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Distortion control of conjugacies between quadratic polynomials

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract We use a new type of distortion control of univalent functions to give an alternative proof of Douady-Hubbard's ray-landing theorem for quadratic Misiurewicz polynomials. The univalent maps arise from Thurston's iterated algorithm on perturbation of such polynomials.

Keywords distortion, conjugacy, polynomial

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1 Introduction

There are many quantities to measure the distance of a univalent function f from Möbius transformations besides of the C^0-topology, for example, the Schwartz derivative S_f, or the nonlinearity f''/f'. In [3], we introduce a new type of distortion control and prove an a priori bound of the distortion by applying this new type of distortion control. Another version of this a priori bound theorem is in particular used for complex dynamics. In this work, we use this theorem to give an alternative proof of Douady-Hubbard's ray-landing theorem for quadratic Misiurewicz polynomials.

1.1 Definition of the distortion

Let $E \subset \overline{\mathbb{C}}$ be an open set and $\phi : E \hookrightarrow \overline{\mathbb{C}}$ be a univalent holomorphic function. Define

$$\mathcal{D}(\phi, E) = \sup\{|\text{mod}(\phi(A)) - \text{mod}(A)|\},$$

where $A \subset \overline{\mathbb{C}}$ are annuli with finite moduli and $\partial A \subset E$, and by abuse of notation $\phi(A)$ denotes the annulus in $\overline{\mathbb{C}}$ bounded by $\phi(\partial A)$ (the map ϕ might not be defined on some points of A).

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1.2 A priori bound

Let g be a *geometrically finite* rational map, i.e., the post-critical set P_g (the closure of the union of all the critical orbits) has a finite (or empty) accumulation set. Assume that the Fatou set of g is non-empty. Let X_0 be the union of finitely many periodic cycles contained in the Julia set of g. It is known that every point in X_0 is either parabolic or repelling. Set $X_1 = g^{-1}(X_0)\setminus X_0$, $X_{n+1} = g^{-n}(X_1)$ and $X = \bigcup_{n\geqslant 0} X_n$.

Now for each $y \in X_0$, choose U_y a simply connected neighborhood of y satisfying the following properties:

(1) These domains U_y are disjoint pairwise.

(2) For each $n \geqslant 1$ and each point $x \in X_n$ with $g^n(x) = y \in X_0$, let U_x be the connected component of $g^{-n}(U_y)$ containing x, and then the map $g^n : U_x \setminus \{x\} \to U_y \setminus \{y\}$ is a covering (this implies that $\overline{U_x} \cap P_g$ is either empty or equal to $\{x\}$).

Denote by $\mathbb{D}$ the unit disc. For any point $y \in X_0$, there is a Riemann mapping χ from U_y to $\mathbb{D}$ with $\chi(y) = 0$. Set $U_y(r) = \chi^{-1}(\{z : |z| < r\})$. For any point $x \in X_n$, set $U_x(r)$ to be the component of $g^{-n}(U_{g^n(x)}(r))$ that contains the point x. Refer to [3] for the next theorem.

Theorem 1.1 (A priori bound). *Let $(g, \{U_x\}_{x\in X})$ be a pair defined as above. Let E be an open set compactly contained in $\overline{\mathbb{C}} \setminus \overline{X}$. Set*

$$V_n(r) = \overline{\mathbb{C}} \Big\backslash \bigcup_{k=0}^{n} \bigcup_{x\in X_k} \overline{U_x(r)}$$

for $n \geqslant 0$. Then there exist a constant $r_0 > 0$ and a positive function $C(r)$ defined on $(0, r_0)$ with $C(r) \to 0$ (as $r \to 0$) such that for any $n \geqslant 0$ and any univalent holomorphic map $\phi : V_n(r) \hookrightarrow \overline{\mathbb{C}}$,

$$\mathcal{D}(\phi, E) \leqslant C(r).$$

1.3 External rays

Now let us consider quadratic polynomials $Q_c(z) = z^2 + c$. Denote by K_c the filled-in Julia set of Q_c. For any $c \in \mathbb{C}$, there exists a conformal map ϕ_c defined on a neighborhood of the infinity such that $\phi_c(\infty) = \infty$, $\phi_c(z)/z \to 1$ as $z \to \infty$ and $\phi_c \circ Q_c(z) = (\phi_c(z))^2$. The map ϕ_c is called the *Böttcher coordinate* of Q_c at the infinity. If K_c is connected, then ϕ_c defines a Riemann mapping $\phi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Set

$$R_c(\theta) = \{\phi_c^{-1}(re^{2\pi i\theta}), r > 1\}.$$

It is called the *dynamical θ-ray* for Q_c. Even if K_c is disconnected, ϕ_c can be extended to a conformal map from a domain U whose boundary passes through the critical point zero to $\{z : |z| > r\}$ for some constant $r > 1$. In particular, the critical value c is contained in U.

Recall that the *Mandelbrot set* is defined by $M = \{c : \{Q_c^n(c)\}_{n\geqslant 0}$ is bounded$\}$. Equivalently, the point c is contained in M if and only if K_c is connected. Define $\Phi(c) = \phi_c(c)$. It turns out that $\Phi : \mathbb{C}\setminus M \to \mathbb{C}\setminus\overline{\mathbb{D}}$ is a conformal map with $\Phi(z)/z \to 1$ as $z \to \infty$ (refer to [4]). Set $R_M(\theta) = \{\Phi^{-1}(re^{2\pi i\theta}), r > 1\}$. It is called the *parameter θ-ray.*

Theorem 1.2 (Pre-periodic external rays landing). *Let $c \in M$ be a parameter such that a strictly pre-periodic dynamical ray $R_c(\theta)$ lands at c. Then the parameter ray $R_M(\theta)$ lands also at c.*

Such a polynomial Q_c is called a *Misiurewicz polynomial*. This theorem is proved by Douady and Hubbard using a perturbation argument (refer to [5, p. 94]). We will reprove this theorem in this paper. Our approach is not known in the literature.

