Stretching rays and their accumulations, following Pia Willumsen
Tan Lei (dedicated to Bodil Branner's 60th birthday), December 1, 2005

Abstract. Pia Willumsen did her graduate work with Bodil Branner. Willumsen has written a very beautiful Ph.D. thesis, [W], with many interesting results. They have, however, remained unpublished up to today. We present here a short account of some of Willumsen's results, sketch some of the proofs, as well as some immediate extensions. The main topic is *stretching rays*, which is the analogue in higher dimensional parameter space of external rays of the Mandelbrot set. In the space of cubic polynomials, the interaction of the two critical points create new and interesting phenomena. A typical case deals with maps with a parabolic basin containing both critical points. The results we present here provide necessary conditions for accumulation and landing of stretching rays to these maps. Also, discontinuity of the wring operator on the cubic (in contrast to quadratic) polynomials is proven.

1 Definitions and statements

All polynomials in this note will be monic and centered, of degree greater than or equal to 2.

Let P be such a polynomial. Let μ be a P-invariant Beltrami form with $\|\mu\|_{L^\infty} \leq 1$. It induces a family of P-invariant Beltrami forms $t \cdot \mu$ for t running through the unit disk $\mathbb{D}$. Obviously $\|t \cdot \mu\|_{L^\infty} \leq |t| < 1$. One can thus apply the Measurable Riemann Mapping theorem to this t-analytic family of Beltrami forms.

For $t \in \mathbb{D}$, we will define χ_t to be the integrating map of $t \cdot \mu$ normalized so that $\chi : \mathbb{D} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic motion and that the new dynamics $P_t := \chi_t \circ P \circ \chi_t^{-1}$ is again a monic centered polynomial for each t.

Thus the pair (P, μ) induces an (analytic, see e.g. [PT]) family of quasi-conformal deformations of P. We are interested in the boundary behavior of such deformations.

A fundamental choice for μ ([W, §5]) is the following: denote by φ_P the Böttcher coordinates defined on a neighborhood U of ∞, normalized to be tangent to the identity at ∞, and by $B(\infty)$ the basin of ∞, define

$$\mu_P(z) := \begin{cases} \left(\log \circ \varphi_P \circ P^n\right)^* \frac{d\bar{z}}{dz} & \text{for } z \in B(\infty) \text{ and for large } n \text{ such that } P^n(z) \in U \\ 0 & \text{for } z \notin B(\infty) \end{cases} .$$

One can check easily that the definition is independent of the choice of n. In this case the holomorphic motion on the dynamical plane $\chi : \mathbb{D} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called **Branner-Hubbard motion**, and the induced operator $S : (t, P) \mapsto P_t$ on the parameter space is called **wring operator**.

Geometrically, $t\mu_P$ corresponds to an ellipse field supported on the basin of ∞ of P, with constant ellipticity, and with the major axis of the ellipse tangent to the external rays when $-1 < t < 0$ but orthogonal to the external rays when $0 < t < 1$. See Figure 1. As $-1 < t < 0$ and $t \searrow -1$, the ratio of major to minor axis tends to ∞. In this case, as the corresponding integrating map χ_t maps these ellipses to circles, it therefore 'pushes' the points closer to the filled Julia set along the external rays.

The real locus $S(P) = \{S(t, P), t \in]-1, 1[\}$ is called the **Stretching ray** (or in short S-ray) through P. As $t \searrow -1$, the eventual escaping critical points of P_t get closer and closer to the

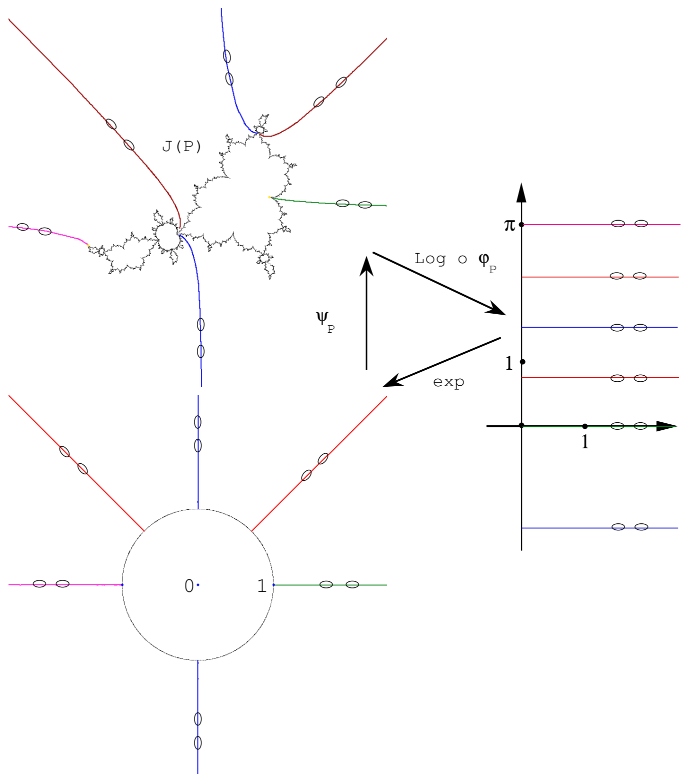


Figure 1: Ellipses of $t\mu_P$ in the dynamical plane, in the Böttcher coordinates and in the log-Böttcher coordinates.

filled Julia set, so that the polynomials P_t accumulate to the connectedness locus of polynomials of the same degree. If an escaping critical point of P sits on a periodic external ray, one should expect a creation of parabolic points as $t \searrow -1$ in order to capture this critical point.

It is quite easy to check that for P in the quadratic family $z \mapsto z^2 + c$ with disconnected Julia set, the stretching ray through P is exactly the external ray through P of the exterior of the Mandelbrot set. And if the escaping critical point of P sits on the 0-external ray, the stretching ray P_t converges to the cauliflower polynomial $Q(z) = z^2 + \frac{1}{4}$.

In general, a stretching ray may or may not land at a point of the connectedness locus. One may thus study the accumulation points of such a ray. This paper focuses on necessary conditions on the pairs (P, Q) such that $S(P)$ accumulates to Q and such that creation of parabolic points occur at Q.

In the quadratic case, if $S(P)$ accumulates to the cauliflower polynomial $Q(z) = z^2 + \frac{1}{4}$, then the dynamical 0-external ray of P must branch at the critical point, consequently $S(P)$ coincides with the 0-external of the Mandelbrot set and $S(P)$ actually lands at Q. This non-trivial fact is closely related to the local connectivity of the Mandelbrot set at $\frac{1}{4}$, and has several proofs in the literature. See for example [T], and the remark below.

The situation in the cubic case is a lot more complicated, due the presence of the two critical points. And this is precisely the focus of the present study.

Denote by $\mathcal{A}$ the set of cubic polynomials such that both critical points are contained in the same immediate basin of a parabolic fixed point of multiplier 1 (therefore with a Jordan curve Julia set and all other periodic points repelling). These polynomials are called **cubic cauliflowers**. Two examples of such Julia sets can be found on the right column of Figure 5. And the parameter set $\mathcal{A}$ is represented, in some suitable parametrization, in Figure 2, as well as in the middle picture of Figure 5.

The main purpose here is to study necessary conditions on a pair (P, Q) of cubic polynomials so that Q belongs to $\mathcal{A}$ and is accumulated by $S(P)$. We study also the (dis)continuity property of the map $(t, P) \mapsto S(t, P)$.

Following A. Epstein, a polynomial $Q \in \mathcal{A}$ is called *parabolic attracting* if any nearby map has an attracting or parabolic fixed point. Such polynomials are represented in Figure 3.

Here is the first main result: denote by $Acc(S(P))$ to be the set of accumulation points of $S(P)$ as $t \searrow -1$.

Theorem 1.1. *Let* (P, Q) *be a pair of cubic polynomials such that* $Q \in Acc(S(P)) \cap \mathcal{A}$ *and* $Q \neq P$. *Then*

I. *All periodic points of* P *are repelling and* Q *is not parabolic attracting ([W, 6.5]).*

II. *The filled Julia set* K_P *of* P *is a Cantor set ([W, 7.13]).*

III. *(exchange* 0 *and* $\frac{1}{2}$ *if necessary) The* 0*-external ray of* P *branches at a critical point of* P *([W, 7.5]); the left and right limit* 0*-ray (see definition below) land at two distinct fixed points of* P*, which collide to the parabolic fixed point of* Q *at the limit; the* $\frac{1}{2}$*-ray of* P *lands at the third fixed point, and at any other periodic point of* P *lands exactly one external ray, which is periodic.*

IV. *For the other critical point of* P*, either*

   a) *it escapes and falls eventually into the* 0*-external ray; or*

   b) *it is the landing or branching point of a* θ*-external ray, with an angle* θ *satisfying* $\{3^k\theta, k \in \mathbb{N}\} \not\ni 0$ *but* $\overline{\{3^k\theta, k \in \mathbb{N}\}} \ni 0$.

   Conjecturally Case b) never occurs.

Two examples of the pair (P, Q) can be found on the top and bottom row of Figure 5. The branching 0-external ray for P, and the 0-external ray for Q are also drawn.

Remark. As mentioned above, there is a similar statement in the quadratic setting (the part IV is void). Only part III is non trivial. Most of the existing proofs in the literature depend on the global combinatorial structure of the quadratic family. But the proof of Part III presented here can be easily adapted to give a purely intrinsic proof of its quadratic parallel.

A theorem of Branner-Hubbard shows the continuity of the wring operator restricted either to the cubic connectedness locus or to the cubic disconnectedness locus ([BH1, §9]), in fact it acts trivially on the connectedness locus. However, the second main result presented here claims:

Theorem 1.2. *The map* $(t, P) \mapsto S(t, P)$ *is continuous in the space of quadratic polynomials, but is discontinuous in the cubics, more precisely at some points in* $\mathbb{D} \times \mathcal{A}$ *([W, 5.7,5.8]).*

Further necessary conditions on landing or accumulation of S-rays to $\mathcal{A}$ in terms of Lavaurs maps will be given in §4, as well as the proof of Theorem 1.2. The proof of Theorem 1.1 is in §3.

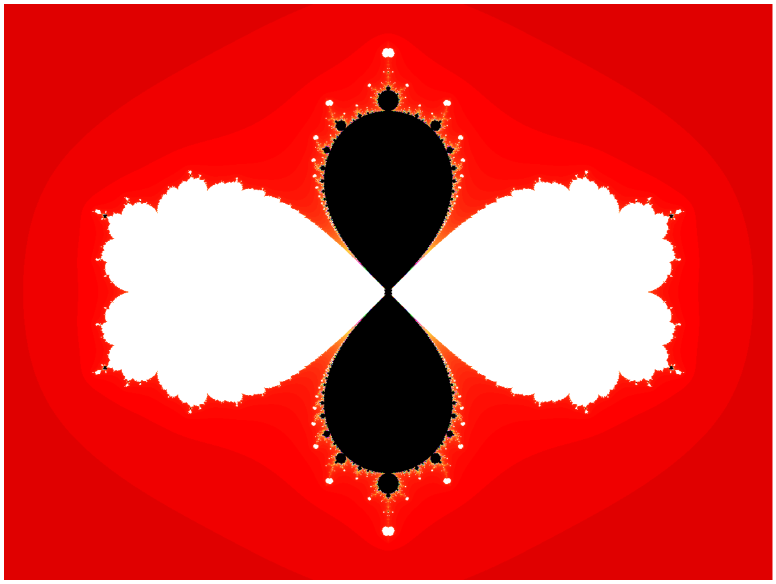


Figure 2: The slice $Per_1(1)$ and its subset $\mathcal{A}$, which is the two large white butterfly wings.

