proof of Theorem 2.1 that for $\alpha(s)$ defined as $\lambda(s) = e^{2\pi i\alpha(s)}$ with $\alpha(s)$ close to 0, the function $\frac{1}{\alpha(s)} - \frac{2\pi i}{\lambda(s)-1} = \frac{1}{\alpha(s)} - \frac{1}{\sigma(s)}$ is continuous in $\Delta r$, in particular has a finite value at $s = 0$.

In [S2], 3.4, Shishikura proved that for any sequence $s_n \to 0$ such that $\frac{1}{\alpha(s_n)} = k_n + \beta_n$ with $k_n \in \mathbb{N}$, $k_n \to +\infty$ and $\beta_n \to \beta \in \mathbb{C}$, the sequence $f_{s_n}^{k_n}|_{\Omega_{-0}}$ converges uniformly to a limit function.

Now assume that $s_n$ is a solution of (5). We have

$$\frac{1}{\alpha(s_n)} = n + \tilde{x}(s_n) - \tilde{y}(s_n) + \left(\frac{1}{\alpha(s_n)} - \frac{1}{\sigma(s_n)}\right)$$

with $\tilde{x}(s_n) - \tilde{y}(s_n) \to \tilde{x}(0) - \tilde{y}(0)$ as $n \to \infty$. So the hypothesis of Shishikura’s theorem is satisfied with $k_n = n$. Therefore $f_{s_n}^n|_{\Omega_{-0}}$ converges uniformly to a limit function $g$. Clearly $g(x) = y$, $g$ is holomorphic and $g \circ f_0 = f_0 \circ g$. ■

### A Non-primitive case

Assume now that $Q_{c_0}$ is a quadratic polynomial with a $k$-periodic point of multiplier $e^{2\pi ip/q}$ with $q > 1$. In this case a similar theory of parabolic implosion can be applied to show:

**Theorem A.1 (Douady-Hubbard, [DH2])**

a) **(landing property)** For the two angles $\theta^\pm$ for which $R_{c_0}(\theta^\pm)$ land at a common parabolic point $z_0$ and are adjacent to the Fatou component containing $c_0$, the $M$-rays $R_M(\theta^\pm)$ land at $c_0$.

b) **(parametrisation)** A germ of $R_M(\theta^\pm)$ can be written as $\bigcup_{n \geq n_0} c_n^\pm([0,1])$ for some $n_0 > 0$ satisfying: $c_n$ injective, $c_{n+1}^\pm(0) = c_n^\pm(1)$, and, for $T(z) = z^2$ and some $C_1, C_2 > 0$,

$$\varphi_M \circ T^{kd} \circ \varphi_M^{-1}(c_n^\pm(t)) = c_{n-1}^\pm(t) \quad \text{and} \quad \frac{C_1}{n} \leq |c_n^\pm(t) - c_0| \leq \frac{C_2}{n}. \quad (8)$$

c) **(transversality)** The local solution of $Q_{c_0}^k(z) - z = 0$ can be written as $(c,h(c))$ with $h$ holomorphic, $h(c_0) = z_0$ and $h'(c_0) \neq 0$. Moreover the multiplier function $\lambda(c) = (Q_{c_0}^k)'(h(c))$ satisfies $\lambda(c_0) = e^{2\pi ip/q}$ and $\lambda'(c_0) \neq 0$.

The inequalities in (1) and (8) can be used to prove a result of the degree of tangency. See Appendix C.

### B Misiurewicz case

A parallel theory for repelling periodic points can be sketched as follows:
Theorem B.1 Let \( f_s(z) = \lambda(s)z + O(z^2) \) be a holomorphic family of maps such that \( |\lambda(0)| > 1 \). Then for \( s \) in a neighbourhood of 0, there are neighbourhood \( U_s \) of 0 and linearising coordinates \( \Phi_s \) such that for any \( z, n \) with \( f^j_s(z) \in U_s, \ j = 0, 1, \ldots, n \), we have
\[
\Phi_s(f^n_s(z)) = \lambda(s)^n \Phi_s(z)
\]
and \( \Phi_s(z) \) depends holomorphically on \( (s, z) \).

