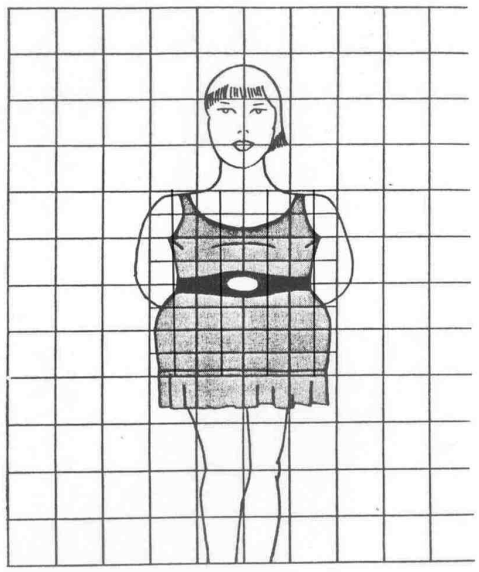
Core entropy of real and complex polynomials

实与复多项式的核拓扑熵

Tan Lei 谭蕾
Université d'Angers

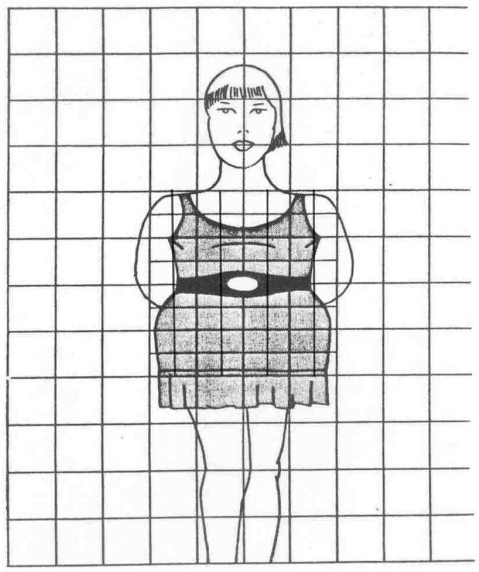
(most figures are provided by Giulio Tiozzo)

Orléans 奥尔良, March 12, 2013



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

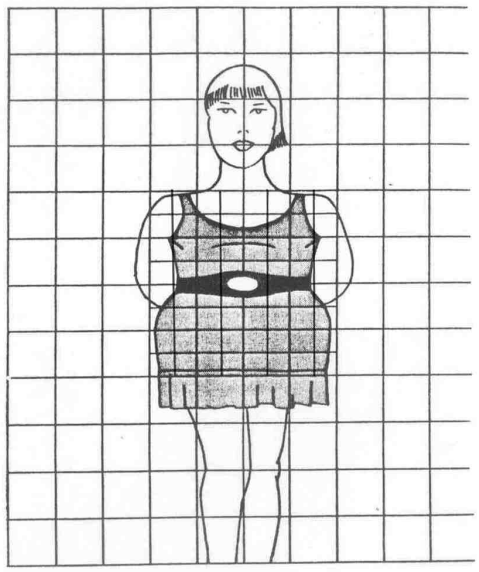
$$h(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{n},$$



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

$$h(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{n},$$

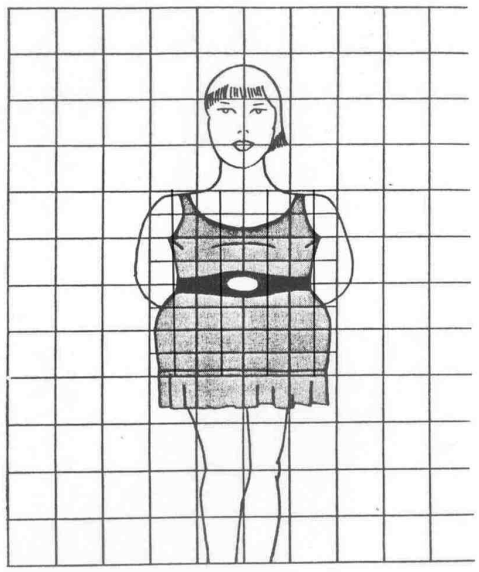
$$d(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{\log \frac{1}{\textit{size}(\text{level-n square})}}$$



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

$$h(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{n},$$

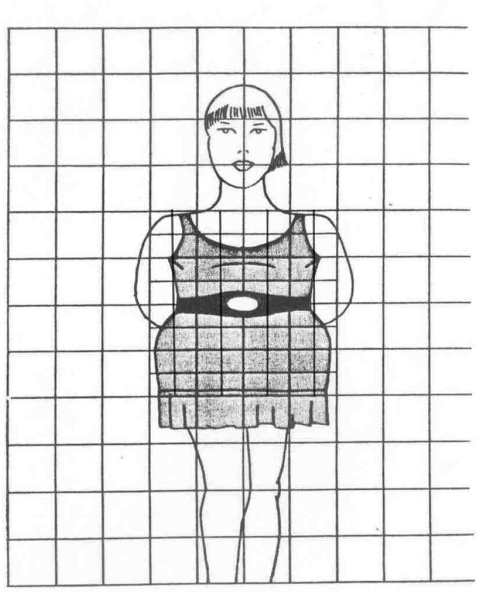
$$d(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{\log \frac{1}{\textit{size}(\text{level-n square})}} = \lim_{n\to\infty} \frac{\log \nu(A, n)}{n \log d}$$



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

$$h(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{n},$$

$$d(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{\log \frac{1}{\textit{size}(\text{level-n square})}} = \lim_{n\to\infty} \frac{\log \nu(A, n)}{n \log d} \overset{\text{trivially}}{=} \frac{h(A)}{\log d}$$



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

$$h(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{n},$$

$$d(A) := \lim_{n\to\infty} \frac{\log \nu(A, n)}{\log \frac{1}{\textit{size}(\text{level-n square})}} = \lim_{n\to\infty} \frac{\log \nu(A, n)}{n \log d} \overset{\text{trivially}}{=} \frac{h(A)}{\log d}$$

Theorem (adapting Furstenberg's similar result on S^1)
For $F : \mathbb{T}^2 \to \mathbb{T}^2, (x, y) \mapsto (dx, dy)$, $F(A) \subset A$, A closed,

- both limits exist
- $h(A) = h_{top}(F|_A)$ (topological entropy, defined for continuous $f : X \to X$ on compact X),
- $d(A) = \text{H.dim}(A)$, $d(A) = \frac{h(A)}{\log d}$.

Existence of the limit

$$\{S_{n+m}\} \hookrightarrow \{(S_n, S_m)\} \qquad \begin{array}{ccc} & & \mathbb{T}^2 \\ & \nearrow_{F^{\circ n}} & \cup \\ S_n & & S_m \\ \cup & \nearrow_{F^{\circ n}} & \\ S_{n+m} & & \end{array}$$

So $\nu(A, n+m) \leq \nu(A, n) \cdot \nu(A, m)$. This sub-multiplicativity implies that $h(A)$ and $d(A)$ exist.

Corollary. For $F : \mathbb{T}^2 \to \mathbb{T}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto d \begin{pmatrix} x \\ y \end{pmatrix}$, $F(A) \subset A$, A closed, all sets $\cdots \supset F^{-n}(A) \supset \cdots \supset F^{-1}(A) \supset A \supset F(A) \supset F^2(A) \supset \cdots$ have the same dimension and entropy, and their union $B(A) = \bigcup_n F^{-n}(A)$ has the same dimension.

Existence of the limit

$$\{S_{n+m}\} \hookrightarrow \{(S_n, S_m)\}$$

$$\begin{array}{ccc} & & \mathbb{T}^2 \\ & \nearrow_{F^{\circ n}} & \cup \\ S_n & & S_m \\ \cup & \nearrow_{F^{\circ n}} & \\ S_{n+m} & & \end{array}$$

So $\nu(A, n+m) \leq \nu(A, n) \cdot \nu(A, m)$. This sub-multiplicativity implies that $h(A)$ and $d(A)$ exist.

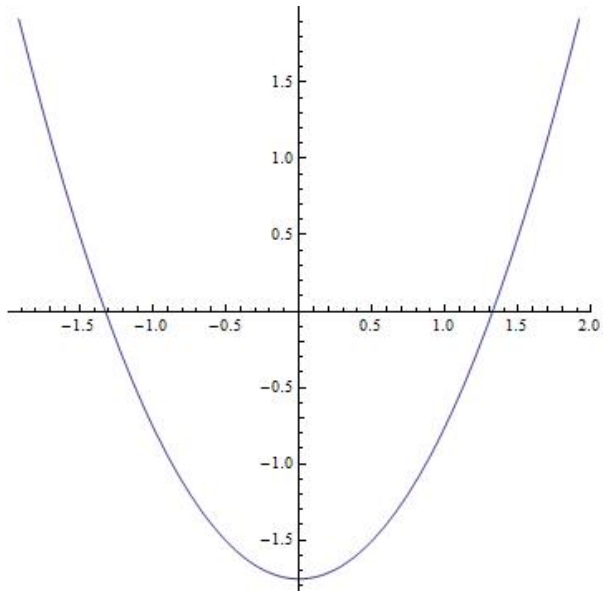
Corollary. For $F : \mathbb{T}^2 \to \mathbb{T}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto d \begin{pmatrix} x \\ y \end{pmatrix}$, $F(A) \subset A$, A closed, all sets $\cdots \supset F^{-n}(A) \supset \cdots \supset F^{-1}(A) \supset A \supset F(A) \supset F^2(A) \supset \cdots$ have the same dimension and entropy, and their union $B(A) = \bigcup_n F^{-n}(A)$ has the same dimension.

We will see that (F, A), $(F, B(A))$ for various A parametrizes combinatorially all degree d polynomials, with $h(A)$ as the core entropy.

Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

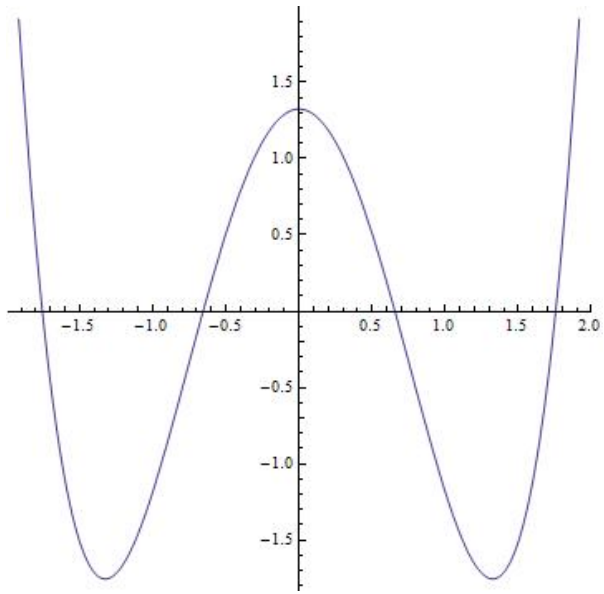
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial $f = \text{entropy}(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

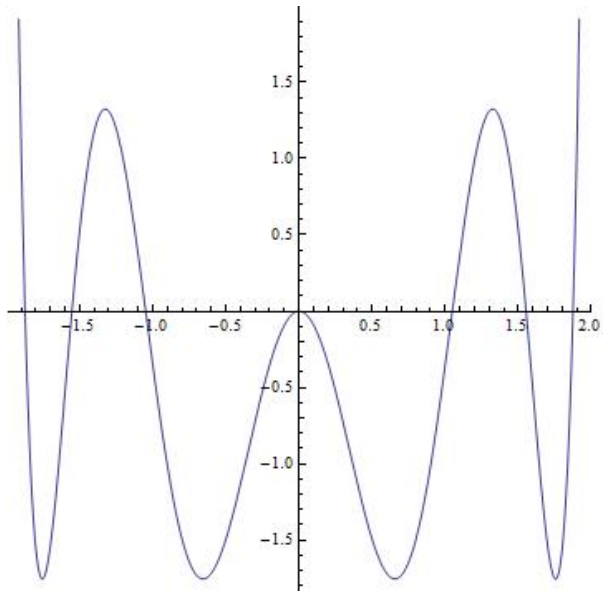
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

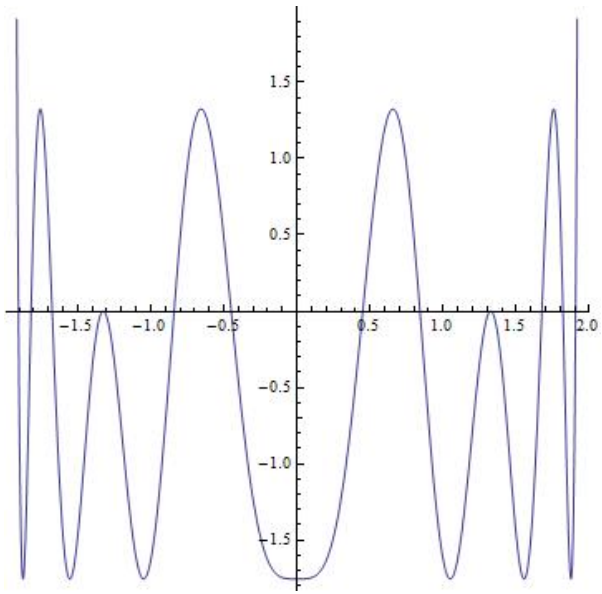
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

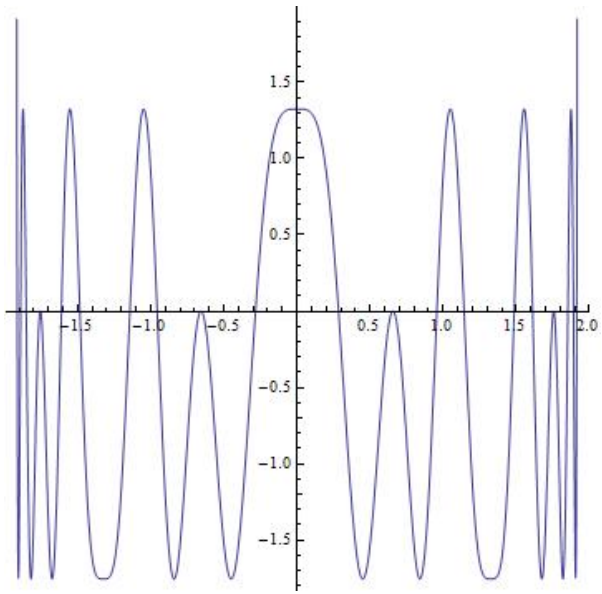
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

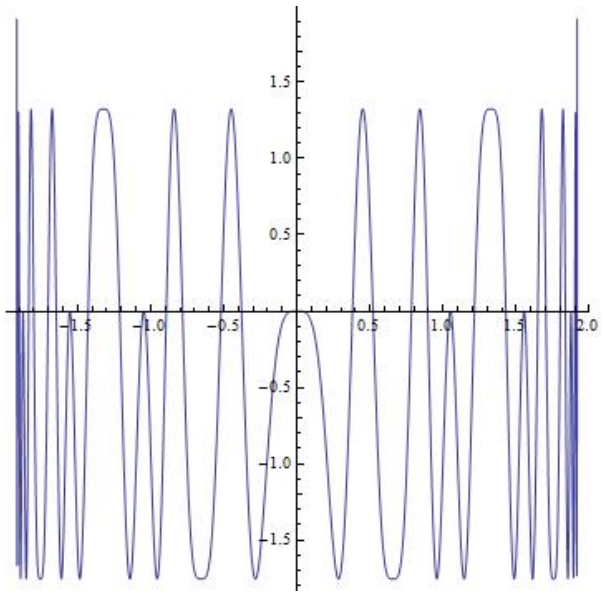
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

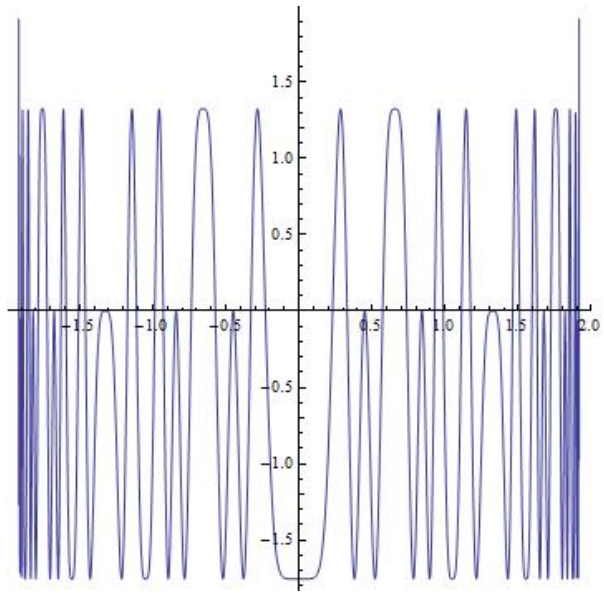
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

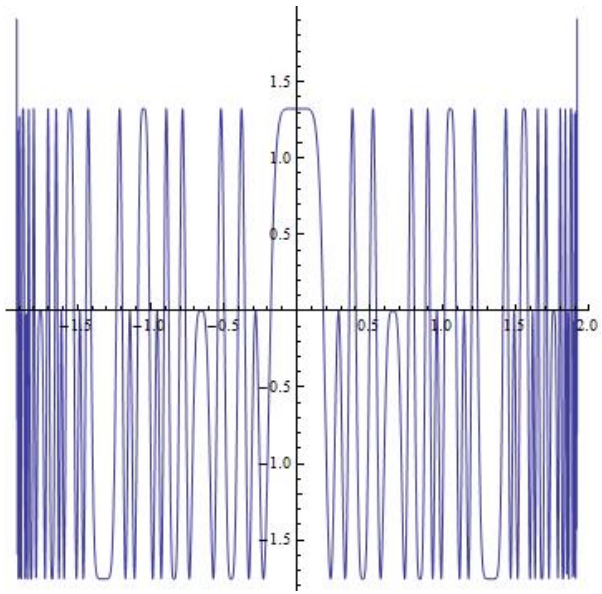
$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Core-entropy of a real polynomial f = entropy$(f|_{\mathbb{R}})$

Let $f : I \to I$ continuous, by Misiurewicz-Szlenk,

$$h_{top}(f, \mathbb{R}) := \lim_{n \to \infty} \frac{\log\{\#\text{laps of } f^n\}}{n}$$



Parameter variation, for the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

Parameter variation, for the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

how to compute $h_{top}(f_c, \mathbb{R}) = h(c)$? How does it change with the parameter c ?

Parameter variation, for the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

how to compute $h_{top}(f_c, \mathbb{R}) = h(c)$? How does it change with the parameter c ? By Milnor-Thurston (77),

- $h(c)$ is continuous and monotone decreasing, $\log 2 \searrow 0$

Parameter variation, for the real quadratic family

$$f_c(z) := z^2 + c \qquad c \in [-2, 1/4]$$

how to compute $h_{top}(f_c, \mathbb{R}) = h(c)$? How does it change with the parameter c ? By Milnor-Thurston (77),

- $h(c)$ is continuous and monotone decreasing, $\log 2 \searrow 0$

- The the kneading series $D_c(t) = \sum_{k \geq 0} \pm t^k$
  $\left[+ \text{ if } f_c^k \text{ has a local minimum at 0, and } - \text{ local maximum} \right]$
  admits $1/e^{h(c)}$ as a zero with least modulus.
- The map $x \mapsto K(x) = \sum_k \pm \lambda^{-k}$ $\left[+ \, (-) \text{ if } f^k|_x \nearrow (\searrow) \right]$,

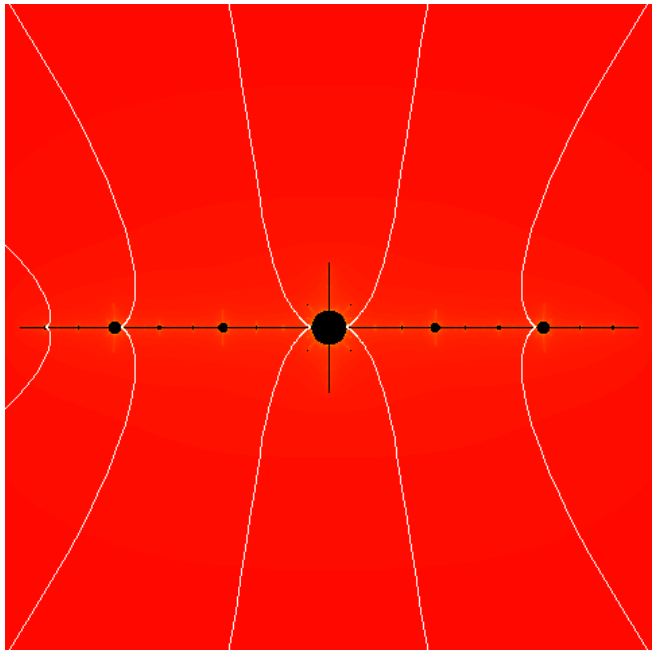
  is continuous iff $\lambda = e^{h(c)}$ and semi-conjugates f to the tent map of slope $e^{h(c)}$.

Complex view of real $z^2 + c$, Douady's circle model

Complex view of real $z^2 + c$, Douady's circle model

Each f_c has a Julia set $K_c := \{z \in \mathbb{C} \mid f_c^{\circ n}(z) \nrightarrow \infty\}$.