The following concept and result will be interesting for further research in the topic, but will not be needed nor proved in the present paper. For a given (polynomial, invariant Beltrami form) pair (P,μ), and χ_t the suitably normalized integrating maps of $t\mu$, the associated initial speed of χ_t and the **ground wind** at P relative to μ are defined by

$$\tau(z) := \frac{d}{dt}\chi_t(z)\bigg|_{t=0}, \quad w(P,\mu) := \frac{d}{dt}P_t\bigg|_{t=0}.$$

Proposition 1.3. *([W, 5.18]) Assume μ agrees with μ_P on the basin of infinity. We have*

$$w(P,\mu) = \tau \circ P - P' \cdot \tau\ .$$

2 Necessary conditions for accumulation

2.1 Generalized external angles and rays

We need to generalize the notion of external rays, angles etc. to polynomials with disconnected Julia set. Let P be such a polynomial, say monic and centered of degree 3. There is a unique univalent map φ_P, the Böttcher coordinate, defined at least in a neighborhood of ∞, conjugating P to $z \mapsto z^3$ and satisfying $\frac{\varphi_P(z)}{z} \to 1$ as $z \to \infty$. Denote by ψ_P the inverse map of φ_P. Then the maximal domain of definition of ψ_P is the complement of $\overline{\mathbb{D}} \cup Y_P$, where Y_P is the union of finitely many radial segments $\{[1, r_\theta] \cdot e^{2\pi i\theta}\}$ together with their successive preimages under the iteration of z^3, and ψ_P extends continuously to the tips of segments in Y_P. For $\theta \in \mathbb{T}$ define the θ-ray by

$$R_P(\theta) = \psi_P(\{]r_\theta, +\infty[\cdot e^{2\pi i\theta}\})$$

where r_θ is the minimal value possible. It is said to be *branched* with branching point $\psi_P(r_\theta \cdot e^{2\pi i\theta})$ if $r_\theta > 1$; *unbranched* otherwise. It *lands* at a Julia point a if $a = \lim_{r\searrow 1} \psi_P(re^{2\pi i\theta})$. Due to the

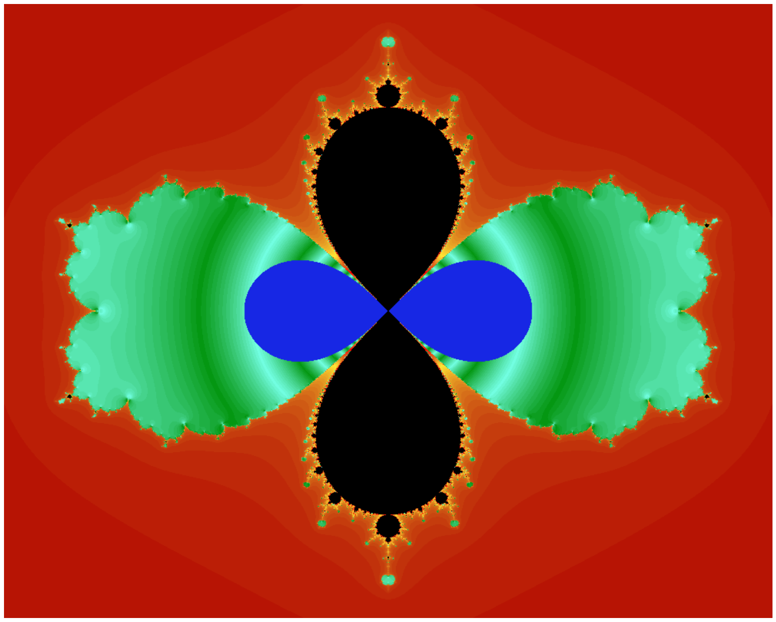


Figure 3: The lemniscate in $\mathcal{A}$ indicates the parabolic attracting locus.

countability of radial segments in Y_P, the following limits exist and are called **left and right limit θ-ray**:

$$R_P^{\pm}(\theta) = \bigcup_{r>1} \{ \lim_{u \to 0^{\pm}} \psi_P(re^{2\pi i(\theta+u)}) \} \ .$$

See the middle top and middle bottom pictures in Figure 5 for examples of left and right limit 0-rays of cubic polynomials.

These limit rays are canonically parametrized by $r > 1$ and preserved by P, in the sense that $P(R_P^+(\theta)) = R_P^+(3\theta)$ and $P(R_P^-(\theta)) = R_P^-(3\theta)$. Now we define the *external angles* $arg(a)$ and the *generalized external angle* $Arg(a)$ by:

$$a \in K_P \quad \left\{ \begin{array}{lcl} arg(a) & := & \{\theta \mid R_P(\theta) \text{ lands at } a\} \\ \cap & & \\ Arg(a) & := & \{\theta \mid R_P^+(\theta) \text{ or } R_P^-(\theta) \text{ lands at } a\} \end{array} \right.$$

$$a \notin K_P \quad \left\{ \begin{array}{lcl} arg(a) & := & \{\theta \mid a \in R_P(\theta) \cup \{\text{branching point}\}\} \\ \cap & & \\ Arg(a) & := & \{\theta \mid a \in R_P^+(\theta) \cup R_P^-(\theta)\} \end{array} \right.$$

Clearly

$$3 \cdot Arg(a) \subset Arg(P(a)) \ . \tag{1}$$

Some of these sets might be empty. In case that K_P is connected, we have $Y_P = \emptyset$, $R_P^{\pm}(\theta) = R_P(\theta)$ for every θ and $arg(a) = Arg(a)$ for every a.

Define also

$$per{-}arg(a) := \{\theta \text{ periodic} \mid R_P(\theta) \text{ lands at } a\} \ . \tag{2}$$

2.2 Fundamental properties of accumulation of a stretching ray

Let (P, Q) be a pair of monic centered polynomials of same degree, with $Q \in Acc(S(P))$. Let $t_n \searrow -1$ be a sequence such that the polynomials $P_n := S(t_n, P)$ converges algebraicly to Q. We compare here critical points and periodic points of P, P_n and Q.

(Most results below are also valid under the more general assumption that $P_n \to Q$ algebraicly and P_n are qc-conjugates of a single map P, not necessarily coming from stretching).

Let α be a periodic point of P. It is associated with a triple (m, λ, k), with :

- $m =$ period of α ;
- $\lambda := (P^m)'(\alpha) =$ the multiplier of α; and
- $k =$ the multiplicity of α, as root of $P^m(z) - z = 0$.

Recall that χ_{t_n} is a quasi-conjugacy from P to P_n. Let $\alpha_n := \chi_{t_n}(\alpha)$ denote the corresponding periodic point for P_n. Its associated triple (m_n, λ_n, k_n) satisfies $m_n = m$ and $k_n = k$, but λ_n might be depending on n.

Taking a subsequence if necessary, we may assume $\alpha_n \to \alpha'$. Clearly α' is again a periodic point of Q, of period a divisor of m. We denote its associated triple by (m', λ', k'). Thus $m'|m$. We look for further relations between these triples. We look also for possible relations between critical orbits of P and those of Q.

We group the various notations in one tableau:

(m, λ)	$(m_n = m, \lambda_n)$		(m', λ')	(period,multiplier)
α	$\alpha_n = \chi_{t_n}(\alpha)$	$\xrightarrow{\text{sub-sequence}}$	α'	periodic point
P	$S(P) \ni P_n$	$\longrightarrow$	Q	
w	$w_n = \chi_{t_n}(w)$	$\xrightarrow{\text{sub-sequence}}$	w_Q	escaping critical point
θ	$\theta_n = \theta$		?	escaping critical angle

See Table 1 for a summery of some of the following result.

Proposition 2.1. *In the above setting,*

1. *The escaping critical angles for P are preserved along $S(P)$, as well as the ratio of escaping critical potential levels (i.e. real part in the $\log \circ \varphi_Q$-coordinate).*
2. *More generally a point z and $\chi_t(z)$ have the same set of generalized external angles.*
3. *If $|\lambda| \leq 1$ and $\lambda \neq 1$, then $(m, \lambda, k) = (m', \lambda', k') = (m, \lambda, 1)$ ([W, 7.1]).*
4. *If $\lambda' = 1$ then $m' = m$, and either $|\lambda| > 1$ or $\lambda = 1$. On the other hand, if $\lambda = 1$ then either $m = m'$ and $\lambda' = 1$, or $m > m'$, $\lambda' \neq 1$ and $\lambda'^{\frac{m}{m'}} = 1$.*
5. *Any critical relation of P is preserved to Q (maybe with a divisor period).*

6. *The map Q has a connected Julia set. All rational rays of Q land. For every parabolic or repelling periodic point z' of Q, we have $arg_Q(z')$ non empty and consisting of only periodic angles (see [Mi] and [Pe]). If P has connected Julia set then $S(P) = \{P\}$ and $\chi_t|_{K_P} = id$ ([BH1, 8.3], see also Lemma 4.1.a) below).*

7. *If $|\lambda'| < 1$ then $(m, \lambda, k) = (m', \lambda', k') = (m, \lambda, 1)$. Consequently if α is repelling then α' can't be attracting.*

8. *If $|\lambda'| > 1$ then $|\lambda| > 1$ and $(m, k) = (m', k') = (m, 1)$ (but maybe $\lambda \neq \lambda'$). Moreover, for the non-generalized external angles:*

$$per-arg_P(\alpha) = per-arg_{P_n}(\alpha_n) \supset arg_Q(\alpha') \neq \emptyset \ .$$

Furthermore, the inclusion $\supset$ becomes an equality iff for any $\theta \in per-arg_P(\alpha)$, the ray $R_Q(\theta)$ for the polynomial Q does not land at a parabolic point of Q.

9. *If a periodic ray $R_P(\theta)$ of P branches then the corresponding ray $R_Q(\theta)$ for Q lands at a parabolic periodic point (see Figure 5 for examples).*

Proof. By the uniform convergence, $Q^m(\alpha') = \alpha'$. So $m'|m$ and

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} (P_n^m)'(\alpha_n) = (Q^m)'(\alpha') = ((Q^{m'})'(\alpha'))^{\frac{m}{m'}} = \lambda'^{\frac{m}{m'}} \ . \tag{3}$$

Points 1 and 2 are due to the definition of the wring operator.

Point 3. The fact that $m_n = m$ is due to the global qc-conjugacy χ_{t_n}. We prove at first $\lambda_n \equiv \lambda$ for $|\lambda| \leq 1$. If $|\lambda| < 1$, α is an attracting periodic point and the complex structure is not deformed in the attracting basin. So χ_{t_n} is a local conformal conjugacy and $\lambda_n = \lambda$. If $|\lambda| = 1$, the map χ_{t_n} is a topological conjugacy and the multiplier is preserved by a (highly non-trivial) theorem of Naishul (see [Na] or [P-M]).

Assume now $|\lambda| \leq 1$ and $\lambda \neq 1$. Then $\lambda'^{\frac{m}{m'}} = \lambda$ by the above argument and (3) , so $\lambda' \neq 1$. By Rouché's theorem, in a neighborhood of α' and for n large, P_n^m has a unique fixed point u_n, and $P_n^{m'}$ has a unique fixed point v_n. But $m'|m$. So $P_n^m(v_n) = v_n$. By unicity, $u_n = v_n = \alpha_n$. But $per(v_n) \leq m' \leq m = per(\alpha_n)$. It follows that $m' = m$ and $\lambda' = \lambda$. The part $k = k' = 1$ is easy.

Point 4. Assume $\lambda' = 1$. By Point 3, either $|\lambda| > 1$ or $\lambda = 1$. We want to show $m' = m$. Set $\alpha' = 0$ for simplicity. Let k be the multiplicity. The local expansions of $Q^{m'}$ and Q^m are:

$$Q^{m'}(z) = z + az^k + \cdots \ ; \quad Q^m(z) = z + \frac{m}{m'} az^k + \cdots \ , \quad a \neq 0 \ .$$

Apply Rouché's theorem again, we get that in a neighborhood of α' and for n large, P_n^m has exactly k fixed points (counting with multiplicity) $u_n^1, u_n^2, \cdots, u_n^k$ (one of them must be α_n). And $P_n^{m'}$ has exactly k fixed points (counting with multiplicity) $v_n^1, v_n^2, \cdots, v_n^k$. But $m'|m$. So $v_n^i = \alpha_n$ for some i. But $per(v_n^i) \leq m' \leq m = per(\alpha_n)$. It follows that $m' = m$.

On the other hand, assume $\lambda = 1$. Then $\lambda_n \equiv 1$ and $\lambda'^{\frac{m}{m'}} = 1$ by the proof in Point 3. If $m' = m$ then $\lambda' = 1$. If $m' < m$ then $\lambda' \neq 1$ by the previous paragraph.

Point 5 is easy.

Point 6. The map Q has no escaping critical points, so has a connected Julia set. The rest are proved in the given references.

Point 7. $|\lambda'| < 1 \Longrightarrow |\lambda_n| < 1 \Longrightarrow \lambda = \lambda_n = \lambda'$ and $m' = m$ (due to Point 3).

Point 8. $|\lambda'| > 1 \Longrightarrow |\lambda_n| > 1 \Longrightarrow |\lambda| > 1$ due to Points 3 and 4. The fact that $m' = m$ is because $\lambda \neq 1$. Now Point 6 gives that $arg_Q(\alpha') \neq \emptyset$ and consists of only periodic angles. The set $\{\alpha'\} \cup \bigcup_{\theta \in arg(\alpha')} R_Q(\theta)$ undergoes a holomorphic motion. So for $\theta \in arg_Q(\alpha')$, for P_n close to Q, and for α_n the perturbed periodic point, $R_{P_n}(\theta)$ continues to land at α_n. But $arg_P(\alpha) = arg_{P_n}(\alpha_n)$. So $per{-}arg_P(\alpha) \supset arg_Q(\alpha')$.