2 Various types of distortions

Let $E \subset \overline{\mathbb{C}}$ be an open set and $\phi : E \hookrightarrow \overline{\mathbb{C}}$ be a univalent holomorphic function with $\mathcal{D}(\phi, E) < \infty$. The next lemma is easy to verify.

Lemma 2.1. (a) $\mathcal{D}(\phi^{-1},\phi(E))=\mathcal{D}(\phi,E)$.

(b) $\mathcal{D}(\gamma\circ\phi\circ\beta,\beta^{-1}(E))=\mathcal{D}(\phi,E)$, *for any Möbius transformations* β *and* γ *of* $\overline{\mathbb{C}}$.

(c) $\mathcal{D}(\phi,E_1)\leqslant\mathcal{D}(\phi,E)$ *if* $E_1\subset E$.

(d) *Assume that* $\phi_n:E\hookrightarrow\overline{\mathbb{C}}$ *is a sequence of univalent functions that converges locally uniformly to a univalent function* ϕ. *Then* $\mathcal{D}(\phi,E)\leqslant\liminf_{n\to\infty}\mathcal{D}(\phi_n,E)$.

2.1 Hyperbolic sup-norm of the Schwarzian derivative

Let $E\subset\overline{\mathbb{C}}$ be a hyperbolic open set. Denote by $\rho_E(z)$ the Poincaré density of E. Let $\phi:E\hookrightarrow\overline{\mathbb{C}}$ be a univalent function. The Schwarzian derivative of ϕ is defined by

$$S_\phi(z)=\frac{\phi'''(z)}{\phi'(z)}-\frac{3}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2.$$

Theorem 2.2. *There is a universal constant* $C_1>0$ *such that*

$$\sup_{z\in E}|S_\phi(z)|\rho_E(z)^{-2}\leqslant C_1\sqrt{\mathcal{D}(\phi,E)}. \tag{1}$$

To prove this theorem, we need the following lemmas. Refer to [3] for the proof of the next lemma.

Lemma 2.3. *Let* $E\subset\overline{\mathbb{C}}$ *be a hyperbolic open set with* $0,1,\infty\in E$. *Let* $\phi:E\hookrightarrow\overline{\mathbb{C}}$ *be a univalent function with* $0,1$ *and* ∞ *fixed. Then* $|\log|\phi'(z)||\leqslant5\pi\mathcal{D}(\phi,E)$ *for any point* $z\in E\setminus\{\infty\}$.

Lemma 2.4. *There is a universal constant* $C_2>0$ *such that*

$$\sup_{z\in E}|S_\phi(z)|\rho_E(z)^{-2}\leqslant C_2. \tag{2}$$

This lemma is a generalization of the classical theorem of Kraus-Nehari (see below), and is first proved by Beardon and Gehring (see [1]) with the explicit constant $C_2=3$. Here we give an independent proof using only the estimation of Poincaré density and Köebe distortion theorem. The following results are known:

(A) (refer to Theorem 2.1 in [6]) For any annulus $A\subset\mathbb{C}$ with $\mathrm{mod}(A)>5\log2/(2\pi)$, there exists an essential round annulus $B\subset A$ (i.e., B separates the boundary components of A) such that

$$\mathrm{mod}(B)\geqslant\mathrm{mod}(A)-5\log2/(2\pi). \tag{3}$$

(B) (Kraus-Nehari theorem, refer to [6, p. 60]) Let D be a round disc in $\mathbb{C}$ with Poincaré density ρ_D and $f:D\hookrightarrow\overline{\mathbb{C}}$ be a univalent function. Then

$$\sup_{z\in D}|S_f(z)|\rho_D^{-2}(z)\leqslant3/2. \tag{4}$$

We say that two positive quantities ρ_1 and ρ_2 are *comparable* (denoted by $\rho_1\sim\rho_2$) if there is a universal constant $C>1$ such that $1/C<\rho_1/\rho_2<C$.

(C) (Beardon-Pommerenke theorem, refer to Theorem 1 in [2] or Theorem 2.3 in [6]) Let U be a hyperbolic open set contained in $\mathbb{C}$ with Poincaré density $\rho_U(z)$, and then

$$1/\rho_U(z)\sim d(z,\partial U)(1+\mathrm{mod}(z,U)), \tag{5}$$

where $d(z,\partial U)$ is the Euclidean distance from z to the boundary of U and $\mathrm{mod}(z,U)$ is the supremum of the moduli of essential round annuli in U whose core curves pass through the point z (if no such annulus exists, set $\mathrm{mod}(z,U)=0$).

Denote by $\rho_{A(R)}$ the Poincaré density of the annulus $A(R)=\{z:1/R<|z|<R\}$.

