Groupes auto-similaires et groupoïdes hyperboliques

Orléans, 21-23 Juin ,2012

Combinatorics of rational dynamical systems

Tan Lei

Université d'Angers

Let $f(z) = \dfrac{p(z)}{q(z)}$ be a rational map. Its iteration generates a dynamical system on the Riemann sphere.

We will only talk about hyperbolic postcritically finite maps : for every critical point c of f, some forward iterate, $f^{\circ k}(c)$ $(k \geq 1)$ is again a critical point. Set

$$P_f := \bigcup \{f^{\circ k}(c),\ c \text{ critical}\}.$$

The same notion exists for orientation preserving branched coverings of S^2. Following Thurston, we say that two hyperbolique postcritically finite branches coverings F and G are combinatorially equivalent if

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{\exists\ \psi \text{ homeo}} & (S^2, P_G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ (S^2, P_F) & \xrightarrow[\exists\ \phi \text{ homeo}]{} & (S^2, P_G) \end{array} \qquad \text{with} \qquad \begin{array}{l} \phi|_{P_F} = \psi|_{P_F} \text{ and} \\ \phi \text{ isotopic to } \psi \\ \text{relative to } P_F. \end{array}$$

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{\exists\ \psi \text{ homeo}} & (S^2, P_G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ (S^2, P_F) & \xrightarrow[\exists\ \phi \text{ homeo}]{} & (S^2, P_G) \end{array} \qquad \text{with} \qquad \begin{array}{l} \phi|_{P_F} = \psi|_{P_F} \text{ and} \\ \phi \text{ isotopic to } \psi \\ \text{relative to } P_F. \end{array}$$

Rigidity Theorem (Thurston) Two combinatorially equivalent hyperbolic postcritically finite rational maps are Möbius conjugate.

(Notice that this is not obvious even if $\psi = \phi$.)

Questions :

1. Find combinatorial invariants of this equivalence relation.
2. Determine which class represents a rational dynamical system.
3. Compute the coefficients of a representative rational map within a prescribed combinatorial class.

The easiest case : real polynomials with simple and real critical points

Questions and answers :

1. Find combinatorial invariants of this equivalence relation.
   — the order of P_f in $\mathbb{R}$ and the induced $f|_{P_f}$, or
   — piecewise linear model map
2. Determine which class represents a polynomial dynamical system.
   — every class
3. Compute the coefficients of a representative polynomial within a prescribed combinatorial class.
   — see de scilab code of HH Rugh.

General case

F branched covering. Nekrashevych introduced :

$$\begin{cases} \text{permutational bimodule } \mathfrak{M}_F & \text{complet invariant} \\ IMG_F, \text{ a selfsimilar group action on a tree} & \text{partial invariant} \end{cases}.$$

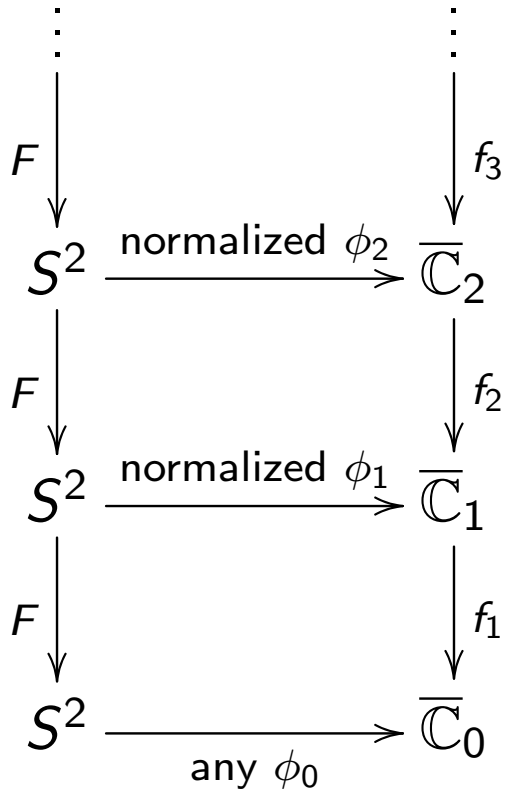
F topological polynomial :

Hubbard trees, characteristic external angles, kneading sequences, internal addresses, critical portraits, fixed points portraits, laminations (Douady, Hubbard, Thurston, Milnor, Schleicher Poirier, Kiwi ...)