Finally for $\theta \in per{-}arg_P(\alpha)$, we know that θ is periodic, thus the ray $R_Q(\theta)$ does not branch and lands at a repelling or parabolic point z' (Snail Lemma, see for example [P-M]). If z' is repelling then by above there is a periodic point z of P with $\theta \in arg_P(z)$. As a consequence $z = \alpha$ and $z' = \alpha'$. So $\theta \in arg_Q(\alpha')$.

Point 9. Again $R_Q(\theta)$ must land and lands at a repelling or parabolic point. The rest follows from Point 8. See Figure 5 for examples of such pairs (P, Q). □

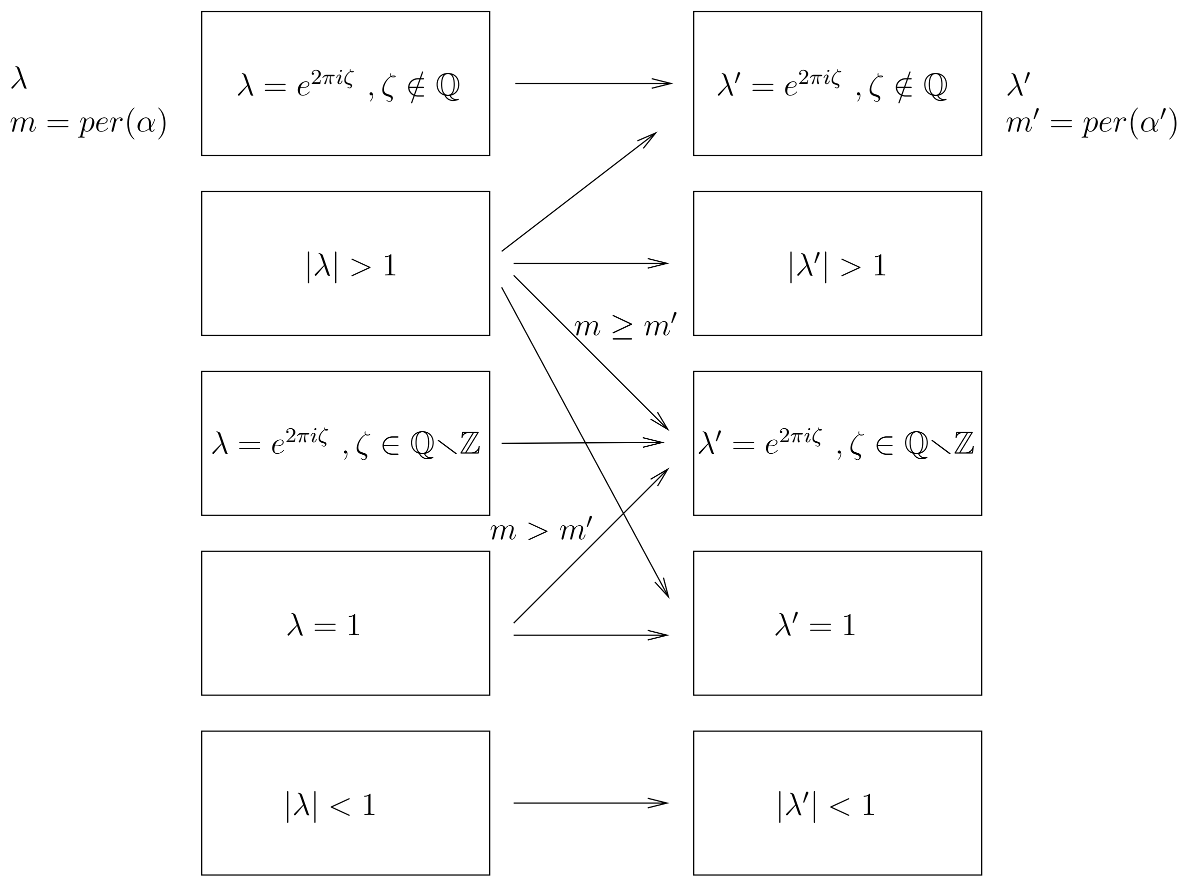


Table 1. In most cases we have $m = m'$. The only exceptions are the two diagonal arrows which are specially labeled.

Table 1 summarizes the results in Proposition 2.1 about relations between a polynomial P with disconnected Julia set (on the left) and an accumulation point Q of its stretching ray (on the right).

The next result establishes Part I of Theorem 1.1:

Corollary 2.2. *Let (P,Q) be a pair of cubic polynomials such that $P \neq Q \in Acc(S(P)) \cap \mathcal{A}$. Then all periodic points of P are repelling, and Q is not parabolic attracting.*

Proof. Note that P can not be in the connectedness locus, for otherwise $S(P) = \{P\} = Acc(S(P)) \not\ni Q$ by Proposition 2.1.(6). We use the fact that Q has a unique non-repelling periodic point β', whose associated triple is $(1,1,2)$, i.e. it is fixed with multiplier 1, and the multiplicity of β' as a root of $Q(z) - z = 0$ is 2 (this follows from the assumption that both critical points of Q are contained in the same attracted basin of β', so that β' has only one Fatou petal).

Let α be a periodic point of P, of associated triple (m, λ, k). We want to prove that $|\lambda| > 1$.

As indicated at the beginning of this section, and due to Proposition 2.1, the point α induces a periodic point α' of Q, whose associated triple (m', λ', k') is related to those of α following Table 1. On the other hand, we know from the property of Q that either $|\lambda'| > 1$ or $\alpha' = \beta'$ and $\lambda' = 1$.

We conclude immediately that either $|\lambda| > 1$ or $\lambda = 1$ by Table 1.

Assume $\lambda = 1$. By Table 1, either $m = m'$ and $\lambda' = 1$, or $m > m'$ and λ' is a non-trivial root of unity. Again the latter case is not possible for our Q. So $m = m'$ and $\lambda' = 1$. This implies that $\alpha' = \beta'$ and $m = m' = 1$. Thus P is in the space $Per_1(1)$ of cubic polynomials having a fixed point of multiplier 1 and P has a disconnected Julia set. But then $S(P)$ accumulates to the boundary of the connected locus in this space, which is disjoint from $\mathcal{A}$. This is a contradiction to the fact that $Q \in Acc(S(P)) \cap \mathcal{A}$.

We conclude then $|\lambda| > 1$. Therefore all periodic points of P are repelling.

Now let $P_n \in S(P)$ with $P_n \to Q$. If Q were parabolic attracting, some P_n would have an attracting or parabolic fixed point. On the other hand, P_n is a qc-deformation of P. So, as P, all periodic points of P_n are repelling. This is a contradiction. □

Lemma 2.3. *If a critical point w of P has a rational generalized external angle θ that is not a preimage of the 0-angle, then $Acc(S(P)) \cap \mathcal{A} = \emptyset$.*

Proof. If $w \in \partial K_P$ then it must be strictly preperiodic. And this critical relation is preserved to any $Q \in Acc(S(P))$, by Proposition 2.1.(5). This implies $Q \notin \mathcal{A}$, for otherwise both critical points of Q would be contained in the attracted basin of a parabolic fixed point and would not be preperiodic.

Assume now $w \notin K_P$. Assume there is $Q \in Acc(S(P)) \cap \mathcal{A}$. The fact that $Q \in \mathcal{A}$ implies that the θ-external ray of Q lands at a prerepelling point. This ray is therefore stable under perturbation, by Proposition 2.1.(8). This means that for any polynomial sufficiently close to Q, its θ-external ray is unbranched and lands at a prerepelling point, in particular it does not contain critical points. On the other hand, the fact that $Q \in Acc(S(P))$ implies the existence of a sequence of polynomials P_n converging to Q, such that the θ-external ray of P_n, as that of P, contains a critical point. This is a contradiction. □

3 Proof of Theorem 1.1

Proof. Assume $P \neq Q \in Acc(S(P)) \cap \mathcal{A}$. Part I is already proved in Corollary 2.2.

II. If both critical points of P escape, then K_P is a Cantor set (this is classical). If instead only one critical point escapes, then, letting L be the filled-Julia-component containing the non-escaping critical point, either K_P is a Cantor set or L is m-periodic for some m, in which case $P^m|_L$ is hybrid equivalent to z^2+c for some c in the Mandelbrot set. This is due to Branner-Hubbard theory for cubics ([BH2, 5.3]).

Assume by contradiction that L is m-periodic.

Then the hybridly equivalent quadratic polynomial z^2+c has a fixed point $\hat{\alpha}$ which is either non-repelling or repelling without 0 as an external angle. This fixed point corresponds to a m-periodic point α for P. Let $P_n = S(t_n, P)$ so that $P_n \to Q$. Then $\alpha_n := \chi_{t_n}(\alpha)$ has a limit α' (taking a subsequence if necessary). Denote by (m', λ', k') the associated triple of α'. It is related to that of α following Table 1.

We use the fact that Q has a unique periodic point β' with associated triple $(1,1,2)$, and all other periodic points are repelling. So either $\alpha' = \beta'$ and $(m', \lambda', k') = (1,1,2)$, or $|\lambda'| > 1$.

Assume at first $m > 1$. By Table 1 we can not have both $m' = 1$ and $\lambda' = 1$. So $\alpha' \neq \beta'$ and $|\lambda'| > 1$. By Table 1 again we have $m' = m$. As the Julia set of Q is a Jordan curve, $arg_Q(\alpha')$ consists of a unique angle θ, which has the same period m. By Proposition 2.1.(8), $\theta \in arg_P(\alpha)$. Now the hybrid conjugacy sends the germ of $R_P(\theta)$ to a fixed external access to $\hat{\alpha}$ for the polynomial z^2+c. This in turn implies (using the Böttcher coordinates for z^2+c) that the 0-external ray lands at $\hat{\alpha}$, contradicting the choice of $\hat{\alpha}$.

Assume now $m = 1$. Assume first $c = \frac{1}{4}$. In this case P is in the $Per_1(1)$ slice and $S(P)$ accumulates to the boundary of the connected locus in this slice, which is disjoint from $\mathcal{A}$, and thus gives a contradiction. Assume now $\hat{\alpha}$ is an attracting or neutral fixed point (with multiplier $\neq 1$). Then α' has the same property by Table 1, and is again not possible for our Q. Assume finally $\hat{\alpha}$ is repelling with at least two external rays. Then α is repelling for P. By Table 1 the point α' is either repelling (and fixed) or parabolic with multiplier 1. The former case is excluded due to the same argument as in the case $m > 1$. So we are left with the case that $\alpha' = \beta'$ is the unique parabolic fixed point of Q. By Theorem 1.(3-4) in [LP], $Arg_P(\alpha) = \{\theta_1, \cdots, \theta_p\}$ with $p \geq 1$ and all θ_i periodic. The case $p = 1$ can be treated as above. If $p > 1$, there is one θ_i that is not the 0-angle, and $R_Q(\theta_i)$ lands at a repelling periodic point $z' \neq \alpha'$. Due to the stability of $\overline{R_Q(\theta_i)}$ under small perturbation, for n large, $R_{P_n}(\theta_i)$ is unbranched and lands at a point z_n tending to z'. On the other hand, an unbranched ray is unique, so $z_n = \chi_{t_n}(\alpha) = \alpha_n \to \alpha'$. This is again a contradiction.

III. We group the various notations in the next tableau.

γ		γ_n		γ' a repelling fixed point
$\overline{\bigcup_{\theta \in A_\eta} R_P(\theta)}$		$\overline{\bigcup_{\theta \in A_\eta} R_{P_n}(\theta)}$		$\overline{\bigcup_{\theta \in A_\eta} R_Q(\theta)}$, $\eta > 0$
does not contain		does not contain		does not contain
α, β	χ_{t_n}	α_n, β_n	$n \to \infty$	β' parabolic fixed point
P	$\mapsto$	P_n	$\longrightarrow$	Q
$z \neq \alpha, \beta$		$\chi_{t_n}(z) \neq \alpha_n, \beta_n$		$z' \neq \beta'$

Choose $P_n = S(t_n, P)$ with $P_n \to Q$ as $t_n \searrow -1$. As Q has a Jordan curve Julia set, every external ray lands, and different rays land at distinct points. Recall that β' denotes the parabolic fixed point of Q with multiplier 1. Assume $R_Q(0)$ lands at β' (otherwise exchange 0 with $\frac{1}{2}$). Denote by γ' the landing point of $R_Q(\frac{1}{2})$.