Lemma 2.5. *For any univalent function* $\psi:A(R)\hookrightarrow\mathbb{C}$ *with* $R>32$, *we have*

$$|S_\psi(1)|\rho_{A(R)}^{-2}(1)\leqslant\left(\frac{2^{12}\log R}{\pi(R-32)}\right)^2.$$

Proof. Fix a constant r with $32 < r < R$. Then there is a quasiconformal map h of $\overline{\mathbb{C}}$ such that $h = \psi$ on the annulus $A(r) = \{z : 1/r < |z| < r\}$. By post-composing with a Möbius transformation, we may assume that h is normalized by fixing $0, 1$ and ∞. Note that the Schwartzian derivative $S_\psi(1)$ remains unchanged after composing a Möbius transformation.

Let h_1 be the normalized quasiconformal map of $\mathbb{C}$ with Beltrami coefficient $\mu_{h_1} = \mu_h$ in $D(r) = \{z : |z| < r\}$ and $\mu_{h_1} = 0$ otherwise. Then h_1 is holomorphic in $\Delta(1/r) = \{z : |z| > 1/r\}$. Set $h_2 = h \circ h_1^{-1}$. It is holomorphic in $h_1(D(r))$. By the composition formula of the Schwarzian derivative, we have

$$S_\psi(1) = S_{h_2 \circ h_1}(1) = S_{h_1}(1) + S_{h_2}(1)(h_1'(1))^2.$$

Since h_1 is univalent in $\Delta(1/r)$, where the Poincaré density is $2r/(r^2|z|^2 - 1)$, by (4) we have

$$|S_{h_1}(1)|\frac{(r^2-1)^2}{4r^2} \leqslant \frac{3}{2}. \tag{6}$$

Since $h_1(\{1 < |z| < r\})$ separates $\{0, 1\}$ from $h_1(\{|z| = r\})$, the Euclidean distance between the point 1 and $h_1(\{|z| = r\})$ is at least $r_1 := 2^{-6}(r - 32)$ by (3). So h_2 is univalent on the disc $D(1, r_1) = \{z : |z - 1| < r_1\}$, where the Poincaré density is $2r_1/(r_1^2 - |z - 1|^2)$. Thus

$$|S_{h_2}(1)|\frac{(r-32)^2}{2^{14}} \leqslant \frac{3}{2} \tag{7}$$

by (4). Now applying Koebe 1/4-theorem to h_1 on $D(1, 1 - 1/r)$, and noticing that its image avoids the origin, we have $|h_1'(1)| \leqslant 4r/(r - 1)$. Now let $r \to R$, and then we have

$$\begin{aligned}|S_\psi(1)| &\leqslant \left(\frac{1}{(R+1)^2} + \frac{4\cdot 2^{14}}{(R-32)^2}\right)\cdot\frac{3}{2}\frac{4R^2}{(R-1)^2}\\ &\leqslant \frac{5\cdot 2^{14}}{(R-32)^2}\cdot 6\frac{32^2}{31^2}.\end{aligned}$$

Note that $\rho_{A(R)}^{-2}(1) = 4(\log R)^2/(\pi^2)$. The lemma is proved.

Proof of Lemma 2.4. For any point $z \in E$, denote by $D \subset E$ the disc centered at z with radius $d(z, \partial E)$, and by A an essential round annulus in E with the maximal modulus whose core curve passes through z (if such annulus exists). Then $1/\rho_E(z) \sim d(z, \partial E)(1 + \mathrm{mod}(A))$ by (5).

Case 1. $\mathrm{mod}(A) \leqslant \log 64/\pi$.

In this case $\rho_E^{-1}(z) \sim d(z, \partial E) \sim \rho_D^{-1}(z)$, and ϕ is univalent in D. So

$$\frac{|S_\phi(z)|}{\rho_E(z)^2} \sim \frac{|S_\phi(z)|}{\rho_D(z)^2} \leqslant \frac{3}{2}.$$

Case 2. $\mathrm{mod}(A) > \log 64/\pi$.

We prove at first that $\rho_E(z)$ is comparable to $\rho_A(z)$. Applying a linear map to E if necessary, we may assume $z = 1$ and $A = A(R)$ for some $R > 1$. As

$$\frac{\log R}{\pi} = \mathrm{mod}(A) = \mathrm{mod}(z, U) > \frac{6\log 2}{\pi},$$

we have actually $R > 64$. Thus $1/64 \leqslant d(1, \partial E) \leqslant 1$. From (5), $\rho_E(z)$ is comparable to $1/\mathrm{mod}(A)$ which is $2\rho_A(z)$. Thus $\rho_E(z)$ is comparable to $\rho_A(z)$. By Lemma 2.5 we have

$$\frac{|S_\phi(z)|}{\rho_E(z)^2} \sim \frac{|S_\phi(z)|}{\rho_A(z)^2} \leqslant \left(\frac{2^{12}\log R}{\pi(R-32)}\right)^2 \leqslant \left(\frac{2^{12}\log 64}{32\pi}\right)^2.$$

This ends the proof.

Proof of Theorem 2.2. Assume that $\mathcal{D}(\phi, E) = \delta < \infty$. By Lemma 2.4 the left-hand side of (1) is bounded by a universal constant. So we only need to consider the case when δ is small. For example we may assume that $\delta < 1/64$. The symbols C_* below denote some universal constants.