(twisted) kneading automata (Nekrashevych, Bartholdi, Dudko, Kelsey ...)

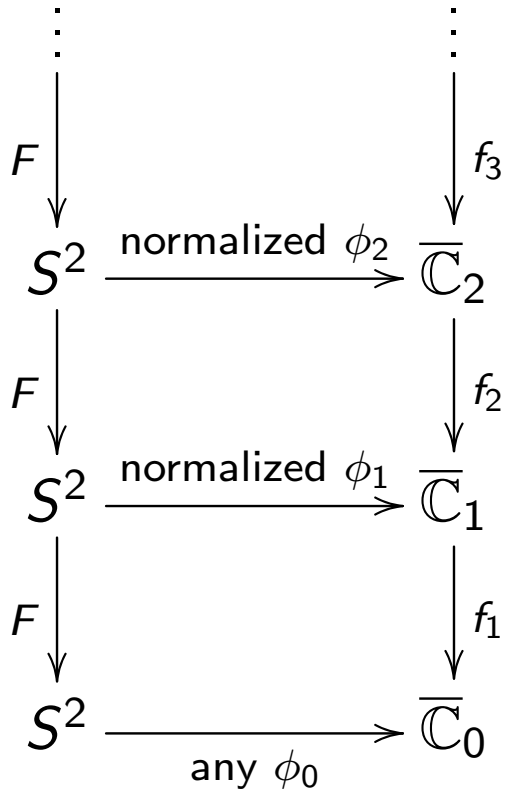
Running algorithms: Hubbard-Schleicher's spider algorithm for $z^2 + c$, Bartholdi-Dudko's Gap codes for IMGs, 'Meduse' Program for matings of quadratic polynomials and Chéritat's movies, HH Rugh's scilab code for real rational maps, Nekrashevych's combinatorial spider algorithm...

Thurston's algorithm



Theorem (Thurston) If F is unobstructed, the sequence $[\phi_n]$ converges to τ in the Teichmüller space $\mathcal{T}_{P_F}$; $\{f_n\}$ converges uniformly to a rational map f combinatorially equivalent to F; and $\phi_n(P_F)$ converges to P_f.

Strategy of implementing Thurston's algorithm



Given a combinatorial invariant, build a branched cover model F, compute the coefficients of f_n (without computing ϕ_n)...

Each step is a decision problem among finitely many possibilities:

Rational maps with prescribed critical values

Two rational maps $f, g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are said to be isomorphic (or covering equivalent) if there is a Möbius transformation M such that $\begin{array}{ccc} \overline{\mathbb{C}} & \stackrel{M}{\to} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array}$. Clearly f and g share the same degree and the same critical value set V.

Theorem (Hurwitz) Fixing the degree d and V, there are only finitely many isomorphic classes.

Questions.

1. How many ?
2. How to distinguish them combinatorially ?
3. How to compute the coefficients of representatives?

This non-dynamical problem is related to :

1. two dimensional-quantum chromodynamics and the related string theories
2. Subgroups of symmetric groups
3. Graph theory, combinatorics
4. Algebraic geometry, singularity theory, Gromov-Witten invariants, complements of discriminants in diverse moduli spaces
5. surface groups and generators
6. 3-orbifolds that are fibered over circles
7. ...

How many ? In the simple critical values case :

1. (Lyashko-Looijenga) For V a set of $d-1$ distinct points in $\mathbb{C}$, the number of isomorphism classes of polynomials with V as the critical value set is equal to d^{d-3}:

$$1, 1, 1, 4, 25, 216, 2401, \cdots .$$

2. (Hurwitz) For V a set of $2d-2$ distinct points in $\overline{\mathbb{C}}$, the number of isomorphism classes of rational maps with V as the critical value set is equal to $\frac{(2d-2)!}{d!}d^{d-3}$:

$$1, 1, 4, 120, 8400, 1088640, 228191040, \cdots .$$

3. Real polynomials with real critical points and simple critical values : with generating function $\sec x + \tan x$:

$$1, 1, 1, 2, 5, 16, 61, 272, \cdots .$$

4. Real rational maps of degree d with real critical points and simple critical values : $\leq 2d-2\,!$.

How to distinguish?, how to compute ?

Lyashko-Looijenga : $\mathcal{LL} : f \mapsto \{\text{critical values}\}$,
$\{$monic centered degree d polynomials with simple critical values$\}$
$\longrightarrow$ Configuration space of unordered $(d-1)$ points in $\mathbb{C}$
is a covering map.

