Local properties of the Mandelbrot set at parabolic points

Tan Lei

Abstract

We formulate the technique of parabolic implosion into an easy-to-use result: Orbit correspondence, and apply it to show that for c_0 a primitive parabolic point, the Mandelbrot set M outside the wake of c_0 is locally connected at c_0. This, combined with known results inside the wake, shows that M is locally connected at c_0. The appendices contain sketches of relative results and their proofs.

1 Introduction

Denote by Q_c the map $z \mapsto z^2 + c$. The Mandelbrot set M is defined to be the set of c such that $\lim_{n\to\infty} Q_c^n(0) \not\to \infty$.

We say that c_0 is a *primitive* (resp. *non-primitive*) parabolic point if Q_{c_0} has a periodic point of multiplier 1 (resp. $e^{2\pi i\theta}$ with $\theta \in \mathbb{Q} \setminus \mathbb{Z}$).

For precise definition of the following notations, see §3.2. Denote by $\varphi_M : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus M$ the conformal isomorphism fixing ∞ and tangent to the identity there. If two M-rays $R_M(\theta^\pm)$ land at the same point c_0 we denote by $wake(\theta^\pm)$ the component of $\mathbb{C} \setminus R_M(\theta^+) \cup R_M(\theta^-) \cup \{c_0\}$ which does not contain 0. Denote by $R_{c_0}(\theta)$ the external ray of Q_{c_0} of angle θ.

Theorem 1.1 *Let $c_0 \neq 1/4$ be a primitive parabolic point. More precisely Q_{c_0} has a k-periodic point of multiplier 1, with $k > 1$.*
a) (landing property) For the two angles $\theta^\pm$ for which $R_{c_0}(\theta^\pm)$ land at a common parabolic point z_0 and are adjacent to the Fatou component containing c_0, the M-rays $R_M(\theta^\pm)$ land at c_0.
b) (parametrisation) A germ of $R_M(\theta^\pm)$ can be written as $\bigcup_{n\ge n_0} c_n^\pm([0,1])$ for some $n_0 > 0$ satisfying: c_n injective, $c_{n+1}^\pm(0) = c_n^\pm(1)$, and, for $T(z) = z^2$ and some $C, C' > 0$,

$$\varphi_M \circ T^k \circ \varphi_M^{-1}(c_n^\pm(t)) = c_{n-1}^\pm(t) \quad \text{and} \quad \frac{C'}{n^2} \le |c_n^\pm(t) - c_0| \le \frac{C}{n^2} \ . \qquad (1)$$

c) (transversality) The local solution of $Q_c^k(z) - z = 0$ can be written as $(c_0+v^2, g_\pm(v))$ with $g_\pm$ holomorphic in v, $g_\pm(0) = z_0$, $g'_\pm(0) \neq 0$. Moreover the multiplier map $\lambda_\pm(v) = (Q^k_{c_0+v^2})'(g_\pm(v))$ satisfies $\lambda_\pm(0) = 1$ and $\lambda'_\pm(0) \neq 0$.
d) (local connectivity) The set $M \setminus wake(\theta^\pm)$ is locally connected at c_0.

e) (no other rays involved) For any other ray $R_M(\zeta)$ outside of the wake, the impression of its corresponding prime end does not contain c_0 (in particular the ray does not land at c_0) and $M \smallsetminus (wake(\theta^{\pm}) \cup \{c_0\})$ is connected.

The last statement says that M has no ghost umbilical cords at c_0, in the terminology of D. Sørensen. We will formulate techniques of parabolic implosion into a parabolic orbit correspondence result and then apply it to construct a sequence of puzzle pieces P_n in $\mathbb{C} \smallsetminus wake(\theta^{\pm})$, with c_0 on the boundary, such that $\varphi_M \circ T^k \circ \varphi_M(\partial P_n) = \partial P_{n-1}$ and $\mathrm{diam}(P_n) \asymp 1/n^2$.

Together with known results inside the wake, see [Hu, Mi, Sø] and Appendix D, we can conclude that M is locally connected at every primitive parabolic points.

A similar statement for non-primitive parabolic points is included in [DH2], [Mi] and [Sø]. In Appendix A and D we give a short account of these results, together with interesting consequences.

Our treatment follows very closely the original approach of Douady and Hubbard ([DH2]), who invented the theory of parabolic implosion while proving landing properties of M-rays.

J. Hubbard in [Hu, Theorem 14.6] (see also Appendix E in the present paper) sketched a proof of the local connectivity of M at a primitive parabolic point, using the theory of Mandelbrot-like families together with the fact that M is locally connected at the cusp $c = 1/4$. This proof is less elementary than the one in the present paper, but has an advantage that it turns the local connectivity property at a cusp into a semi-direct consequence of the existence of sub-Mandelbrot-like families (semi-direct means that extra combinatorial information is needed), and thus can be adopted to more general settings.

We show in Appendix B that the similar statement for repelling orbit correspondence can be applied to show that M is locally connected at Misiurewicz points (i.e. c values such that $Q_c(0)$ is strictly preperiodic).

For a more combinatorial approach of landing properties and local connectivity of M at various points, see [Sc1] and [Sc2].

Acknowledgements. Discussions with A. Douady and M. Shishikura have been very fruitful. J. Milnor has helped to largely improve both the global structure and local details.

2 Orbit correspondence in parabolic implosion

We will use the terminology and results in [S1].

Definition. By a *Fatou coordinate*, associated with a rational map f on a region U, we will mean a univalent map $\Phi : U \to \mathbb{C}$ which satisfies the following condition for every $z \in U$,

$$f(z) \in U \iff \Phi(z) + 1 \in \Phi(U) \iff \Phi(f(z)) = \Phi(z) + 1 \ ,$$

and moreover satisfies the convexity condition that whenever both w and $w+n$ belong to $\Phi(U)$ the intermediate points $w+1$, $w+2$, $\dots$, $w+(n-1)$ must also belong belong to $\Phi(U)$. Note that if Φ exists f is also univalent in U.

The set up. Fix $r > 0$ small. Let $\Delta_r \subset \mathbb{C}$ be a bounded connected open set with 0 on the boundary, U an open neighbourhood of 0 and $f_s(z)$ be a family of holomorphic maps on U satisfying:
1. $(s,z) \mapsto f_s(z)$ is well defined, continuous on $\overline{\Delta_r} \times U$ and holomorphic in z, with $f_s(z) = \lambda(s)z + O(z^2)$;
2. $(s,z) \mapsto f_s(z)$ is holomorphic on $\Delta_r \times U$;
3. $f_0''(0) \neq 0$;
4. Set $\sigma(s) = (\lambda(s) - 1)/(2\pi i)$. Then σ is continuous on $\overline{\Delta_r}$ with $\sigma(0) = 0$, and σ maps Δ_r univalently onto $\{z \mid |z| < r,\ |\arg z| < \pi/4\}$.

Remark. In practice $f_s(z)$ and $\sigma(s)$ are often defined in a larger sector neighbourhood of $s = 0$ than Δ_r. But we are only interested in the subsector of parameters which is mapped by $\sigma(s)$ bijectively onto $\{z \mid |z| < r,\ |\arg z| < \pi/4\}$. Here the width $\pi/4$ is not really relevant, it can be any angle in the interval $(0, \pi)$.

Theorem 2.1 (Douady-Lavaurs-Shishikura): *If r is sufficiently small, then there exist the followings (see Figure 1):*

For $s = 0$ there are open Jordan domains $\Omega_{-,0}$ and $\Omega_{+,0}$ which are respectively an attracting and a repelling petal for f_0, satisfying $\overline{\Omega_{-,0}} \cap \overline{\Omega_{+,0}} = \{0\}$, with associated Fatou coordinates $\Phi_{\pm,0} : \Omega_{\pm,0} \to \mathbb{C}$.

For $s \in \overline{\Delta_r} \smallsetminus \{0\}$ there is a Jordan domain Ω_s containing two fixed points for f_s on its boundary and two Fatou coordinates $\Phi_{\pm,s} : \Omega_s \to \mathbb{C}$ with

$$\Phi_{-,s}(z) - \frac{1}{\sigma(s)} \ = \ \Phi_{+,s}(z) \ . \tag{2}$$

Taking Ω_0 to be $\Omega_{-,0} \cup \Omega_{+,0}$, both the closure $\overline{\Omega_s}$ and the complement $\hat{\mathbb{C}} \smallsetminus \Omega_s$ depend continuously on $s \in \overline{\Delta_r}$, using the Hausdorff metric for the space of compact subsets of the Riemann sphere, and the correspondence $(s,z) \mapsto \Phi_{\pm,s}(z)$ is continuous in both variables for either choice of sign, wherever it is defined. Moreover $\Phi_{\pm,s}(z)$ is holomorphic in s for $s \in \Delta_r$.

Proof. Let $\alpha(s) \in \mathbb{C}$ be defined so that $\lambda(s) = e^{2\pi i \alpha(s)}$ and $|\Re(\alpha(s))| < 1/2$. This theorem was proved in [S1], Proposition 3.2.2 and Appendix, with $\frac{1}{\alpha(s)}$ in place of $\frac{1}{\sigma(s)}$, and with $|\arg(\alpha(s))| < \pi/4 + \varepsilon$ in place of $s \in \Delta_r$. For $s \in \Delta_r$

we have $|\arg\alpha(s)| \approx |\arg\sigma(s)| < \frac{\pi}{4}$. Using the Taylor series expansion of e^x one can see easily that the difference $\frac{1}{\sigma(s)} - \frac{1}{\alpha(s)}$ is continuous on $\overline{\Delta_r}$. Since the Fatou coordinates are defined up to addition of constants (relative to z), one can add the above difference to $\Phi_{-,s}$ in Shishikura's normalisation without alerting the continuous dependence on (s,z). Therefore (2) holds together with the rest part of the theorem. ∎

Using this result, we can restate and prove the "Tour de Valse" theorem of Douady and Sentenac ([DH2], exposé n°XI) as follows.

Figure 1 : Parabolic orbit correspondence

Proposition 2.2 (Parabolic orbit correspondence). *Let $X \subset \Omega_{-,0}$ and $Y \subset \Omega_{+,0}$ be compact sets. If r is sufficiently small, then given continuous functions $x : \overline{\Delta_r} \times [0,1] \to X$ and $y : \overline{\Delta_r} \times [0,1] \to Y$, holomorphic in s for $s \in \Delta_r$, the equations*

$$f_s^n(x(s,t)) = y(s,t) \quad \text{and} \quad \Phi_{-,s}(x(s,t)) + n - \frac{1}{\sigma(s)} \;=\; \Phi_{+,s}(y(s,t)) \qquad (3)$$

have at least one common solution $s = s_n(t)$ in Δ_r, which depends continuously on t, provided that n is sufficiently large. Moreover

$$\lim_{n\to\infty} n - \frac{1}{\sigma(s_n(t))} = C_0(t) \quad \text{and} \quad \lim_{n\to\infty} \left|1 - \frac{1}{n\sigma(s_n(t))}\right| = 0 \quad \text{as} \quad n \to \infty \qquad (4)$$

where $C_0(t)$ is finite and continuous on t, and the convergences are uniform on t (see Figure 1).