For any $z \in E$, choose $z_1 \in E$ such that $\rho_{E_1}(z) \leqslant 2\rho_E(z)$ for $E_1 := E \setminus \{z_1\}$. Choose now two Möbius transformations β and γ such that $\beta(z_1) = \infty$, $\beta(z) = 1$, $d(1, \partial\beta(E_1)) = 2$ such that $\psi := \gamma \circ \phi \circ \beta^{-1}$ fixes $0, 1$ and ∞. Set $G := \beta(E_1)$. It is contained in $\mathbb{C}$. We have

$$\frac{|S_\phi(z)|}{\rho_E^2(z)} \leqslant 4\frac{|S_\phi(z)|}{\rho_{E_1}^2(z)} = 4\frac{|S_\psi(1)|}{\rho_G^2(1)} . \tag{8}$$

Since $\delta < 1/64$, we have $\delta(1 + |\log\delta|)^2 \leqslant 2\sqrt{\delta}$. Thus in order to prove (1), we only need to show that

$$|S_\psi(1)|\rho_G^{-2}(1) \leqslant C_3\delta(1 + |\log\delta|)^2. \tag{9}$$

Case A. $\mathrm{mod}(1, G) \leqslant |\log\delta|/\pi$.

By our choice of G, we have $d(1, \partial G) = 2$. By (5), $\rho_G(1)^{-2} \sim (1+\mathrm{mod}(1, G))^2 \leqslant (1+|\log\delta|)^2$. Applying Lemma 2.3 to ψ, we get $|\log|\psi'(z)|| \leqslant 5\pi\delta$ for $z \in G$. This implies that ψ' maps $\{z : |z - 1| < 2\}$ into $A_\delta := \{e^{-6\pi\delta} \leqslant |w| \leqslant e^{6\pi\delta}\}$. Therefore $\psi'(\{z : |z-1| \leqslant 1\})$ is contained in $B_\delta := \{z \in A_\delta : d_p(z, \psi'(1)) \leqslant d_0(0, 1/2)\}$ by Ahlfors-Schwarz lemma, where d_p is the Poincaré metric on A_δ, and d_0 is the Poincaré metric on the unit disc. One can then check easily that the Euclidean diameter of B_δ is less than $C_4\delta$ for a universal constant $C_4 > 0$. Thus

$$|\psi'(z) - \psi'(1)| \leqslant C_4\delta$$

for $|z - 1| \leqslant 1$. Now we may apply the Cauchy integral formula to $\psi'(z)$ on the circle $|z - 1| = 1$ to get that $|\psi''(1)|, |\psi'''(1)| \leqslant C_5\delta$. Combining with $\psi'(1) \in A_\delta$, we get $|S_\psi(1)| \leqslant C_6\delta$. Therefore $|S_\psi(1)|\rho_G(1)^{-2} \leqslant C_7\delta(1 + |\log\delta|)^2$.

Case B. $\mathrm{mod}(1, G) > |\log\delta|/\pi > 6\log 2/\pi$.

Let A be an essential round annulus in G whose core curve passes through the point 1 with modulus equal to $\mathrm{mod}(1, G)$, which is equal to $\log R/\pi$ for some constant $R > 0$. Then

$$\log R/\pi = \mathrm{mod}(A) = \mathrm{mod}(1, G) > |\log\delta|/\pi.$$

So $R > 1/\delta > 64$. From Lemma 2.5, we have

$$\frac{|S_\psi(1)|}{\rho_G^2(1)} \overset{\text{Case 2}}{\sim} \frac{|S_\psi(1)|}{\rho_A^2(1)} \leqslant \left(\frac{2^{12}\log R}{\pi(R - 32)}\right)^2 \leqslant \left(\frac{2^{12}|\log\delta|}{\pi(1/\delta - 32)}\right)^2 < C_8\delta(1 + |\log\delta|)^2.$$

Hence in both cases (9), (1) is proved.

2.2 Controlling distortions of conjugations

Denote $Q_c(z) = z^2 + c$. Denote by ϕ_c the Böttcher coordinate of Q_c at the infinity, M the Mandelbrot set, and $\Phi(c) = \phi_c(c)$. Then $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is a conformal map. Note that if $|\Phi(c)| \leqslant 2$, then ϕ_c^{-1} is univalent in $\{z : |z| > 2\}$ and hence ϕ_c is univalent in $E = \{z : |z| > 8\} \cup \{\infty\}$ by Koebe distortion theorem.

Lemma 2.6. *There is a universal constant $C_0 > 0$, such that for any points $c_1, c_2 \in \mathbb{C} \setminus M$ with $|\Phi(c_i)| \leqslant 2$ $(i = 1, 2)$, we have*

$$|c_1 - c_2| \leqslant C_0\sqrt{\mathcal{D}(\psi, E)},$$

where $\psi = \phi_{c_2}^{-1} \circ \phi_{c_1}$.

Proof. Let $\phi_c(z) = z+b_0+b_1/z+\cdots$ be the expansion at the infinity. By the formula $\phi_c\circ Q_c(z) = \phi_c(z)^2$, one may check that $b_0 = 0$ and $b_1 = c/2$. Therefore $\psi(z)$ has the expansion $\psi(z) = z+(c_1-c_2)/(2z)+\cdots$ at the infinity.