Proposition B.2 (Repelling orbit correspondence). In the same setting, let \( x(s), y(s) \) be two holomorphic functions with \( x(0) = 0 \), \( x(s) \not\equiv 0 \) and \( y(0) \in U_0 \setminus \{0\} \). Then there is a sequence \( s_n \to 0 \) for \( n > N \) such that
\[
f^{n}_{s_{n}}(x(s_{n})) = y(s_{n}) \quad \text{and} \quad f^{j}_{s_{n}}(x(s_{n})) \in U_{s_{n}} \quad \text{for} \quad j = 0, 1, \ldots, n.
\]
Moreover, if \( \lambda'(0) \neq 0 \), there are constants \( C_1, C_2 \) such that
\[
\frac{C_1}{|\lambda(0)|^n} < |s_n| < \frac{C_2}{|\lambda(0)|^n}.
\]

Proof. Set \( \tilde{x}(s) = \Phi_s(x(s)), \tilde{y}(s) = \Phi_s(y(s)) \). They are holomorphic functions with \( \tilde{x}(0) = 0, \tilde{x}(s) \not\equiv 0 \) and \( \tilde{y}(0) \not\equiv 0 \). Given \( n \), we seek for solution of
\[
\lambda(s)^n \tilde{x}(s) = \tilde{y}(s).
\]
With the help of power series expansions one can carry out a similar proof as the one of Proposition 2.2.

One can find extensive applications of Proposition B.2 in [DH3] and [Mc]. We present here an application to the Mandelbrot set (Parts 1) and 2) are due to Douady and Hubbard, Parts 3) and 4) are due to Tan Lei):

Theorem B.3 Let \( c_0 \) be a Misiurewicz point, more precisely there are \( k \geq 0, \ l \geq 0 \) minimal such that \( Q_{c_0}^{k+l}(0) = Q_{c_0}^l(0) \) and \( l > 0 \). Then
1) The angles of \( M \)-rays landing at \( c_0 \) coincide with the angles of \( Q_{c_0} \)-rays landing at \( c_0 \)
2) \( \frac{d}{dc}(Q_{c_0}^{k+l}(0) - Q_{c_0}^l(0))|_{c=c_0} \neq 0. \)
3) \( M \) is locally connected at \( c_0 \).
4) The impression of the prime end of any other \( M \)-ray does not contain \( c_0 \).

Details of proofs can be found in [T].

C degree

The following illustrates a quick application of the inequalities in Theorems 1.1, A.1, although the statement is not in its sharpest form.

Theorem C.1 For \( c_0 \neq 1/4 \) a primitive parabolic point, there is a unique hyperbolic component \( W \) attached to \( c_0 \), and for the two rays \( R_M(\theta^\pm) \) landing
at \( c_0 \), we have that the degree of tangency at \( c_0 \) of \( R_M(\theta^+) \) and \( R_M(\theta^-) \) is \( 2^+ \) (i.e. is larger than or equal to 2, but smaller than \( 2 + \mu \) for any \( \mu > 0 \)), as well as the degree of tangency at \( c_0 \) of \( \tilde{R}_M(\theta^i) \) and \( \partial M \), \( i = + \) or \( - \). For \( c_0 \) a non-primitive parabolic point, there are exactly two hyperbolic components \( W_0 \) and \( W_1 \) attached to \( c_0 \) and the two rays \( R_M(\theta^\pm) \) landing at \( c_0 \) separate them, moreover the degree of tangency at \( c_0 \) of \( R_M(\theta^i) \) and \( \partial W_j \) is \( 2^+ \), \( i = + \) or \( - \), \( j = 0, 1 \), as well as the degree of tangency at \( c_0 \) of \( \partial W_0 \) and \( \partial W_1 \).

**Proof.** For the existence of such hyperbolic components see Corollary D.4 below.

Here we only prove the part about degree of tangency, which is due to Tan Lei.

Assume at first \( c_0 \) is non-primitive. We need an inequality between hyperbolic metric and Euclidean metric: Let \( U \) be a simply connected domain in \( \mathbb{C} \). Then for \( x, y \in U \), we have

\[
|y - x| \leq e^{2d_U(x,y)} \cdot d(x, \partial U) ,
\]

where \( d_U \) denotes the hyperbolic metric of \( U \). For a proof, see Appendix A of [ST].

Let \( W_0, W_1 \) be the two hyperbolic components attached to \( c_0 \). We are going to show for \( n \) large enough, and \( c_n^+ = c_n^+(0) \) (in the notation of Theorem A.1),

\[
|c_n^+ - c_{n+1}^+| \leq B \cdot d(c_n^+, W_0 \cup W_1) , \quad |c_n^- - c_{n+1}^-| \leq B \cdot d(c_n^-, W_0 \cup W_1) , \quad (10)
\]

where \( B \) is a constant.