So the fundamental group, i.e. the braid group, has a monodromy action on the space of polynomials.

The scilab code interactively lifts paths traced by the mouse, in the configuration space. The mouse moves one critical value at a time. The traced path induces a vector field and lifting the path becomes a trajectory of a non-autonomous ODE...

There is a similar code for rational maps, even though the $\mathcal{LL}$ is not explicit.

Observe: 3-cycle monodromy, expansion/isometry of $\mathcal{LL}$ in the polynomial/rational map cases...

Pullback action on multicurves

Recall that F and G are combinatorially equivalent if

$$\begin{array}{ccc} (S^2, P_F) & \xrightarrow{\exists\ \psi, \approx} & (S^2, P_G) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ (S^2, P_F) & \xrightarrow[\exists\ \phi, \approx]{} & (S^2, P_G) \end{array}$$

with $\phi|_{P_F} = \psi|_{P_F}$ and ϕ isotopic to ψ relative to P_F.

This implies that F and G induces conjugate pullback actions on the set of multicurves. One can encode this action by

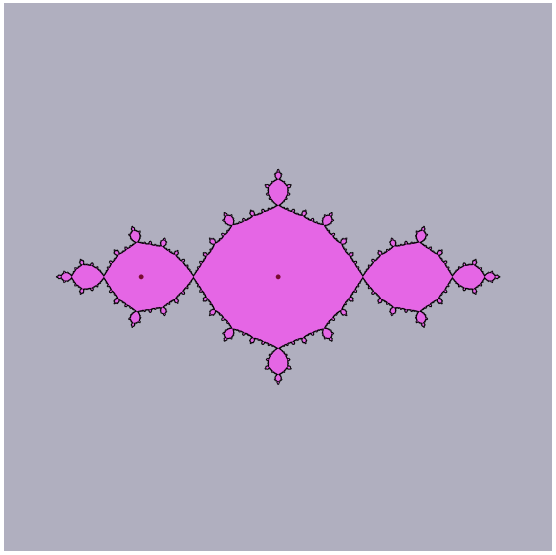
$$F^*[\gamma] = \sum_{[\eta]} \left(\sum_{\delta \subset F^{-1}(\gamma), \delta \sim \eta} \frac{1}{\deg_f(\delta)} \right) [\eta] \ .$$

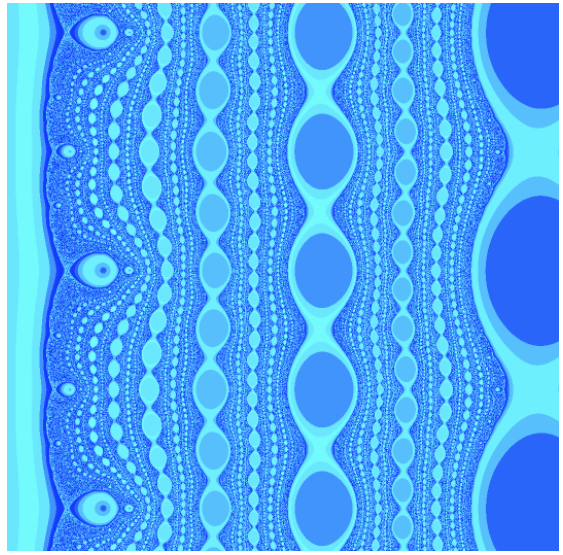
A multicurve Γ is F-invariant if $F^*(\mathbb{R}^\Gamma) \subset \mathbb{R}^\Gamma$, a Thurston obstruction if furthermore $sp(F^* : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma) \geq 1$.

Applications.

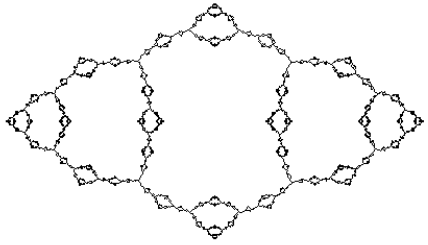
1. If f is a hyperbolic postcritically finite rational map, then it has no obstruction. Conversely, a hyperbolic postcritically finite branched covering F with out obstructions is combinatorially equivalent to a rational map (Thurston's criterion).
2. can be used to combinatorially distinguish different maps (twisted rabbit problems and other twist problems, Bartholdi-Nekreshevych, Pilgrim, Lodge...)
3. can be used to decompose a rational map dynamics into smaller and simpler pieces (renormalizations):
   – reduce disconnected Julia set cases (necessarily postcritically infinite) to postcritically finite ones (Cui-T., Godillon, Wang...)
   – decompose a postcritically finite map along a Cantor multicurve (Cui-Peng-T.)
   – decompose a postcritically finite map along an equator into two polynomials (the inverse of matings) (lots of names to be put here) ... ...