Let $\Psi(z) = 1/\psi(1/z) = z + a_2z^2 + a_3z^3 + \cdots$. Using $\psi(1/z)\Psi(z) \equiv 1$ one obtains that $a_2 = 0$ and $a_3 = (c_2 - c_1)/2$. This implies that $S_\Psi(0) = 6a_3 = 3(c_2 - c_1)$. Let $\tilde{\rho}(z)|dz|$ be the Poincaré metric on $\{z : |z| < 1/8\}$. Then $\tilde{\rho}(0) = 16$. Denote by $\rho(z)|dz|$ the Poincaré metric on E. Then

$$\lim_{z\to\infty} \frac{S_\psi(z)}{\rho^2(z)} = \frac{S_\Psi(0)}{\tilde{\rho}^2(0)} = \frac{3(c_2 - c_1)}{256}.$$

Therefore by Theorem 2.2,

$$|c_1 - c_2| = \frac{256}{3} \lim_{z\to\infty} \frac{|S_\psi(z)|}{\rho^2(z)} \leqslant C_0\sqrt{\mathcal{D}(\psi, E)}.$$

3 Thurston algorithm

We will apply our distortion control to univalent maps arising naturally from the Thurston algorithm of perturbations of rational maps.

3.1 c-Equivalence between semi-rational maps

Let $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a branched covering with $\deg F \geqslant 2$. Its post-critical P_F is defined to be the closure of the forward orbits of the critical points. Denote by P'_F the accumulation set of P_F. We say that F is a *sub-hyperbolic semi-rational map* if P'_F is finite and either $P'_F = \emptyset$ or F is holomorphic in a neighborhood of P'_F and each cycle in P'_F is either attracting or super-attracting.

Two sub-hyperbolic semi-rational maps F_1 and F_2 are *c-equivalent* if there is a pair of homeomorphisms (ϕ_0, ϕ_1) of $\overline{\mathbb{C}}$ such that (a) $\phi_0 \circ F_1 = F_2 \circ \phi_1$; (b) the two maps ϕ_0 and ϕ_1 are isotopic rel P_{F_1}; and (c) ϕ_0 is holomorphic in a neighborhood of P'_{F_1} (hence ϕ_1 is also holomorphic and coincides with ϕ_0 in a neighborhood of P'_{F_1}).

3.2 Thurston algorithm

Let $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a sub-hyperbolic semi-rational map with P'_F non-empty. Denote by P''_F the union of P'_F with all the periodic cycles in P_F which meet critical points. Assume that F is holomorphic in a neighborhood of P''_F. Pick three distinct points in P_F. In this section, we say that a homeomorphism of $\overline{\mathbb{C}}$ is *normalized* if it fixes these three points.

Assume that the sub-hyperbolic semi-rational map F is c-equivalent to a rational map f via a pair of normalized homeomorphisms (ϕ_0, ϕ_1). Since F is holomorphic in a neighborhood of P''_F, there is a pair of normalized homeomorphisms (ξ_0, ξ_1) of $\overline{\mathbb{C}}$ in the isotopy class of ϕ_0 rel P_F, such that they are holomorphic and coincide with each other in a neighborhood of P''_F, and $\xi_0 \circ F = f \circ \xi_1$. Furthermore, for $n \geqslant 2$, there is a normalized homeomorphism ξ_n of $\overline{\mathbb{C}}$ in the isotopy class of ξ_0 rel P_F such that $\xi_{n-1} \circ F = f \circ \xi_n$ and ξ_n coincides with ξ_0 in a neighborhood of P''_F.

Lemma 3.1. *The sequence $\{\xi_n\}$ is uniformly convergent.*

This lemma is proved by Shishikura and Rees in the case that P_F is finite (refer to [7]). One may check that their proof works in the case that F is a sub-hyperbolic semi-rational map.

Assume that η_0 is a normalized homeomorphism of $\overline{\mathbb{C}}$ such that η_0 is holomorphic in a neighborhood of P''_F. Then there is a unique normalized homeomorphism η_1 of $\overline{\mathbb{C}}$ such that $f_1 := \eta_0 \circ F \circ \eta_1^{-1}$ is holomorphic. Obviously, η_1 is also holomorphic in a neighborhood of P''_F. Recursively, there is a unique normalized homeomorphism η_n of $\overline{\mathbb{C}}$ such that $f_n := \eta_{n-1} \circ F \circ \eta_n^{-1}$ is holomorphic. Then η_n is also holomorphic in a neighborhood of P''_F. The sequence of rational maps $\{f_n\}$ is called a *Thurston sequence* of F.

Theorem 3.2 (Convergence of the Thurston algorithm). *Assume the sub-hyperbolic semi-rational map F is c-equivalent to a rational map f via a pair of normalized homeomorphisms. Then the Thurston sequence $\{f_n\}$ converges uniformly to the rational map f. Moreover, the sequence $\{\eta_n\}$ is also uniformly convergent.*

Proof. By the definition we know that (a) $\phi_0 \circ F = f \circ \phi_1$; (b) the two maps ϕ_0 and ϕ_1 are isotopic rel P_F; and (c) both ϕ_0 and ϕ_1 are holomorphic in a neighborhood of P'_F. Since F is holomorphic in a neighborhood of P''_F, we may modify ϕ_0 in its homotopy class such that both ϕ_0 and ϕ_1 are holomorphic and coincide in a neighborhood of P''_F. As above, we have a sequence of normalized homeomorphisms

$\{\phi_n\}_{n\geqslant 1}$ of $\overline{\mathbb{C}}$ in the isotopy class of ϕ_0 rel P_F, such that $\phi_{n-1}\circ F = f\circ\phi_n$ and $\phi_n = \phi_0$ in a neighborhood of P''_F.