To prove this estimate, we need the relationship between \( c_n^+ \) in (8) and the help of hyperbolic metric. We will only work on the + case. The other one is similar. Take a simply connected domain \( V \) in the model plane so that \( T^{-kq}(V) \) has a component \( V' \) contained in \( V \) and that the two sets \( U = \varphi_M(V) \), and \( U' = \varphi_M(V') \) both contain \( c_n^+ \) for \( n \geq N \). Therefore, for \( V_n \) the component of \( T^{-km}(V) \) with \( e^{2n\theta^+} \) on the boundary, and \( U_n = \varphi_M(V_n) \subset U \),

\[
d_U(c_n^+, c_n^{+1}) \leq d_U(c_n^+, c_n^{+1}) = d_U(c_n^+, c_n^{+1}) = A .
\]

Now apply (9) we get

\[
|c_n^+ - c_{n+1}^+| \leq e^{2d_U(c_n^+, c_n^{+1})} \cdot d(c_n^+, \partial U) \leq B \cdot d(c_n^+, \partial U) .
\]

In particular, since \((W_0 \cup W_1) \cap U = \emptyset\),

\[
|c_n^+ - c_{n+1}^+| \leq B \cdot d(c_n^+, W_0 \cup W_1)
\]

for any large \( n \). This is (10).
Assume by contradiction that the $\theta^+$-ray is tangent to $\partial W_0$ with a degree of tangency at least $2 + \mu$, $\mu > 0$. Then

$$d(c_n^+, W_0 \cup W_1) \leq d(c_n^+, W_0) \leq C \cdot |c_n^+ - c_0|^{2+\mu}. \quad (11)$$

So

$$\frac{C_1}{n} \leq |c_n^+ - c_0| \leq \sum_{k \geq n} |c_k^+ - c_{k+1}^+| \leq B \cdot \sum_{k \geq n} d(c_k^+, W_0 \cup W_1)$$

$$\leq B' \cdot \sum_{k \geq n} |c_k^+ - c_0|^{2+\mu} \leq B'' \cdot \sum_{k \geq n} \frac{1}{k^{2+\mu}}$$

where $B', B''$ are constants, the first and the last inequalities is due to (8), the second one is trivial, the third one is due to (10) and the fourth one is due to the assumption (11). Therefore

$$\frac{C_1}{B''} \leq \sum_{k \geq n} \frac{n}{k^{2+\mu}} \leq \sum_{k \geq n} \frac{1}{k^{1+\mu}}.$$

This is a contradiction to the fact that $\sum \frac{1}{k^{1+\mu}}$ converges.

This shows that the rays have degree $2^+$ tangency to each $\partial W_0$ and $\partial W_1$. A similar calculation using (1) would show that in the primitive case, the two rays $R_M(\theta^\pm)$ are tangent to each other at $c_0$ with degree $2^+$ tangency.

## D A short account of relative results

(One can find in [Mi], [Sc1] and [Sc2] radically different approaches of results in this appendix).

Here we give a short exposition of a proof for

**Theorem D.1** For $c$ Misiurewicz or on the boundary of a hyperbolic component, $M$ is locally connected at $c$.

The part concerning Misiurewicz points is already established in Theorem B.3. We will only deal with the remaining cases.

We will sketch a proof of a stronger result (Theorem D.3) and show how to deduce from it the above result. We will also give a list of important corollaries.

For $c_0 \neq 1/4$ a parabolic point, define $\mathrm{Ang}(c_0)$ to be the set of two angles such that the corresponding dynamical rays for $Q_{c_0}$ land at the parabolic point $z_0$ on the boundary of the Fatou component containing $c$, and adjacent to this Fatou component. For $c_0 = 1/4$ we set by convention $\mathrm{Ang}(c_0) = \{0, 1\}$.

Our starting point is the following combination of Theorems 1.1.a and A.1.a, proved by parabolic implosion technique in the case $c_0 \neq 1/4$ (in the case $c_0 = 1/4$ it can be checked by hand):
Theorem D.2 Let $c_0$ be a parabolic point. Denote by $\theta^\pm$ the two angles in $\text{Ang}(c_0)$. Then the $M$-rays $R_M(\theta^\pm)$ land at $c_0$.

Now define $\text{wake}(c_0)$ to be the connected component not containing 0 of $\mathbb{C} \setminus (R_M(\theta^+) \cup R_M(\theta^-) \cup \{c_0\})$.