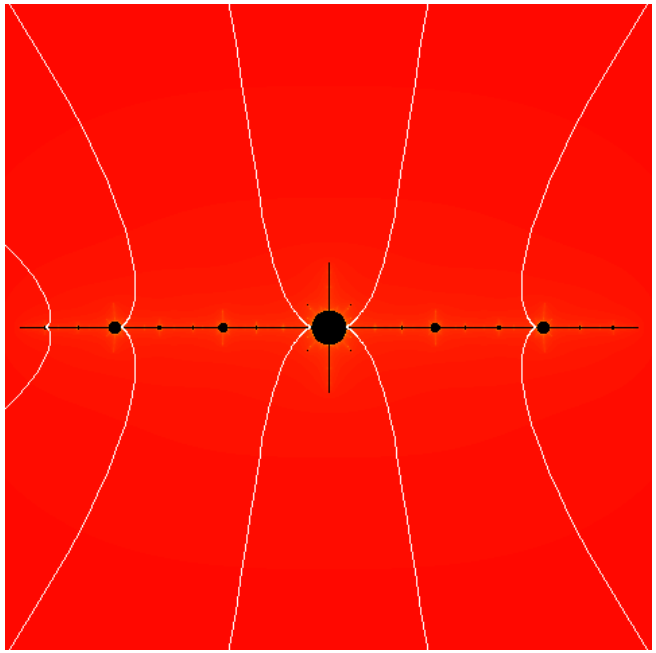
For $c \in [-2, \frac{1}{4}]$, the set $\hat{\mathbb{C}} \setminus K_c$ is simply-connected, there is a conformal map $\Phi_c : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K_c$ semi-conjugating z^2 to f_c. The external rays $\{R_c(t), t \in S^1\}$ coming from radial rays outside $\overline{\mathbb{D}}$, foliate the outside of K_c, and are invariant by f : $f_c(R_c(t)) = R_c(2t)$. This translates the core entropy to an entropy of the **symbolic dynamics** :



Complex view of real $z^2 + c$, Douady's circle model

Each f_c has a Julia set $K_c := \{z \in \mathbb{C} \mid f_c^{\circ n}(z) \nrightarrow \infty\}$.

For $c \in [-2, \frac{1}{4}]$, the set $\hat{\mathbb{C}} \setminus K_c$ is simply-connected, there is a conformal map $\Phi_c : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K_c$ semi-conjugating z^2 to f_c. The external rays $\{R_c(t), t \in S^1\}$ coming from radial rays outside $\overline{\mathbb{D}}$, foliate the outside of K_c, and are invariant by f : $f_c(R_c(t)) = R_c(2t)$. This translates the core entropy to an entropy of the **symbolic dynamics** :

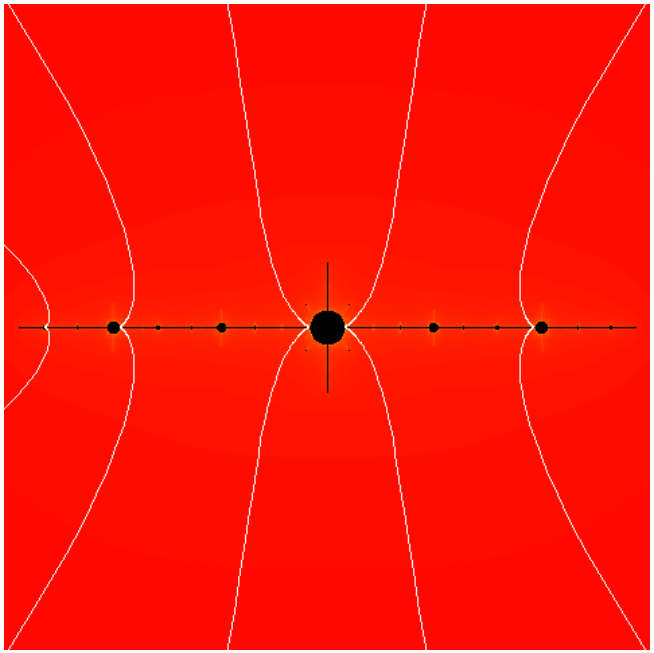


Douady's formula ('93) : $Y_c := \{t \in S^1 \ : \ R_c(t) \text{ lands on } \mathbb{R}\}$ is a closed invariant subset of the angle doubling map

$$q : t \mapsto 2t, \ S^1 \to S^1, \quad \text{and}$$

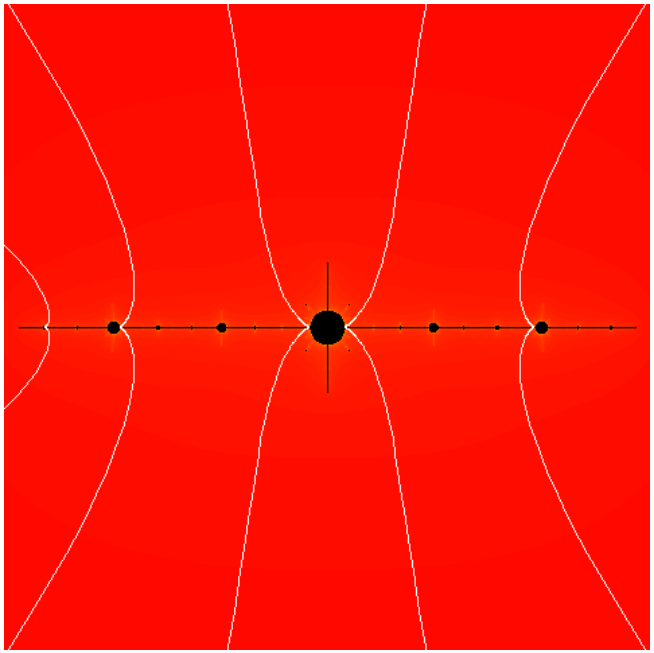
$$h_{top}(q, Y_c) = h_{top}(f_c, \mathbb{R}) =: \text{Core-entropy}(f_c).$$

Thurston's bi-circle (torus) model, '2001



The set $A_c := \{(t, 1-t) \in \mathbb{T}^2, t \in Y_c\}$, consisting of bi-angles landing at a common point on $\mathbb{R}$, is a closed forward invariant set of $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto 2\begin{pmatrix} s \\ t \end{pmatrix}$, $\mathbb{T}^2 \to \mathbb{T}^2$ with the same entropy.

Thurston's bi-circle (torus) model, '2001

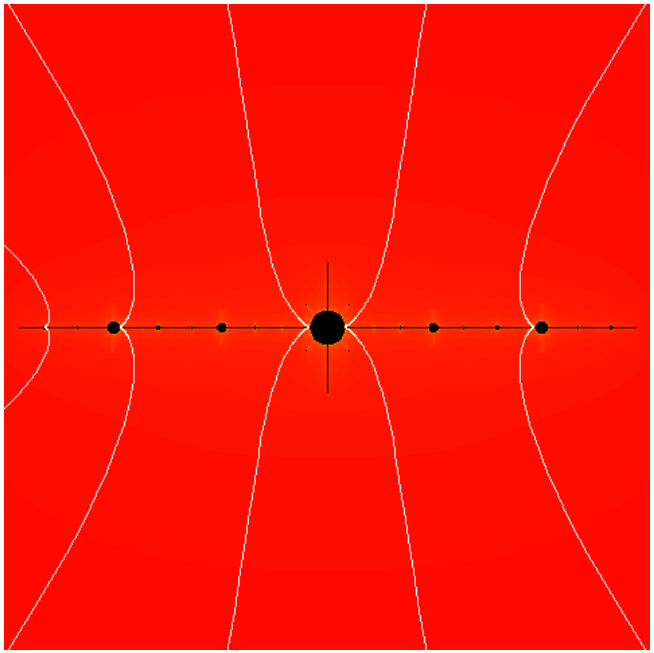


The set $A_c := \{(t, 1-t) \in \mathbb{T}^2, t \in Y_c\}$, consisting of bi-angles landing at a common point on $\mathbb{R}$, is a closed forward invariant set of $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto 2\begin{pmatrix} s \\ t \end{pmatrix}$, $\mathbb{T}^2 \to \mathbb{T}^2$ with the same entropy.

One can then connect entropy to dimension via Furstenberg :

$$\text{H.dim}(A_c) \overset{\text{Thurston}}{=} \frac{\text{Core-entropy}(f_c)}{\log 2} \overset{\text{Tiozzo}}{=} \text{H.dim}(Y_c) \ .$$

Thurston's bi-circle (torus) model, '2001



The set $A_c := \{(t, 1-t) \in \mathbb{T}^2, t \in Y_c\}$, consisting of bi-angles landing at a common point on $\mathbb{R}$, is a closed forward invariant set of $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto 2\begin{pmatrix} s \\ t \end{pmatrix}$, $\mathbb{T}^2 \to \mathbb{T}^2$ with the same entropy.

One can then connect entropy to dimension via Furstenberg :

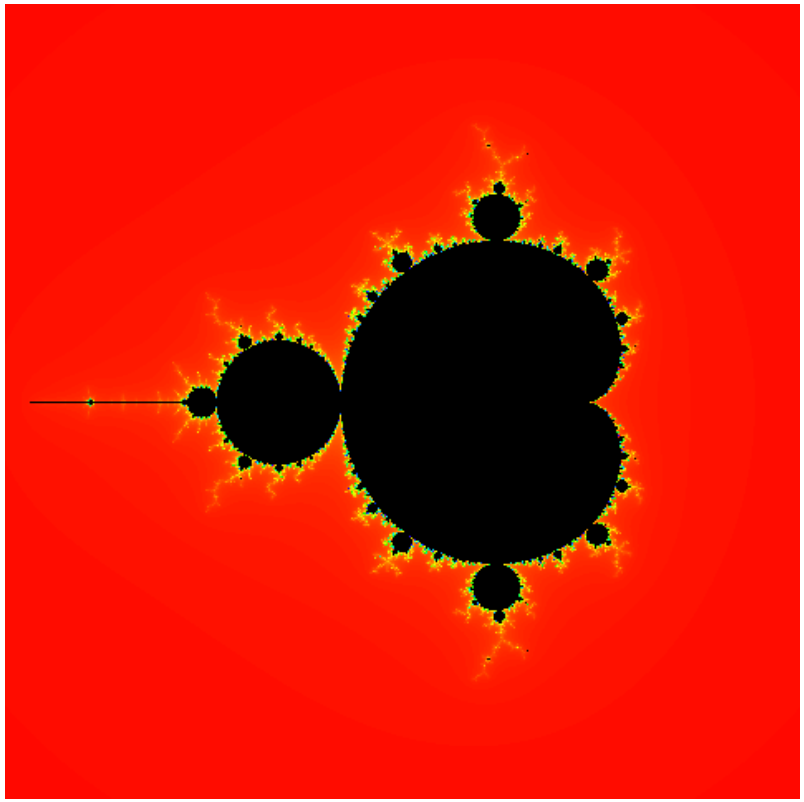
$$\text{H.dim}(A_c) \overset{\text{Thurston}}{=} \frac{\text{Core-entropy}(f_c)}{\log 2} \overset{\text{Tiozzo}}{=} \text{H.dim}(Y_c) \ .$$

Consequently, the set $B(A_c) := \bigcup_n F^{-n}(A_c)$, consisting of bi-angles landing at a common point on the Julia set, has the same dimension as A_c.