Before giving the proof we give a quick application showing that the Julia sets move discontinuously:

Application I. Let $f_s(z) = (1+s)z + z^2$. For Δ_r a sector neighbourhood of 0 containing the direction of $i\mathbb{R}$ the family satisfies all the required conditions. The Julia set $J(f_0)$ of f_0 is a cauliflower (see the left picture in Figure 1 of [S1], in this volume). Let x be any point in $\Omega_{-,0}$.

So $x \notin J(f_0)$. We will show that there is a sequence $s_n \in \Delta_r$ converging to 0 such that $x \in J(f_{s_n})$ for all n. It follows that for any converging subsequence $J(f_{s_{n_k}})$, $\lim J(f_{s_{n_k}}) \neq J(f_0)$.

In order to find s_n, we define at first an appropriate function $y(s)$. On $J(f_0)$ there is a repelling periodic cycle of period two. Let $y(0)$ be a preimage of this cycle such that $y(0) \in \Omega_{+,0}$. For any s near 0, there is a unique

$y(s)$ which is holomorphic on s and is mapped by some iterates of f_s onto a repelling periodic cycle of period two. Clearly $y(s) \in J(f_s)$.

Now we can apply Proposition 2.2 for $x(s) \equiv x$ to conclude that there is a sequence $s_n \in \Delta_r$, $s_n \to 0$, such that $f_{s_n}^n(x) = y(s_n)$. As a consequence $x \in J(f_{s_n})$ for every large n.

Proof of Proposition 2.2. First note that we only need to find a solution for the right equation in (3) , since combining it with (2) we get $\Phi_{+,s}(x(s,t)) + n = \Phi_{+,s}(y(s,t))$. By definition of Fatou coordinates we have $\Phi_{+,s}(f_s^n(x(s,t))) = \Phi_{+,s}(x(s,t)) + n = \Phi_{+,s}(y(s,t))$. Since $\Phi_{+,s}$ is univalent, we have $f_s^n(x(s,t)) = y(s,t)$. This is the left equation in (3) .

To simplify the situation, assume at first $x(s,t) = x(s)$ and $y(s,t) = y(s)$ (i.e. they are independent of t). Set $\tilde{x}(s) = \Phi_{-,s}(x(s))$ and $\tilde{y}(s) = \Phi_{+,s}(y(s))$. Then $\tilde{x}(s)$ and $\tilde{y}(s)$ are continuous in $\overline{\Delta_r}$. For n large, we are seeking for solutions of the following:

$$\tilde{x}(s) + n - \frac{1}{\sigma(s)} = \tilde{y}(s) \ . \tag{5}$$

Let $\tilde{x}(s) - \tilde{y}(s) = C_0 + H(s)$, with $C_0 = \tilde{x}(0) - \tilde{y}(0)$, $H(s)$ continuous in $\overline{\Delta_r}$ and $H(0) = 0$. We may assume r is small so that $|H(s)|_{\overline{\Delta_r}} \leq 1/4$.

Set $F(s) = C_0 + n - \frac{1}{\sigma(s)} = \frac{1}{z_n} - \frac{1}{\sigma(s)}$ with $z_n = 1/(C_0 + n)$, and $G(s) = F(s) + H(s)$. Then (5) becomes $G(s) = 0$. We will at first find a solution for $F(s) = 0$ and then use it to approximate a solution of $G = 0$.

For n large enough,

$$z_n \in \overline{D}(z_n, |z_n|^2) \subset \{z \mid |z| < r,\ |\arg(z)| < \frac{\pi}{4}\} \ .$$

Let $\overline{D} = \sigma^{-1}(\overline{D}(z_n, |z_n|^2)) \subset \Delta_r$. By univalence of σ, the set $\overline{D}$ is a closed topological disc. Moreover there is a unique $s'_n \in D$ such that $\sigma(s'_n) = z_n$, i.e. $F(s'_n) = 0$. Now

$$|F(s)|_{\partial D} = \left|\frac{1}{z_n} - \frac{1}{\sigma(s)}\right|_{\partial D} = \left.\frac{|\sigma(s) - z_n|}{|z_n \sigma(s)|}\right|_{\partial D} \geq \frac{|z_n|^2}{2|z_n|^2} \geq \frac{1}{2} \quad \text{and}$$

$$|F(s) - G(s)|_{\partial D} = |H(s)|_{\partial D} \leq \frac{1}{4} \ .$$

Therefore $|F - G|_{\partial D} < |F|_{\partial D}$. By Rouché's theorem we conclude that there is a unique $s_n \in D$ such that $G(s_n) = 0$. (Here is another argument showing the existence of s_n: We have $|\arg(F(s)) - \arg(G(s))|_{s \in \partial D} < \pi/4$. Since $\arg(F)|_{\partial D}$ is of degree one so is $\arg(G)|_{\partial D}$. Now G must have at least one zero in D for otherwise $\arg(G(s))$ would extend to a continuous, homotopically non-trivial map from $\overline{D}$ to $\mathbb{R}/2\pi\mathbb{Z}$. This is impossible.)

Therefore s_n is a common solution of the equations in (5) and (3) . Furthermore, by definition of C_0 and (5) ,

$$\lim_{n\to\infty} n - \frac{1}{\sigma(s_n)} = C_0 \ .$$

This is part of (4) , the other part of (4) easily follows.

Finally assume that $x(s,t)$ and $y(s,t)$ do depend on t. Set as above $\tilde{x}(s,t) = \Phi_{-,s}(x(s,t))$ and $\tilde{y}(s,t) = \Phi_{+,s}(y(s,t))$. Then $\tilde{x}(s,t) - \tilde{y}(s,t) = C_0(t) + H(s,t)$, with $C_0(t) = \tilde{x}(0,t) - \tilde{y}(0,t)$, $H(s,t)$ continuous on $\overline{\Delta_r} \times [0,1]$ and $H(0,t) = 0$. By uniform continuity of $H(s,t)$ we may choose r small such that $|H(s,t)| \le 1/4$ on $\overline{\Delta_r} \times [0,1]$. The same argument above would find a unique $s_n(t)$ as zero of $G(s,t)$. This implies that $s_n(t)$ is continuous in t. ∎

3 Proof of Theorem 1.1

In this section let (c_0, z_0) be such that z_0 is a k-periodic point $(k > 1)$ of Q_{c_0} with multiplier 1 and it is on the boundary of the Fatou component of Q_{c_0} containing c_0. We would like to find an appropriate family $f_s(z)$ and two appropriate functions $x(s,t)$ and $y(s,t)$ in order to apply Proposition 2.2.

3.1 The family f_s relative to $\vec{w}$

We will at first define holomorphic functions $g_\pm(v), \sigma_\pm(v)$ for $v = \sqrt{c - c_0}$, define a holomorphic map $I(c)$, choose a vector $\vec{w}$ with certain properties and then define a family of maps f_s relative to $\vec{w}$ satisfying the set up of §2.

Lemma 3.1 *Set $F(c,z) = Q_c^k(z) - z$. Then $(Q_{c_0}^k)''(z_0) \neq 0$ and $F(c,z) = 0$ has two local solutions which can be written as $(c_0 + v^2, g_\pm(v))$ with $g_\pm(v)$ holomorphic.*

Proof. By assumption $Q_{c_0}^k$ is the first return map about z_0 with derivative 1 at z_0. It follows that each parabolic basin attached to the point z_0 is fixed by $Q_{c_0}^k$. In other words, different parabolic basins attached to z_0 belong to different orbits under the iteration of Q_{c_0}. On the other hand, Q_{c_0} has only one critical point so there can be at most one periodic cycle of parabolic basins. It follows that there is exactly one such basin attached to z_0. As a consequence $(Q_{c_0}^k)''(z_0) \neq 0$ (by local theory of Fatou flowers).

Now consider $F(c,z)$. We have

$$F(c_0,z_0) = 0, \quad \left.\frac{\partial F}{\partial z}\right|_{(c_0,z_0)} = 0, \quad \left.\frac{\partial^2 F}{\partial^2 z}\right|_{(c_0,z_0)} = (Q_{c_0}^k)''(z_0) \neq 0 \ .$$

By Weierstrass preparation theorem the local solutions of $F(c,z)=0$ coincide with the solution of

$$(z-z_0)^2+(c-c_0)\cdot A(c-c_0)\cdot(z-z_0)+(c-c_0)\cdot B(c-c_0)=0 \qquad (6)$$

where $A(*),B(*)$ are holomorphic functions. Moreover the analytic set $\{c \mid F(c,z)$ has multiple solutions$\}$ can not have dimension one, so it has dimension zero, hence is equal to the singleton $\{c_0\}$ in a neighbourhood of c_0.

Now replace $c-c_0$ by v^2 in (6) , we get

$$(z-z_0)^2+v^2A(v^2)\cdot(z-z_0)+v^2B(v^2)=0\ .$$

As the discriminant is not constantly 0, it is in the form $v^{2m}H(v^2)$ with $H(0)\neq 0$. Therefore $\sqrt{H(v^2)}$ is a locally well defined holomorphic function. So the solutions of (6) (and of $F(c,z)=0$) are the two holomorphic functions $g_\pm(v)=-v^2A(v^2)\pm v^m\sqrt{H(v^2)}$. (We will show later that $g_\pm'(0)\neq 0$ therefore $m=1$.) ■

Set

$$\sigma_\pm(v)=\frac{(Q^k_{c_0+v^2})'(g_\pm(v))-1}{2\pi i}\ .$$

Lemma 3.2 *For $c=c_0+v^2$ set $I(c)=\sigma_+(v)\sigma_-(v)$. The correspondence $c\mapsto I(c)$ is holomorphic throughout a neighbourhood of c_0 with $I(c_0)=0$. Furthermore, given a direction $\vec{w}$ such that $I(c_0+hw)/|I(c_0+hw)|\to -1$ as $h\searrow 0$, and a small number $r>0$, there is an open connected set Δ_r containing $\{hw,\ h>0 \text{ small}\}$ and with 0 on the boundary, such that σ_+ maps a choice Δ of $\sqrt{\Delta_r}$ univalently onto $\{z\mid |z|<r, |\arg(z)|<\pi/4\}$, and maps $\{h\sqrt{w},\ h>0 \text{ small}\}$ onto an arc tangent to $\mathbb{R}^+$ at 0 (see Figure 2).*

Proof. Since $z-Q^k_{c_0+v^2}(z)=0$ has only two solutions $g_\pm(v)$, we have (by residue theorem)

$$\begin{aligned}\int_{|z-z_0|=t}\frac{1}{Q^k_{c_0+v^2}(z)-z}dz &= \frac{2\pi i}{(Q^k_{c_0+v^2})'(g_+(v))-1}+\frac{2\pi i}{(Q^k_{c_0+v^2})'(g_-(v))-1}\\ &= \frac{1}{\sigma_+(v)}+\frac{1}{\sigma_-(v)}\ .\end{aligned}$$

But the left hand side is holomorphic on v and has finite value at $v=0$. So

$$\lim_{v\to 0}\frac{1}{\sigma_+(v)}+\frac{1}{\sigma_-(v)}\quad \text{is finite.} \qquad (7)$$

Set $\sigma_\pm(v)=r_\pm(v)e^{i\eta_\pm(v)}$. Then (7) gives $\eta_-(v)-\eta_+(v)\to\pi$ as $v\to 0$ (exchange g_+ and g_- if necessary).