Given any hyperbolic component $W$ of $M$ it is quite easy to see (with the help to quasi-conformal surgery) that the multiplier $\rho(c)$ of an attracting periodic point $z(c)$ for each $c \in W$ realizes a conformal homeomorphism from $W$ onto the unit disc $\mathbb{D}$, and extends to a homeomorphism of the closure (see [DH2], pp.108-109). Denote by $\rho^{-1}(1)$ the root of $W$. The parabolic points on $\partial W$ are exactly $\rho^{-1}(e^{2\pi i q})$.

Theorem D.3 Let $c_0$ be a parabolic point with $\text{Ang}(c_0) = \{\theta^-, \theta^+\}$.

a) There is a holomorphic function $\beta_{c_0}(c)$ defined on $\text{wake}(c_0)$ such that $\beta_{c_0}(c)$ is the common landing point of $R_c(\theta^+)$ and of $R_c(\theta^-)$.

b) There is a hyperbolic component $W(c_0)$ of $M$ with root $c_0$ and $W(c_0) \subset \text{wake}(c_0)$. Moreover for the common landing point $z_0$ of $R_{c_0}(\theta^\pm)$, 

$$\lim_{c \in \text{wake}(c_0), c \to c_0} \beta(c) = z_0.$$  

c) Structure of sub-wakes of $W(c_0)$ (no ghost limbs):

$$M \cap \overline{\text{wake}(c_0)} \setminus \bigcup_{c' \in \partial W \setminus \{c_0\}, \text{parabolic}} \text{wake}(c') = \overline{W(c_0)}.$$  

(12)

d) $M$ is locally connected at $c$ for any $c \in \partial W(c_0)$ which is non-parabolic; $M \cap \overline{\text{wake}(c_0)}$ is locally connected at $c_0$; and $M \setminus \text{wake}(c')$ is locally connected at $c'$ for any parabolic $c' \in \partial W(c_0) \setminus \{c_0\}$.

Proof:

a) can be found in [Mi], Theorem 3.1. It is proved by an elementary argument. A sketch is included in Theorem D.8 below.

b) is due to a parabolic perturbation argument (in the non-imploded direction). See [Mi], section 4.

c) We will follow essentially the route of D. Sørensen ([Sø]).

At first we apply a) and b) to every parabolic point on the boundary of $W(c_0)$. Combining with the tuning algorithm (cf. for example [Ha1, §7] and [Sø]), we obtain the following: for any parabolic $c' \in \partial W(c_0) \setminus \{c_0\}$ and any $c \in \text{wake}(c')$, the common landing point $\beta_{c'}(c)$ of $R_c(\theta^\pm(c'))$ has a constant and non-trivial combinatorial rotation number $\hat{\rho}(c')$. Moreover, if we choose a converging non-constant sequence $c'_n$ of such parabolic points, $\hat{\rho}(c'_n) \to \infty$. This allows us to apply the powerful Yoccoz inequality (see for example [Hu] and [Sø]), to show that

$$\text{diam}(M \cap \overline{\text{wake}(c'_n)}) \to 0 \quad \text{as } n \to \infty.$$  

(13)
Two more ingredients are needed to end the proof: A. The parabolic points are dense in \( \partial W(c_0) \), moreover, a consequence of the tuning algorithm gives \( \sum_{c' \in \partial W(c_0) \setminus \{c_0\}, \text{parabolic}} \theta^+(c') - \theta^-(c') = \theta^+ - \theta^- \). B. Every point of \( \partial M \) is accumulated by points in the hyperbolic components (by a normal family argument, cf. Branner [Br]).

Denote by \( M' \) the difference of the left set in (12) to the right set. Assume it is not empty. Let \( c \in \partial M' \setminus \partial W(c_0) \). Part A says that there is no room left for \( M' \) to contain hyperbolic components. Therefore by Part B the point \( c \) must be the limit of some \( c_n \in \text{wake}(c_n') \), for some non-constant converging parabolic sequence \( c_n' \in \partial W(c_0) \setminus \{c_0\} \). Applying (13), we get \( c \in \partial W(c_0) \). A contradiction.

d) This is a trivial topological consequence of c): Let \( c \in \partial W(c_0) \) be non-parabolic. Choose \( c_n', c''_n \) two sequences of parabolic points on \( \partial W(c_0) \) tending to \( c \) from opposite sides, with \( \theta^+(c_n') < \theta^-(c''_n) \). Then the short arc in \( \partial W(c_0) \) connecting \( c_n' \) and \( c''_n \) together with the two \( M \)-rays \( R_M(\theta^+(c_n')) \), \( R_M(\theta^-(c''_n)) \) bounds a simply connected region \( U_n \). Let \( M_n = M \cap U_n \). Then \( \{M_n\} \) is a nested sequence of connected neighbourhoods in \( \partial M \) of \( c \). As \( \bigcap_n M_n \) is disjoint from any of the sub-wakes, and \( \bigcap_n M_n \subset M \) (as \( M \) is closed) we conclude from (12) that \( \bigcap_n M_n = \{c\} \).

