Breaking parabolic points along stable directions

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Motivations

We want to generalize the following results to rational maps:

In the quadratic family $\{z \mapsto z^2 + c\}$,

1. the fat rabbit parameter is accessible from both the main cardioid and the rabbit hyperbolic component. In the main cardioid direction, the parabolic point 'breaks' into an attracting fixed point and a period-3 repelling cycle, whereas in the rabbit direction, the parabolic point 'breaks' into a repelling fixed point and a period-3 attracting cycle.

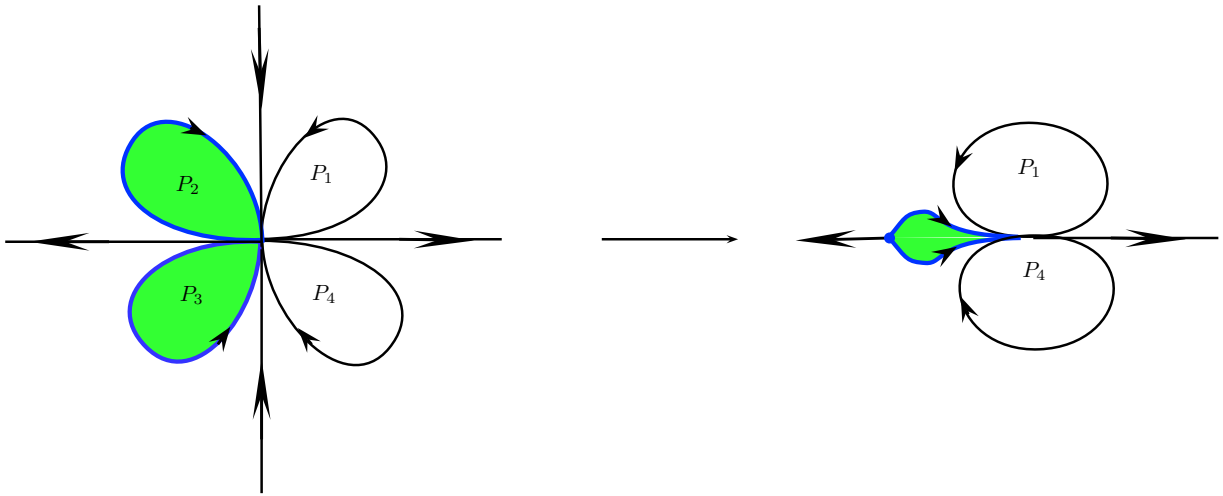
These two types of nearby dynamical systems are kind of 'stable' (as opposite to 'implosive') perturbations of the parabolic rabbit dynamical system.

2. a Misiurewicz parameter is accessible from the escape hyperbolic component.

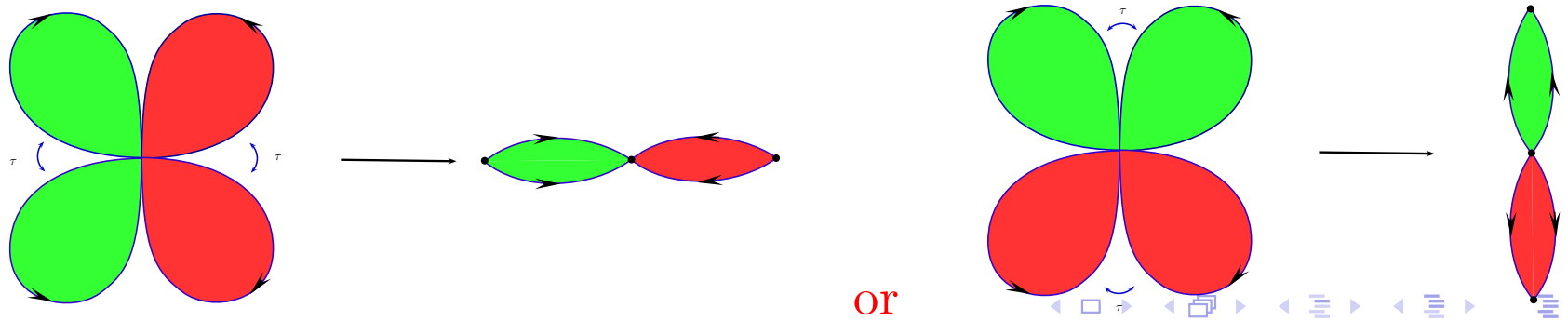
We want to construct such accesses for more general maps, using intrinsic dynamical perturbations, rather than extrinsic parameter parametrizations.