Parameter view of real $z^2 + c$

The Mandelbrot set $\mathcal{M}$ is defined by

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^{\circ n}(0) \nrightarrow \infty\}$$

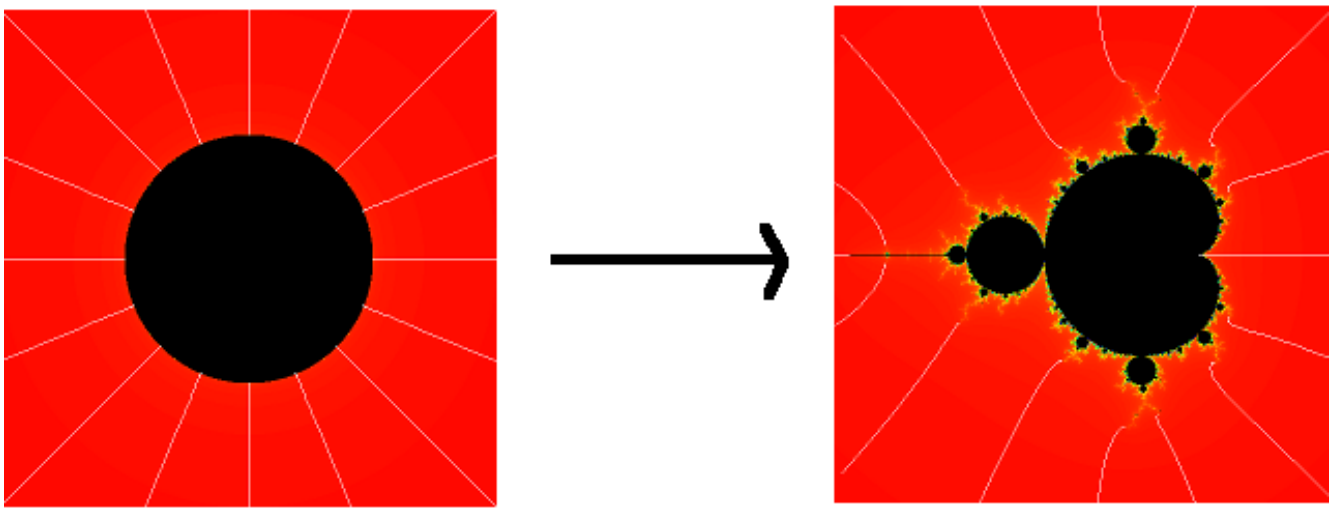


External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected (by Douady-Hubbard and Sibony), there is a conformal map :

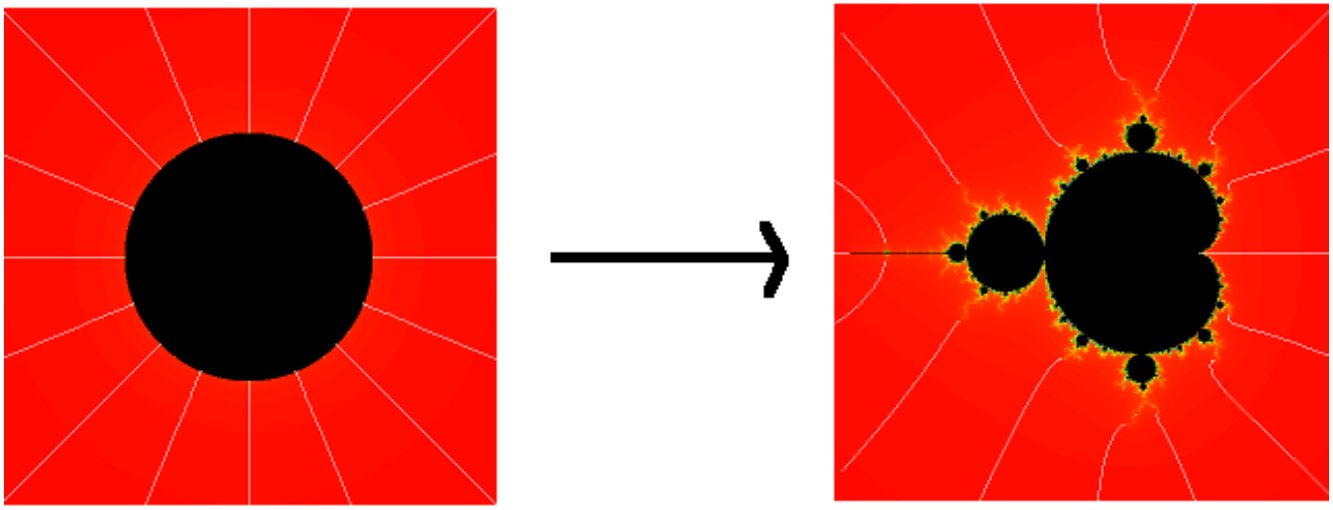
$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

parametrizing the outside and the boundary of $\mathcal{M}$ by external angles.



Almost all (and conjecturally all) rays land at some point on the boundary of $\mathcal{M}$.

Douady's monotonicity argument ('93)



The following chain of associations are trivially monotone :

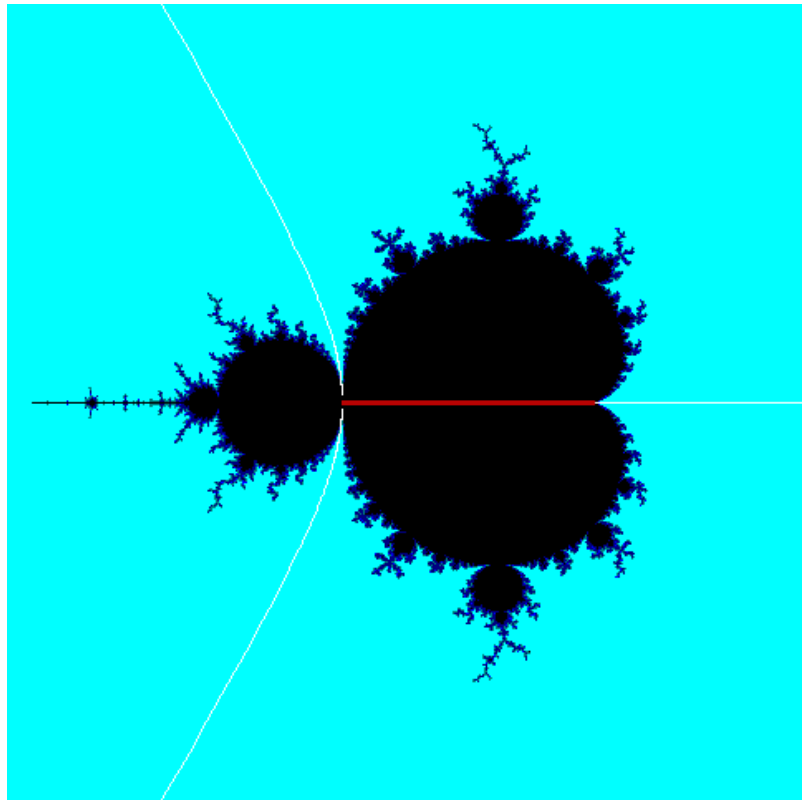
$$[-2, \frac{1}{4}] \ni c \quad \overset{\text{external angle}}{\longrightarrow} \quad \theta \in [0, \frac{1}{2}] \to \text{forbidden region }]\theta, 1-\theta[$$

$$\longrightarrow \quad X_\theta := \{t \in S^1 \mid 2^n t \notin\ ]\theta, 1-\theta[\,\}$$

$$\longrightarrow \quad h_{top}(q, X_\theta) \underset{\text{less trivial}}{\overset{q(Y_c)=X_\theta}{=}} \text{core-entropy}(f_c).$$

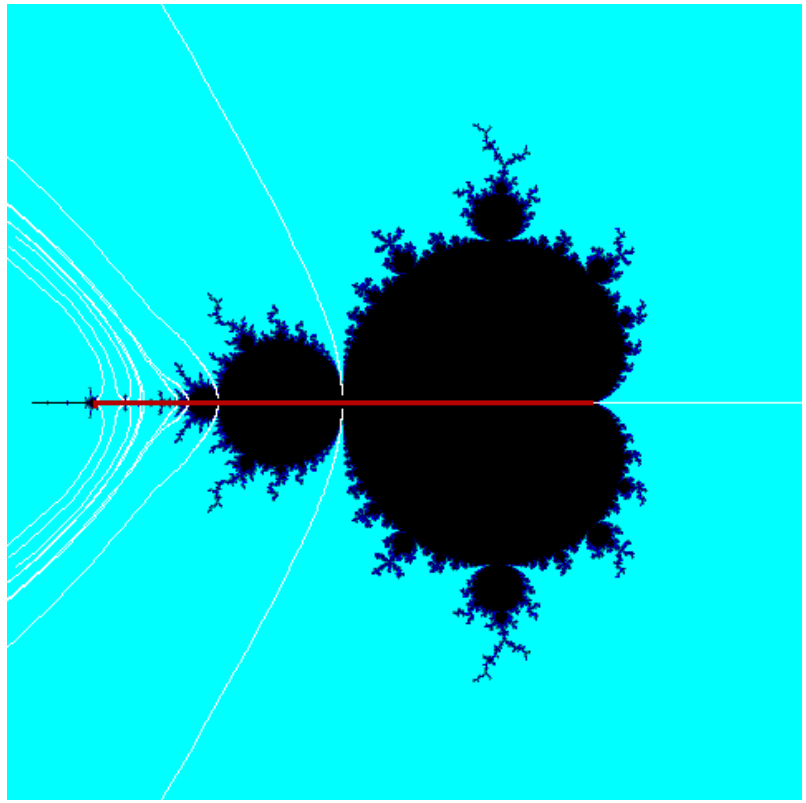
Tiozzo's section theorem ('2012)

Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



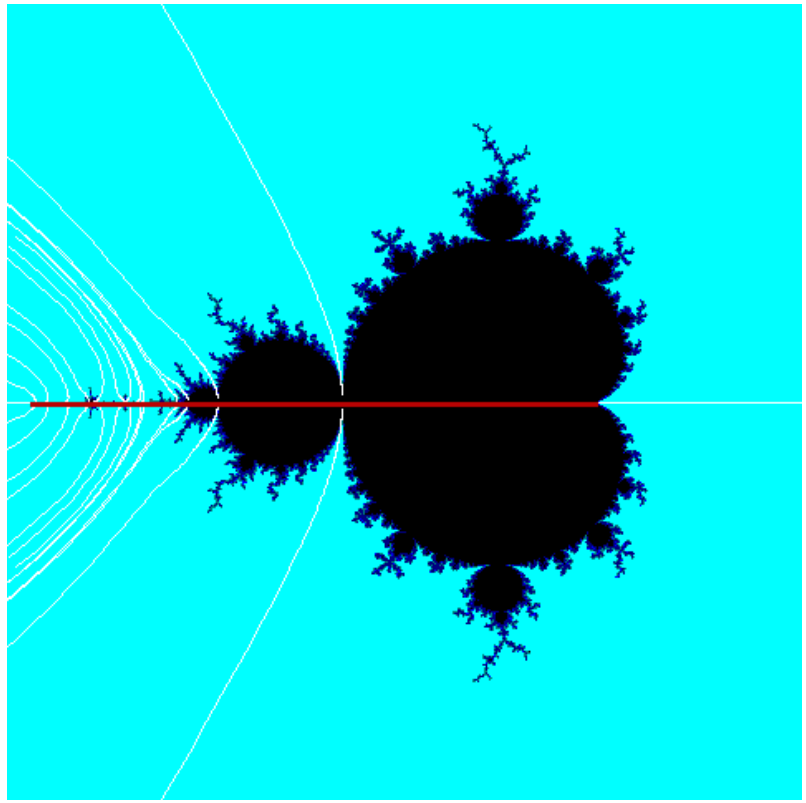
Tiozzo's section theorem ('2012)

Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



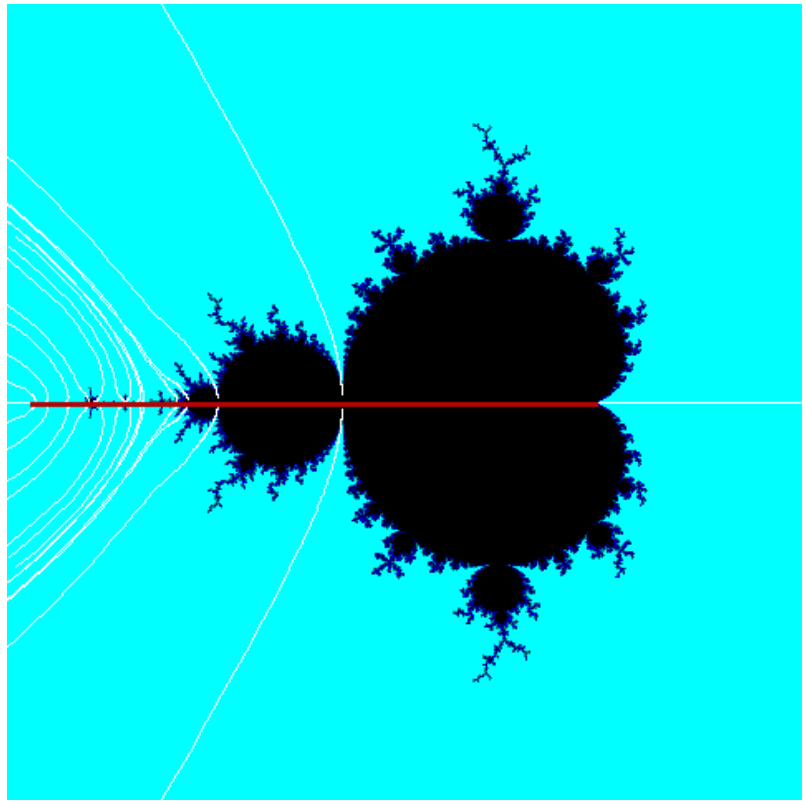
Tiozzo's section theorem ('2012)

Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



Tiozzo's section theorem ('2012)

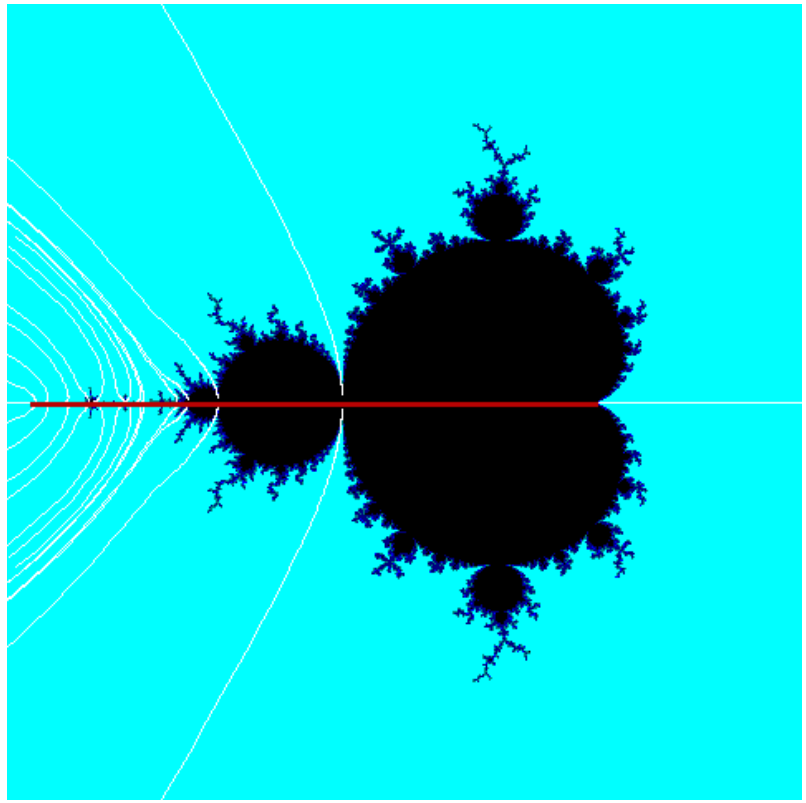
Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



Then,

Tiozzo's section theorem ('2012)

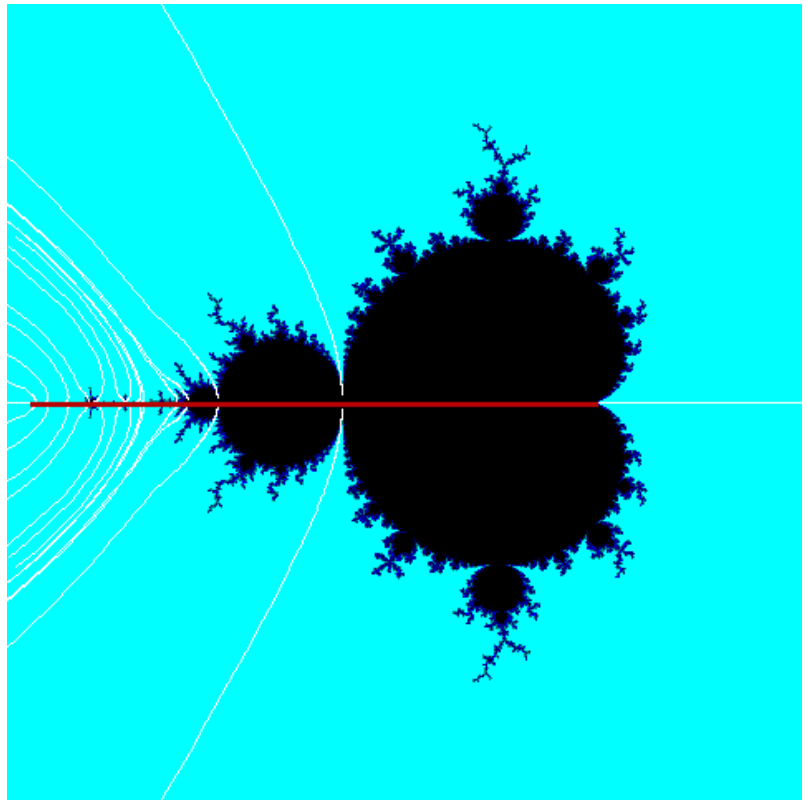
Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



Then, $\dfrac{\text{core-entropy}(f_c)}{\log 2} = \text{H.dim } P_c$.

Tiozzo's section theorem ('2012)

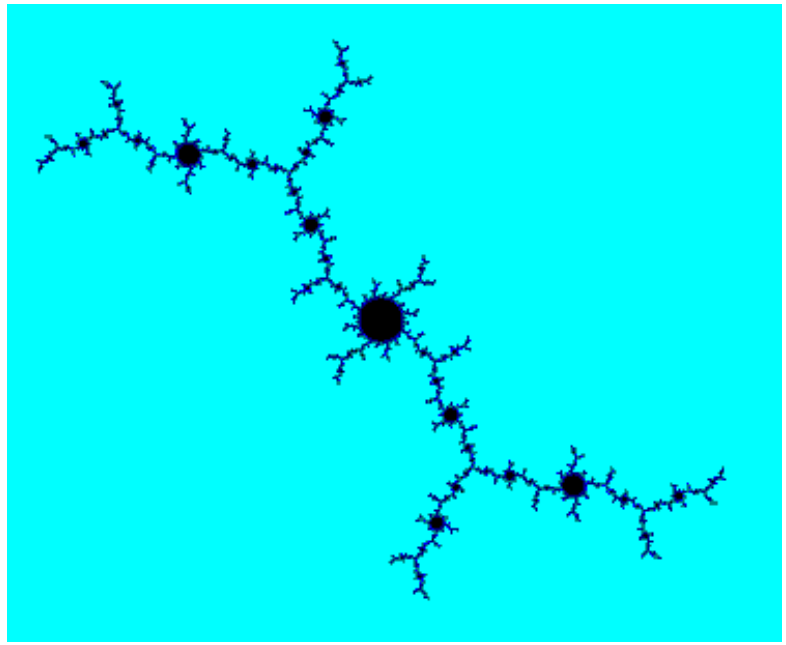
Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



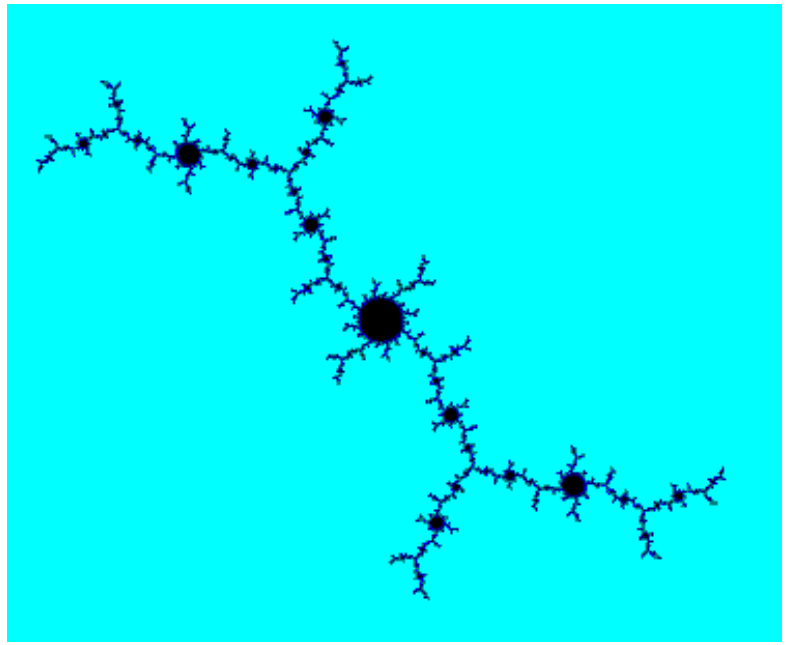
Then, $$\frac{\text{core-entropy}(f_c)}{\log 2} = \text{H.dim } P_c \,.$$

The proof uses techniques coming from α-continued fractions.