For \( c = c_0 \), the points \( c_n', c''_n \) can be chosen such that \( \theta^+(c_n') \nearrow \theta^+ \) and \( \theta^-(c''_n) \searrow \theta^- \), and \( U_n \) the union of two simply connected regions bounded by the short arc in \( \partial W(c_0) \) connecting \( c_n' \) and \( c''_n \) and the four \( M \)-rays \( R_M(\theta^+(c_n')) \), \( R_M(\theta^-(c''_n)) \) and \( R_M(\theta^+) \). The rest follows.

Similar argument can be carried out for \( c = c' \in \partial W(c_0) \) parabolic. ■

Proof of Theorem D.1. Let \( W \) be a hyperbolic component with root \( c_0 \).

Assume at first that \( c_0 \) is primitive. Theorem D.3.d) applied to \( \text{wake}(c_0) \) shows that \( M \cap \text{wake}(c_0) \) is locally connected at \( c_0 \). Theorem 1.1.d shows that \( M \setminus \text{wake}(c_0) \) is locally connected at \( c_0 \). As a consequence \( M \) is locally connected at \( c_0 \).

Another consequence of this is that \( W = W(c_0) \), for \( W(c_0) \) given by Theorem D.3.b). In other words \( W(c_0) \) is the unique hyperbolic component with root \( c_0 \). We claim that this is also true whether \( c_0 \) is primitive or not.

Proof. We proceed by induction on the period of \( W \). For period 1, there is a unique hyperbolic component which is the central cardioid. Its root 1/4 is primitive and the assertion is true. Assume it is true for all hyperbolic components of period up to \( k \). Assume \( W \) is of period \( k+1 \). We want
to show $W = W(c_0)$. If $c_0$ is primitive we are done. Assume $c_0$ is non-primitive. Then by a perturbation argument as in [Mi], section 4 (see also [Ha2]), one can show that there is a hyperbolic component $W_1$ with root $c_1$, with $c_0 \in \partial W_1 \setminus \{c_1\}$ and with period less than or equal to $k$. By induction $W_1 = W(c_1)$. If $W \neq W(c_0)$, Theorem D.3.c) applied to $\text{wake}(c_0)$ shows that $W \cap \text{wake}(c_0) = \emptyset$, and applied to $\text{wake}(c_1)$ shows that $W \cap (M \setminus \text{wake}(c_0)) = \emptyset$. This is a contradiction.

Coming back to the proof of Theorem D.1, we may already conclude that $M$ is locally connected at any $c \in \partial W$ which is non-parabolic, by applying Theorem D.3.d) to $W = W(c_0)$.

We may assume now that $c_0$ is non-primitive. By the above argument $c_0 \in \partial W(c_1)$ for some different parabolic point $c_1$. Theorem D.3.d) applied to $\text{wake}(c_0)$ shows that $M \cap \text{wake}(c_0)$ is locally connected at $c_0$; and then applied to $\text{wake}(c_1)$ shows that $M \setminus \text{wake}(c_0)$ is locally connected at $c_0$. As a consequence $M$ is locally connected at $c_0$.

For any $c' \in \partial W$ parabolic, it is the root of a hyperbolic component $W'$ by Theorem D.3.b). We can then repeat the same argument above with $W$ replaced by $W'$, and conclude that $M$ is locally connected at $c'$.

Let us restate a few important corollaries in the proof:

**Corollary D.4** Every parabolic point is the root of a unique hyperbolic component. Moreover $c_0$ is on the boundary of precisely two (resp. one) hyperbolic components if $c_0$ is non-primitive (resp. primitive).

**Corollary D.5** For $c_0$ a parabolic point $M \setminus \{c_0\}$ has exactly two connected components.

**Corollary D.6** For $c_0$ a parabolic point and $\theta$ any angle not in $\text{Ang}(c_0)$. The impression of the prime end corresponding to $R_M(\theta)$ does not contain $c_0$. In particular $c_0$ is the landing point of precisely two $M$-rays, i.e. those with angle in $\text{Ang}(c_0)$.

Note that the same result as Corollary D.6 for $c_0$ a Misiurewicz point is stated in Theorem B.3.