Clearly $\gamma' \neq \beta'$ and γ' is a repelling fixed point. Apply Proposition 2.1.(8) to γ', one proves that $R_P(\frac{1}{2})$ lands at some point γ, and $\chi_{t_n}(\gamma) =: \gamma_n$ converges to γ'.

By part II, K_P is a cantor set. Thus all fixed points of P are simple. There are therefore $\alpha \neq \beta$ two fixed points of P distinct from γ. Set $\alpha_n = \chi_{t_n}(\alpha)$ and $\beta_n = \chi_{t_n}(\beta)$. They are distinct from γ_n and are bounded away from ∞. Further the limit of any convergent subsequence of α_n must be a fixed point of Q, distinct from γ' (as in a neighborhood of γ' there can be only one fixed point of P_n, which is γ_n). But the only fixed points of Q are β' and γ'. Thus the entire sequence α_n converges to β'. We have $Arg_P(\alpha) = Arg_{P_n}(\alpha_n)$ due to the stretching properties of χ_t. Similarly for β,β_n. By Corollary 2.2 both α, β are repelling.

Claim 1. The set $Arg_P(\alpha)$ is non-empty, closed, and satisfies: $3 \cdot Arg_P(\alpha) \subset Arg_P(\alpha)$.

Proof. The inclusion is due to the fact that $P(\alpha) = \alpha$ and (1) . The rest follows from [LP, 2.1] (or [W, 2.10]) (in fact when K_P is a Cantor set, every left or right limit ray lands, and every Julia point has a non-empty compact set of generalized external angles, by pulling back disks with equipotential boundaries).

Claim 2. We have $0 \in Arg_P(\alpha)$.

Proof. For any fixed $\eta > 0$ set

$$A_\eta = \{\theta \in \mathbb{T} \mid d_{\mathbb{T}}(3^k\theta, 0) \geq \eta \text{ for all } k\} ,$$

et let X_η be the set of landing points of $R_Q(\theta)$ for all $\theta \in A_\eta$. Then $\beta' \notin X_\eta \cup \bigcup_{\theta\in A_\eta} R_Q(\theta)$ and X_η is a compact hyperbolic subset. Furthermore $X_\eta \cup \bigcup_{\theta\in A_\eta} R_Q(\theta)$ undergoes a holomorphic motion over a small neighborhood of Q, preserving the dynamics. Therefore for n large, the ray $R_{P_n}(\theta)$ is unbranched for all $\theta \in A_\eta$ and

$$\alpha_n, \beta_n \notin \overline{\bigcup_{\theta\in A_\eta} R_{P_n}(\theta)} .$$

(And the right hand set does not contain critical points of P_n). So

$$\left(\bigcup_{\eta>0} A_\eta\right) \cap Arg_{P_n}(\alpha_n) = \emptyset .$$

The same is true if we replace $Arg_{P_n}(\alpha_n)$ by $Arg_P(\alpha)$ as the two sets are equal. But $Arg_P(\alpha)$ is closed, non-empty and forward invariant by angle tripling (by Claim 1). So

$$\theta \in Arg_P(\alpha) \Longrightarrow 3^k\theta \in Arg_P(\alpha) \Longrightarrow \overline{\{3^k\theta, k \in \mathbb{N}\}} \subset Arg_P(\alpha) \Longrightarrow 0 \in Arg_P(\alpha) ,$$

where the last implication is due to the fact that $\theta \notin \bigcup_{\eta>0} A_\eta$. This ends the proof of Claim 2.

Similarly one shows $0 \in Arg(\beta)$. So 0 is a generalized external angle for both α and β. This means that the 0-ray of P is branched at a critical point, that $R_P^{\pm}(0)$ land at two distinct fixed points which collide to the parabolic fixed point of Q in the limit.

Now let $z \neq \alpha, \beta$ be any periodic point of P. Let z' be a limit point of $\chi_{t_n}(z)$. It is a periodic point for Q. We now show that $z' \neq \beta'$. If $per(z) = 1$, then $z = \gamma$ and $z' = \gamma' \neq \beta'$. So we may assume $m := per(z) > 1$. Either $per(z') < m$ then z' is parabolic but with multiplier distinct from 1 (Table 1). This is not possible for Q. Or $per(z') = m > 1$. In this case $z' \neq \beta'$.

As all periodic points of Q other than β' are repelling, so is z'. As $Q|_{J_Q}$ is topologically conjugate to $z^3|_{S^1}$ we have $arg_Q(z') = \{\theta'\}$ and θ' is m-periodic.

Now let $\theta \in arg_P(z)$ be periodic, that is the ray $R_P(\theta)$ does not branch and lands at z. Clearly $\theta \neq 0$ as the 0-ray branches. But $R_Q(\theta)$ always lands and lands at a distinct point than the landing point of $R_Q(0)$, which is β'. Therefore $R_Q(\theta)$ lands at a non parabolic point. So the equality in Propostion 2.1.(8) holds for z and z', i.e.

$$\{\theta \in arg_P(z),\ \theta \text{ periodic}\} = \{\theta'\}\ .$$

To prove that $arg_P(z) = \{\theta'\}$ we just need to apply the (non-trivial) result [LP, Thm. 1.4].

IV. Let w be a critical point of P, escaping or not. The fact that K_P is a Cantor set ensures that $Arg_P(w) \neq \emptyset$ (see Claim 1). Let $\theta \in Arg_P(w)$.

Case 1. There is N such that $3^N\theta = 0$. Then w must escape for otherwise $P^{N+1}(w) = P^N(w)$ and this critical relation would have persisted to Q. This is Case a) of the theorem.

Case 2. There is $\eta > 0$ such that $\theta \in A_\eta$. This is not possible due to the argument in III, similar to the proof of Lemma 2.3.

Case 3. The orbit of θ does not meet 0 but accumulates to 0. In this case either $w \notin K_P$, in which case the θ-ray branches at w, or $w \in K_P$ in which case $Arg(w) = arg(w) \neq \emptyset$. This is Case b) of the theorem. Conjecturally this case never occurs. □

4 Lavaurs maps, Fatou vectors and Theorem 1.2

4.1 Lavaurs maps and enriched Branner-Hubbard motion

In this subsection, denote by Q a monic centered polynomial of degree d, with connected Julia set and with a parabolic fixed point β' of multiplier 1. We will define the following objects related to Q:

$$\left(Q, \beta', B(\beta'), K_Q, B_Q(\infty), \varphi_Q, \psi_Q, \Phi_Q^{\pm}, \Psi_Q^{+}, g_{\tilde{\sigma}}\right) \tag{4}$$

where:

- $B(\beta')$ is an immediate basin of β'.
- K_Q is the filled Julia set.
- $B_Q(\infty)$ is the basin of ∞.
- φ_Q denotes the Böttcher coordinates of Q near ∞, tangent to the identity at ∞. As K_Q is connected, the map φ_Q extends to a conformal homeomorphism from $B_Q(\infty)$ to $\mathbb{C}\setminus\overline{\mathbb{D}}$.
- $\psi_Q : \mathbb{C}\setminus\overline{\mathbb{D}} \to B_Q(\infty)$ denotes the inverse of φ_Q.

- Φ_Q^- denotes the attracting Fatou coordinates. More precisely, it is at first defined and univalent on an attracting petal of β' satisfying $\Phi_Q^- \circ Q = T_1 \circ \Phi_Q^-$. It is then extended to the entire basin $B(\beta')$ using the functional equation (and is no more univalent).
- Φ_Q^+ denotes the repelling Fatou coordinates. It is at first defined and univalent on a repelling petal of β' satisfying $\Phi_Q^+ \circ Q = T_1 \circ \Phi_Q^+$.
- Ψ_Q^+ denotes the inverse of this local Φ_Q^+. It is then extended to the entire plane $\mathbb{C}$ using the functional equation $Q \circ \Psi_Q^+ = \Psi_Q^+ \circ T_1$ (and is not globally univalent).
- T_* denotes the translation $z \mapsto z + *$, in particular T_1 is the translation by 1.
- For $\tilde\sigma \in \mathbb{C}$, the **Lavaurs map** $g_{\tilde\sigma}$ of lifted phase $\tilde\sigma$, is by definition $\Psi_Q^+ \circ T_{\tilde\sigma} \circ \Phi_Q^-$. It satisfies $g_{\tilde\sigma} \circ Q = Q \circ g_{\tilde\sigma}$.

The Fatou coordinates $\Phi_Q^\pm$ are uniquely defined up to post-composition of a translation. We may for example choose the following normalizations:

Φ_Q^- has an inverse branch mapping the right half plane univalently onto a region U_Q whose boundary contains β' and a critical point w_0 (in some sense w_0 is the 'closest' attracted critical point), and $\Phi_Q^-(w_0) = 0$;

$\Psi_Q^+(0) = \psi_Q(e^x)$ for some $x > 0$ (in other words $\Psi_Q^+(0)$ is a point on the 0-ray of Q), and Ψ_Q^+ is univalent on the left half plane.

The various maps and change of coordinates are sketched in the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & \mathbb{C}/\mathbb{Z} & \xrightarrow{T_\sigma} & \mathbb{C}/\mathbb{Z} & & \\
 & & \pi \uparrow & & \pi \uparrow & & \\
\mathbb{C}^- & \xrightarrow{T_1} & \mathbb{C}^- & \xrightarrow{T_{\tilde\sigma}} & \mathbb{C}^+ & \xrightarrow{T_1} & \mathbb{C}^+ \\
\Phi_Q^- \uparrow & & \Phi_Q^- \uparrow & & \Psi_Q^+ \downarrow & & \downarrow \Psi_Q^+ \\
B(\beta') & \xrightarrow[Q]{} & B(\beta') & \xrightarrow[g_{\tilde\sigma}]{} & \mathbb{C} & \xrightarrow[Q]{} & \mathbb{C} \\
 & & & & \bigcup & & \bigcup \\
 & & & & B_Q(\infty) & \xrightarrow[Q]{} & B_Q(\infty) \\
 & & & & \psi_Q \uparrow & & \uparrow \psi_Q \\
 & & & & \mathbb{C}\smallsetminus\overline{\mathbb{D}} & \xrightarrow[z^d]{} & \mathbb{C}\smallsetminus\overline{\mathbb{D}} \\
 & & & & l_t \downarrow & & \downarrow l_t : z \mapsto z \cdot |z|^{\frac{2t}{1-t}} \\
 & & & & \mathbb{C}\smallsetminus\overline{\mathbb{D}} & \xrightarrow[z^d]{} & \mathbb{C}\smallsetminus\overline{\mathbb{D}}
\end{array}
$$

In the above diagram, the sets $\mathbb{C}^\pm$ are two different copies of the complex plane. The map π denotes the natural projection from $\mathbb{C}$ to $\mathbb{C}/\mathbb{Z}$. For σ the class of $\tilde\sigma$ in $\mathbb{C}/\mathbb{Z}$, the map T_σ denotes simply the quotient of $T_{\tilde\sigma}$. For $t \in \mathbb{D}$, the map $l_t : z \mapsto z \cdot |z|^{\frac{2t}{1-t}}$, $\mathbb{C}\smallsetminus\overline{\mathbb{D}} \to \mathbb{C}\smallsetminus\overline{\mathbb{D}}$, is a quasi-conformal map commuting with z^d.

We may now describe the Branner-Hubbard motion of Q as follows: One checks easily (see for example [PT]) that the complex structure $t\mu_Q$ defined in §1 coincides with the pull-back of the standard structure by $l_t \circ \varphi_Q$. We use as in §1 the map χ_t to denote the integrating map of $t\mu_Q$, uniquely normalized so that $\chi_t Q \chi_t^{-1}$ is again a monic centered polynomial.

Part a) of the following Lemma in contained in [BH1, 8.3] (we include a sketch of its proof for completeness). The rest part of the following two lemmas are inspired from [PT, §4.2].