Now we turn to $I(c)$. As a symmetric function of the roots of (6) , $I(c)$ is a function of the coefficients of (6) . So $c \mapsto I(c)$ is holomorphic in a neighbourhood of c_0. Clearly $I(c_0) = 0$. It has then a power series expansion $I(c_0 + s) = as^\nu + \cdots$ with $\nu \geq 1$. Thus there are ν distinct directions $\vec{w}$ such that $I(c_0 + hw)/|I(c_0 + hw)| \to -1$ as $h \searrow 0$.

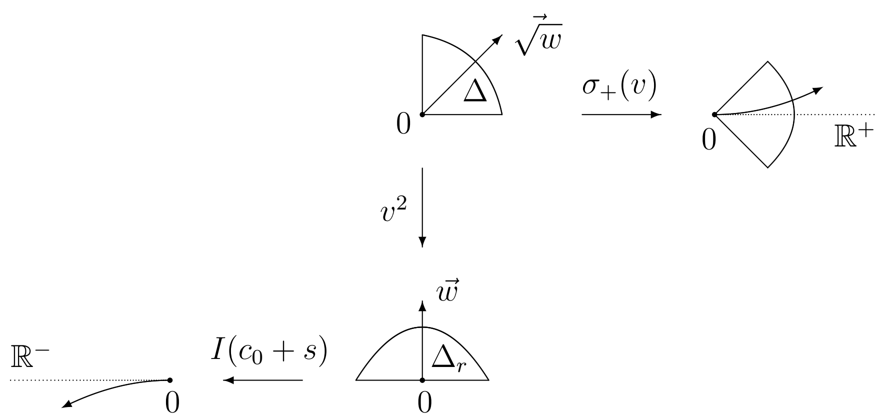

Figure 2 : Finding $\vec{w}$, Δ and Δ_r, in the case $\sigma'_+(0) \neq 0$

Fix a choice of w and a choice of $\sqrt{w}$. Let $v = v(h) = h \cdot \sqrt{w}$. Then $I(c_0 + v^2) = \sigma_+(v)\sigma_-(v)$ and $\frac{I}{|I|} = e^{i(\eta_+(v)+\eta_-(v))}$. So $\eta_+(v) + \eta_-(v) \to \pi$ as $h \searrow 0$.

Combining the two equations on $\eta_\pm(v)$ we conclude that, as $v = h\sqrt{w}$, $\lim_{h\searrow 0} \eta_+(v)$ exists and is equal to 0. In other words, $\sigma_+(v)$ maps $\{h\sqrt{w},\ h > 0 \text{ small}\}$ onto an arc tangent to $\mathbb{R}^+$ at 0. (Similarly $\sigma_-(v)$ maps it onto an arc tangent to $\mathbb{R}^-$ at 0.)

As a holomorphic function fixing 0, $\sigma_+(v) = b(v)^p$ in a neighbourhood of 0, with b univalent, $b(0) = 0$ and $p > 0$ some integer. There is therefore a (almost) sector neighbourhood Δ of 0 containing the direction of $\sqrt{\vec{w}}$ which is mapped by $\sigma_+(v)$ univalently onto $\{z \mid |z| < r, |\arg(z)| < \pi/4\}$. Since Δ has an opening angle close to $\pi/(4p) \leq \pi/4$, the map v^2 is univalent on Δ, whose image will be the required set Δ_r. ■

Definition of f_s relative to $\vec{w}$. Fix a choice of $\vec{w}$ such that $I(c_0+hw)/|I(c_0+hw)| \to -1$ as $h \searrow 0$. Lemma 3.2 provides two sets $\overline{\Delta}$ and $\overline{\Delta_r}$. For $v \in \overline{\Delta}$ and $s = v^2 \in \overline{\Delta_r}$, set $g(s) = g_+(v)$ and define $f_s(z)$ to be $Q^k_{c_0+s}(z + g(s)) - g(s)$.

Corollary 3.3 *The family f_s satisfies the hypothesis in the set up of §2 with $(s, \sigma(s))$ parametrised by $(v^2, \sigma_+(v))$.*

Proof. The conditions 1 and 2 about regularities of $f_s(z)$ are easily verified, with $\lambda(s) = f'_s(0) = (Q^k_{c_0+s})'(g(s)) = (Q^k_{c_0+s})'(g_+(v))$. So $\sigma(s) := \frac{\lambda(s)-1}{2\pi i} =$

$\sigma_+(v)$. For condition 3, we have $f_0''(0) = (Q_{c_0}^k)''(z_0) \neq 0$ (Lemma 3.1). Condition 4 follows from Lemma 3.2. ■

3.2 The angles $\theta^\pm$, $\eta_n^\pm$ and the functions $x(s,t)$, $y(s,t)$

Here we need to understand more about the dynamics of Q_{c_0} and of Q_c for nearby c. We will at first recall the classic theory of Riemann representations, external rays, state two results about the dynamics of Q_{c_0}, and then use perturbations to construct $x(s,t)$ and $y(s,t)$.

We consider a *Model plane* as a complex plane together with the action of $q(= Q_0) : z \mapsto z^2$. This plane is used to give (external) coordinates in the dynamical plane of Q_c as well as in the parameter plane in the following sense:

For $c \in \mathbb{C}$ define φ_c from a subset of the model plane onto a subset of the dynamical plane of Q_c such that φ_c conjugates q to Q_c, fixes ∞ and is tangent to the identity there (this is often called the Böttcher coordinates). It is known that the maximal domain of definition of φ_c is $\overline{\mathbb{C}} \smallsetminus X_c$, with X_c a bounded star-like compact set containing the closed unit disc. Moreover, if $c \in M$, the set X_c coincides with the closed unit disc, and if $c \notin M$, there is $z_c \in \mathbb{C} \smallsetminus X_c$ such that $\varphi_c(z_c) = c$.

Define φ_M from $\{|z| > 1\}$ (a subset of the model plane) onto $\mathbb{C} \smallsetminus M$ (a subset of the parameter plane) by $\varphi_M(z_c) = c$. Due to a result of Douady-Hubbard, and independently Sibony, we know that φ_M is a well defined conformal isomorphism, fixing ∞ and tangent to the identity there.

The rays $R_M(\theta)$, $R_c(\theta)$ are defined to be $\varphi_M(\{e^{\mu+2\pi i\theta} \mid 0 < \mu\})$ and $\varphi_c(\{e^{\mu+2\pi i\theta} \mid 0 < \mu\})$ respectively. Such a ray is said to *land* at a point z if the limit, as $\mu \searrow 0$, exists and is equal to z. We say also that z has θ as *external angle.*

The great importance of these rays are:

1. they are preserved by the dynamics;

2. when two or more rays land at the same point, they provide a clean cut of the plane, and, together with equipotentials, provide *puzzle pieces* which are candidates of local connected neighbourhoods;

3. they often depend analytically on the parameter.

Concerning our polynomial Q_{c_0}, we have the following known result:

Lemma 3.4 *(Douady-Hubbard) The Julia set of Q_{c_0} is locally connected. The convex hull of the critical orbit forms a topologically finite tree (Hubbard tree) with the critical value c_0 as an extremity. The point z_0, as the unique parabolic point on the boundary of the Fatou component U containing c_0, has two external angles adjacent to U. These angles are denoted by $\theta^\pm$ (with $\theta^\pm \in (0,1)$ and $\theta^- < \theta^+$). The period of them by angle doubling is equal to $per(z_0)$.*

For a proof, see [DH2], pages 51-52. From this one derives easily

Lemma 3.5 *In the same setting, there is a sequence of prerepelling (i.e. eventually periodic repelling) points z_n in the Hubbard tree such that $z_n \to z_0$, z_n has two external angles $\eta_n^\pm$ with $\eta_n^- \nearrow \theta^-$, $\eta_n^+ \searrow \theta^+$, as $n \to \infty$, and, for k the period of z_0, $Q_{c_0}^k(z_n) = z_{n-1}$, $2^k\theta^+ = \theta^+$ (mod 1) and $2^k\theta^- = \theta^-$ (mod 1).*

Proof. In the segment $[\alpha, c_0]$ of the Hubbard tree (where α is the fixed point of Q_{c_0} with a non-zero rotation number), choose a point x so that in $[x, z_0] \smallsetminus \{z_0\}$ there is no branching points of the tree (this is possible since there are only finitely many branching points), and no points in the basin of attraction of the parabolic orbit (this is possible as we have assumed that c_0 is primitive). By expansion property on the Julia set, there is a minimal N such that $Q_{c_0}^N$ restricted to $[x, z_0]$ is no more injective. This implies that there is $z' \in [x, z_0]$ such that $Q_{c_0}^{N-1}(z') = 0$ and $Q_{c_0}^N(z') = c_0$. By minimality of N, $Q_{c_0}^N$ restricted to $[z', z_0]$ is a homeomorphism onto its image. As $Q_{c_0}^N(z_0)$ is a point in the parabolic orbit, distinct from z_0, and is on the Hubbard tree, by the choice of x we have $[Q_{c_0}^N(z_0), z_0] \supset [x, z_0] \supset [z', z_0]$. Therefore

$$Q_{c_0}^N([z_0, z']) = [Q_{c_0}^N(z_0), Q_{c_0}^N(z')] = [Q_{c_0}^N(z_0), c_0] \supset [z', z_0] \ .$$

As $Q_{c_0}^N$ reverses the orientation, there is a point $z_1 \in [z', z_0]$ fixed by $Q_{c_0}^N$, i.e. z_1 is a periodic point. It must be repelling as it is not in the parabolic orbit. It must have exactly two external rays, as it is in the Hubbard tree, and is not a branching or extremal point. By expansion again the local inverse branch g of $Q_{c_0}^k$ mapping z_0 to z_0 maps $[z_1, z_0]$ into itself. Denote by z_n the orbit of z_1 under g. They satisfy the desired property of the Lemma. See Figure 3. ■

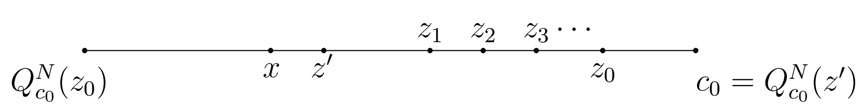

Figure 3 : The segment $[Q_{c_0}^N(z_0), c_0]$ in the Hubbard tree of Q_{c_0}

A proof of a similar statement in the case that 0 is preperiodic for Q_c can be found in [T], Lemma 2.3(2), and in the case 0 is periodic for Q_c can be found in [Sc2], Lemma 5.1.