§1. Breaking parabolic points along stable directions

A parabolic point takes the form $z(1+z^n+o(z^n))$, $n \geq 1$ (after passing to some iterates) and has a flower P of $2n$ sepals such that $f(P) = P$. A nearby map will break the fixed point 0 into a set of nearby fixed points with total multiplicity p following some compatible combinatorics.



In this case, P_2 and P_3 merge after the breaking. You can choose to merge any pair of neighboring sepals, or choose to merge simultaneously two pairs:



In general, a compatible stable breaking combinatorics is a choice of pairs of sepals to be merged, so that:

- distinct pairs do not cross
- each group of consecutive un-paired sepals has an even number of sepals
- compatible with the dynamics (rotation, period).

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Theorem (breaking parabolic points)

Let g be a rational map with parabolic cycles Y and with at most finite orbited critical points on the Julia set. Given any compatible stable breaking combinatorics of Y, $\exists$ a continuous path of rational maps $\{f_r\}_{0<r\leq r_0}$ realizing the combinatorics and converging dynamically to g.

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Let g be a rational map with parabolic cycles Y and with at most finite orbited critical points on the Julia set. Given any compatible stable breaking combinatorics of Y, $\exists$ a continuous path of rational maps $\{f_r\}_{0<r\leq r_0}$ realizing the combinatorics together with qc-conjugacies ϕ_r from f_{r_0} to f_r such that:

$$\begin{array}{ccc} f_r & \overset{\text{uniform}}{\underset{r\to 0}{\longrightarrow}} & g \\ \phi_r & & \phi \end{array} \quad \text{and} \quad (f_{r_0}, \mathcal{J}_{f_{r_0}}) \overset{\phi \text{ semi-conjugate}}{\underset{\text{reassembling the breaking}}{\longrightarrow}} (g, \mathcal{J}_g).$$

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A particular case concerns Milnor's conjecture in the geometrically finite setting: that parabolic cycles of a rational map can always be converted to attracting cycles without changing the topology of the Julia set (see also P. Haissinsky and T. Kawahira).

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The case with rotations seems more complicated ...

§2. Misiurewicz polynomials are accessible by external rays

Douady-Hubbard (generalized by Kiwi to higher degree polynom.)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

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We will see a new proof of this theorem that works for more general settings.

§3. A preliminary convergence criterion

Let: g be a rational map without critical points[1] on the Julia set;
X_0 be a finite collection of parabolic and repelling cycles;
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a) Let $p_{r,n}$ normalized univalent maps outside $\bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}$, and $V \subset\subset \left(\overline{X}\right)^c$. Then $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

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b) Let $f_{r,n}$ rational map of the same degree, $g(V) \subset V$ and $g^{-1}(z) \subset V$ for some z, s.t.

$$\begin{array}{ccc} V & \overset{p_{r,n+1}}{\longrightarrow} & \\ g\downarrow & & \downarrow f_{r,n} \\ V & \underset{p_{r,n}}{\longrightarrow} & \end{array}$$

Then $f_{r,n} \underset{r\to 0}{\longrightarrow} g$ uniformly on n.

The part a)$\Longrightarrow$b) is easy. So the main point is a).

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§4. Application to $g(z) = z^2 + c$ with 0 preperiodic

1. Replace g by a (quasi-regular) map F_r which is equal to g outside a small neighborhood W_r of 0 but $F_r(0) = z(r)$ on a ray $R_c(\theta)$ landing at c (of potential e^r). This new map is not holomorphic in W_r.

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2. (this does not work) Pullback the standard complex structure infinitely many times to get an F_r-invariant structure and integrate using Measurable Riemann Mapping Theorem.