How to define the core entropy for a non-real polynomial P ?

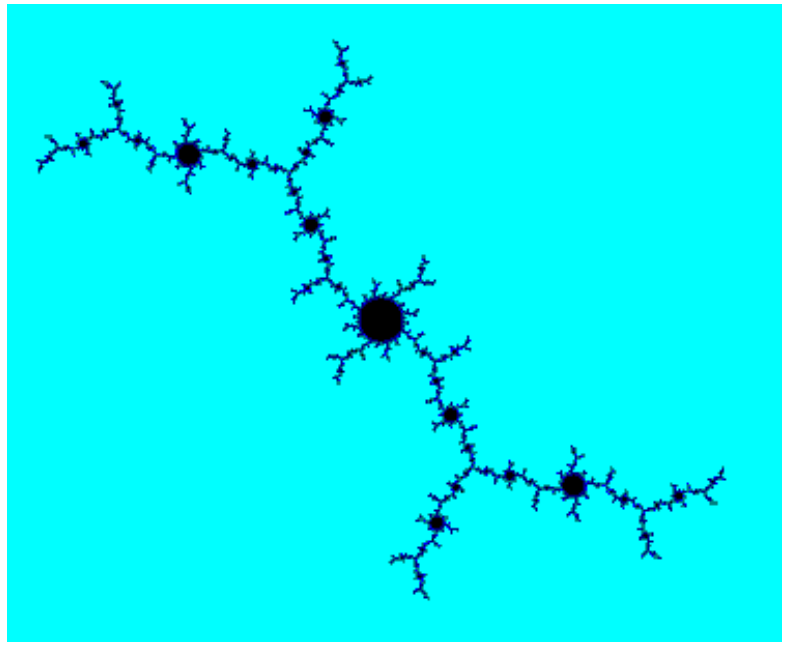


How to define the core entropy for a non-real polynomial P ?



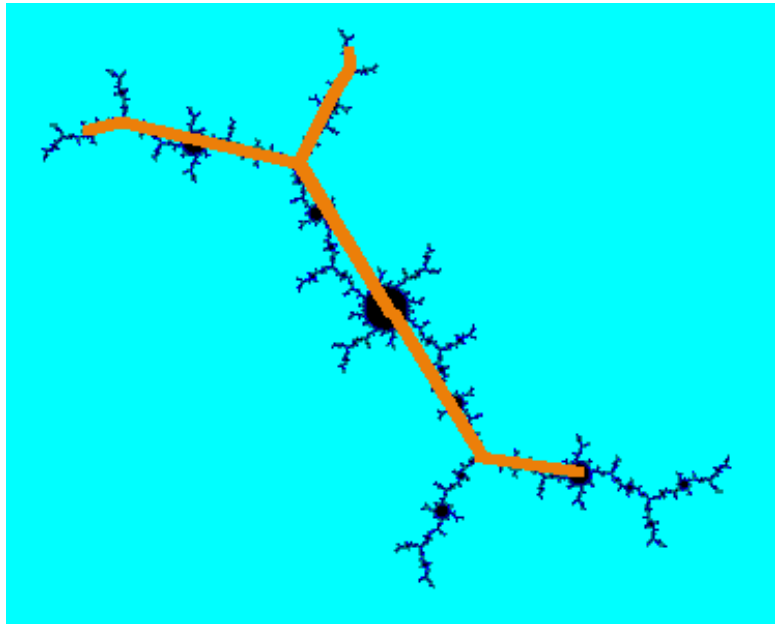
One can define it to be H.dim$(B(P)) \cdot$ log degree, for $B(P) :=$ the set of bi-angles landing at a common point, this is well defined for any P, or

How to define the core entropy for a non-real polynomial P ?

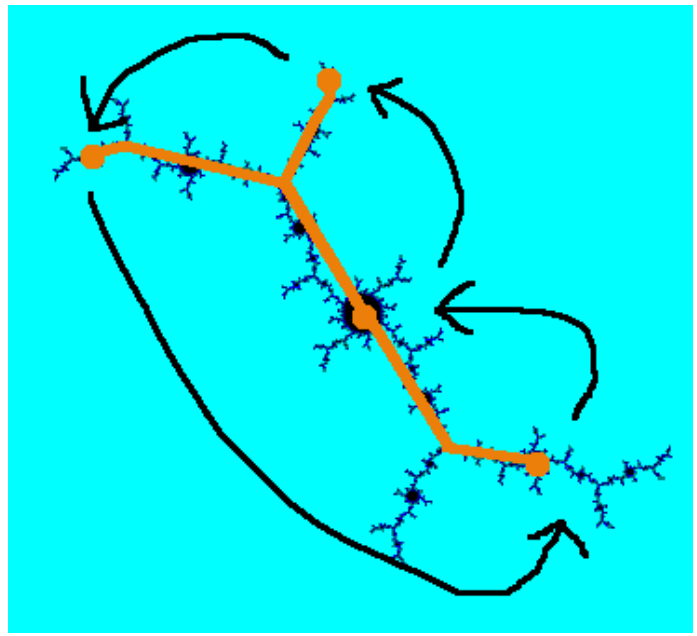


One can define it to be H.dim$(B(P)) \cdot$ log degree, for $B(P) :=$ the set of bi-angles landing at a common point, this is well defined for any P, or $h_{top}(P, H)$, if there exists a forward invariant tree H playing the role of the real line ...

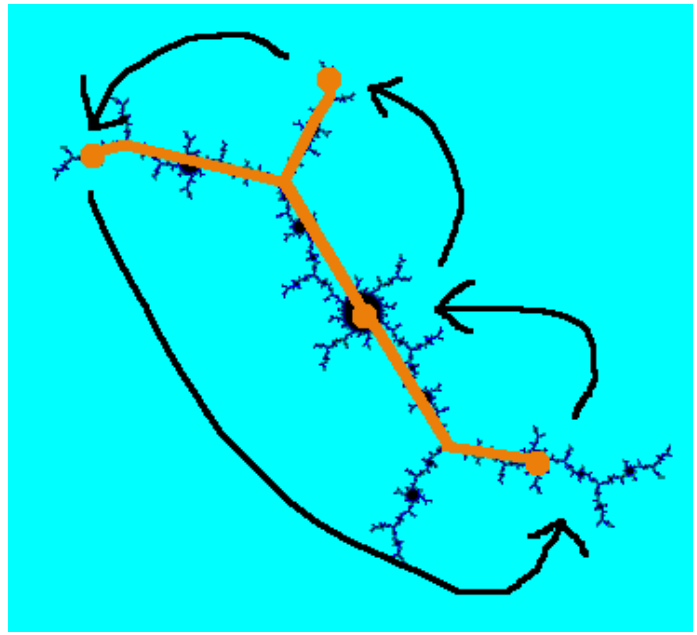
The Hubbard tree H_c of $f_c : z \mapsto z^2 + c$, if exists, is the smallest topologically finite tree in K_c containing the orbit of the critical point 0. It is automatically forward invariant.



The Hubbard tree H_c of $f_c : z \mapsto z^2 + c$, if exists, is the smallest topologically finite tree in K_c containing the orbit of the critical point 0. It is automatically forward invariant.



The Hubbard tree H_c of $f_c : z \mapsto z^2 + c$, if exists, is the smallest topologically finite tree in K_c containing the orbit of the critical point 0. It is automatically forward invariant.



For many "good" values of c, the Hubbard tree, and many larger forward invariant trees, exist, and all give the same entropy (see Tiozzo's preprint). For even better values of c (such that 0 has a finite orbit), the tree admits a Markov partition and matrix and

$$\text{Core-entropy}(f_c) = \log \lambda_{\text{leading}}(\text{Markov matrix}).$$

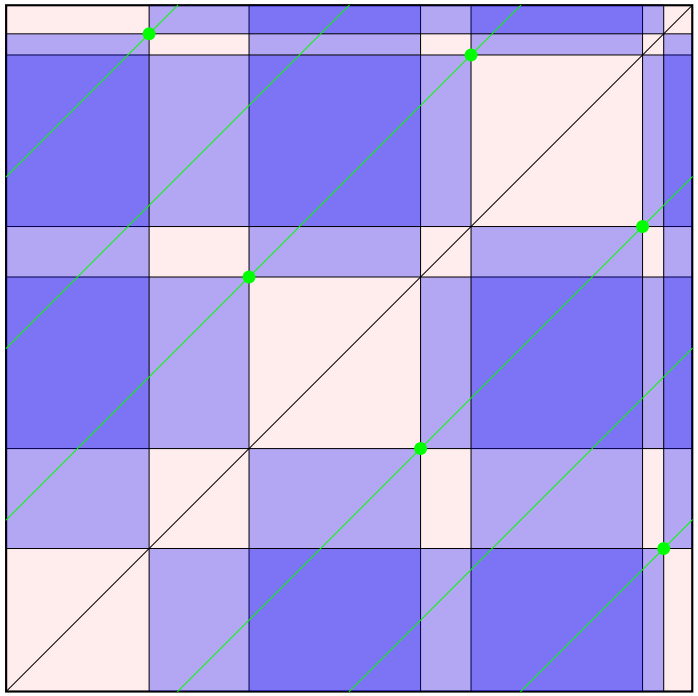
Other models

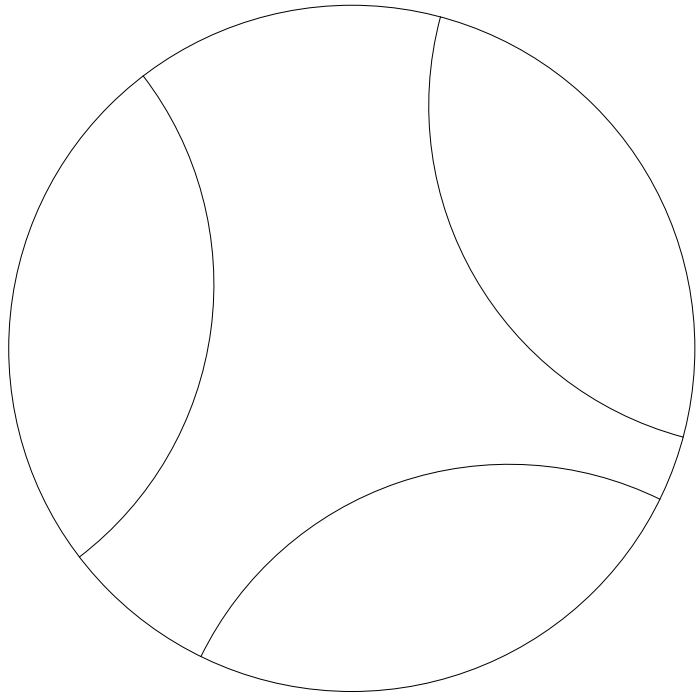
There are other means to encode the core entropy, such as Bartholdi-Dudko-Nekrashevych automata model, Tiozzo's parameter section model, Milnor-Thurston's kneading determinant and uniform-expander model, etc. Relating them all and studying parameter variations is part of the current research. See work of Alsedà-Fagella, Bartholdi-Dudko-Nekrashevych, Branner-Hubbard, Bruin-Schleicher, Dujardin-Favre, Gao Y. 高延, Li T. 李涛, Milnor-Tesser, Penrose, Poirier, Thurston, Tiozzo, among others...