The following important result comes as a corollary: For any $\theta \in \mathbb{T}$, define $L(\theta)$ to be the landing point of $R_M(\theta)$ if it exists. For any $c$ a Misiurewicz point, define $\text{Ang}(c)$ to be the set of angles such that the corresponding dynamical ray for $Q_c$ lands at $c$.

**Corollary D.7** (Douady-Hubbard-landing-theorem)

1. The map $L$ is well defined on the set of rationals with even denominators and maps it onto the set of Misiurewicz parameters. Moreover for any $c$
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Misiurewicz $L^{-1}(c) = \text{Ang}(c)$, and $\text{Ang}(c)$ consist of finitely many rationals with even denominators.

2. The map $L$ is well defined on the set of rationals with odd denominators and maps it onto the set of parabolic parameters. Moreover for any $c$ parabolic $L^{-1}(c) = \text{Ang}(c)$.

Proof. By an elementary argument it is quite easy to see that $L$ is well defined on the set of rationals and maps it into the set of Misiurewicz and parabolic parameters (cf. [PR] in this volume or [GM]).

Dynamical properties tell us that $\text{Ang}(c)$ consists of non-empty but finitely many rationals with even denominators for $c$ Misiurewicz; and of two rationals with odd denominators for $c$ parabolic.

Our theorem B.3 establishes $L^{-1}(c) = \text{Ang}(c)$ for $c$ Misiurewicz, and Corollary D.6 establishes $L^{-1}(c) = \text{Ang}(c)$ for $c$ parabolic.

Remark. One can find a more combinatorial approach of Corollary D.7 in [Mi] and [Sc1]. One hidden difficulty in proving Corollary D.7.1 is that for $\theta$ a rational with even denominator, $L(\theta)$ must be a Misiurewicz point, or equivalently can not be a parabolic point. Our proof, as well as the original proof of Douady-Hubbard, relies on results about parabolic points. However, there are ways to avoid this, for example one can use Thurston’s algorithm to construct a parameter $c$ such that $\theta \in \text{Ang}(c)$ (a rather involved method), or one can apply the pretty elementary method given by Petersen and Ryd (cf. [PR], in this volume).

The following is a refinement of Theorem D.3.a) and gives a dynamical description of $\text{wake}(c_0)$:

**Theorem D.8** Let $c_0$ be a parabolic point with $\text{Ang}(c_0) = \{\theta^-, \theta^+\}$. Then $\text{wake}(c_0)$ coincides with the set of $c \in \mathbb{C}$ such that $R_c(\theta^-)$ and $R_c(\theta^+)$ both land, and land at the same repelling periodic point.

Proof. The main ideas are the following: At first this is true for $c \in \mathbb{C} \setminus M$, by a combinatorial study conducted by Milnor ([Mi], Lemmas 2.6, 2.9 and 2.11). An argument of P. Haïssinsky ([Ha1], Proposition A.1) shows that the landing points $z_\pm(c)$ of $R_c(\theta^\pm)$ are holomorphic in $c$, on any connected open set where they are both defined, and moreover on this connected open set either $z_+(c) \equiv z_-(c)$ or $z_+(c) \neq z_-(c)$ for any $c$. We can then conclude easily by combining these two arguments. For details, see [Mi], Theorem 3.1.

The following function is as important as the function $\beta_{c_0}(c)$ defined in Theorem D.3.a):

**Theorem D.9** Let $W$ be a hyperbolic component with root $c_0$. Let $\alpha_{c_0}(c)$ be an attracting periodic point for $c \in W$ depending holomorphically on $c$. Then
α_{c_0}(c) extends to a holomorphic map on the entire wake(c_0), is a periodic point of Q_c with constant period, repelling for c ∈ wake(c_0) \ W, and coincides, in each sub-wake of W, i.e. wake(c'), c' ∈ ∂W parabolic, with the holomorphic function β_{c'}(c) (which is the landing point of R_c(θ^±(c'))).

**Proof.** This is basically a corollary of Theorem D.3. By the property of the multiplier function ρ(c) and the implicit function theorem, α_{c_0}(c) extends holomorphically to a neighbourhood of W \ {c_0} in wake(c_0). Now fix c' ∈ ∂W \ {c_0} parabolic. Let W' be the unique hyperbolic component with root c' (Corollary D.4). Theorem D.3.b) applied to W' = W(c') says α_{c_0}(c) = β_{c'}(c) on W'. Since β_{c'}(c) is defined and holomorphic on the entire sub-wake wake(c'), so is α_{c_0}(c), and they coincide throughout the sub-wake.