In order to define an appropriate $y(s, t)$, we define at first $\Gamma_0(t)$ for $t \in [0, 1]$ in the log-model plane as follows (we have chosen deliberately $\log e^{2\pi i\theta^+}$ to be $2\pi i\theta^+ - 2\pi i$ for the purpose to give a similar picture as in the dynamical plane and in the parameter plane, Figures 4, 5 and 6): Fix $\hat{\mu} > 0$ small and

n_1 large, the map $t \mapsto \Gamma_0(t)$ should be piecewise linear and continuous on $[0,1/2)$ and $(1/2,1]$, with

$$\begin{array}{llllll}
\Gamma_0(0) & = & \hat\mu/2^k + 2\pi i\theta^+ - 2\pi i\ , & \Gamma_0(1/6) & = & \hat\mu + 2\pi i\theta^+ - 2\pi i\ , \\
\Gamma_0(1/3) & = & \hat\mu + 2\pi i\eta^+_{n_1} - 2\pi i\ , & \Gamma_0(\frac{1}{2}^-) & = & 2\pi i\eta^+_{n_1} - 2\pi i\ , \\
\Gamma_0(\frac{1}{2}^+) & = & 2\pi i\eta^-_{n_1}\ , & \Gamma_0(2/3) & = & \hat\mu + 2\pi i\eta^-_{n_1}\ , \\
\Gamma_0(5/6) & = & \hat\mu + 2\pi i\theta^-\ , & \Gamma_0(1) & = & \hat\mu/2^k + 2\pi i\theta^-\ .
\end{array}$$

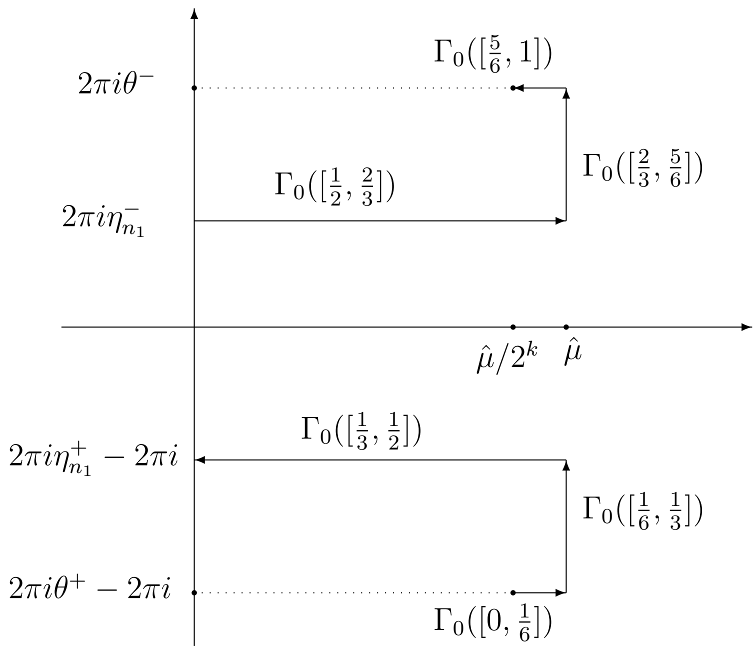

Figure 4 : The curve $\Gamma_0(t)$ and its direction of parametrisation

We then define $y(0,t) = \varphi_{c_0}(e^{\Gamma_0(t)})$. Although Γ_0 has a discontinuity at $t = 1/2$, by the choice of $\eta^\pm_{n_1}$, the map $y(0,t)$ is continuous everywhere and parametrises a Jordan arc.

The function $y(0,t)$ is stable in a neighbourhood of c_0 (cf. [DH1], last chapter), in other words, $y(s,t) = \varphi_{c_0+s}\varphi^{-1}_{c_0}(y(0,t)) = \varphi_{c_0+s}(e^{\Gamma_0(t)})$ is well defined and is an analytic continuation of $y(0,t)$. One may choose $\hat\mu$ small enough and n_1 large enough so that $y(s,t)$ belongs to a compact set Y in $\Omega_{+,0}$ for all $(s,t) \in \overline{\Delta_r} \times [0,1]$.

Define $x(s,t) = x(s)$ to be simply $Q^{kl}_{c_0+s}(c_0+s)$, where l is the minimal integer such that $x(0) \in \Omega_{-,0}$. We may choose a small enough neighbourhood of c_0 so that $x(s)$ belongs to a compact set X in $\Omega_{-,0}$.

We now define $\Gamma_n(t)$ for $n \geq 0$ inductively as follows: choose $\Theta \in (\theta^-,\theta^+)$. Let τ be the map from $[\Theta - 2\pi i, \Theta)$ to its self mapping z to $2z$ (mod $2\pi i\mathbb{Z}$). Then $2\pi i\theta^-$ and $2\pi i\theta^+ - 2\pi i$ are fixed points of τ^k. Set

$$\Gamma_n(0) = \frac{\hat\mu}{2^{n(k+1)}} + 2\pi i\theta^+ - 2\pi i\ ,\quad \Gamma_n(1) = \frac{\hat\mu}{2^{n(k+1)}} + 2\pi i\theta^-\ ,$$

and define $\Gamma_n([0,1/2))$ (resp. $\Gamma_n((1/2,1])$) to be the lift by τ^{-k} of the arc $\Gamma_{n-1}([0,1/2))$ (resp. $\Gamma_{n-1}((1/2,1])$) with the above initial condition. Note that

$$\Gamma_n(\frac{1}{6}) = \Gamma_{n-1}(0), \quad \Gamma_n(\frac{5}{6}) = \Gamma_{n-1}(1),$$

$$\lim_{t\to\frac{1}{2}^-} \Gamma_n(t) = 2\pi i\eta^+_{n_1+n} - 2\pi i \text{ and } \lim_{t\to\frac{1}{2}^+} \Gamma_n(t) = 2\pi i\eta^-_{n_1+n} \ .$$

Moreover

$$\bigcup_{n\geq 0} \Gamma_n([0,\frac{1}{6}]) = \{\mu + 2\pi i\theta^+ - 2\pi i \mid 0 < \mu \leq \hat{\mu}\}$$

$$\bigcup_{n\geq 0} \Gamma_n([\frac{5}{6},1]) = \{\mu + 2\pi i\theta^- \mid 0 < \mu \leq \hat{\mu}\} \ .$$

Note also that for $q : z \mapsto z^2$ in the model plane we have $q^k(e^{\Gamma_n(t)}) = e^{\Gamma_{n-1}(t)}$.

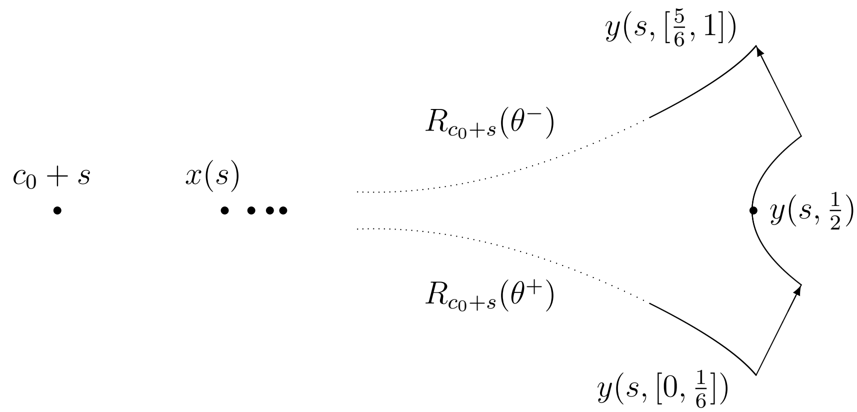

Figure 5 : The dynamical plane of Q_{c_0+s}

3.3 Application II of Proposition 2.2 for a fixed $\vec{w}$

Fix a choice of $c_0 + \vec{w}$ along which $I(c)/|I(c)| \to -1$, as in Lemma 3.2.

We can now apply Proposition 2.2 to the family f_s relative to $\vec{w}$ and the functions $x(s) - g(s)$ and $y(s,t) - g(s)$ defined in the previous sub-sections. There exists then $n_0 > 0$ and, for every $n \geq n_0$, $t \in [0,1]$, a point $s_n(t) \in \Delta_r$ such that $f^n_{s_n(t)}(x(s_n(t)) - g(s_n(t))) = y(s_n(t),t) - g(s_n(t))$. In other words,

$$Q^{kn}_{c_0+s_n(t)}(x(s_n(t)) = y(s_n(t),t) \ .$$

Lemma 3.6 *We have* $c_n(t) := c_0 + s_n(t) = \varphi_M(e^{\Gamma_{kl+n}(t)})$ *for* $n \geq n_0$ *and* $t \in [0,1]$.

Proof. Fix $n \geq n_0$ so that $s_n(t)$ exists.

We claim at first that $x(s_n(t)) = \varphi_{c_0+s_n(t)}(e^{\Gamma_n(t)})$ for $t \in [0,1]$ (cf. Figure 5). Set $Q = Q_{c_0+s_n(t)}$ and $\varphi = \varphi_{c_0+s_n(t)}$. Define $y_j(t) = Q^{k(n-j)}(x(s_n(t)))$ for $j = 0, \cdots, n$. We want to show $y_j(t) = \varphi \circ \exp(\Gamma_j(t))$ for $t \in [0,1]$ and $j \in \{0, 1, \cdots, n\}$, i.e. that the following diagram commutes for $j = 0, 1, \cdots, n-1$:

$$\begin{array}{ccc} \Gamma_{j+1}([0,1]) & \xrightarrow{\varphi\circ\exp} & y_{j+1}([0,1]) \\ {\scriptstyle \tau^k}\big\downarrow & & \big\downarrow{\scriptstyle Q^k} \\ \Gamma_j([0,1]) & \xrightarrow{\varphi\circ\exp} & y_j([0,1]). \end{array}$$

Since Q^k is univalent where the Fatou coordinates are defined and maps both $y_1(1/6)$ and $y_0(0)$ to $y_0(1/6)$, we must have $y_1(1/6) = y_0(0) = \varphi \circ \exp(\Gamma_0(0)) = \varphi \circ \exp(\Gamma_1(1/6))$. So the diagram commutes for $t = 1/6$, $j = 0$. The rest follows easily by lifting and by induction.

Let $c_n(t) = c_0 + s_n(t)$. Recall that $x(s) = Q^{kl}_{c_0+s}(c_0+s)$. We want to show now $c_n(t) = \varphi_{c_n(t)}(e^{\Gamma_{kl+n}(t)})$ (cf. Figure 5).

One can choose $c = c_0 + s$ close enough to c_0 so that Ω_s (given by Theorem 2.1) is disjoint from $c, Q_c(c), \cdots, Q_c^{kl-1}(c)$. This last set is the set of critical values of Q_c^{kl}. As a consequence, any arc γ in Ω_s from $Q_c^{kl}(x(s))$ to $x(s)$ would lift by Q_c^{-kl} to an arc starting from $x(s)$ and ending at a point $w(c)$ which is independent of the choice of γ. We claim now $w(c) = c$: On one hand, $w(c_0) = c_0$, which is an easy consequence of the pulling back of the Fatou coordinates $\Phi_{-,0}$. On the other hand, $w(c)$ is continuous on c, and is an element of $Q_c^{-kl}(Q_c^{kl}(c))$. As a consequence, $w(c) = c$ provided that c is close enough to c_0.

We can show that $w(c_n(t)) = \varphi_{c_n(t)}(e^{\Gamma_{kl+n}(t)})$ using the lifting argument as above. Therefore $c_n(t) = \varphi_{c_n(t)}(e^{\Gamma_{kl+n}(t)})$.