Lemma 4.1. *Fix any* $t \in \mathbb{D}$.

a) We have $\chi_t \circ Q \circ \chi_t^{-1} = Q$, $\chi_t|_{K_Q} = id$, *and* $\chi_t = id$ *on the ideal boundary of* $B_Q(\infty)$.

b) $\chi_t|_{B_Q(\infty)} = \psi_Q \circ l_t \circ \psi_Q^{-1}$, $\chi_t \circ \psi_Q = \psi_Q \circ l_t$, *and* $\Phi_Q^- \circ \chi_t|_{B(\beta')} = \Phi_Q^-$.

c) Denote by $\mu_t^+ := (\Psi_Q^+)^*(t\mu_Q)$, *then there is a unique integrating map* ξ_t *of* μ_t^+ *such that*

$$\xi_t \circ T_1 = T_1 \circ \xi_t \ , \quad \chi_t \circ \Psi_Q^+ = \Psi_Q^+ \circ \xi_t \ . \tag{5}$$

d) Denote by $\widetilde{K}(Q)$, *resp.* $\widetilde{B}_Q(\infty)$, *the preimage by* Ψ_Q^+ *of* K_Q, *resp.* $B_Q(\infty)$. *Then* $\xi_t|_{\widetilde{K}_Q} = id$, *the map* $\Psi_Q^+ : \widetilde{B}_Q(\infty) \to B_Q(\infty)\smallsetminus\{\infty\}$ *is a universal covering, and* $\xi_t = id$ *on the ideal boundary of* $\widetilde{B}_Q(\infty)$.

Proof. a) This part uses only the fact that Q has a connected Julia set, as stated in Proposition 2.1.(6) above. To prove it, define

$$h_t = \begin{cases} \psi_Q \circ l_t \circ \psi_Q^{-1} & \text{on } B_Q(\infty) \\ id & \text{on } K_Q \end{cases} .$$

Then one checks easily, using the explicit formula of l_t, that h_t satisfies a) in place of χ_t, and h_t is a global homeomorphism. One needs to apply Rickman's gluing lemma to prove that h_t is actually quasi-conformal, therefore an integrating map of $t\mu_Q$.

Now $h_t \circ \chi_t^{-1} : B_Q(\infty) \to B_Q(\infty)$ is analytic on z, conjugates Q to itself, equals to the identity for $t = 0$ and depends continuously on t. As there are only finitely many such conformal selfconjugacies we conclude that $h_t = \chi_t$ for all t. This settles a).

b) The first two equalities following from the explicit construction of χ_t above. The last one is also trivial as $\chi_t|_{B(\beta')} = id$ by a).

c) As $t\mu_Q$ is Q-invariant, the complex structure μ_t^+ is T_1-invariant. Denote by ζ_t the unique integrating map of μ_t^+ fixing $0, 1, \infty$. Then $\zeta_t \circ T_1 \circ \zeta_t^{-1}$ is a global conformal homeomorphism of $\mathbb{C}$, is fixed point free, and maps 0 to 1 by the normalization of ζ_t. So $\zeta_t \circ T_1 \circ \zeta_t^{-1} = T_1$.

Now $\chi_t \circ \Psi_Q^+ \circ \zeta_t^{-1}$ is analytic in z and conjugates T_1 to Q. So it is a repelling Fatou coordinate. By unicity up to additif constant of such coordinates, there is $a(t)$ such that $\Psi_Q^+ \circ T_{a(t)} \circ \zeta_t = \chi_t \circ \Psi_Q^+$. Set $\xi_t = T_{a(t)} \circ \zeta_t$. It is again an integrating map of μ_t^+ and satisfies (5) . The unicity of such a map is an easy exercise and is left to the reader.

d) Fix $z \in K_Q$ and $\widetilde{z} \in (\Psi_Q^+)^{-1}(z)$. Now the map $t \mapsto \xi_t(\widetilde{z})$ is analytic on t and equals to $\widetilde{z}$ at $t = 0$. On the other hand, $\chi_t(z) \equiv z$ for all t by a), so $\xi_t(\widetilde{z}) \in (\Psi_Q^+)^{-1}(z)$ for all t. But the latter set is discrete. Consequently $\xi_t(\widetilde{z}) \equiv \widetilde{z}$ for all t. Argueing similarly on the accesses of $\partial\widetilde{B}_Q(\infty)$ from $\widetilde{B}_Q(\infty)$, one proves that ξ_t is the identity on the ideal boundary of $\widetilde{B}_Q(\infty)$. The universal covering property of $\Psi_Q^+ : \widetilde{B}_Q(\infty) \to B_Q(\infty)\smallsetminus\{\infty\}$ is easy.

The relations of various maps are illustrated in the following commutative diagrams:

$$
\begin{array}{ccc}
\mathbb{C}^-,0 & \xleftarrow{\;id\;} & \mathbb{C}^-,0 \\
\Phi_Q^- \uparrow & & \uparrow \Phi_Q^- \\
B(\beta'),w_0 & \xleftarrow[\chi_t=id]{} & B(\beta'),w_0
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C}^+,0 & \xrightarrow{\;\xi_t\;} & \mathbb{C}^+,a(t) \\
\Psi_Q^+ \downarrow & & \downarrow \Psi_Q^+ \\
\mathbb{C},\psi_Q(e^x) & \xrightarrow[\chi_t]{} & \mathbb{C},\psi_Q(e^{\frac{1+t}{1-t}x}) \\
\psi_Q \uparrow & & \uparrow \psi_Q \\
\mathbb{C}\smallsetminus\overline{\mathbb{D}},e^x & \xrightarrow[l_t]{} & \mathbb{C}\smallsetminus\overline{\mathbb{D}},e^{\frac{1+t}{1-t}x}
\end{array}
$$

□

However, the Lavaurs map $g_{\tilde{\sigma}}$ behaves badly under the above Branner-Hubbard motion. In fact the conjugated map $\chi_t \circ g_{\tilde{\sigma}} \circ \chi_t^{-1}|_{B(\beta')} = \chi_t \circ g_{\tilde{\sigma}}|_{B(\beta')}$ is no more analytic in z. For this reason, following Douady and Lavaurs we introduce a new Q-invariant Beltrami form which will be also $g_{\tilde{\sigma}}$-invariant, as follows: define

$$B_\sigma(\infty) := B_Q(\infty) \cup \{z|\ \exists n\ g_{\tilde{\sigma}}^n(z) \in B_Q(\infty)\}\ .$$

It depends only on the class σ of $\tilde{\sigma}$ in $\mathbb{C}/\mathbb{Z}$. Set

$$\mu_{Q,\sigma} := \begin{cases} (g_{\tilde{\sigma}}^n)^*\mu_Q = \left(\log \circ \varphi_Q \circ g_{\tilde{\sigma}}^n\right)^* \frac{d\bar{z}}{dz} & \text{for } z \in B_\sigma(\infty) \text{ and for } n \text{ such that } g_{\tilde{\sigma}}^n(z) \in B_Q(\infty) \\ 0 & \text{for } z \notin B_\sigma(\infty) \end{cases} .$$

Fix now $t \in \mathbb{D}$. Note that $g_{\tilde{\sigma}}^*(t\mu_{Q,\sigma}) = t\mu_{Q,\sigma}$, $Q^*(t\mu_{Q,\sigma}) = t\mu_{Q,\sigma}$, and $t\mu_{Q,\sigma}$ depends only on the class σ of $\tilde{\sigma}$. We use $\chi_{t,\sigma}$ to denote the integrating map of $t\mu_{Q,\sigma}$, uniquely normalized so that the conjugated maps, $Q_{t,\sigma} := \chi_{t,\sigma} Q \chi_{t,\sigma}^{-1}$ is again a monic centered polynomial. Clearly $Q_{t,\sigma}$ has again a connected Julia set, and a parabolic fixed point $\beta'_{t,\sigma} := \chi_{t,\sigma}(\beta')$ of multiplier 1, with $B_{t,\sigma} := \chi_{t,\sigma}(B(\beta'))$ as an immediate basin. We may thus define the corresponding objects in the list (4) for $Q_{t,\sigma}$, in particular the corresponding Böttcher/Fatou coordinates $\varphi_{t,\sigma}$, $\psi_{t,\sigma}$, $\Phi_{t,\sigma}^{\pm}$, $\Psi_{t,\sigma}^{+}$. We choose the normalization so that $\varphi_{t,\sigma}$ is again tangent to the identity at ∞, $\Phi_{t,\sigma}^{-}(\chi_{t,\sigma}(w_0)) = 0$ and $\Psi_{t,\sigma}^{+}(0) = \psi_{t,\sigma}(e^x)$, with $x > 0$ independent of t, σ.

Denote by τ_0 the standard complex structure. These maps are related as indicated in the following diagram:

$$
\begin{array}{cccccc}
Q \hookrightarrow & t\mu_{Q,\sigma},\mathbb{C} & \stackrel{\langle g_{\tilde{\sigma}},Q\rangle}{\longrightarrow} & t\mu_Q,\mathbb{C} \supset B_Q(\infty) & \stackrel{\varphi_Q}{\longrightarrow} & \mathbb{C}\smallsetminus\overline{\mathbb{D}} \\
 & \chi_{t,\sigma}\downarrow & & \downarrow \chi_t & & \downarrow l_t : z \mapsto z \cdot |z|^{\frac{2t}{1-t}} \\
Q_{t,\sigma} \hookrightarrow & \tau_0,\mathbb{C} & & \tau_0,\mathbb{C} & \underset{\varphi_Q}{\longrightarrow} & \mathbb{C}\smallsetminus\overline{\mathbb{D}}
\end{array}
$$

Denote by $\mu_{t,\sigma}^{+} := (\Psi_Q^{+})^*(t\mu_{Q,\sigma})$ and $\mu_{t,\sigma}^{-} := T_{\tilde{\sigma}}^{*}\mu_{t,\sigma}^{+}$.

Lemma 4.2. *Fix $t \in \mathbb{D}$ and $\sigma \in \mathbb{C}/\mathbb{Z}$. The objects of $Q_{t,\sigma}$ and those of Q are related as follows:*

a) $\psi_{t,\sigma} = \chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ on $\mathbb{C}\smallsetminus\overline{\mathbb{D}}$;

b) For $\eta_{t,\sigma}^{-}$ the integrating map of $\mu_{t,\sigma}^{-}$ fixing $0, 1$ and ∞, we have $\Phi_{t,\sigma}^{-} = \eta_{t,\sigma}^{-} \circ \Phi_Q^{-} \circ \chi_{t,\sigma}^{-1}$;

c) There is a unique integrating map η_t^+ of $\mu_{t,\sigma}^+$, commuting with T_1 and satisfying $\Psi_{t,\sigma}^+ = \chi_{t,\sigma} \circ \Psi_Q^+ \circ (\eta_{t,\sigma}^+)^{-1}$;

d) the map $\eta_{t,\sigma}^+ \circ T_{\tilde\sigma} \circ (\eta_{t,\sigma}^-)^{-1}$ is a translation and $\chi_{t,\sigma} g_{\tilde\sigma} \chi_{t,\sigma}^{-1}$ is a Lavaurs map of $Q_{t,\sigma}$.

e) If, for some t, σ, we have $Q_{t,\sigma} = Q$, then $\chi_{t,\sigma} = \chi_t$ on $B_Q(\infty)$ and $\chi_{t,\sigma} = id$ on the ideal boundary of $B_Q(\infty)$.

f) If $Q_{t,\sigma} = Q$ for all $t \in \mathbb{D}$, then $\eta_{t,\sigma}^+$ coincides with ξ_t of Lemma 4.1.c) on $\widetilde{B}_Q(\infty)$ and in particular $\eta_{t,\sigma}^+ = id$ on the ideal boundary of $\widetilde{B}_Q(\infty)$.