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2. Instead pullback step by step the standard complex structure and

integrate, to get

$$\begin{array}{ccc} & \xrightarrow[p_{r,n+1}]{} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow{p_{r,n}} & \end{array}$$

with $p_{r,n+1}$ univalent outside

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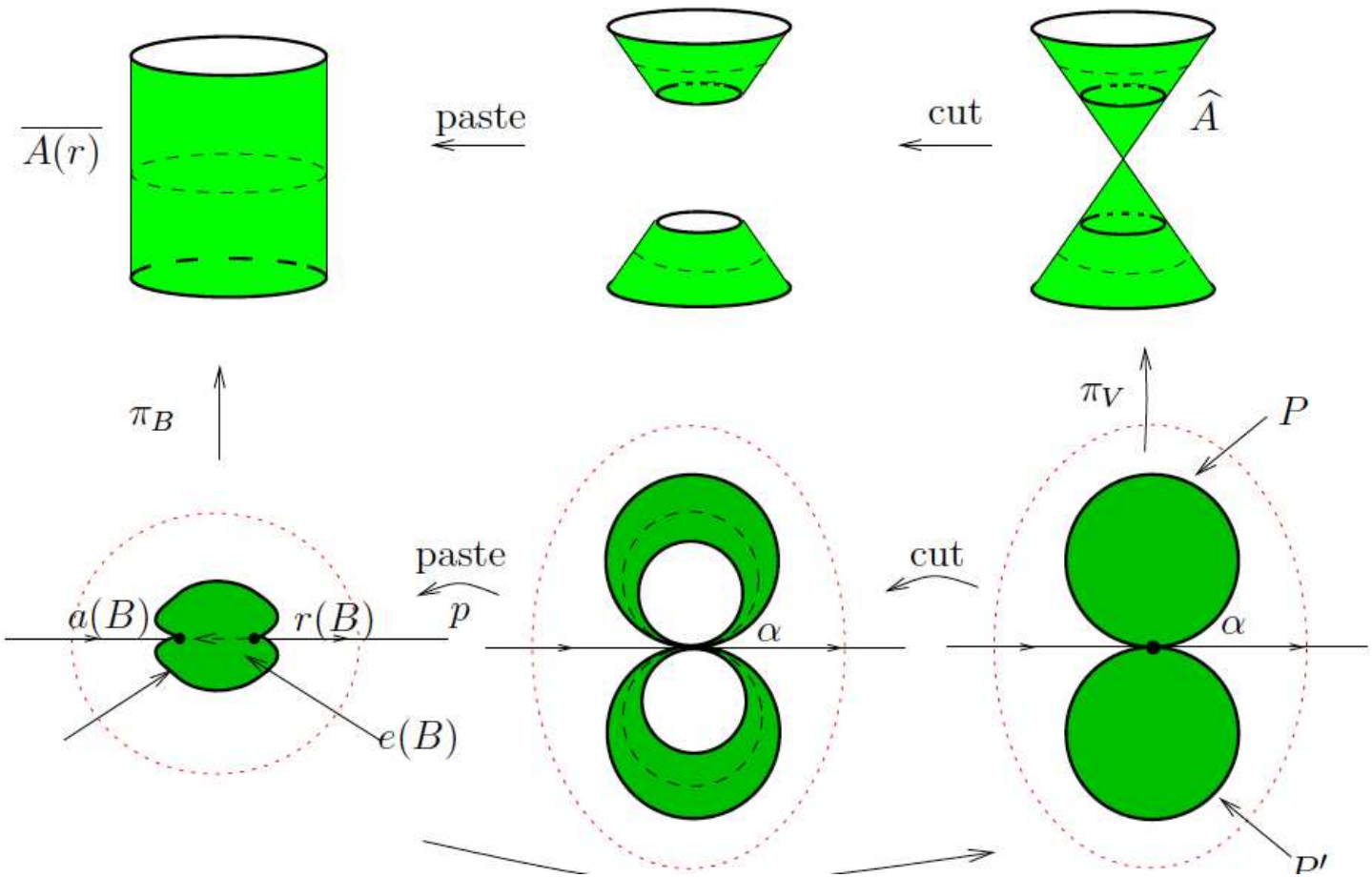
§5 Application to parabolic points: plumbing surgery



This surgery makes a new Riemann surface, and a new local holomorphic dynamics with the desired combinatorices.

two ways to get a global map

Given a rational map g with a parabolic sepal flower of size r,

1. use a quasi-regular modification in a neighborhood of the strict first preimage of the flower, to get a branched covering F_r realizing holomorphically (locally) the prescribed combinatorics, with F_r holomorphic except on $\bigcup_{y \in g^{-1}(X_0) \smallsetminus X_0} U_y(r)$.