Hence $c_n(t) = \varphi_M(e^{\Gamma_{kl+n}(t)})$ for $t \in [0,1]$. ∎

3.4 Conclusions

Lemma 3.7 *(= Theorem 1.1.a)) The M-ray of angle $\theta^\pm$ lands at c_0 and is tangent to $c_0 + \vec{w}$ at c_0, for each direction $c_0 + \vec{w}$ along which $I(c)/|I(c)|$ converges to -1,*

Proof. Note that for $c_n(t)$ defined in Lemma 3.6,

$$\gamma^+ := \bigcup_{n \geq n_0} c_n([0, 1/6]) = \varphi_M(\exp(\{\mu + 2\pi i\theta^+ \mid 0 < \mu \leq \hat{\mu}\}))$$

is a germ of $R_M(\theta^+)$. By Lemma 3.2 and Corollary 3.3, the arc $\sigma(\{hw,\ h > 0 \text{ small}\})$ is tangent to $\mathbb{R}^+$ at 0. On the other hand, since $\frac{1}{n}/\sigma(s_n(t)) \to 1$ uniformly (by (4) of Proposition 2.2), the arc $\sigma(T_{-c_0}\gamma^+) = \sigma(\bigcup_{n\geq 0} s_n([0,1/6])$

is also tangent to $\mathbb{R}^+$ at 0 (where T_{-c_0} denotes the translation $z \mapsto z - c_0$). Therefore $\sigma(T_{-c_0}\gamma^+)$ and $\sigma(\{hw,\ h > 0 \text{ small}\})$ are tangent. So $R_M(\theta^+)$ lands at c_0 and is tangent to $c_0 + \vec{w}$ at c_0.

Similarly $R_M(\theta^-)$ lands at c_0 and is tangent to $c_0 + \vec{w}$. ■

Corollary 3.8 *(= Theorem 1.1.c)) The functions $g_\pm(v)$ defined in Lemma 3.1 and their multiplier functions $\lambda_\pm(v)$ satisfy $(g_\pm)'(0) \neq 0$ and $\lambda'_\pm(0) \neq 0$.*

Proof. Relying on the fact that φ_M is a conformal isomorphism we know that there is only one M-ray of angle θ^+. Combining with Lemma 3.7 we conclude that there is a unique $\vec{w}$ such that $I(c)/|I(c)|$ converges to -1 along $c_0 + \vec{w}$. It follows that $I(c)$ is univalent.

If $\sigma_\pm(v)$ were holomorphic on c, the function $I(c) = \sigma_+(v)\sigma_-(v)$ could not be univalent (since $\sigma_\pm(0) = 0$). Therefore none of $\sigma_\pm(v)$ is holomorphic on c. It follows that none of $g_\pm(v)$ can be holomorphic on c.

Since $I(c)$ is univalent, $I(c_0 + v^2)$ as a function of v has a double zero at 0. It follows that each $\sigma_\pm(v)$ has a simple zero at 0, i.e. $\sigma'_\pm(0) \neq 0$. It follows easily that $\lambda'_\pm(0) \neq 0$, as $\lambda_\pm(v) = 1 + 2\pi i \sigma_\pm(v)$. Finally

$$\lambda'_\pm(0) = \frac{d}{dv}(Q^k_{c_0+v^2})'(g_\pm(v))\Big|_{v=0} =$$

$$\frac{\partial}{\partial c}(Q^k_c)'(z)\Big|_{(c_0,z_0)} \cdot \frac{d}{dv}(c_0 + v^2)\Big|_{v=0} + \frac{\partial}{\partial z}(Q^k_c)'(z)\Big|_{(c_0,z_0)} \cdot \frac{d}{dv} g_\pm(v)\Big|_{v=0} =$$

$$= (Q^k_{c_0})''(z_0) \cdot g'_\pm(0)$$

so $g'_\pm(0) \neq 0$. ■

Corollary 3.9 *There exist constants $C, C' > 0$ independent of n, t such that $C'/n^2 \leq |c_n(t) - c_0| \leq C/n^2$, for all $t \in [0,1]$ and all $n \geq n_0$.*

Proof. The curve $(s, \sigma(s))$ is parametrised by $(v^2, \sigma_+(v))$. By the left limit in (4) of Proposition 2.2 and the fact that $C_0(t)$ is continuous on t, we know that $\sigma(s_n(t))/\frac{1}{n} \to 1$ uniformly. Now $\sigma(s_n(t)) = \sigma_+(v_n(t))$, where $v_n(t) = \sqrt{s_n(t)} \in \Delta$, and σ_+ is univalent by the above corollary, so $\sigma_+(v_n(t))/(b \cdot v_n(t)) \to 1$ (for some $b \neq 0$) uniformly. Therefore $b \cdot v_n(t)/\frac{1}{n} \to 1$ uniformly. Hence

$$\frac{c_n(t) - c_0}{1/(bn)^2} = \frac{v_n(t)^2}{1/(bn)^2} \to 1$$

uniformly. In particular the constants C, C' required in the Corollary exist. ■

Proof of Theorem 1.1.b), d) and e).

b) To get $c_n^+([0,1])$ and $c_n^-([0,1])$ for $n \geq n_0$ we just need to reparametrise $c_n([0,1/6])$ and $c_n([5/6,1])$ respectively. The rest follows from Lemma 3.6 and Corollary 3.9.

d) By construction and the fact that $R_M(\theta^\pm)$ land at c_0, for $n \geq n_0$, the set

$$c_n([\frac{1}{6},\frac{5}{6}]) \cup \bigcup_{m\geq n} c_m([0,\frac{1}{6}]) \cup \bigcup_{m \geq n} c_m([\frac{5}{6},1]) \cup \{c_0\}$$

bounds a Jordan domain (called a puzzle piece, Figure 6). Let us denote it by P_n. We have $\mathrm{diam}(P_n) \leq C/n^2$ due to the above corollary. Moreover $\partial P_n \cap M$ has only two points c_0 and $c_n(1/2)$. So $M \cap \overline{P_n}$ is connected (this is a general fact, cf. for example Lemma 2.1 of [T]). Therefore $M \setminus wake(\theta^\pm)$ is locally connected at c_0 and $M \setminus (wake(\theta^\pm) \cup \{c_0\})$ is connected.

e) Given any $R_M(\zeta)$ outside the wake, there is n such that $\zeta > \eta_n^+$ or $\zeta < \eta_n^-$. So the corresponding impression is contained in $(M \setminus wake(\eta_n^\pm)) \cup \{c_n(1/2)\}$, hence is disjoint from c_0. ∎

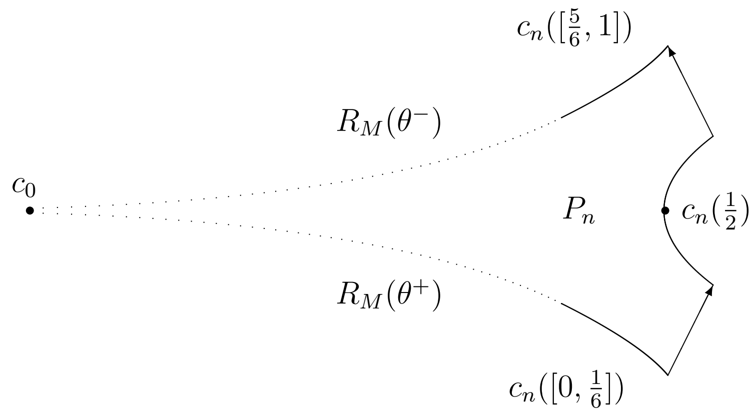

Figure 6 : The puzzle piece P_n

4 Application III of Proposition 2.2

Here is a qualitative version and the most common form of parabolic implosion:

Corollary 4.1 *In the set up of §2, for any $x \in \Omega_{-,0}$, $y \in \Omega_{+,0}$, there are $s_n \to 0$ and $k_n \to \infty$ such that $f_{s_n}^{k_n}$ converges uniformly on compact sets of $\Omega_{-,0}$ to a holomorphic map g with $g(x) = y$.*

Proof. Apply Proposition 2.2 to $x(s,t) \equiv x$ and $y(s,t) \equiv y$, we find a sequence $s_n \to 0$ such that $f_{s_n}^n(x) = y$. We will see that $f_{s_n}^n|_{\Omega_{-,0}}$ converges uniformly on compact sets to a function g satisfying $g(x) = y$ and $g \circ f_0 = f_0 \circ g$.

Recall from the proof of Theorem 2.1 that for $\alpha(s)$ defined as $\lambda(s) = e^{2\pi i\alpha(s)}$ with $\alpha(s)$ close to 0, the function $\frac{1}{\alpha(s)} - \frac{2\pi i}{\lambda(s)-1} = \frac{1}{\alpha(s)} - \frac{1}{\sigma(s)}$ is continuous in $\overline{\Delta_r}$, in particular has a finite value at $s = 0$.

In [S2], 3.4, Shishikura proved that for any sequence $s_n \to 0$ such that $\frac{1}{\alpha(s_n)} = k_n + \beta_n$ with $k_n \in \mathbb{N}$, $k_n \to +\infty$ and $\beta_n \to \beta \in \mathbb{C}$, the sequence $f_{s_n}^{k_n}|_{\Omega_{-,0}}$ converges uniformly to a limit function.

Now assume that s_n is a solution of (5) . We have

$$\frac{1}{\alpha(s_n)} = n + \tilde{x}(s_n) - \tilde{y}(s_n) + (\frac{1}{\alpha(s_n)} - \frac{1}{\sigma(s_n)})$$

with $\tilde{x}(s_n) - \tilde{y}(s_n) \to \tilde{x}(0) - \tilde{y}(0)$ as $n \to \infty$. So the hypothesis of Shishikura's theorem is satisfied with $k_n = n$. Therefore $f_{s_n}^n|_{\Omega_{-,0}}$ converges uniformly to a limit function g. Clearly $g(x) = y$, g is holomorphic and $g \circ f_0 = f_0 \circ g$. ■

A Non-primitive case

Assume now that Q_{c_0} is a quadratic polynomial with a k-periodic point of multiplier $e^{2\pi ip/q}$ with $q > 1$. In this case a similar theory of parabolic implosion can be applied to show:

Theorem A.1 *(Douady-Hubbard, [DH2])*
a) (landing property) For the two angles $\theta^\pm$ for which $R_{c_0}(\theta^\pm)$ land at a common parabolic point z_0 and are adjacent to the Fatou component containing c_0, the M-rays $R_M(\theta^\pm)$ land at c_0.
b) (parametrisation) A germ of $R_M(\theta^\pm)$ can be written as $\bigcup_{n\geq n_0} c_n^\pm([0,1])$ for some $n_0 > 0$ satisfying: c_n injective, $c_{n+1}^\pm(0) = c_n^\pm(1)$, and, for $T(z) = z^2$ and some $C_1, C_2 > 0$,

$$\varphi_M \circ T^{kq} \circ \varphi_M^{-1}(c_n^\pm(t)) = c_{n-1}^\pm(t) \quad \text{and} \quad \frac{C_1}{n} \leq |c_n^\pm(t) - c_0| \leq \frac{C_2}{n} . \qquad (8)$$

c) (transversality) The local solution of $Q_c^k(z) - z = 0$ can be written as $(c, h(c))$ with h holomorphic, $h(c_0) = z_0$ and $h'(c_0) \neq 0$. Moreover the multiplier function $\lambda(c) = (Q_c^k)'(h(c))$ satisfies $\lambda(c_0) = e^{2\pi ip/q}$ and $\lambda'(c_0) \neq 0$.