Proof. The proof is very similar to that of Lemma 4.1. We give at first a sketch in the following commutative diagram.

$$
\begin{array}{ccccccc}
\mathbb{C}^-, 0 & \xleftarrow{\eta_{t,\sigma}^-} & \mathbb{C}^-, 0 & \xrightarrow{T_{\tilde\sigma}} & \mathbb{C}^+, 0 & \xrightarrow{\eta_{t,\sigma}^+} & \mathbb{C}^+, b(t) \\
\Phi_{t,\sigma}^- \uparrow & & \Phi_Q^- \uparrow & & \downarrow \Psi_Q^+ & & \downarrow \Psi_{t,\sigma}^+ \\
B_{t,\sigma}, \chi_{t,\sigma}(w_0) & \xleftarrow[\chi_{t,\sigma}]{} & B(\beta'), w_0 & \xrightarrow[g_{\tilde\sigma}]{} & \mathbb{C}, \psi_Q(e^x) & \xrightarrow[\chi_{t,\sigma}]{} & \mathbb{C}, \psi_{t,\sigma}(e^{\frac{1+t}{1-t}x}) \\
 & & & & \psi_Q \uparrow & & \uparrow \psi_{t,\sigma} \\
 & & & & \mathbb{C} \setminus \overline{\mathbb{D}}, e^x & \xrightarrow[l_t]{} & \mathbb{C} \setminus \overline{\mathbb{D}}, e^{\frac{1+t}{1-t}x}
\end{array}
\qquad (6)
$$

In this diagram, $x > 0$ is the real number such that $\Psi_Q^+(0) = \psi_Q(e^x)$; the maps $g_{\tilde\sigma}$ and $T_{\tilde\sigma}$ do not necessarily preserve the base points, whereas the other maps do.

a) One checks easily that $\chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ is conformal, conjugates z^d to $Q_{t,\sigma}$, depends continuously on t, is tangent to the identity at ∞ when $t = 0$, and tangent to a $(d-1)$th-root of unity at ∞ for any t. One concludes then for every t, the map $\chi_{t,\sigma} \circ \psi_Q \circ l_t^{-1}$ is tangent to the identity at ∞, coincides thus with $\psi_{t,\sigma}$.

b) and c). As in the proof of the previous lemma, the structure $\mu_{t,\sigma}^+$ is T_1-invariant. So $\mu_{t,\sigma}^-$ is $T_{\tilde\sigma}^{-1} T_1 T_{\tilde\sigma}$-invariant, thus is T_1-invariant. Argueing as for ζ_t, ξ_t in the previous Lemma, one concludes that $\eta_{t,\sigma}^- \circ \Phi_Q^- \circ \chi_{t\sigma}^{-1}$ coincides with the attracting Fatou coordinates of $Q_{t,\sigma}$, whereas $\chi_{t,\sigma} \circ \Psi_Q^+ \circ (\eta_{t,\sigma}^+)^{-1}$ coincides with the repelling Fatou coordinates of $Q_{t,\sigma}$, after suitable normalization of the integrating map $\eta_{t,\sigma}^+$.

d) The map $\eta_{t,\sigma}^+ \circ T_{\tilde\sigma} \circ (\eta_{t,\sigma}^-)^{-1}$ is a conformal automorphism of $\mathbb{C}$, as $T_{\tilde\sigma}^* \mu_{t,\sigma}^+ = \mu_{t,\sigma}^-$. It is therefore of the form $az + c$. But it commutes with T_1, so it is of the form $z + c$, i.e. a translation. It follows by definition that $\chi_{t,\sigma} g_{\tilde\sigma} \chi_{t,\sigma}^{-1}$ is a Lavaurs map of $Q_{t,\sigma}$.

e) Assume that for some given t, σ we have $Q_{t,\sigma} = Q$. This implies $B_{t,\sigma} = B(\beta')$, $\psi_{t,\sigma} = \psi_Q$, $\Psi^+_{t,\sigma} = \Psi^+_Q$ and $\Phi^-_{t,\sigma} = \Phi^-_Q$. So the diagram (6) becomes

$$\begin{array}{ccccccc}
\mathbb{C}^-, 0 & \xleftarrow{\eta^-_{t,\sigma}} & \mathbb{C}^-, 0 & \xrightarrow{T_{\tilde\sigma}} & \mathbb{C}^+, 0 & \xrightarrow{\eta^+_{t,\sigma}} & \mathbb{C}^+, b(t) \\
\Phi^-_Q \uparrow & & \Phi^-_Q \uparrow & & \Psi^+_Q \downarrow & & \downarrow \Psi^+_Q \\
B(\beta'), w_0 & \xleftarrow[\chi_{t,\sigma}]{} & B(\beta'), w_0 & \xrightarrow[g_{\tilde\sigma}]{} & \mathbb{C}, \psi_Q(e^x) & \xrightarrow[\chi_{t,\sigma}]{} & \mathbb{C}, \psi_Q(e^{\frac{1+t}{1-t}x}) \\
 & & & & \psi_Q \uparrow & & \uparrow \psi_Q \\
 & & & & \mathbb{C} \smallsetminus \overline{\mathbb{D}}, e^x & \xrightarrow[l_t]{} & \mathbb{C} \smallsetminus \overline{\mathbb{D}}, e^{\frac{1+t}{1-t}x}
\end{array} \qquad (7)$$

It follows from a) that $\chi_{t,\sigma} = \psi_Q \circ l_t \circ \psi_Q^{-1}$ on $B_Q(\infty)$. By Lemma 4.1.a), $\chi_{t,\sigma}$ coincides with χ_t on $B_Q(\infty)$ and is the identity on the ideal boundary.

f) Argueing as in the proof of Lemma 4.1.d), one proves that $\eta^+_{t,\sigma}$ is the identity on the ideal boundary of $\widetilde{B}_Q(\infty)$. Now $\eta^+_{t,\sigma} \circ \xi_t^{-1}$ on $\widetilde{B}_Q(\infty)$ is a conformal automorphism (as they integrate the same complex structure) and is the identity on the ideal boundary. It follows that $\xi_t = \eta^+_{t,\sigma}$ on $\widetilde{B}_Q(\infty)$. $\square$

Although χ_t leaves Q invariant, $\chi_{t,\sigma}$ may deform Q to nearby maps. Even in the case $\chi_{t,\sigma}$ leaves Q invariant, it often deforms $g_{\tilde\sigma}$ to other Lavaurs maps (see below).

4.2 Fatou vectors for $Q \in \mathcal{A}$.

In this subsection, fix $Q \in \mathcal{A}$, i.e. a cubic cauliflower. Denote as before by β' the unique parabolic fixed point of multiplier 1, and by $B(\beta')$ the unique immediate basin of β'. The following is classical:

Lemma 4.3. *For any polynomial P close to Q, there are exactly two fixed points α_1, α_2 (counting with multiplicity) of P close to β'. For $j = 1, 2$ denote by λ_j the multiplier of P at α_j and $\tilde\sigma_j = \frac{2\pi i}{1-\lambda_j}$. Assume that both α_1 and α_2 are repelling. Then there are two constants $C \in \mathbb{C}$ and $C' \in \mathbb{R}^-$ depending on Q such that*

a) $\tilde\sigma_1 + \tilde\sigma_2 \to C$ as $P \to Q$;

b) $|\tilde\sigma_i| \to \infty$, as $P \to Q$, for $i = 1, 2$;

c) $C' < \Im(\tilde\sigma_i) < \pi$, for $i = 1, 2$;

d) exchanging the labeling of α_1, α_2 if necessary, $\Re(\tilde\sigma_1) \to -\infty$ as $P \to Q$.

Proof. a) is due to the continuity of holomorphic indices. See e.g. [Mi].

b) is due to the fact that $\lambda_i \to 1$ as $P \to Q$.

c). By assumption $|\lambda_i| > 1$, i.e. $\lambda_i \in \overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. The Möbius map $h(w) = \frac{2\pi i}{1-w}$ maps $\overline{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ onto $\{\Im(w) < \pi\}$. Therefore $\Im(\tilde\sigma_i) = \Im h(\lambda_i) < \pi$. On the other hand, by a), $\Im(\tilde\sigma_1) + \Im(\tilde\sigma_2)$ remains bounded. So each $\Im(\tilde\sigma_i)$ is bounded from below.

d) Combining a), b) and c), we conclude that $\Re(\tilde\sigma_1) + \Re(\tilde\sigma_2)$ remains bounded and $|\Re(\tilde\sigma_i)| \to \infty$, for $i = 1, 2$. Therefore one of $\Re(\tilde\sigma_1)$, $\Re(\tilde\sigma_2)$ tends to $-\infty$ as $P \to Q$. $\square$

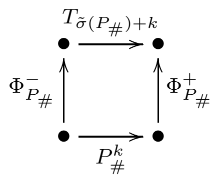
Definition. For $P_\#$ a perturbed map of Q without attracting fixed points, we define **the lifted phase** $\tilde\sigma(P_\#)$ to be one of the $\tilde\sigma_i$ with large negative real part, as indicated in Lemma 4.3. Denote by $\sigma(P_\#)$ its class in $\mathbb{C}/\mathbb{Z}$.

Corollary 4.4. *Assume P_n is a sequence of cubic polynomials without attracting fixed points, and converging algebraicly to Q. Then, taking a subsequence if necessary, we have $\sigma(P_n) \to \sigma$ in $\mathbb{C}/\mathbb{Z}$.*

Proof. By Lemma 4.3 we have $\tilde\sigma(P_n) = a_n + ib_n$ with $C' < b_n < \pi$ and $a_n \to -\infty$. Denote by $\{a_n\}$ the fractional part of a_n. Taking a subsequence if necessary we have $\{a_n\} \to a \in [0, 1[$ and $b_n \to b \in [C', \pi]$. Consequently $\sigma(P_n)$ converges to the class of $a + ib$ in $\mathbb{C}/\mathbb{Z}$. □

The following is due to Lavaurs-Douady:

Lemma 4.5. *Another meaning of $\tilde\sigma(P_\#)$ is as follows: there exist Fatou coordinates $\Phi^{\pm}_{P_\#}$ for $P_\#$, and they can be normalized suitablely so that they depend continuously on $P_\#$ and that $\Phi^+_{P_\#}(z) - \Phi^-_{P_\#}(z) \equiv \tilde\sigma(P_\#)$. In other words, for suitably chosen z, and some N depending on $P_\#$ (tending to $+\infty$ as $P_\# \to Q$), we have $P^k_\#(z) = (\Phi^+_{P_\#})^{-1} \circ T_{\tilde\sigma(P_\#)+k} \circ \Phi^-_{P_\#}(z)$ for $k = 0, 1, \cdots, N$.*



In case $P_n \to Q$ and $\sigma(P_n) \to \sigma$, for $\tilde\sigma \in \mathbb{C}$ a lift of σ and $k_n \in \mathbb{N}$ such that $\tilde\sigma(P_n) + k_n \to \tilde\sigma$, we have $T_{\tilde\sigma(P_n)+k_n} \to T_{\tilde\sigma}$, $\Phi^{\pm}_{P_n} \to \Phi^{\pm}_Q$, and therefore $P_n^{k_n} \to g_{\tilde\sigma}$ uniformly on compact sets of K_Q. Based on this, we have

Theorem 4.6. *(Key continuity theorem, [W, 8.2]). Denote by $K(Q, \sigma)$, $J(Q, \sigma)$ the enriched filled Julia set and the enriched Julia set. More precisely $K(Q, \sigma) = K_Q \setminus B_\sigma(\infty)$ and $J(Q, \sigma) = \partial K(Q, \sigma)$ (see [D2] for more details). Assume*

C0. $(Q, \sigma) \in \mathcal{A} \times \mathbb{C}/\mathbb{Z}$ and $g_{\tilde\sigma}$ sends the two critical points of Q outside K_Q.

A. $P_n \to Q \in \mathcal{A}$, and $P_n \notin \mathcal{A}$.

B. $\sigma(P_n) \to \sigma \in \mathbb{C}/\mathbb{Z}$.

These conditions imply C1. $K_{P_n} \to K(Q, \sigma)$, $J_{P_n} \to J(Q, \sigma)$ and $mes(J(Q, \sigma)) = 0$.

Assume furthermore D. $t_n \to t_0 \in \mathbb{D}$. Then

$A, B, C1$ and $D \Longrightarrow$

E. $t_n \mu_{P_n} \longrightarrow_{a.e.} t_0 \mu_{Q,\sigma}$; $\Longrightarrow$

F. $\chi_{t_n, P_n} \to \chi_{t_0,\sigma}$ uniformly on $\overline{\mathbb{C}}$; $\Longrightarrow$

G. $S(t_n, P_n) \to Q_{t_0,\sigma}$.

Sketch of a proof: Part $A+B+C0 \Longrightarrow C1$ is a theorem of Douady-Lavaurs. Part $E \Longrightarrow F$ can be found in [L, Thm. 4.6]. The remaining part should be checked by hand. $\square$

Definition (three half neighborhoods of β' and $-\infty \in \mathbb{C}^+$)

• Define $V_Q = \Psi_Q^+(\{\Re w < 0\})$. In other words Φ_Q^+ is well defined and univalent on V_Q, mapping it onto the left half plane.

• Define $L_Q = \psi_Q(e^R)$, where $R = \{s + i\theta \mid 0 < s < s_0, -\theta_0 < \theta < \theta_0\}$ is a small rectangle so that $L_Q \subset V_Q$.

• Define $M < 0$ large enough so that

$$\{\Re w < M\} \cap \widetilde{B}_Q(\infty) \subset \Phi_Q^+(L_Q) \subset \{\Re w < 0\}, \quad \text{and } \Psi_Q^+(\{\Re w < M\}) \cap B_Q(\infty) \subset L_Q \subset V_Q \ .$$

Definition. Define the **Fatou vector** $v(Q)$ to be $\Phi_Q^-(w_1) - \Phi_Q^-(w_0)$, the difference of the two critical points in the attracting Fatou coordinates. It depends on the labelling of the critical points, but not on the normalization of Φ_Q^-.