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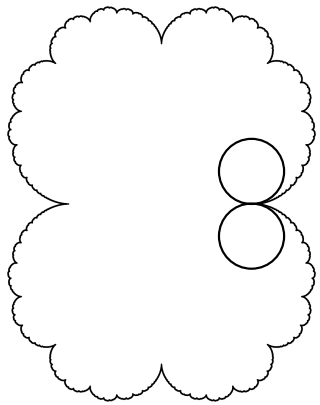
2. use a consecutive sequence of plumbing surgery to get a pair of sequences $(f_{r,n}, p_{r,n})_n$ realizing holomorphically (locally) the prescribed combinatorics (see next slice).
3. prove a generalized Thurston theorem for parabolic maps, use it to conclude that each F_r is c-equivalent to a rational map f_r, and its Thurston's algorithm gives the same sequence $f_{r,n}$, so $\begin{array}{c} f_r \\ \uparrow \\ f_{r,n} \end{array}$ as $\begin{array}{c} \infty \\ \uparrow \\ n \end{array}$.
4. Apply the convergence criterion to $(f_{r,n}, p_{r,n})$ to get $f_{r,n} \to g$ uniformly on n. Therefore

$$\begin{array}{ccc} \infty & f_r & \\ \uparrow & \uparrow & \text{so} \searrow \\ n & f_{r,n} & \underset{r \to 0}{\longrightarrow} \quad g \text{ unif. on } n. \end{array}$$

Consecutive plumbing surgery



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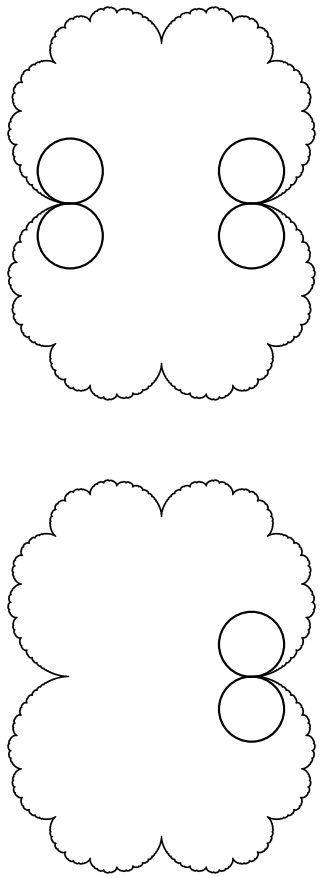

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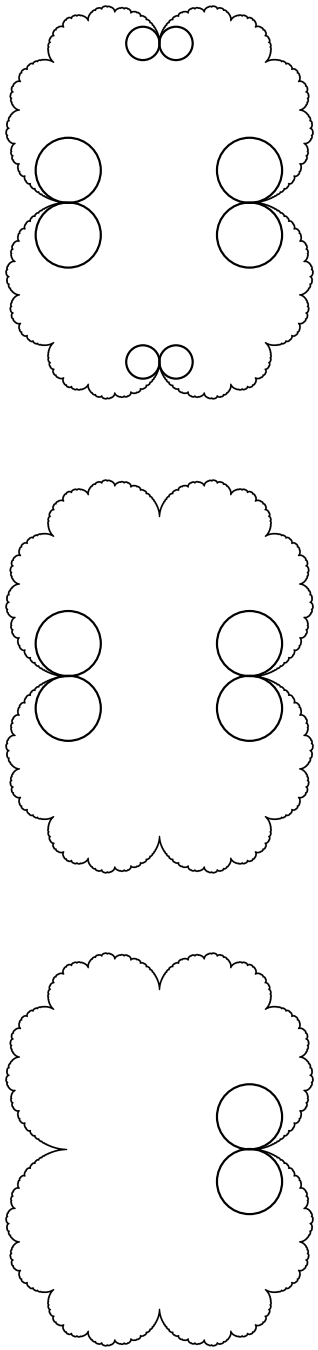