The inequalities in (1) and (8) can be used to prove a result of the degree of tangency. See Appendix C.

B Misiurewicz case

A parallel theory for repelling periodic points can be sketched as follows:

Theorem B.1 *Let $f_s(z) = \lambda(s)z + O(z^2)$ be a holomorphic family of maps such that $|\lambda(0)| > 1$. Then for s in a neighbourhood of 0, there are neighbourhood U_s of 0 and linearising coordinates Φ_s such that for any z, n with $f_s^j(z) \in U_s$, $j = 0, 1, \cdots, n$, we have*

$$\Phi_s(f_s^n(z)) = \lambda(s)^n \Phi_s(z)$$

and $\Phi_s(z)$ depends holomorphically on (s, z).

Proposition B.2 (Repelling orbit correspondence). *In the same setting, let $x(s)$, $y(s)$ be two holomorphic functions with $x(0) = 0$, $x(s) \not\equiv 0$ and $y(0) \in U_0 \smallsetminus \{0\}$. Then there is a sequence $s_n \to 0$ for $n > N$ such that $f_{s_n}^n(x(s_n)) = y(s_n)$ and $f_{s_n}^j(x(s_n)) \in U^{s_n}$ for $j = 0, 1, \cdots, n$. Moreover, if $\lambda'(0) \neq 0$, there are constants C_1, C_2 such that $\frac{C_1}{|\lambda(0)|^n} < |s_n| < \frac{C_2}{|\lambda(0)|^n}$.*

Proof. Set $\tilde{x}(s) = \Phi_s(x(s))$, $\tilde{y}(s) = \Phi_s(y(s))$. They are holomorphic functions with $\tilde{x}(0) = 0$, $\tilde{x}(s) \not\equiv 0$ and $\tilde{y}(0) \neq 0$. Given n, we seek for solution of

$$\lambda(s)^n \tilde{x}(s) = \tilde{y}(s) \ .$$

With the help of power series expansions one can carry out a similar proof as the one of Proposition 2.2. ∎

One can find extensive applications of Proposition B.2 in [DH3] and [Mc]. We present here an application to the Mandelbrot set (Parts 1) and 2) are due to Douady and Hubbard, Parts 3) and 4) are due to Tan Lei):

Theorem B.3 *Let c_0 be a Misiurewicz point, more precisely there are $k \geq 0$, $l \geq 0$ minimal such that $Q_{c_0}^{k+l}(0) = Q_{c_0}^l(0)$ and $l > 0$. Then*
1) The angles of M-rays landing at c_0 coincide with the angles of Q_{c_0}-rays landing at c_0
2) $\frac{d}{dc}(Q_c^{k+l}(0) - Q_c^l(0))|_{c=c_0} \neq 0$.
3) M is locally connected at c_0.
4) The impression of the prime end of any other M-ray does not contain c_0.

Details of proofs can be found in [T].

C degree

The following illustrates a quick application of the inequalities in Theorems 1.1, A.1, although the statement is not in its sharpest form.

Theorem C.1 *For $c_0 \neq 1/4$ a primitive parabolic point, there is a unique hyperbolic component W attached to c_0, and for the two rays $R_M(\theta^\pm)$ landing* *at c_0, we have that the degree of tangency at c_0 of $R_M(\theta^+)$ and $R_M(\theta^-)$ is 2^+ (i.e. is larger than or equal to 2, but smaller than $2+\mu$ for any $\mu > 0$), as well as the degree of tangency at c_0 of $R_M(\theta^i)$ and ∂M, $i = +$ or $-$. For c_0 a non-primitive parabolic point, there are exactly two hyperbolic components W_0 and W_1 attached to c_0 and the two rays $R_M(\theta^\pm)$ landing at c_0 separate them, moreover the degree of tangency at c_0 of $R_M(\theta^i)$ and ∂W_j is 2^+ , $i = +$ or $-$, $j = 0, 1$, as well as the degree of tangency at c_0 of ∂W_0 and ∂W_1.*

Proof. For the existence of such hyperbolic components see Corollary D.4 below.

Here we only prove the part about degree of tangency, which is due to Tan Lei.

Assume at first c_0 is non-primitive. We need an inequality between hyperbolic metric and Euclidean metric: Let U be a simply connected domain in $\mathbb{C}$. Then for $x, y \in U$, we have

$$|y - x| \le e^{2d_U(x,y)} \cdot d(x, \partial U) \ , \tag{9}$$

where d_U denotes the hyperbolic metric of U. For a proof, see Appendix A of [ST].

Let W_0, W_1 be the two hyperbolic components attached to c_0. We are going to show for n large enough, and $c_n^\pm = c_n^\pm(0)$ (in the notation of Theorem A.1),

$$|c_n^+ - c_{n+1}^+| \le B \cdot d(c_n^+, W_0 \cup W_1), \quad |c_n^- - c_{n+1}^-| \le B \cdot d(c_n^-, W_0 \cup W_1) \ , \tag{10}$$

where B is a constant.

To prove this estimate, we need the relationship between $c_n^\pm$ in (8) and the help of hyperbolic metric. We will only work on the $+$ case. The other one is similar. Take a simply connected domain V in the model plane so that $T^{-kq}(V)$ has a component V' contained in V and that the two sets $U = \varphi_M(V)$, and $U' = \varphi_M(V')$ both contain c_n^+ for $n \ge N$. Therefore, for V_n the component of $T^{-kqn}(V)$ with $e^{2\pi i \theta^+}$ on the boundary, and $U_n = \varphi_M(V_n) \subset U$,

$$d_U(c_{n+N}^+, c_{n+N+1}^+) \le d_{U_n}(c_{n+N}^+, c_{n+N+1}^+) = d_U(c_N^+, c_{N+1}^+) = A \ .$$

Now apply (9) we get

$$|c_{n+N}^+ - c_{n+N+1}^+| \le e^{2d_U(c_{n+N}^+, c_{n+N+1}^+)} \cdot d(c_{n+N}^+, \partial U) \le B \cdot d(c_{n+N}^+, \partial U) \ .$$

In particular, since $(W_0 \cup W_1) \cap U = \emptyset$,

$$|c_n^+ - c_{n+1}^+| \le B \cdot d(c_n^+, W_0 \cup W_1)$$

for any large n. This is (10) .

Assume by contradiction that the θ^+-ray is tangent to ∂W_0 with a degree of tangency at least $2+\mu$, $\mu > 0$. Then

$$d(c_n^+, W_0 \cup W_1) \le d(c_n^+, W_0) \le C \cdot |c_n^+ - c_0|^{2+\mu} \ . \tag{11}$$

So

$$\frac{C_1}{n} \le |c_n^+ - c_0| \le \sum_{k\ge n} |c_k^+ - c_{k+1}^+| \le B \cdot \sum_{k\ge n} d(c_k^+, W_0 \cup W_1)$$
$$\le B' \cdot \sum_{k\ge n} |c_k^+ - c_0|^{2+\mu} \le B'' \cdot \sum_{k\ge n} \frac{1}{k^{2+\mu}}$$

where B', B'' are constants, the first and the last inequalities is due to (8) , the second one is trivial, the third one is due to (10) and the fourth one is due to the assumption (11) . Therefore

$$\frac{C_1}{B''} \le \sum_{k\ge n} \frac{n}{k^{2+\mu}} \le \sum_{k\ge n} \frac{1}{k^{1+\mu}} \ .$$

This is a contradiction to the fact that $\sum \frac{1}{k^{1+\mu}}$ converges.

This shows that the rays have degree 2^+ tangency to each ∂W_0 and ∂W_1. A similar calculation using (1) would show that in the primitive case, the two rays $R_M(\theta^\pm)$ are tangent to each other at c_0 with degree 2^+ tangency.■

D A short account of relative results

(One can find in [Mi], [Sc1] and [Sc2] radically different approaches of results in this appendix).

Here we give a short exposition of a proof for

Theorem D.1 *For c Misiurewicz or on the boundary of a hyperbolic component, M is locally connected at c.*

The part concerning Misiurewicz points is already established in Theorem B.3. We will only deal with the remaining cases.

We will sketch a proof of a stronger result (Theorem D.3) and show how to deduce from it the above result. We will also give a list of important corollaries.

For $c_0 \ne 1/4$ a parabolic point, define $Ang(c_0)$ to be the set of two angles such that the corresponding dynamical rays for Q_{c_0} land at the parabolic point z_0 on the boundary of the Fatou component containing c, and adjacent to this Fatou component. For $c_0 = 1/4$ we set by convention $Ang(c_0) = \{0, 1\}$.

Our starting point is the following combination of Theorems 1.1.a and A.1.a, proved by parabolic implosion technique in the case $c_0 \ne 1/4$ (in the case $c_0 = 1/4$ it can be checked by hand):

Theorem D.2 *Let c_0 be a parabolic point. Denote by $\theta^{\pm}$ the two angles in $Ang(c_0)$. Then the M-rays $R_M(\theta^{\pm})$ land at c_0.*

Now define $wake(c_0)$ to be the connected component not containing 0 of $\mathbb{C} \setminus (R_M(\theta^+) \cup R_M(\theta^-) \cup \{c_0\})$.

Given any hyperbolic component W of M it is quite easy to see (with the help to quasi-conformal surgery) that the multiplier $\rho(c)$ of an attracting periodic point $z(c)$ for each $c \in W$ realizes a conformal homeomorphism from W onto the unit disc $\mathbb{D}$, and extends to a homeomorphism of the closure (see [DH2], pp.108-109). Denote by $\rho^{-1}(1)$ the root of W. The parabolic points on ∂W are exactly $\rho^{-1}(e^{2\pi i \mathbb{Q}})$.

Theorem D.3 *Let c_0 be a parabolic point with $Ang(c_0) = \{\theta^-, \theta^+\}$.*

a) There is a holomorphic function $\beta_{c_0}(c)$ defined on $wake(c_0)$ such that $\beta_{c_0}(c)$ is the common landing point of $R_c(\theta^+)$ and of $R_c(\theta^-)$.

b) There is a hyperbolic component $W(c_0)$ of M with root c_0 and $W(c_0) \subset wake(c_0)$. Moreover for the common landing point z_0 of $R_{c_0}(\theta^{\pm})$,

$$\lim_{c \in wake(c_0),\, c \to c_0} \beta(c) = z_0 \ .$$

c) Structure of sub-wakes of $W(c_0)$ (no ghost limbs):

$$M \cap \overline{wake(c_0)} \ \setminus \bigcup_{c' \in \partial W \setminus \{c_0\},\ \text{parabolic}} wake(c') \ = \ \overline{W(c_0)} \ . \qquad (12)$$

d) M is locally connected at c for any $c \in \partial W(c_0)$ which is non-parabolic; $M \cap \overline{wake(c_0)}$ is locally connected at c_0; and $M \setminus wake(c')$ is locally connected at c' for any parabolic $c' \in \partial W(c_0) \setminus \{c_0\}$.