Thurston's parametrization of degree d polynomials by primitive majors (critical portrait) (partially proved by Gao and T.)

$$
\begin{array}{rl}
P \to & \text{Critical portrait } m \to \text{Forbidden regions} \\
\longrightarrow & \mathbb{T}^2 \supset K(m) := \text{the non escaping set} \\
\longrightarrow & B(m) := K(m) \setminus \text{diagonal} \\
\underset{\text{For good } m}{\longrightarrow} & B(m) = \bigcup_n F^{-n}(A(m)), A(m) \text{ closed and invariant} \\
\underset{\text{For rational } m}{\overset{\text{computable}}{\longrightarrow}} & \text{matrix } \Gamma_m \text{ whose } \lambda_{\text{leading}} \overset{\text{Thm}}{\underset{\text{Gao}}{=}} \exp(\text{entropy}).
\end{array}
$$

From a primitive major m to its binding set $B(m)$



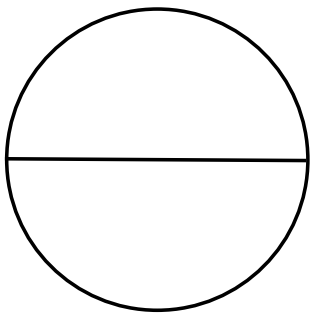


A primitive major m of degree d is a finite lamination in $\mathbb{D}$ so that each region of $\mathbb{D} \setminus m$ touches S^1 in a union of one or more closed intervals $J_1 \cup \cdots \cup J_k$ of total length $1/d$. The product $(J_1 \cup \cdots \cup J_k) \times (J_1 \cup \cdots \cup J_k)$ is a union of light pink rectangles.

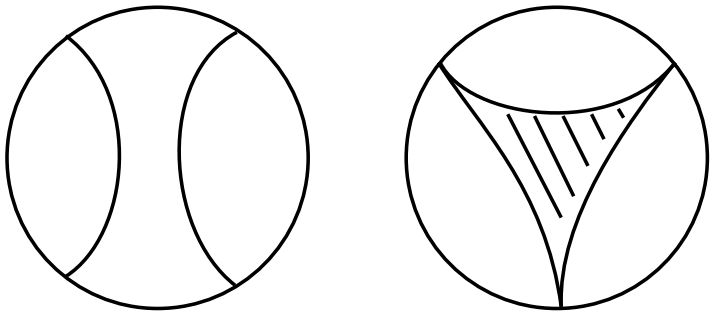
The set $K(m)$ is the non-escaping locus under $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto d \begin{pmatrix} s \\ t \end{pmatrix}$.

Combinatorics of primitive majors (work of Tomasini)

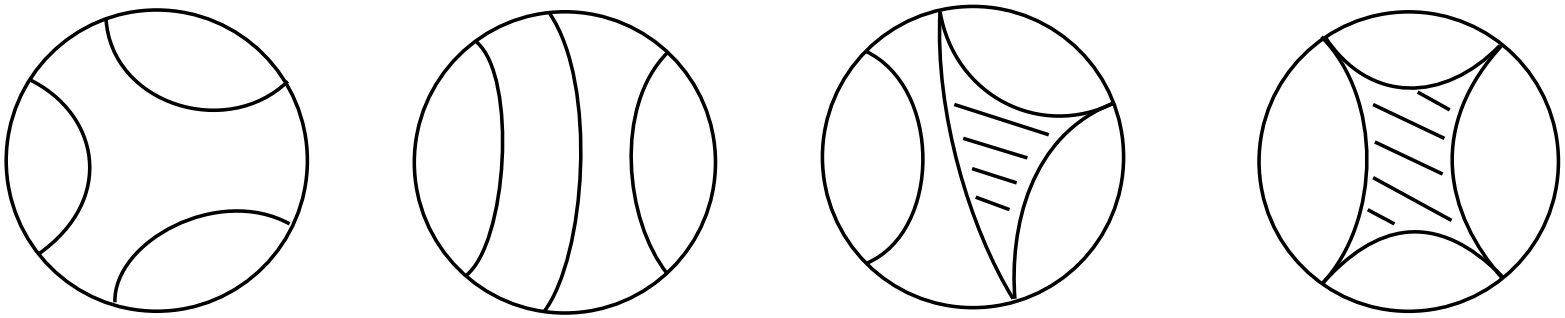
$d = 2$



$d = 3$

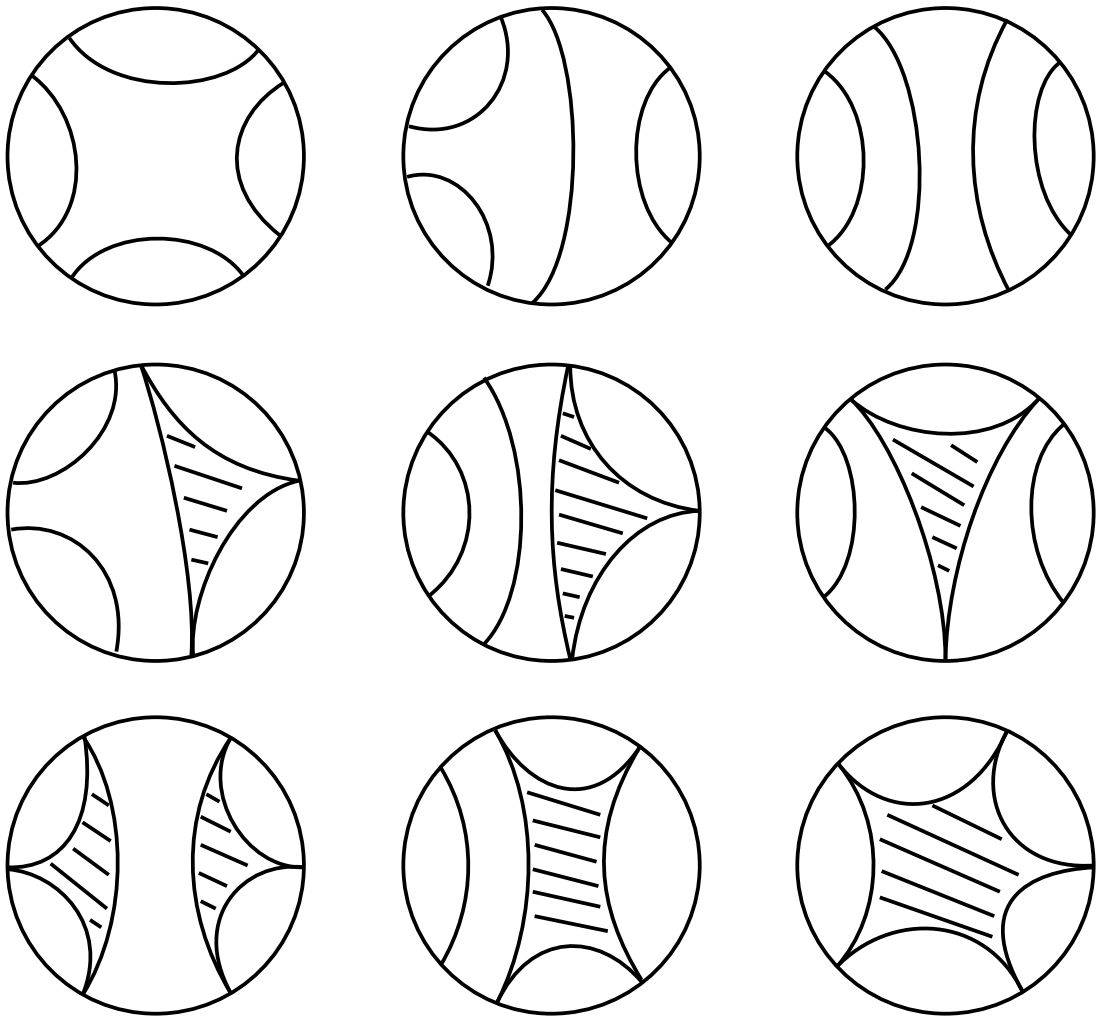


$d = 4$



Combinatorics of primitive majors (work of Tomasini)

$d = 5$



The total number $\tilde{p}_d$ is $\sum_{k=0}^{d-1} \tilde{p}_d^k$ (by Tomasini, '2013), where

$$p_d^k = \frac{1}{d} \sum_{i=k}^{d-1} \binom{i}{k} \binom{d-1+i}{i} \binom{d}{d-1-i} (-1)^{d-1-i}$$