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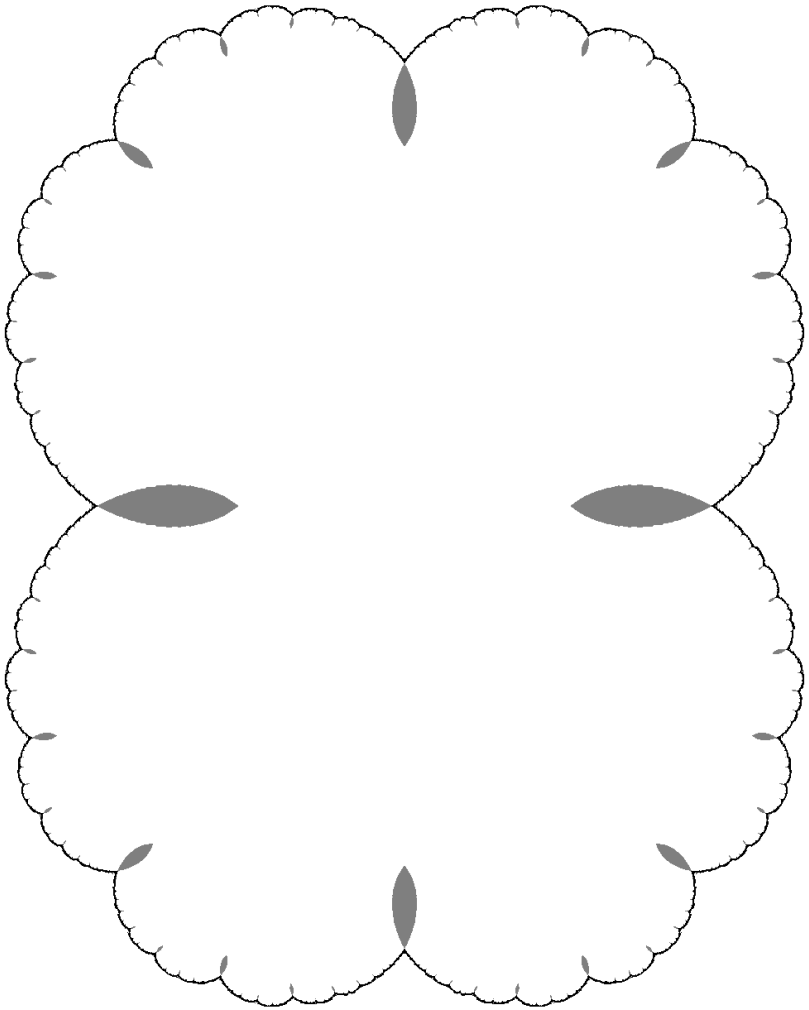

§6. Proof of the convergence criterion (sketch)

Theorem (a convergence criterion)

a) Let $p_{r,n}$ normalized univalent maps outside $\bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}$, and $V \subset\subset \left(\overline{X}\right)^c$. Then $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

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By control of degree, $f_{r,n}|_{\widehat{\mathbb{C}}} \underset{r\to 0}{\longrightarrow} g$ uniformly on n.

In order to prove a), we first forget the normalization and control the distortion of $p_{r,n}$ from being globally conformal (i.e. Möbius transformation). We then use the normalization to control their spherical distance to the identity.