Set $\psi_n = \eta_n\circ\phi_n^{-1}$. Then ψ_n is holomorphic in a neighborhood of P''_f. We may assume furthermore that ψ_0 is a K-quasiconformal map of $\overline{\mathbb{C}}$ by modifying ϕ_0 in its isotopy class. Then ψ_n is also K-quasiconformal. It is easy to check that

$$\psi_{n-1}\circ f = f_n\circ\psi_n.$$

See the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \to & \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} & \to \cdots \to & \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} \\
 & & \uparrow \phi_n & & \uparrow \phi_{n-1} & & \uparrow \phi_2 & & \uparrow \phi_1 & & \uparrow \phi_0 \\
\cdots & \to & \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} & \to \cdots \to & \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} & \xrightarrow{F} & \overline{\mathbb{C}} \\
 & & \downarrow \eta_n & & \downarrow \eta_{n-1} & & \downarrow \eta_2 & & \downarrow \eta_1 & & \downarrow \eta_0 \\
\cdots & \to & \overline{\mathbb{C}} & \xrightarrow{f_n} & \overline{\mathbb{C}} & \to \cdots \to & \overline{\mathbb{C}} & \xrightarrow{f_2} & \overline{\mathbb{C}} & \xrightarrow{f_1} & \overline{\mathbb{C}}\,.
\end{array}$$

Since ψ_0 is a K-quasiconformal map of $\overline{\mathbb{C}}$ and is holomorphic in a neighborhood W of P''_f (we may choose W such that it is contained in the Fatou set of f and $f^{-1}(W)\supset W$), by the equation $\psi_{n-1}\circ f = f_n\circ\psi_n$, we see that ψ_n is also a K-quasiconformal map of $\overline{\mathbb{C}}$ and holomorphic in $f^{-n}(W)$. Therefore there is a subsequence of $\{\psi_n\}$ which converges uniformly to a limit quasiconformal map ψ of $\overline{\mathbb{C}}$. Moreover, ψ is holomorphic in $\cup f^{-n}(W)$ which is the Fatou set. Thus ψ is holomorphic on $\overline{\mathbb{C}}$ since the measure of the Julia set of f is zero. Combining with the normalization condition, we know that ψ is the identity. Therefore the entire sequence $\{\psi_n\}$ converges uniformly to the identity. It follows that the Thurston sequence $\{f_n\}$ converges uniformly to the rational map f .

Because $\eta_n = \psi_n\circ\phi_n$, from Lemma 3.1, we know that the sequence $\{\phi_n\}$, and therefore the sequence $\{\eta_n\}$ is uniformly convergent.

4 Misiurewicz-hyperbolic deformation

We now begin to prove Theorem 1.2. Recall that ϕ_c is the Böttcher coordinate of the quadratic polynomial $Q_c(z) = z^2 + c$ at infinity. For $c\in M$, the map ϕ_c defines a Riemann representation $\phi_c : \mathbb{C}\setminus K_c \to \mathbb{C}\setminus\overline{\mathbb{D}}$. The dynamical θ-ray is defined by

$$R_c(\theta) = \{\phi_c^{-1}(re^{2\pi i\theta}), r > 1\}.$$

For the Mandelbrot set M, the Douady-Hubbard Riemann representation $\Phi : \mathbb{C}\setminus M \to \mathbb{C}\setminus\overline{\mathbb{D}}$ is defined by $\Phi(c) = \phi_c(c)$. The parameter θ-ray is defined by

$$R_M(\theta) = \{\Phi^{-1}(re^{2\pi i\theta}), r > 1\}.$$

Let $c\in M$ be a parameter such that a strictly pre-periodic dynamical ray $R_c(\theta)$ lands at c. We want to show that the parameter ray $R_M(\theta)$ lands also at c.

Set $c(t) = \Phi^{-1}(te^{2\pi i\theta})$ for $t > 1$. In other words, we will prove that $c(t)\to c$ as $t\to 1$. Denote by

$$\mathcal{U} = \{z : |\phi_c(z)| > t\},\ \mathcal{U}_1 = \{z : |\phi_{c(t)}(z)| > t\},\ \mathcal{V} = \{z : |\phi_c(z)| > t^2\} \quad\text{and}\quad \mathcal{V}_1 = \{z : |\phi_{c(t)}(z)| > t^2\}.$$

Then they are Jordan domains. Both $Q_c : \mathcal{U}\to\mathcal{V}$ and $Q_{c(t)} : \mathcal{U}_1\to\mathcal{V}_1$ are proper. The critical value $c(t)$ of $Q_{c(t)}$ lies on the boundary of $\mathcal{U}_1$ and the post-critical set of $Q_{c(t)}$ is contained in the closure of $\mathcal{U}_1$. Set $\psi = \phi_{c(t)}^{-1}\circ\phi_c$. Then $\psi(\mathcal{U}) = \mathcal{U}_1$ and $\psi(\mathcal{V}) = \mathcal{V}_1$. Moreover, the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\psi} & \mathcal{U}_1 \\
Q_c\downarrow & & \downarrow Q_{c(t)} \\
\mathcal{V} & \xrightarrow{\psi} & \mathcal{V}_1.
\end{array}$$

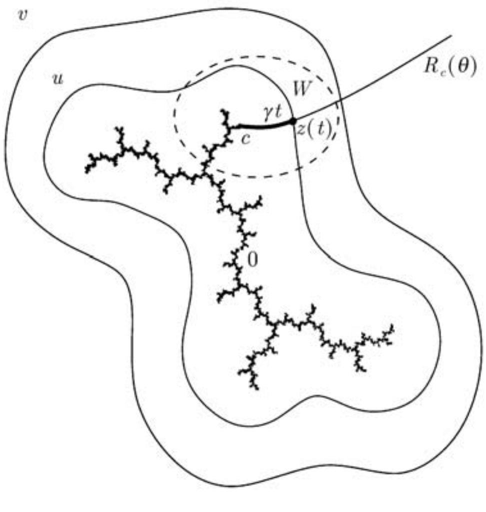

Figure 1 Perturbation F of Q_c

Step 1. Construction of a topological perturbation F of Q_c.