4.3 Discontinuity of the wring operator

Proposition 4.7. *I. ([W, 8.12]) There are $Q \in \mathcal{A}$ with $R_Q(0)$ landing at β' such that, for any given labeling w', w'' of the critical points of Q, there is $\tilde{\sigma} \in \mathbb{C}$, such that, for L_Q defined as above, we have $T_{\tilde{\sigma}} \circ \Phi_Q^-(w'), T_{\tilde{\sigma}} \circ \Phi_Q^-(w'') \in \Phi_Q^+(L_Q)$, $g_{\tilde{\sigma}}(w') \in L_Q \cap R_Q(0)$ and $g_{\tilde{\sigma}}(w'') \in L_Q \smallsetminus R_Q(0)$.*

For any such couple $(Q, \tilde{\sigma})$, and for σ the class in $\mathbb{C}/\mathbb{Z}$ of $\tilde{\sigma}$,

II. σ satisfies C0 and $t \mapsto Q_{t,\sigma}$ is non constant (a sufficient condition for this is that the ground wind $w(Q, \sigma)$ does not vanish, [W, 8.12]).

III. $(t, P) \mapsto S(t, P)$ is discontinuous at (t, Q) for any $t \in \mathbb{D}^*$ with $Q_{t,\sigma} \neq Q$ ([W, 8.5, 8.9]).

Proof.

I. In the quotient repelling Ecalle cylinder, the quotient $[R_Q(0)]$ of the 0-ray is the core curve of the annulus $[B_Q(\infty)]$, of modulus $\frac{\pi}{\log 3}$ (independent of Q, this can be seen in the Böttcher coordinate). For any real polynomial $Q \in \mathcal{A}$ with $v(Q) \neq 0$, the maps Φ_Q^+ is real, $R_Q(0) \subset \mathbb{R}$ and K_Q is symmetric with respect to $\mathbb{R}$. There is a universal constant h_0 such that Ψ_Q^+ maps the strip $\{u+iy \mid |y| \leq h_0\}$ into $B_Q(\infty)$. Note that Ψ_Q^+ maps $\mathbb{R}$ onto $R_Q(0)$ and maps $\mathbb{R}^-$ onto $R_Q(0) \cap V_Q$. We may then choose a real Q such that $|\Im v(Q)| \leq h_0$ and $\Im v(Q) \neq 0$ (such map exists, see §5). Define the three half neighborhood of β' of Q as in §4.2. We have a fourth half neighborhood consisting of $\Psi_Q^+(\{u + iy | u < M, |y| < h_0\})$, which is contained in $B_Q(\infty) \cap L_Q \subset V_Q$.

Fix a labeling w', w'' of the critical points of Q.

Now choose a translation $T_{\tilde{\sigma}}$ so that $a := T_{\tilde{\sigma}}(\Phi_Q^-(w'))$ is a large negative real number satisfying

$$\max\{a, a \pm \Re v(Q)\} < M \ .$$

Then

$$T_{\tilde{\sigma}}(\Phi_Q^-(w'')) = T_{\tilde{\sigma}}\left(\Phi_Q^-(w') \pm v(Q)\right) = T_{\tilde{\sigma}}(\Phi_Q^-(w')) \pm v(Q) = a \pm \Re v(Q) \pm i \Im v(Q) \ ,$$

So both $g_{\tilde\sigma}(w') = \Psi_Q^+ \circ T_{\tilde\sigma} \circ \Phi_Q^-(w')$ and $g_{\tilde\sigma}(w'') = \Psi_Q^+ \circ T_{\tilde\sigma} \circ \Phi_Q^-(w'')$ are contained in the fourth half neighborhood of β', satisfying in particular $g_{\tilde\sigma}(w') \in L_Q \cap R_Q(0)$ and $g_{\tilde\sigma}(w'') \in L_Q \setminus R_Q(0)$.

II. Clearly σ satisfies C0, that is, $g_{\tilde\sigma}$ maps the two critical points of Q outside K_Q.

Assume by contradiction that $Q_{t,\sigma} \equiv Q$ for all $t \in \mathbb{D}$. Then $\chi_{t,\sigma}$ commutes with Q, in particular $\chi_{t,\sigma}$ maps critical points of Q to critical points of Q. But $t \mapsto \chi_{t,\sigma}(z)$ is analytic on t and is equal to z at $t = 0$. We conclude that $\chi_{t,\sigma}$ is the identity on the grand orbit of the critical points. On the other hand, by Lemma 4.2.e) and f),

$$\begin{array}{l} \chi_{t,\sigma} = \chi_t = \psi_Q \circ l_t \circ \psi_Q^{-1} \quad \text{on} \quad B_Q(\infty), \quad \text{and} \\ \eta^+_{t,\sigma} = \xi_t \quad \text{on} \quad \widetilde{B}_Q(\infty) \end{array} \quad . \tag{8}$$

Fix $-1 < t < 0$. By Lemma 4.2.d) the map $\eta^+_{t,\sigma} \circ T_{\tilde\sigma} \circ (\eta^-_{t,\sigma})^{-1}$ is some translation T_c. So

$$\begin{aligned} \pm v(Q) = \Phi_Q^-(w') - \Phi_Q^-(w'') &= T_c(\Phi_Q^-(w')) - T_c(\Phi_Q^-(w'')) \\ &= \eta^+_{t,\sigma} \circ T_{\tilde\sigma} \circ (\eta^-_{t,\sigma})^{-1}(\Phi_Q^-(w')) - \eta^+_{t,\sigma} \circ T_{\tilde\sigma} \circ (\eta^-_{t,\sigma})^{-1}(\Phi_Q^-(w'')) \\ &= \eta^+_{t,\sigma} \circ T_{\tilde\sigma}(\Phi_Q^-(w')) - \eta^+_{t,\sigma} \circ T_{\tilde\sigma}(\Phi_Q^-(w'')) \\ &= \xi_t \circ T_{\tilde\sigma}(\Phi_Q^-(w')) - \xi_t \circ T_{\tilde\sigma}(\Phi_Q^-(w'')) \\ &= \Phi_Q^+ \circ \chi_t \circ g_{\tilde\sigma}(w') - \Phi_Q^+ \circ \chi_t \circ g_{\tilde\sigma}(w'') \ , \end{aligned}$$

where the second equality is due to the fact that T_c is a translation, the third is due to the previous paragraph, the fourth is due to the fact that $\chi^{-1}_{t,\sigma}$ is the identity on the critical points, the fifth is due to (8) , and finally the last is due to the facts that $g_{\tilde\sigma}(w'), g_{\tilde\sigma}(w'') \in L_Q \subset V_Q$ (by the construction of $\tilde\sigma$ and L_Q, V_Q) and that L_Q is invariant by χ_t, $-1 < t < 0$ (by the choice of L_Q).

However we claim that the right hand side tends to ∞ as $-1 < t < 0$, $t \searrow -1$. Because, one of the points $\chi_t \circ g_{\tilde\sigma}(w')$, $\chi_t \circ g_{\tilde\sigma}(w'')$, say the first one, tends to the landing point β' of $R_Q(0)$ whereas the other tends to the landing point γ_θ of $R_Q(\theta)$ for some $\theta \neq 0$. So $\lim_{t\searrow -1} \Phi_Q^+(\chi_t \circ g_{\tilde\sigma}(w')) = -\infty$ whereas $\lim_{t\searrow -1} \Phi_Q^+(\chi_t \circ g_{\tilde\sigma}(w'')) = \Phi_Q^+(\gamma_\theta) \in \mathbb{C}$.

This is a contradiction.

III. Fix $t \in \mathbb{D}$ such that $Q_{t,\sigma} \neq Q$. Let R' be a compact set contained in the rectangle R such that

$$g_{\tilde\sigma}(w'), g_{\tilde\sigma}(w'') \in \psi_Q(e^{R'}) \subset \psi_Q(e^R) = L_Q \ .$$

By continuous dependence of $\psi_P|_{R'}$ on P and continuous dependence of $\Phi_P^{\pm}$, one concludes that there is a sequence $P_n \to Q$, $k_n \in \mathbb{N}$ such that $P_n^{k_n}(w'_n), P_n^{k_n}(w''_n) \in \psi_{P_n}(e^{R'})$. Therefore J_{P_n} is a Cantor set and all periodic points are repelling. We may thus apply Corollary 4.4 to conclude that $\sigma(P_n) \to \sigma$ (taking a subsequence if necessary). Set $t_n \equiv t \in \mathbb{D}^*$. Then the conditions A,B,C0,D of Theorem 4.6 are satisfied. So $S(t, P_n) \to Q_{t,\sigma} \neq Q \equiv S(t, Q)$, where the last equality is due to Proposition 2.1.(6). Therefore the wring operator $S(t, P)$ is not continuous at (t, Q). □

This proves in particular the part of Theorem 1.2 about the discontinuity of the wring operator on the cubic polynomials.

An interesting necessary condition for a S-ray $S(P)$ to land at a $Q \in \mathcal{A}$ is the following: Assume $S(t, P) := P_t$ converges to $Q \in \mathcal{A}$ as $t \searrow -1$ (with $P \neq Q$). By Theorem 1.1.III, the ray

$R_{P_t}(0)$ branches at a critical point denoted by w_t. These points w_t have a limit w as $t \searrow -1$, which is necessarily a critical point of Q. Define

$$\Sigma(w) = \left\{\sigma \in \mathbb{C}/\mathbb{Z} \mid \exists\, \tilde\sigma,\ g_{\tilde\sigma}(w) \in R_Q(0) \cap \psi_Q\left([e^s, e^{3s}]\right)\right\} \text{ and}$$
$$\Sigma^t = \left\{\sigma \in \mathbb{C}/\mathbb{Z} \mid T_\sigma([w_t]) \in \left[R_{P_t}(0) \cap \psi_{P_t}\left([e^s, e^{3s}]\right)\right]\right\}\ .$$

They are two closed loops in $\mathbb{C}/\mathbb{Z}$.

Lemma 4.8. *([W, 8.7, 8.8]) In the above setting (landing of an S-ray), assume furthermore $\sigma(P_t) \in \Sigma^t$. Then $d(\sigma(P_t), \Sigma(w)) \to 0$ as $t \searrow -1$. Moreover, if a $\sigma \in \Sigma(w)$ satisfies C0 (i.e. mapping both critical points outside K_Q) then $Q_{t,\sigma} \equiv Q$.*

Proof. By continuity of all involved maps, points, one sees easily that $\Sigma^t \to \Sigma(w)$ in the Hausdorff topology. But $\sigma(P_t) \in \Sigma^t$ by assumption. So $d(\sigma(P_t), \Sigma(w)) \to 0$.

Since the real part of $\tilde\sigma(P_t)$ tends to $-\infty$ as $t \searrow -1$, the phase map $t \mapsto \sigma(P_t)$ spirals asymptotically to the loop $\Sigma(w)$. As a consequence, for any $\sigma \in \Sigma(w)$, there is a sequence $t_n \searrow -1$ such that $\sigma(P_{t_n}) \to \sigma$.

Assume a $\sigma \in \Sigma(w)$ satisfies C0. Then by Theorem 4.6, for $-1 < t < 1$,

$$Q \overset{n\to\infty}{\longleftarrow} S_{t\star t_n}(P) = S_t(P_{t_n}) \overset{n\to\infty}{\longrightarrow} Q_{t,\sigma}\ ,$$

where $t\star t'$ denotes the group structure in $\mathbb{D}$ related to the operator S. Therefore $Q_{t,\sigma}$ is constant for t real. But it is analytic on t (cf. e.g. [PT, Thm. 2.7]). So it is also constant for all $t \in \mathbb{D}$. □

As for the continuity part of Theorem 1.2, we may apply the following Lemma to the quadratic family (this proof is somewhat different from the two proofs of Willumsen).