Proof.

a) can be found in [Mi], Theorem 3.1. It is proved by an elementary argument. A sketch is included in Theorem D.8 below.

b) is due to a parabolic perturbation argument (in the non-imploded direction). See [Mi], section 4.

c) We will follow essentially the route of D. Sørensen ([Sø]).

At first we apply a) and b) to every parabolic point on the boundary of $W(c_0)$. Combining with the tuning algorithm (cf. for example [Ha1, §7] and [Sø]), we obtain the following: for any parabolic $c' \in \partial W(c_0) \setminus \{c_0\}$ and any $c \in wake(c')$, the common landing point $\beta_{c'}(c)$ of $R_c(\theta^{\pm}(c'))$ has a constant and non-trivial combinatorial rotation number $\hat{\rho}(c')$. Moreover, if we choose a converging non-constant sequence c'_n of such parabolic points, $\hat{\rho}(c'_n) \to \infty$. This allows us to apply the powerful Yoccoz inequality (see for example [Hu] and [Sø]), to show that

$$\operatorname{diam}(M \cap \overline{wake(c'_n)}\,) \to 0 \quad \text{as } n \to \infty \ . \qquad (13)$$

Figure 7 : The sub-wake structure of $W(c_0)$

Two more ingredients are needed to end the proof:
A. The parabolic points are dense in $\partial W(c_0)$, moreover, a consequence of the tuning algorithm gives $\sum_{c' \in \partial W(c_0) \smallsetminus \{c_0\},\ \text{parabolic}} \theta^+(c') - \theta^-(c') = \theta^+ - \theta^-$;
B. Every point of ∂M is accumulated by points in the hyperbolic components (by a normal family argument, cf. Branner [Br]).

Denote by M' the difference of the left set in (12) to the right set. Assume it is not empty. Let $c \in \partial M' \smallsetminus \overline{W(c_0)}$. Part A says that there is no room left for M' to contain hyperbolic components. Therefore by Part B the point c must be the limit of some $c_n \in wake(c'_n)$, for some non-constant converging parabolic sequence $c'_n \in \partial W(c_0) \smallsetminus \{c_0\}$. Applying (13) , we get $c \in \partial W(c_0)$. A contradiction.

d) This is a trivial topological consequence of c): Let $c \in \partial W(c_0)$ be non-parabolic. Choose c'_n, c''_n two sequences of parabolic points on $\partial W(c_0)$ tending to c from opposite sides, with $\theta^+(c'_n) < \theta^-(c''_n)$. Then the short arc in $\partial W(c_0)$ connecting c'_n and c''_n together with the two M-rays $R_M(\theta^+(c'_n))$, $R_M(\theta^-(c''_n))$ bounds a simply connected region U_n. Let $M_n = M \cap \overline{U_n}$. Then $\{M_n\}$ is a nested sequence of connected neighbourhoods in ∂M of c. As $\bigcap_n M_n$ is disjoint from any of the sub-wakes, and $\bigcap_n M_n \subset M$ (as M is closed) we conclude from (12) that $\bigcap_n M_n = \{c\}$.

For $c = c_0$, the points c'_n, c''_n can be chosen such that $\theta^+(c'_n) \nearrow \theta^+$ and $\theta^-(c''_n) \searrow \theta^-$, and U_n the union of two simply connected regions bounded by the short arc in $\partial W(c_0)$ connecting c'_n and c''_n and the four M-rays $R_M(\theta^+(c'_n))$, $R_M(\theta^-(c''_n))$ and $R_M(\theta^\pm)$. The rest follows.

Similar argument can be carried out for $c = c' \in \partial W(c_0)$ parabolic. ■

Proof of Theorem D.1. Let W be a hyperbolic component with root c_0.

Assume at first that c_0 is primitive. Theorem D.3.d) applied to $wake(c_0)$ shows that $M \cap \overline{wake(c_0)}$ is locally connected at c_0. Theorem 1.1.d shows that $M \smallsetminus wake(c_0)$ is locally connected at c_0. As a consequence M is locally connected at c_0.

Another consequence of this is that $W = W(c_0)$, for $W(c_0)$ given by Theorem D.3.b). In other words $W(c_0)$ is the unique hyperbolic component with root c_0. We claim that this is also true whether c_0 is primitive or not.
Proof. We proceed by induction on the period of W. For period 1, there is a unique hyperbolic component which is the central cardioid. Its root $1/4$ is primitive and the assertion is true. Assume it is true for all hyperbolic components of period up to k. Assume W is of period $k + 1$. We want

to show $W = W(c_0)$. If c_0 is primitive we are done. Assume c_0 is non-primitive. Then by a perturbation argument as in [Mi], section 4 (see also [Ha2]), one can show that there is a hyperbolic component W_1 with root c_1, with $c_0 \in \partial W_1 \setminus \{c_1\}$ and with period less than or equal to k. By induction $W_1 = W(c_1)$. If $W \neq W(c_0)$, Theorem D.3.c) applied to $wake(c_0)$ shows that $W \cap wake(c_0) = \emptyset$, and applied to $wake(c_1)$ shows that $W \cap (M \setminus wake(c_0)) = \emptyset$. This is a contradiction. ■

Coming back to the proof of Theorem D.1, we may already conclude that M is locally connected at any $c \in \partial W$ which is non-parabolic, by applying Theorem D.3.d) to $W = W(c_0)$.

We may assume now that c_0 is non-primitive. By the above argument $c_0 \in \partial W(c_1)$ for some different parabolic point c_1. Theorem D.3.d) applied to $wake(c_0)$ shows that $M \cap \overline{wake(c_0)}$ is locally connected at c_0; and then applied to $wake(c_1)$ shows that $M \setminus wake(c_0)$ is locally connected at c_0. As a consequence M is locally connected at c_0.

For any $c' \in \partial W$ parabolic, it is the root of a hyperbolic component W' by Theorem D.3.b). We can then repeat the same argument above with W replaced by W', and conclude that M is locally connected at c'. ■

Let us restate a few important corollaries in the proof:

Corollary D.4 *Every parabolic point is the root of a unique hyperbolic component. Moreover c_0 is on the boundary of precisely two (resp. one) hyperbolic components if c_0 is non-primitive (resp. primitive).*

Corollary D.5 *For c_0 a parabolic point $M \setminus \{c_0\}$ has exactly two connected components.*

Corollary D.6 *For c_0 a parabolic point and θ any angle not in $Ang(c_0)$. The impression of the prime end corresponding to $R_M(\theta)$ does not contain c_0. In particular c_0 is the landing point of precisely two M-rays, i.e. those with angle in $Ang(c_0)$.*

Note that the same result as Corollary D.6 for c_0 a Misiurewicz point is stated in Theorem B.3.

The following important result comes as a corollary: For any $\theta \in \mathbb{T}$, define $L(\theta)$ to be the landing point of $R_M(\theta)$ if it exists. For any c a Misiurewicz point, define $Ang(c)$ to be the set of angles such that the corresponding dynamical ray for Q_c lands at c.

Corollary D.7 (Douady-Hubbard-landing-theorem)

1. The map L is well defined on the set of rationals with even denominators and maps it onto the set of Misiurewicz parameters. Moreover for any c

Misiurewicz $L^{-1}(c) = Ang(c)$, and $Ang(c)$ consist of finitely many rationals with even denominators.

2. The map L is well defined on the set of rationals with odd denominators and maps it onto the set of parabolic parameters. Moreover for any c parabolic $L^{-1}(c) = Ang(c)$.

Proof. By an elementary argument it is quite easy to see that L is well defined on the set of rationals and maps it into the set of Misiurewicz and parabolic parameters (cf. [PR] in this volume or [GM]).

Dynamical properties tell us that $Ang(c)$ consists of non-empty but finitely many rationals with even denominators for c Misiurewicz; and of two rationals with odd denominators for c parabolic.

Our theorem B.3 establishes $L^{-1}(c) = Ang(c)$ for c Misiurewicz, and Corollary D.6 establishes $L^{-1}(c) = Ang(c)$ for c parabolic. ■

Remark. One can find a more combinatorial approach of Corollary D.7 in [Mi] and [Sc1]. One hidden difficulty in proving Corollary D.7.1 is that for θ a rational with even denominator, $L(\theta)$ must be a Misiurewicz point, or equivalently can not be a parabolic point. Our proof, as well as the original proof of Douady-Hubbard, relies on results about parabolic points. However, there are ways to avoid this, for example one can use Thurston's algorithm to construct a parameter c such that $\theta \in Ang(c)$ (a rather involved method), or one can apply the pretty elementary method given by Petersen and Ryd (cf. [PR], in this volume).

The following is a refinement of Theorem D.3.a) and gives a dynamical description of $wake(c_0)$:

Theorem D.8 *Let c_0 be a parabolic point with $Ang(c_0) = \{\theta^-, \theta^+\}$. Then $wake(c_0)$ coincides with the set of $c \in \mathbb{C}$ such that $R_c(\theta^-)$ and $R_c(\theta^+)$ both land, and land at the same repelling periodic point.*

Proof. The main ideas are the following: At first this is true for $c \in \mathbb{C} \setminus M$, by a combinatorial study conducted by Milnor ([Mi], Lemmas 2.6, 2.9 and 2.11). An argument of P. Haïssinsky ([Ha1], Proposition A.1) shows that the landing points $z_\pm(c)$ of $R_c(\theta^\pm)$ are holomorphic in c, on any connected open set where they are both defined, and moreover on this connected open set either $z_+(c) \equiv z_-(c)$ or $z_+(c) \neq z_-(c)$ for any c. We can then conclude easily by combining these two arguments. For details, see [Mi], Theorem 3.1. ■

The following function is as important as the function $\beta_{c_0}(c)$ defined in Theorem D.3.a).

Theorem D.9 *Let W be a hyperbolic component with root c_0. Let $\alpha_{c_0}(c)$ be an attracting periodic point for $c \in W$ depending holomorphically on c. Then*

$\alpha_{c_0}(c)$ extends to a holomorphic map on the entire $wake(c_0)$, is a periodic point of Q_c with constant period, repelling for $c \in wake(c_0) \setminus \overline{W}$, and coincides, in each sub-wake of W, i.e. $wake(c')$, $c' \in \partial W$ parabolic, with the holomorphic function $\beta_{c'}(c)$ (which is the landing point of $R_c(\theta^\pm(c'))$).

Proof. This is basically a corollary of Theorem D.3. By the property of the multiplier function $\rho(c)$ and the implicit function theorem, $\alpha_{c_0}(c)$ extends holomorphically to a neighbourhood of $\overline{W} \setminus \{c_0\}$ in $wake(c_0)$. Now fix $c' \in \partial W \setminus \{c_0\}$ parabolic. Let W' be the unique hyperbolic component with root c' (Corollary D.4). Theorem D.3.b) applied to $W' = W(c')$ says $\alpha_{c_0}(c) = \beta_{c'}(c)$ on W'. Since $\beta_{c'}(c)$ is defined and holomorphic on the entire sub-wake $wake(c')$, so is $\alpha_{c_0}(c)$, and they coincide throughout the subwake.