Denote by $z(t) = \phi_c^{-1}(te^{2\pi i\theta})$. Then $z(t) = \psi^{-1}(c(t)) \in \partial\mathcal{U}$. Let $\gamma_t = \phi_c^{-1}(\{re^{2\pi i\theta}, 1 \leqslant r \leqslant t\})$. It is a closed arc connecting the point c with the point $z(t)$, whose interior is contained in $R_c(\theta) \cap (\overline{\mathbb{C}} \setminus \mathcal{U})$.

Let W be a Jordan domain in $\overline{\mathbb{C}} \setminus \mathcal{V}$ such that $\gamma_t \subset W$. Choose $\zeta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a homeomorphism that is the identity outside W, with $\zeta(c) = z(t)$. Set $F := \zeta \circ Q_c$. Then the critical points of F are $\{0, \infty\}$ with $F(0) = z(t)$. Therefore $F^n(z(t)) = Q_c^n(z(t)) \to \infty$ as $n \to \infty$. So $P_F = \{z(t), F(z(t)), \ldots\} \cup \{\infty\}$ and $P'_F = P''_F = \{\infty\}$. As F is holomorphic in a neighborhood of the infinity which is a super-attracting fixed point of F, we conclude that F is a sub-hyperbolic semi-rational map.

Lemma 4.1. *The sub-hyperbolic semi-rational map F is c-equivalent to $Q_{c(t)}$.*

Proof. Let ψ_0 be a homeomorphism of $\overline{\mathbb{C}}$ such that $\psi_0|_{\mathcal{U}} = \psi|_{\mathcal{U}}$. Then $\psi_0(P_F) = P_{Q_{c(t)}}$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\psi_0} & \mathcal{U}_1 \\ F \downarrow & & \downarrow Q_{c(t)} \\ \mathcal{V} & \xrightarrow{\psi_0} & \mathcal{V}_1. \end{array}$$

The homeomorphism $\psi_0 : \overline{\mathbb{C}} \setminus \mathcal{V} \to \overline{\mathbb{C}} \setminus \mathcal{V}_1$ maps the critical value $z(t)$ of F to the critical value $c(t)$ of $Q_{c(t)}$. Thus there is a unique lift $\psi_1 : \overline{\mathbb{C}} \setminus \mathcal{U} \to \overline{\mathbb{C}} \setminus \mathcal{U}_1$ of ψ_0, such that $\psi_1|_{\partial\mathcal{U}} = \psi|_{\partial\mathcal{U}}$. Obviously, as a lift of ψ_0, the map ψ_1 satisfies that $\psi_1(0) = 0$, and the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathbb{C}} \setminus \mathcal{U} & \xrightarrow{\psi_1} & \overline{\mathbb{C}} \setminus \mathcal{U}_1 \\ F \downarrow & & \downarrow Q_{c(t)} \\ \overline{\mathbb{C}} \setminus \mathcal{V} & \xrightarrow{\psi_0} & \overline{\mathbb{C}} \setminus \mathcal{V}_1. \end{array}$$

Now extend the map ψ_1 to a homeomorphism of $\overline{\mathbb{C}}$ by $\psi_1|_{\mathcal{U}} = \psi|_{\mathcal{U}}$. Then $\psi_0 \circ F = Q_{c(t)} \circ \psi_1$. Since $\psi_0|_{\mathcal{U}} = \psi_1|_{\mathcal{U}} = \psi|_{\mathcal{U}}$ and $P_F \subset \overline{\mathcal{U}}$, by Alexander trick, any homeomorphism of a topological disc which is the identity on the boundary is isotopic to the identity rel the boundary. We know that ψ_1 and ψ_0 are isotopic rel P_F. This proves that F and $Q_{c(t)}$ are c-equivalent.

Step 2. Application of the Thurston algorithm to F.

Let η_0 be a Möbius transformation of $\overline{\mathbb{C}}$ such that it is normalized by mapping the triple $(0, F^2(z(t)), \infty)$ to the triple $(0, Q^2_{c(t)}(c(t)), \infty)$. As in Section 3, there is a unique homeomorphism η_1 of $\overline{\mathbb{C}}$ with the same normalization, such that $f_1 := \eta_0 \circ F \circ \eta_1^{-1}$ is holomorphic on the Riemann sphere $\overline{\mathbb{C}}$. Since F is holomorphic except on $F^{-1}(W) = Q_c^{-1}(W)$, we see that η_1 is holomorphic except on $F^{-1}(W)$.