We need now the help of a third holomorphic function ξ(c) which is defined to be α_{c_0}(c_1) for some c_1 ∈ wake(c') \ M (with c' as above), and to be its analytic continuation throughout C \ (M ∪ R_M(0)) following the holomorphic motion of the Cantor Julia set. Clearly α_{c_0}(c) = ξ(c) on a open set. So α_{c_0}(c) extends to C \ (M ∪ R_M(0)) and coincides with ξ(c) there. In particular α_{c_0}(c) extends to wake(c_0) \ M.

Finally by (12) every point c ∈ wake(c_0) is either outside of M, or in one of the sub-wakes, or in W. Therefore the theorem. □

Generally speaking α_{c_0}(c), as a periodic point of Q_c, can be analytically continued along any path in C \ {finitely many parabolic points}. But we may loose tract of the rays landing at α_{c_0}(c), it may become non-repelling, and we do not know a priori whether these parabolic points present real branch points or not. Theorem D.9 answers these questions at least in wake(θ^±(c_0)).

**Remark.** In case W is the central cardioid of M, the proof of D.3, D.9 can be made more elementary and thus lead to a simpler proof of local connectivity of M at c_0 = 1/4. See [GM], Appendix C.

Our final corollary is the following Lavaurs continuity result ([DH2], Chapter XVII: Une propriété de continuité), which is a stronger version of Theorem D.3.b):

**Theorem D.10 (Lavaurs)** For θ a rational angle, denote by γ_θ(c) the landing point of R_c(θ) if it exists. Then γ_θ(c) is continuous on C \ ∪_{k≥1} R_M(2^kθ). Here the rays R_M(η) do not include their landing points. This in particular says that γ_θ(c) is continuous at the landing points of these rays.

**Proof.** We will only treat the case θ has odd denominator. The other case is a much easier exercise.

On the open set

\[ C \setminus \bigcup_{k≥1} R_M(2^kθ) \]
\( \gamma_0(c) \) is well defined (Douady-Hubbard) and holomorphic ([Ha1], appendix). So we just need to show that \( \gamma_0(c) \) is continuous at \( L(\theta) = c_0 \), the landing point of \( R_M(\theta) \) (as this will imply that \( \gamma_{2\theta}(c) \) for \( l \geq 1 \) is continuous at \( L(\theta) \) and as a consequence \( \gamma_0(c) \) is continuous at \( L(2^m\theta) \) for \( m \geq 1 \)).

Let \( U \) be a neighbourhood of \( c_0 \) such that \( U \setminus R_M(\theta) \), \( U \cap \text{wake}(c_0) \) and \( U \setminus \text{wake}(c_0) \) are all simply connected. Set \( \gamma(c) = \gamma_0(c) \). Denote as before, by \( z_0 \) the parabolic point of \( Q_{c_0} \) on the boundary of the Fatou component of \( c_0 \).

Case 1. The point \( c_0 \) is primitive. In this case the other angle \( \theta' \) in \( \text{Ang}(c_0) \) is not in the form \( 2^l\theta \). So \( \gamma(c) \) is well defined in \( U \setminus R_M(\theta) \). But \( \gamma(c) = \beta_{c_0}(c) \) for \( c \in U \cap \text{wake}(c_0) \) (Theorem D.3.a)) and \( \beta_{c_0}(c) = g_i(v) \), \( i = + \) or \( - \), where \( v = \sqrt{c - c_0} \) and \( g_\pm(v) \) is defined in Section 3.1 (they are the two local periodic points perturbed from \( z_0 \)). As \( U \setminus R_M(\theta) \) is simply connected, \( \gamma(c) \equiv g_i(\sqrt{c - c_0}) \) on \( U \setminus R_M(\theta) \) and is continuous at \( c_0 \) as \( g_i(\sqrt{c - c_0}) \) is.

Case 2. The point \( c_0 \) is non-primitive. This is the case where the two angles \( \theta, \theta' \) in \( \text{Ang}(c_0) \) are in the same orbit by angle doubling and \( \gamma(c) \) is defined on \( U \setminus (R_M(\theta) \cup R_M(\theta')) \), and is holomorphic on \( U \setminus (R_M(\theta) \cup R_M(\theta') \cup \{c_0\}) \) ([Ha1], appendix).

In \( U \cap \text{wake}(c_0) \) we have \( \gamma(c) = \beta_{c_0}(c) \) and \( \gamma(c_0) = z_0 \) (Theorem D.3.a and b). So \( \gamma \) is continuous at \( c_0 \) on \( (U \cap \text{wake}(c_0)) \cup \{c_0\} \).