Lemma 4.9. *([BH1, 7.2] + [PT]) For $\mathcal{F}$ an analytic family of polynomials, if S maps $\mathbb{D} \times \mathcal{F}$ into $\mathcal{F}$, then S is a parameter holomorphic motion. It is continuous if in addition $\mathcal{F} \subset \overline{\mathbb{C}}$ (by Słodkovski's theorem).*

In the same spirit as Proposition 4.7, one can show:

Proposition 4.10. *Assume $S(P) \ni P_n \to Q \in \mathcal{A}$, $P \neq Q$, $\sigma(P_n) \to \sigma$, and $g_{\tilde\sigma}(w) \in B_Q(\infty)$ for a lift $\tilde\sigma$ of σ and for w a critical point of Q. Then, for any lift $\tilde\sigma$ of σ, we have $g_{\tilde\sigma}(w) \in R_Q(0)$ and $P_n^l(w(P_n)) \in R_{P_n}(0)$ for some l independent of n.*

A consequence of this is (improved [W, 8.10]): Assume $(Q, \sigma) \in \mathcal{A} \times \mathbb{C}/\mathbb{Z}$ such that for one critical point w' of Q, we have $g_{\tilde\sigma}(w') \in R_Q(\theta')$, with $\theta' \neq 0$. Assume furthermore $P_n \to Q$, $\sigma(P_n) \to \sigma$. Then $\{P_n\}$ can not belong to a single S-ray.

Proof. All periodic points of P_n are repelling, by Theorem 1.1.I. So we may apply Corollary 4.4 to conclude that there is a subsequence such that $\sigma(P_n)$ converges in $\mathbb{C}/\mathbb{Z}$ to some σ. Fix a large integer l so that $Q^l(w)$ is in an attracting petal, where Φ_Q^- is injective.

Assume $g_{\tilde\sigma}(w) \in B_Q(\infty)$ (this is independent of the lift). Choose $\tilde\sigma \in \mathbb{C}$ a lift of σ such that $T_{\tilde\sigma+l}(\Phi_Q^-(w))$ has a negative real part. There is therefore a θ such that $T_{\tilde\sigma+l}(\Phi_Q^-(w)) \in \Phi_Q^+(R_Q(\theta))$.

There are integers $k_n \to +\infty$ such that $\tilde\sigma(P_n) + k_n \to \tilde\sigma$ and that $P_n^{k_n}(P_n^l(w(P_n))) \to g_{\tilde\sigma}(Q^l(w))$ (cf. [D2, 18.2]). By semi-continuity of K_{P_n} on n one concludes that $P_n^{k_n}(P_n^l(w(P_n))) \notin$

K_{P_n}, and is therefore contained in some $\theta(n)$-ray. Hence the $\frac{\theta(n)}{3^{k_n}}$-ray contains $P_n^l(w(P_n))$ (cf. [Ta, §2], beware that the σ there is different from the σ here). But $P_n \in S(P)$. By Proposition 2.1.(2), applied to $z = P_n^l(w(P_n))$, we have $\arg_{P_n}(P_n^l(w(P_n))) = \arg_P(P^l(w(P)))$ and is independent of n. We conclude that $\theta(n) = 0$ for any n. But $\theta(n) \to \theta$. So $\theta = 0$.

From $T_{\tilde{\sigma}+l}(\Phi_Q^-(w)) \in \Phi_Q^+(R_Q(0))$ on concludes easily that $g_{\tilde{\sigma}+N}(w) = \Psi_Q^+(T_{\tilde{\sigma}+N}(\Phi_Q^-(w))) \in R_Q(0)$ for any integer $N \in \mathbb{Z}$. □

5 Parameter interpretation, comments

The space $Per_1(1)$ of cubic polynomials with a fixed point of multiplier 1 modulo affine conjugacy can be parametrized by $b^2 \in \mathbb{C}$ with $\tilde{Q}_b(z) = z^3 + bz^2 + z$, as two such maps are conjugate iff $b = -b'$. Figure 2 represents the b-plane, centered at 0. The set $\mathcal{A}$ in this plane consists of the two large butterfly wings. The locus of parabolic attracting maps in $\mathcal{A}$ is the lemniscate $\{b, |b^2 - \frac{1}{2}| < \frac{1}{2}\}$ (Figure 3). The parameter interpretation of Theorem 1.1.I is that no S-ray accumulates at this lemniscate.

Figure 4 represent the right wing of $\mathcal{A}$ in the b-plane. It is quite easy to check that in this plane $\mathcal{A} \cap \mathbb{R}^+ = \,]0, 2[$. The examples constructed in Proposition 4.7 correspond to $b \in]0, \sqrt{3}[$, close to $\sqrt{3}$ (see Figure 4). The conclusion is that no S-rays lands at such b.

Komori and Nakane have proved the following closely related results (see [KN]):

There is a monotonic sequence of points $b_n \in [\sqrt{3}, 2[$, $b_0 = \sqrt{3}$, $b_n \nearrow 2$ with integer Fatou vectors. Each of them is the landing point of some S-ray. The in-between real S-rays accumulate but do not land in $[\sqrt{3}, 2]$.

The paper [PT2] contains the following parameterization of each wing of $\mathcal{A}$:

Denote by B the parabolic basin of $F : z \mapsto z^2 + \frac{1}{4}$, by $\Phi^- : B \to \mathbb{C}$ the attracting Fatou coordinates normalized so that $\Phi^-(0) = 0$. Then $(\Phi^-)^{-1}(\mathbb{R})$ decomposes B into a chess board structure, mapping each open chess square bi-holomorphically onto the upper half or the lower half plane, and mapping the borders onto the real line. Denote by U the component of $(\Phi^-)^{-1}(\mathbb{H}_r)$ containing the parabolic fixed point on the boundary. Define $\sim$ on ∂U by $z \sim \bar{z}$.

There exists a bi-holomorphic map $\mathcal{H}$ from each of the two wings of $\mathcal{A}$ onto $B \smallsetminus U/\sim$, recording the position of the second attracted critical point (the quotient is meant to cancel the confusing case when both critical points can be considered as 'first'). One proves easily $v(Q) = \Phi^- \circ \mathcal{H}(Q)$. There is therefore a chess board structure on $\mathcal{A}$ (see Figure 4 and 5). Moreover $\Phi^- \circ \mathcal{H}$ maps each of the two main chess squares (the ones in Figure 4 with $]0, \sqrt{3}[$ as an edge) bi-holomorphically onto the upper-left quarter and the lower-left quarter plane, and maps each of the other chess squares onto the upper half or the lower half plane. This implies in particular the surjectivity of $Q \mapsto v(Q), \mathcal{A} \to \mathbb{C}$.

The results listed in this article lead naturally to the following questions:

1. Are the following conditions equivalent? Assume $P \notin \mathcal{A}$.

- $Acc(S(P)) \cap Per_1(1) \neq \emptyset$,
- $Acc(S(P)) \subset Per_1(1)$, and

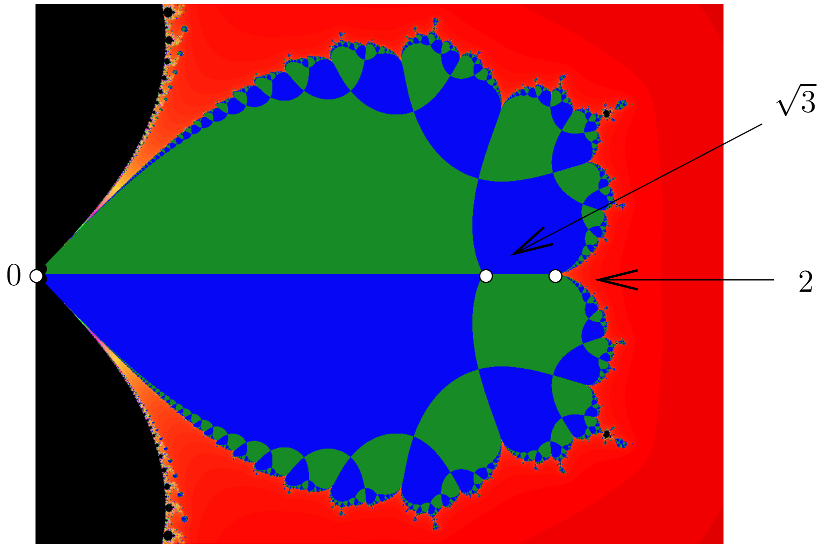


Figure 4: The chess board structure in $\mathcal{A}$.

- either $P \in Per_1(1)$ or a fixed ray of P branches.

2. $Acc(S(P)) \cap \mathcal{A} \neq \emptyset \Longleftrightarrow Acc(S(P)) \subset \mathcal{A} \Longleftrightarrow$ the set $\bigcup_n P^{-n}(R_P(0))$ (or $\frac{1}{2}$)) is connected, and necessarily contains both critical points.

3. In the setting of Lemma 4.8, is the assumption $\sigma(P_t) \in \Sigma^t$ always satisfied?

Comments. This is a free extraction of some of the results in Pia Willumsen's thesis, with a different organization and sometimes a different proof. It contains also some generalizations of the original results. I take the responsibility for all eventual introduced errors. The reader is strongly advised to read the original document, which includes a lot more related results, background material, complete proofs as well as many impressive figures.

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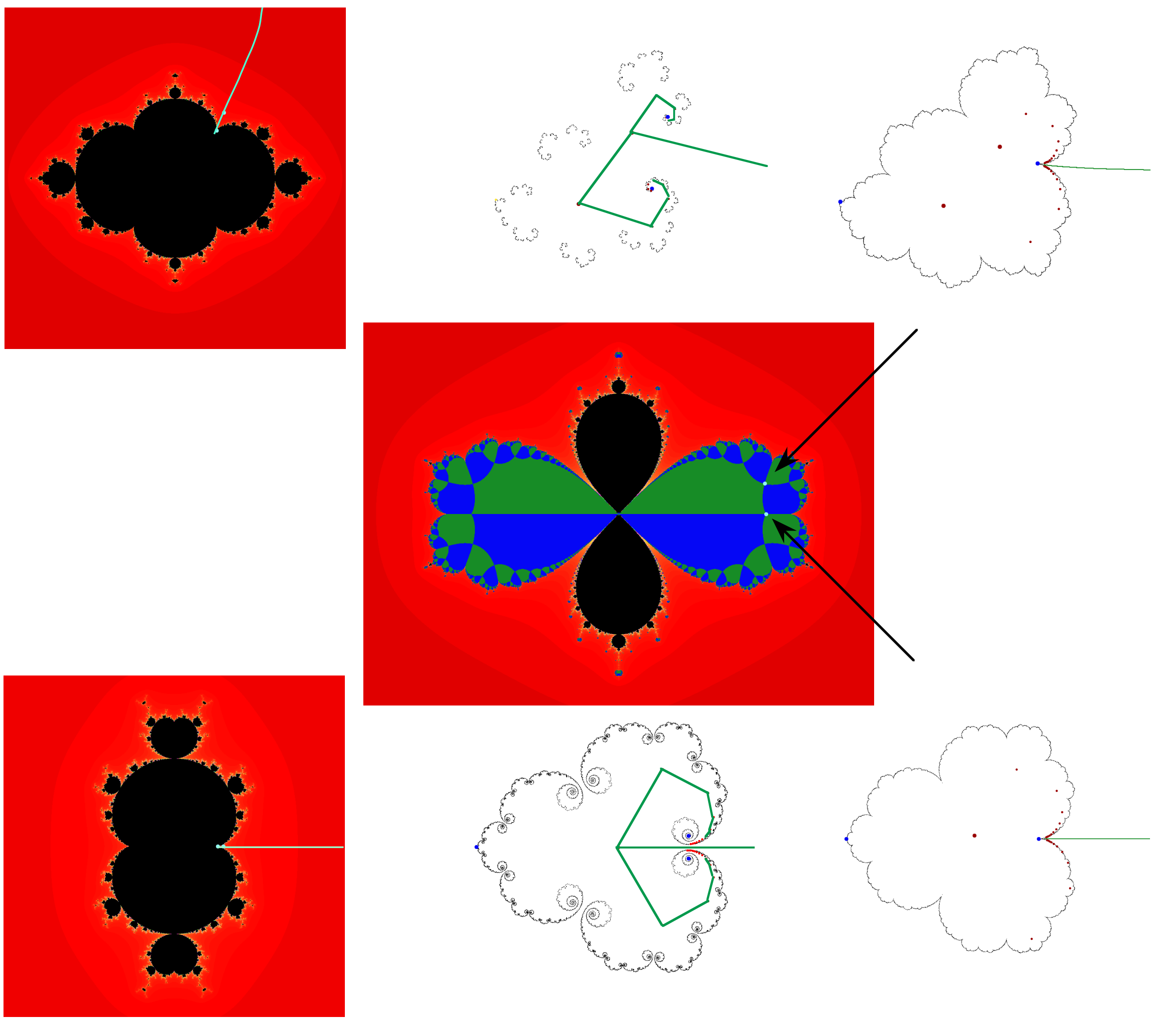

Figure 5: Julia sets for pairs (P, Q), left and right limit 0-rays for P and the 0-ray for Q, and parameter slices containing these maps.

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