§7. Univalent maps and Möbius transformations

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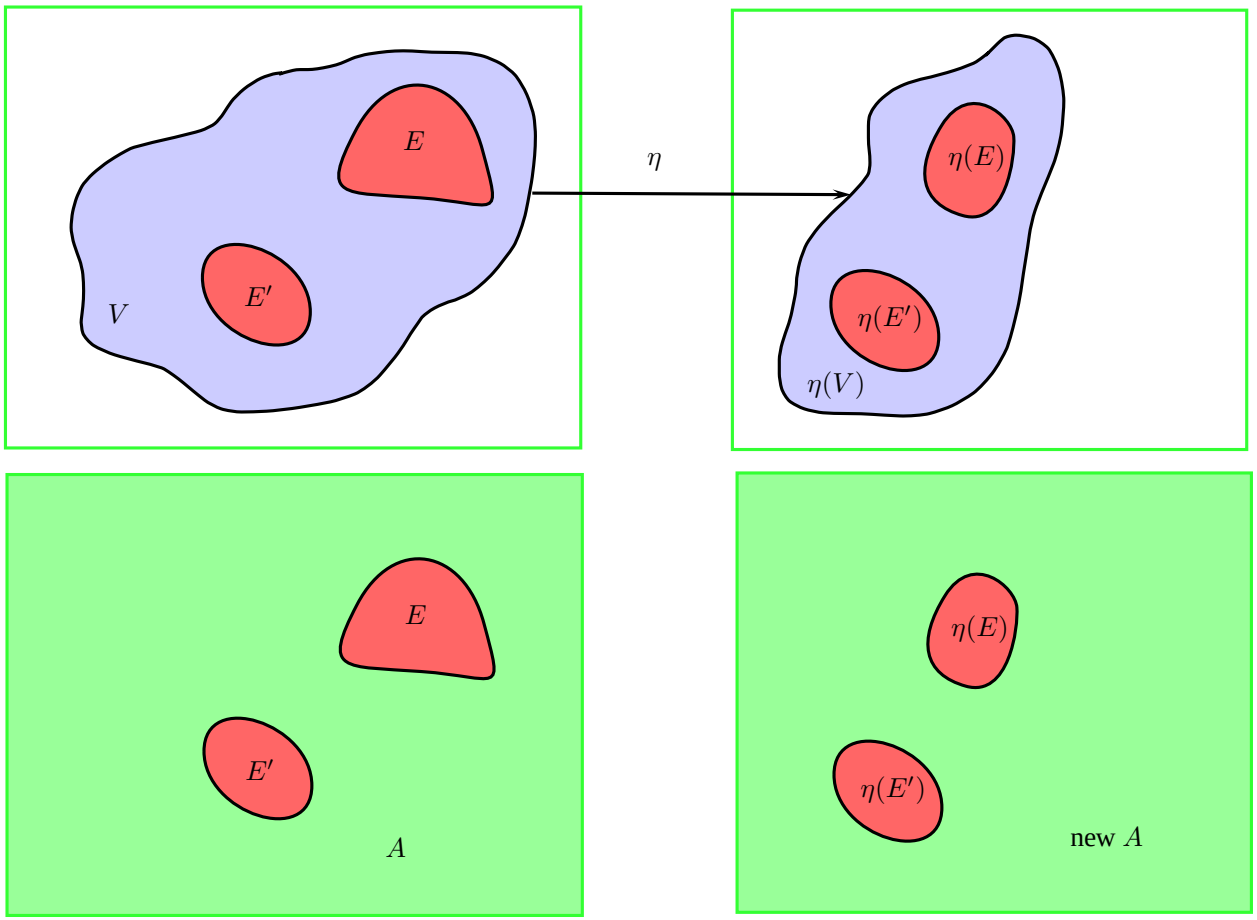

We choose to define
$\mathcal{D}(\eta, V) = \sup\{|\text{mod}\mathbf{A} - \text{mod}\mathbf{newA}|, E, E' \text{ disjoint full continua in } V\}$.

Let:
g be a rational map without critical points[3] on the Julia set;
X_0 be a finite collection of parabolic and repelling cycles;
X be the grand orbit of X_0; $\Delta =$ the unit disc;
$\{U_x(r)\}_{0<r<1, x\in X}$ be families small concentric disc-neighborhoods.

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Theorem (conformal distortion of $p_{r,n}$)

Let $V \subset\subset (\overline{X})^c$. Then $\exists\, C(r)$ for small r with $C(r) \underset{r\to 0}{\longrightarrow} 0$ such that

$$\forall n \geq 0, \quad \forall p_{r,n} \text{ univalent outside } \bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}, \quad \mathcal{D}(p_{r,n}, V) \leq C(r).$$

Clearly only the overlapping properties of the $U_y(r)$'s will play a role, but not the dynamical relations between them.