In \( U \setminus \text{wake}(c_0) \), let \( W_1 \) the hyperbolic component containing \( c_0 \) on the boundary. As \( c \in U \cap W_1 \) tending to \( c_0 \) radically, the same perturbative argument of Milnor ([Mi], section 4) shows that \( \gamma(c) \) is repelling for \( Q_c \) and tends to \( z_0 \). Let \( z(c) \) be this repelling periodic point. It has analytic continuation along any path in \( U \setminus \{c_0\} \) (shrinking \( U \) if necessary) and is periodic of constant period for \( Q_c \). It is therefore a well defined holomorphic function in \( U \setminus \text{wake}(c_0) \) as this last set is simply connected. As \( \gamma(c) \) is holomorphic, we have \( \gamma(c) \equiv z(c) \) on \( (U \setminus \text{wake}(c_0)) \cup \{c_0\} \) and continuous at \( c_0 \), as \( z(c) \) is.

\section{Local connectivity via Mandelbrot-like families (following Hubbard)}

Here we sketch a proof of the local connectivity of \( M \) at a primitive parabolic point using the theory of Mandelbrot-like families and the fact that \( M \) is locally connected at \( 1/4 \). We follow essentially the route indicated by Hubbard ([Hu]).

Let \( c_0 \) be a primitive parabolic point, more precisely \( Q_{c_0} \) has a \( k \) periodic point of multiplier 1. In [Ha1] in this volume, it is shown that there are open discs \( U_0, U_1 \) containing \( c_0 \), open discs \( B_c \) (resp. \( B_c' \)) whose boundary undergo a holomorphic motion, such that \( f = \{Q^k_c : B_c' \to B_c, c \in U_0\} \)
forms a Mandelbrot-like family, with $\overline{U_1} \subset U_0$, $U_1$ is the set of $c$ such that the critical point of $Q_c^k$ in $B'_c$ does not escape $B'_c$ after one iteration, $\overline{U_1} \cap M$ is connected (it is in fact a puzzle piece in the sense of Yoccoz), and, for $\M$ the connectedness locus and $\chi : M_f \to M$ the straightening map, $M_f \subset M \cap U_0$ and $\chi$ is a homeomorphism with $\chi(c_0) = 1/4$.

Let $U_n$ to be the set of $c \in U_0$ such that the critical point of $Q_c^k$ in $B'_c$ does not escape $B'_c$ after $n$ iterations. Then $U_n$ is again a puzzle piece and $\bigcap_n U_n = M_f$.

Denote by $W$ the unique hyperbolic component with $c_0$ as root (Corollary D.4). Choose $c'_n, c''_n$ two sequences of parabolic points on $\partial W$ converging to $c_0$ such that $\theta^+(c'_n) < \theta^+$ and $\theta^-(c''_n) > \theta^-$. Denote by $V_n$ the simply connected domain containing $R_M(\theta^+)$ bounded by the short arc in $\partial W$ connecting $c'_n$ and $c''_n$ and the two rays $R_M(\theta^+(c'_n))$ and $R_M(\theta^-(c''_n))$ (here we need Theorem A.1 for the landing of these rays). Denote by $M_n$ the set $\chi(V_n \cap M_f)$.

Set $L_n = \overline{V_n} \cap U_n$. Then $L_n \subset L_{n-1}$ and $L_n \cap \partial M$ is a connected neighbourhood (in $\partial M$) of $c_0$. We now show that $\bigcap_n L_n = \{c_0\}$:

$$c_0 \in \bigcap_n L_n = \left( \bigcap_n L_n \right) \cap M_f = \bigcap_n (L_n \cap M_f) \subset \bigcap_n (\overline{V_n} \cap M_f) =$$

$$\bigcap_n \chi^{-1}(M_n) = \chi^{-1}(\{1/4\}) = \{c_0\},$$

where the first equality is due to $\bigcap_n L_n \subset \bigcap_n U_n = M_f$, the second is trivial, the set inclusion is because $L_n \subset \overline{V_n}$, the next equality is by definition of $M_n$ and the fact that $\chi$ is a homeomorphism, the following is due to the facts that $\lim_{n \to \infty} \chi(c'_n) = \lim_{n \to \infty} \chi(c''_n) = 1/4$ (as $\chi$ is continuous) and that $M$ is locally connected at $1/4$ (see [GM] Appendix C for an elementary proof), and the final inequality is by properties of $\chi$.

**References**


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