We need now the help of a third holomorphic function $\xi(c)$ which is defined to be $\alpha_{c_0}(c_1)$ for some $c_1 \in wake(c') \setminus M$ (with c' as above), and to be its analytic continuation throughout $\mathbb{C} \setminus (M \cup R_M(0))$ following the holomorphic motion of the Cantor Julia set. Clearly $\alpha_{c_0}(c) = \xi(c)$ on a open set. So $\alpha_{c_0}(c)$ extends to $\mathbb{C} \setminus (M \cup R_M(0))$ and coincides with $\xi(c)$ there. In particular $\alpha_{c_0}(c)$ extends to $wake(c_0) \setminus M$.

Finally by (12) every point $c \in wake(c_0)$ is either outside of M, or in one of the sub-wakes, or in $\overline{W}$. Therefore the theorem. ■

Generally speaking $\alpha_{c_0}(c)$, as a periodic point of Q_c, can be analytically continued along any path in $\mathbb{C} \setminus \{\text{finitely many parabolic points}\}$. But we may loose tract of the rays landing at $\alpha_{c_0}(c)$, it may become non-repelling, and we do not know a priori whether these parabolic points present real branch points or not. Theorem D.9 answers these questions at least in $wake(\theta^\pm(c_0))$.
Remark. In case W is the central cardioid of M, the proof of D.3, D.9 can be made more elementary and thus lead to a simpler proof of local connectivity of M at $c_0 = 1/4$. See [GM], Appendix C.

Our final corollary is the following Lavaurs continuity result ([DH2], Chapter XVII: Une propriété de continuité), which is a stronger version of Theorem D.3.b):

Theorem D.10 (Lavaurs) *For θ a rational angle, denote by $\gamma_\theta(c)$ the landing point of $R_c(\theta)$ if it exists. Then $\gamma_\theta(c)$ is continuous on $\mathbb{C} \setminus \bigcup_{k\geq 1} R_M(2^k\theta)$. Here the rays $R_M(\eta)$ do not include their landing points. This in particular says that $\gamma_\theta(c)$ is continuous at the landing points of these rays.*

Proof. We will only treat the case θ has odd denominator. The other case is a much easier exercise.

On the open set

$$\mathbb{C} \setminus \overline{\bigcup_{k\geq 1} R_M(2^k\theta)}$$

$\gamma_\theta(c)$ is well defined (Douady-Hubbard) and holomorphic ([Ha1], appendix). So we just need to show that $\gamma_\theta(c)$ is continuous at $L(\theta) = c_0$, the landing point of $R_M(\theta)$ (as this will imply that $\gamma_{2^l\theta}(c)$ for $l \geq 1$ is continuous at $L(\theta)$ and as a consequence $\gamma_\theta(c)$ is continuous at $L(2^m\theta)$ for $m \geq 1$).

Let U be a neighbourhood of c_0 such that $U \setminus R_M(\theta)$, $U \cap wake(c_0)$ and $U \setminus \overline{wake(c_0)}$ are all simply connected. Set $\gamma(c) = \gamma_\theta(c)$. Denote as before, by z_0 the parabolic point of Q_{c_0} on the boundary of the Fatou component of c_0.

Case 1. The point c_0 is primitive. In this case the other angle θ' in $Ang(c_0)$ is not in the form $2^l\theta$. So $\gamma(c)$ is well defined in $U \setminus R_M(\theta)$. But $\gamma(c) = \beta_{c_0}(c)$ for $c \in U \cap wake(c_0)$ (Theorem D.3.a)) and $\beta_{c_0}(c) = g_i(v)$, $i = +$ or $-$, where $v = \sqrt{c - c_0}$ and $g_\pm(v)$ is defined in Section 3.1 (they are the two local periodic points perturbed from z_0). As $U \setminus R_M(\theta)$ is simply connected, $\gamma(c) \equiv g_i(\sqrt{c - c_0})$ on $U \setminus R_M(\theta)$ and is continuous at c_0 (as $g_i(\sqrt{c - c_0})$ is).

Case 2. The point c_0 is non-primitive. This is the case where the two angles θ, θ' in $Ang(c_0)$ are in the same orbit by angle doubling and $\gamma(c)$ is defined on $U \setminus (R_M(\theta) \cup R_M(\theta'))$, and is holomorphic on $U \setminus (R_M(\theta) \cup R_M(\theta') \cup \{c_0\})$ ([Ha1], appendix).

In $U \cap wake(c_0)$ we have $\gamma(c) = \beta_{c_0}(c)$ and $\gamma(c_0) = z_0$ (Theorem D.3.a) and b)). So γ is continuous at c_0 on $(U \cap wake(c_0)) \cup \{c_0\}$.

In $U \setminus \overline{wake(c_0)}$, let W_1 the hyperbolic component containing c_0 on the boundary. As $c \in U \cap W_1$ tending to c_0 radically, the same perturbative argument of Milnor ([Mi], section 4) shows that $\gamma(c)$ is repelling for Q_c and tends to z_0. Let $z(c)$ be this repelling periodic point. It has analytic continuation along any path in $U \setminus \{c_0\}$ (shrinking U if necessary) and is periodic of constant period for Q_C. It is therefore a well defined holomorphic function in $U \setminus \overline{wake(c_0)}$ as this last set is simply connected. As $\gamma(c)$ is holomorphic, we have $\gamma(c) \equiv z(c)$ on $(U \setminus \overline{wake(c_0)}) \cup \{c_0\}$ and continuous at c_0, as $z(c)$ is. ∎

E Local connectivity via Mandelbrot-like families (following Hubbard)

Here we sketch a proof of the local connectivity of M at a primitive parabolic point using the theory of Mandelbrot-like families and the fact that M is locally connected at 1/4. We follow essentially the route indicated by Hubbard ([Hu]).

Let c_0 be a primitive parabolic point, more precisely Q_{c_0} has a k periodic point of multiplier 1. In [Ha1] in this volume, it is shown that there are open discs U_0, U_1 containing c_0, open discs B_c (resp. B'_c) whose boundary undergo a holomorphic motion, such that $\mathbf{f} = \{Q_c^k : B'_c \to B_c,\ c \in U_0\}$

forms a Mandelbrot-like family, with $\overline{U_1} \subset U_0$, U_1 is the set of c such that the critical point of Q_c^k in B_c' does not escape B_c' after one iteration, $\overline{U_1} \cap M$ is connected (it is in fact a puzzle piece in the sense of Yoccoz), and, for $M_{\mathbf{f}}$ the connectedness locus and $\chi : M_{\mathbf{f}} \to M$ the straightening map, $M_{\mathbf{f}} \subset M \cap U_0$ and χ is a homeomorphism with $\chi(c_0) = 1/4$.

Let U_n to be the set of $c \in U_0$ such that the critical point of Q_c^k in B_c' does not escape B_c' after n iterations. Then U_n is again a puzzle piece and $\bigcap_n U_n = M_{\mathbf{f}}$.

Denote by W the unique hyperbolic component with c_0 as root (Corollary D.4). Choose c_n', c_n'' two sequences of parabolic points on ∂W converging to c_0 such that $\theta^+(c_n') < \theta^+$ and $\theta^-(c_n'') > \theta^-$. Denote by V_n the simply connected domain containing $R_M(\theta^+)$ bounded by the short arc in ∂W connecting c_n' and c_n'' and the two rays $R_M(\theta^+(c_n'))$ and $R_M(\theta^-(c_n''))$ (here we need Theorem A.1 for the landing of these rays). Denote by M_n the set $\chi(\overline{V_n} \cap M_{\mathbf{f}})$.

Set $L_n = \overline{V_n \cap U_n}$. Then $L_n \subset L_{n-1}$ and $L_n \cap \partial M$ is a connected neighbourhood (in ∂M) of c_0. We now show that $\bigcap_n L_n = \{c_0\}$:

$$c_0 \in \bigcap_n L_n = (\bigcap_n L_n) \cap M_{\mathbf{f}} = \bigcap_n (L_n \cap M_{\mathbf{f}}) \subset \bigcap_n (\overline{V_n} \cap M_{\mathbf{f}}) =$$

$$\bigcap_n \chi^{-1}(M_n) = \chi^{-1}(\{\frac{1}{4}\}) = \{c_0\} ,$$

where the first equality is due to $\bigcap_n L_n \subset \bigcap_n U_n = M_{\mathbf{f}}$, the second is trivial, the set inclusion is because $L_n \subset \overline{V_n}$, the next equality is by definition of M_n and the fact that χ is a homeomorphism, the following is due to the facts that $\lim_{n\to\infty} \chi(c_n') = \lim_{n\to\infty} \chi(c_n'') = 1/4$ (as χ is continuous) and that M is locally connected at $1/4$ (see [GM] Appendix C for an elementary proof), and the final inequality is by properties of χ.

References

[Br] B. Branner, *The Mandelbrot set*, Proc. Symp. Appl. Math., AMS 39 (1989), 75-105.

[DH1] A. Douady & J. Hubbard, *Etudes dynamique des polynômes complexes*, I, Pub. Math. d'Orsay, 84-02.

[DH2] A. Douady & J. Hubbard, *Etudes dynamique des polynômes complexes*, II, Pub. Math. d'Orsay, 85-05.

[DH3] A. Douady & J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. ENS, vol. 18, pp. 287-343, 1985.

[GM] L. Goldberg & J. Milnor, *Fixed points of polynomial maps*, part II, Ann. Sci. Ens. 26 (1993), 51-98.

[Ha1] P. Haïssinsky, *Modulation dans l'ensemble de Mandelbort*, in this volume.

[Ha2] P. Haïssinsky, *Déformation J-équivalente de polynômes géométriquement finis - Pincements de polynômes*, ENS Lyon, preprint no. 235, 1998.

[Hu] Hubbard, *Local connectivity of Julia sets and bifurcation loci*, in Topological Methods in Modern Mathematics, ed. by L. Goldberg and A. Phillips, Publish or Perish, Inc. 1993.

[Mc] C. McMullen, *The Mandelbrot set is universal*, in this volume.

[Mi] J. Milnor, *Periodic Orbits, External Rays and the Mandelbrot set: An expository account*, Stony Brook IMS preprint #1999/3, Astérisque, to appear.

[PR] C. Petersen & G. Ryd, *Convergence of rational rays in parameter spaces*, in this volume.

[S1] M. Shishikura, *Bifurcation of parabolic fixed points*, in this volume.

[S2] M. Shishikura, *The boundary of the Mandelbrot set has Hausdorff dimension two*, S.M.F. Astérisque 222, 1994.

[Sc1] D. Schleicher, *Rational external rays of the Mandelbrot set*, Astérisque, to appear.

[Sc2] D. Schleicher, *On fibers and local connectivity of Mandelbrot and multibrot sets*, Stony Brook IMS preprint #1998/13.

[Sø] D. Sørensen, *Complex Dynamical Systems: Rays and Non-Local Connectivity*, Ph.D. thesis, DTU Denmark, 1994.

[ST] M. Shishikura & Tan L., *An alternative proof of Mañé's theorem on non-expanding Julia sets*, in this volume.

[T] Tan L, *Voisinages connexes des points de Misiurewicz*, Ann. Inst. Fourier, Grenoble, 42, 4, 707-735, 1992.

Department of mathematics, University of Warwick, Coventry CV4 7AL, United Kingdom
e-mail: tanlei@maths.warwick.ac.uk