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§8. Univalent maps outside overlapped disks

Let $X \subset \mathbb{C}$ be a finite set together with a family $\{D_x(r)\}$ of concentric disc-neighborhoods.

Let $\lambda \in [0,1)$ be a constant and $d(r) : (0, r_0] \to (0,1]$ is a function with $d(r) \underset{r\to 0}{\longrightarrow} 0$. We will say that the discs $\{D_x\}_{x\in X}$ are $(\lambda, d(r))$-**nested** if :

(1) Any two are either disjoint or one contains the other.

(2) If $D_y \cap D_x(r) \neq \emptyset$, then $D_y \subset D_x(d(r))$ (near the center they are not too long)

(3) $\text{area}(\bigcup_{y\in D_x \smallsetminus \{x\}} D_y) \leq \lambda \, \text{area} D_x$.

Theorem (a priori bound, no dynamics involved)

There exists a function $C(r) \underset{r\to 0}{\longrightarrow} 0$ depending only on λ and $d(r)$, such that for any set $\{D_x\}_{x\in X}$ of $(\lambda, d(r))$-nested discs and any disjoint $\overline{V}$,

$$\forall \text{ univalent map } \phi \text{ outside } \bigcup \overline{D_x(r)}, \quad \mathcal{D}(\phi, V) \leq C(r) \cdot M(\cup D_x \subset (\overline{V})^c),$$

where $M(D \subset W)$ is a constant depending only on D and W.

§9. Proof of the apriori bound

We use an extremal length argument.

Let A be an annulus with $A^c \subset V$. Let $\rho_0(z)$ be the extremal metric on A so that

$$\ell_{width}(\rho_0, A) = 1, \quad \ell_{height}(\rho_0, A) = \text{area}_{\rho_0}(A) = \text{mod}(A).$$

Key lemma

Let $K \subset A$ be compact and φ be univalent on $A - K$. The for any length increasing conformal metric $\rho(z)|dz|$ supported on $A \setminus K$,
$|\text{mod}\mathbf{A} - \text{mod}\mathbf{newA}| < area_\rho(A) - area_{\rho_0}(A)$.

§9. Proof of the apriori bound

For $k \geq 1$ let I_k, $I_k(r)$ be the union of D_x and $D_x(r)$ of nesting depth k respectively.

Fix $r > 0$ small enough. A disk D_x of depth k is **off centered** if $D_x \cap I_l(r) = \emptyset$ for $l < k$.

Let I'_k , $I'_k(r)$ be the union of off centered D_x and $D_x(r)$ of depth k respectively.

Define $\rho_k(z)$ inductively on k by

$$\rho_k(z) = \begin{cases} \rho_{k-1}(z) & \text{on } A - I'_k, \\ \rho_0(z)(1 - \sqrt{d(r)})^{-k} & \text{on } I'_k - I'_k(d(r)), \\ 0 & \text{on } I'_k(d(r)). \end{cases}$$

Then after finitely many depths, the metric stabilize to a metric ρ, which increases length

$$\ell_{height}(\rho, A) \geq \ell_{hight}(\rho_0, A) = \mathrm{mod} A, \quad \ell_{width}(\rho, A) \geq \ell_{width}(\rho_0, A) = 1$$

and $0 \leq \mathrm{area}_\rho(A) - \mathrm{area}_{\rho_0}(A) \leq C(r) M(D \subset W)$.

§9. Proof of the apriori bound

Let now ϕ a univalent map outside $\cup D_x(r)$, A an annulus whose complement is in the domain of definition of ϕ. Let $A' \subset \widehat{\mathbb{C}}$ be the annulus whose complement is $\phi(\widehat{\mathbb{C}} \setminus A)$ and ρ the conformal metric defined above.

$$|\mathrm{mod}(A') - \mathrm{mod}(A)| \overset{\text{Key lemma}}{\leq} \mathrm{area}_\rho(A) - \mathrm{area}_{\rho_0}(A) \leq C(r) M(D \subset W).$$

So

$$\mathcal{D}_0(\phi, V) \leq C(r) M(D \subset W).$$

□

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——— Other issues:

To check (use the work of Buff-T.) the convergence of the dimension along the paths...

Study the strata structure of these nearby 'hyperbolic components'.

Grazie !

Thank you !