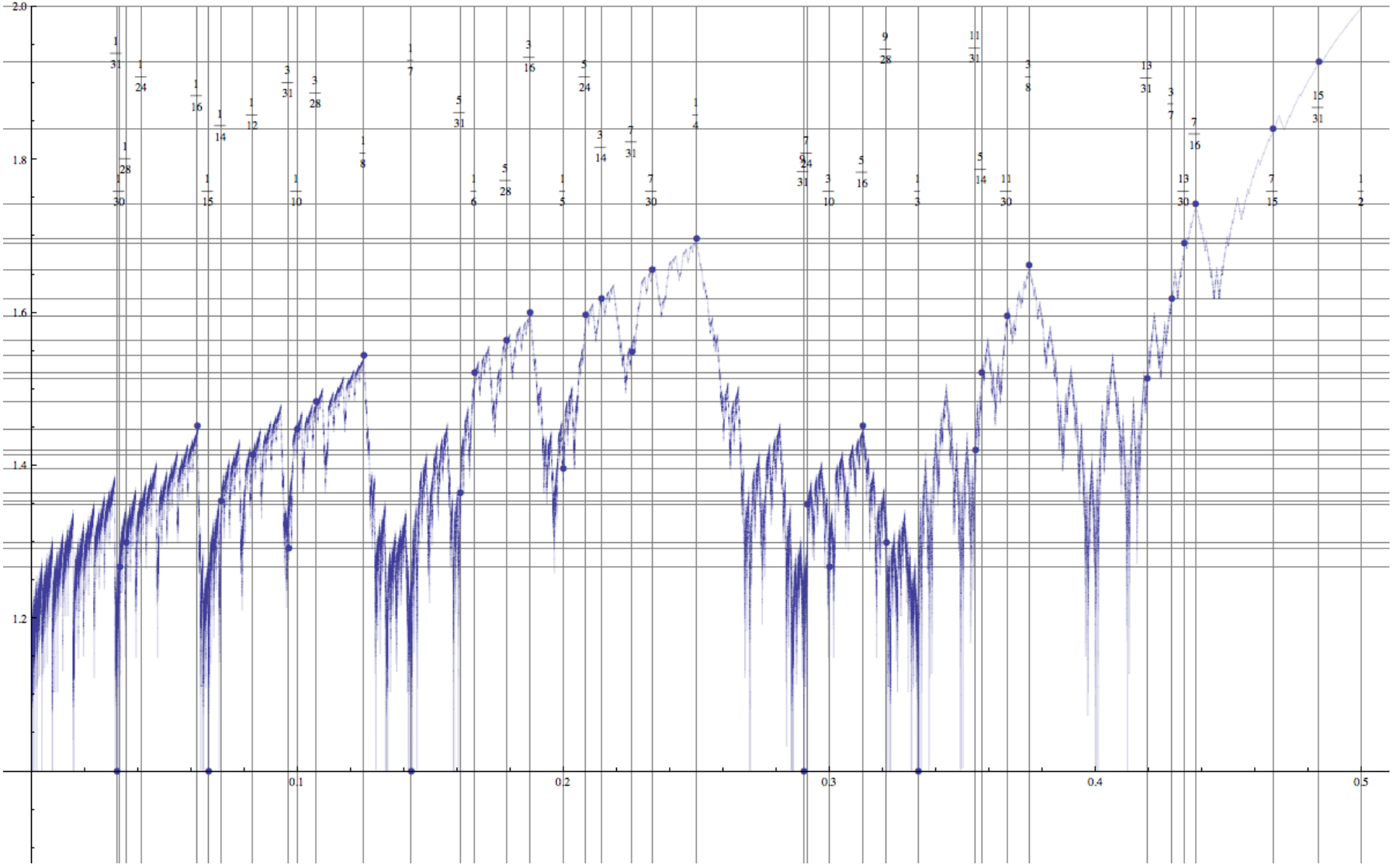
$$\tilde{p}_d^k = \frac{1}{d+k-1} \left[p_d^k + \sum_{\substack{\ell \geq 2 \\ \ell \mid d-1 \\ \ell \mid k}} \varphi(\ell) \left(\frac{d-1}{\ell} + 1 \right) p_{(d-1)/\ell+1}^{k/\ell} + \right.$$

$$\left. \sum_{\substack{\ell \geq 2 \\ \ell \mid d \\ \ell \mid k-1}} \varphi(\ell) \left(\left(\frac{k-1}{\ell} + 1 \right) p_{d/\ell}^{(k-1)/\ell+1} + \left(\frac{d+k-1}{\ell} \right) p_{d/\ell}^{(k-1)/\ell} \right) \right]$$

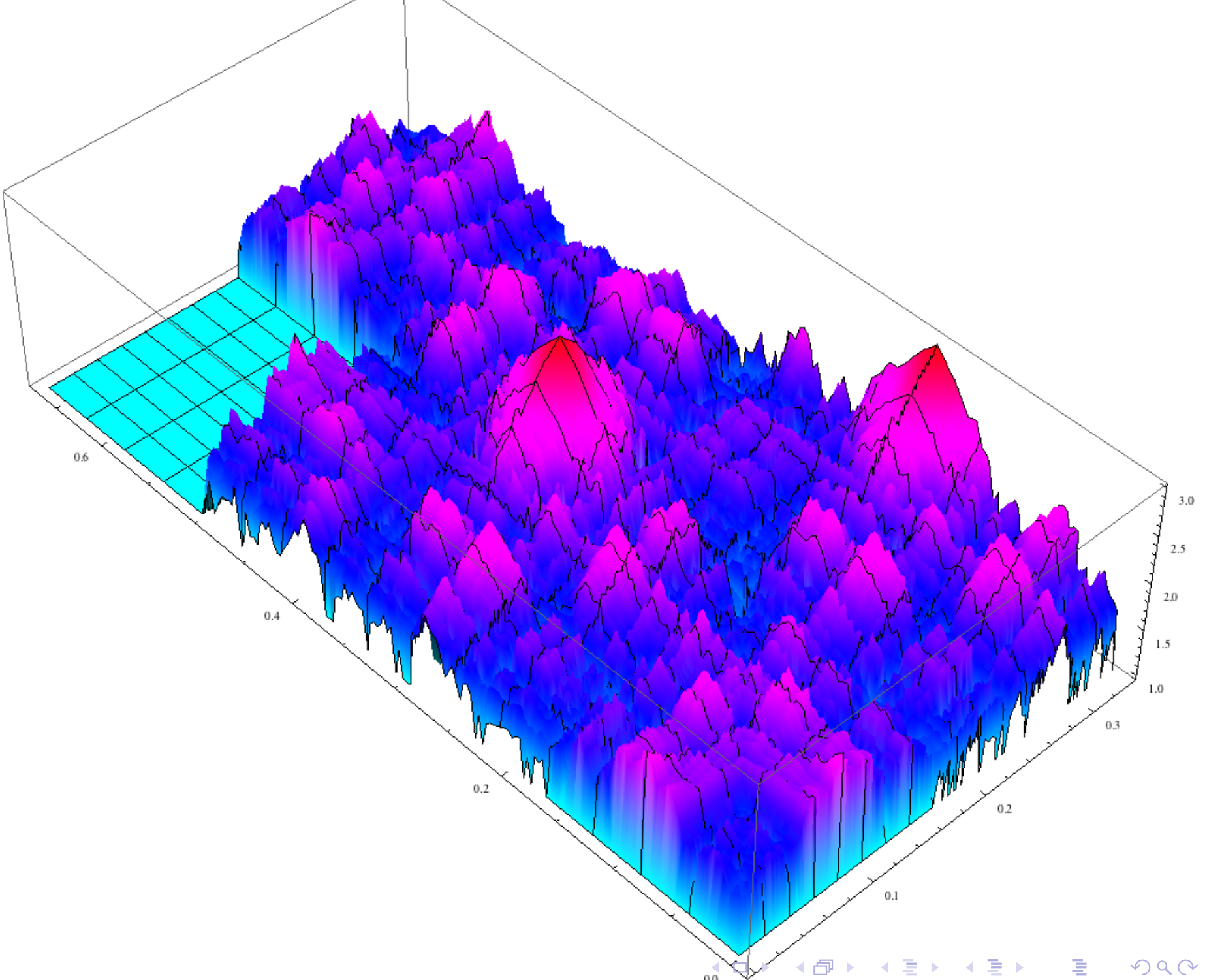
d	2	3	4	5	6	7	8	9	10	11	12	
$\tilde{p}_d$	1	2	4	9	27	94	364	1529	6689	30230	140114	$\cdots$

Thurston's mathematica animations

$\mathbb{Q} \ni \theta \mapsto$ leading eigenvalue of m_θ, plot of Thurston

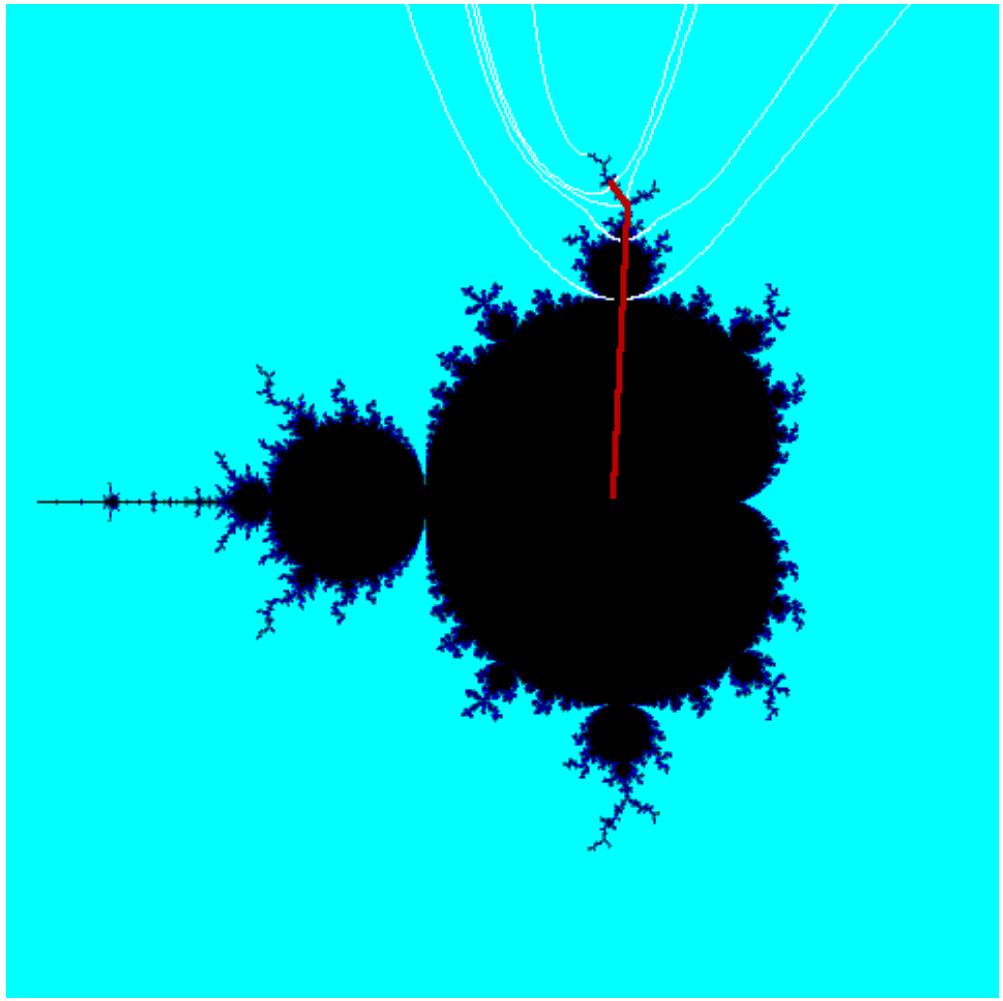


Is this function continuous ?


0.6
0.4
0.2
0.0
0.1
0.2
0.3
1.0
1.5
2.0
2.5
3.0

Monotonicity and continuity along the veins

The Mandelbrot set has a tree-like structure of veins, with the real segment as a particular vein.



See works of Li and Tiozzo ...

Thank you !

Merci !

谢谢 !