Recursively, there is a unique homeomorphism η_n of $\overline{\mathbb{C}}$ with the same normalization, such that $f_n := \eta_{n-1} \circ F \circ \eta_n^{-1}$ is holomorphic on $\overline{\mathbb{C}}$. Moreover η_n is holomorphic except on $\bigcup_{k=1}^n F^{-k}(W)$. Noticing that $F = Q_c$ except on $F^{-1}(W) = Q_c^{-1}(W)$, we have

$$\bigcup_{k=1}^{n} F^{-k}(W) = \bigcup_{k=1}^{n} Q_c^{-k}(W).$$

By Theorem 3.2 and Lemma 4.1, we see that the sequence $\{f_n\}$ converges uniformly to the quadratic polynomial $Q_{c(t)}$ and the sequence $\{\eta_n\}$ uniformly converges to a continuous map η of $\overline{\mathbb{C}}$. Note that η_n is holomorphic in $\mathcal{V}$ and is normalized by mapping the triple $(0, F^2(z(t)), \infty)$ to the triple $(0, Q^2_{c(t)}(c(t)), \infty)$ for all $n \geqslant 0$. Thus η is univalent on $\mathcal{V}$ and mapping the pair $(F^2(z(t)), \infty)$ to the pair $(Q^2_{c(t)}(c(t)), \infty)$ (note that $F^2(z(t)) = Q_c^2(z(t)) \in \mathcal{V}$). In particular, $\eta^{-1} \circ Q_{c(t)} \circ \eta = F = Q_c$ on $\mathcal{V}$. Therefore $\eta|_{\mathcal{V}} = \psi|_{\mathcal{V}}$. This is because that the holomorphic conjugation between two super-attracting fixed points of degree two is unique.

Proof of Theorem 1.2. The critical point zero is pre-periodic for Q_c. Set X_0 to be the unique periodic cycle in P_{Q_c}. We define X_k and X as in Section 1.

Note that $E := \{z : |z| > 8\} \cup \{\infty\}$ is compactly contained in $\mathcal{V}$ for $t \in (1, \sqrt{2})$. For each $y \in X_0$ we choose a simply connected neighborhood U_y such that $U_y \cap \overline{E} = \emptyset$ and $U_y \setminus \{y\}$ is disjoint from P_{Q_c}. Define U_x and $U_x(r)$ for any point $x \in X$ and any constant $r < 1$ as in Section 1. In particular, as $c \in X_k$ for some $k \geqslant 1$, U_c is a neighborhood of c which is a component of $Q_c^{-k}(U_y)$ for $y = Q^k(c)$.

Recall that $\gamma_t = \phi_c^{-1}(\{re^{2\pi i\theta}, 1 \leqslant r \leqslant t\})$. Obviously the diameter diam$\gamma_t \to 0$ as $t \to 1$. Thus there are a constant $t_0 \in (1, \sqrt{2})$ and a positive function $r(t)$ on $(1, t_0)$ with $r(t) \to 0$ as $t \to 1$ such that $\gamma_t \subset U_c(r(t))$ for $t \in (1, t_0)$.

Recall that the homeomorphism η_n defined in Step 2 is holomorphic on the complement of the closure of $\bigcup_{k=1}^n Q_c^{-k}(W)$. For $t \in (1, t_0)$, since $\gamma_t \subset U_c(r(t))$, we may require that the Jordan domain W chosen in Step 1 satisfies that $W \subset U_c(r(t))$. Then the homeomorphism η_n is holomorphic on the complement of the closure of $\bigcup_{k=1}^n Q_c^{-k}(U_c(r(t)))$.

Now applying Theorem 1.1 to the pair $(Q_c, \{U_x\}_{x\in X})$ and the map η_n, we have a constant $r_0 > 0$ and a positive function $C(r)$ defined on $r \in (0, r_0)$ with $C(r) \to 0$ (as $r \to 0$) such that for any $t > 1$ with $r(t) < r_0$ and any integer $n \geqslant 1$,

$$\mathcal{D}(\eta_n, E) \leqslant C(r(t)).$$

Since $r(t) \to 0$ as $t \to 1$, there exists a constant $t_0 > 1$ such that $r(t) < r_0$ for $t \in (1, t_0)$. So we have $\mathcal{D}(\eta_n, E) \leqslant C(r(t))$ for $t \in (1, t_0)$ and any integer $n \geqslant 1$. In particular $\mathcal{D}(\eta, E) \leqslant C(r(t))$ for $t \in (1, t_0)$ by Lemma 2.1(d).

Recall that $\eta = \psi = \phi_{c(t)} \circ \phi_c^{-1}$ on $\mathcal{V}$, therefore on E. By Lemma 2.6, there is a universal constant $C_0 > 0$ such that

$$|c - c(t)| \leqslant C_0\sqrt{\mathcal{D}(\eta, E)} \leqslant C_0\sqrt{C(r(t))}.$$

Therefore $|c - c(t)| \to 0$ as $t \to 1$ since $C(r) \to 0$ as $r \to 0$ and $r(t) \to 0$ as $t \to 1$. Now the theorem is proved.

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References

1 Beardon A, Gehring F W. Schwarzian derivatives, the Poincare metric and the kernel function. Comment Math Helv, 1980, 55: 50–64

2 Beardon A, Pommerenke C. The Poincaré metric of plane domains. J London Math Soc, 1978, 18: 475–483

3 Cui G Z, Tan L. A priori bound for the distortion of univalent functions. Preprint
4 Douady A, Hubbard J H. Etude dynamique des polynômes complexes (première partie). Publications Mathématiqus d'Orsay, 84-02, 1984
5 Lehto O. Univalent Functions and Teichmüller Spaces. Berlin: Springer-Verlag, 1987
6 McMullen C. Complex Dynamics and Renormalizations. Annals of Mathematics Studies 135. Princeton: Princeton University Press, 1994
7 Shishikura M. On a theorem of Mary Rees for matings of polynomials. In: Tan L, ed. The Mandelbrot Set, Theme and Variations. LMS Lecture Note Series 274. Cambridge: Cambridge University Press, 2000, 289–305