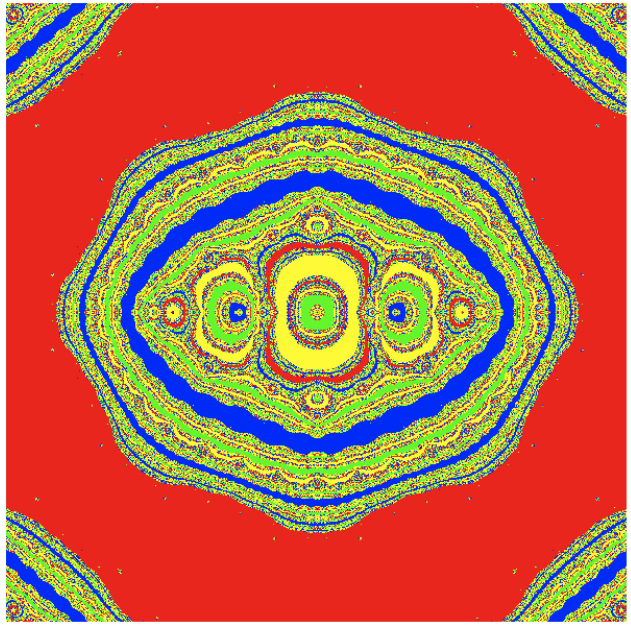
Examples.





Pilgrim-T.



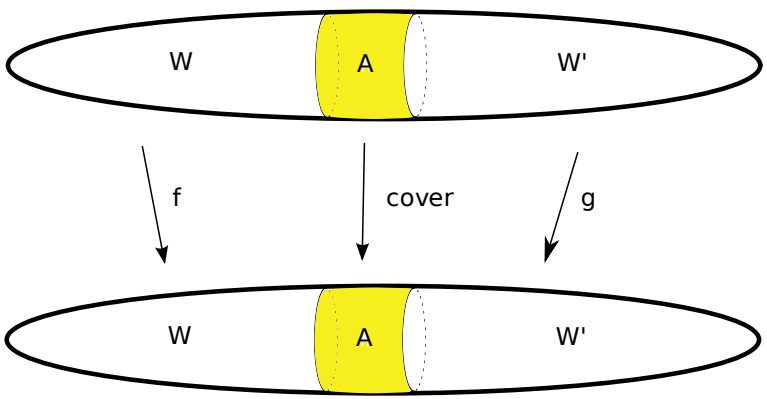
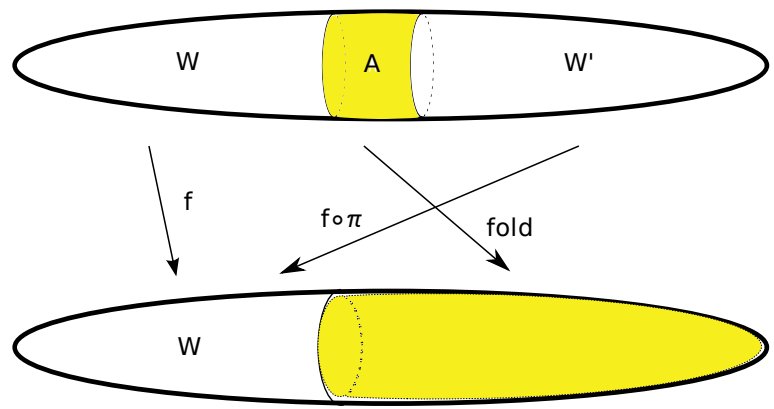


Godillon

Equators and Cantor multicurves

- $\overline{W} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta}, \theta \in \mathbb{R}\}$,
  $\overline{W'} = \mathbb{C}' \cup \{(\infty \cdot e^{2i\pi\theta})', \theta \in \mathbb{R}\}$,
- $A = [-1, 1] \times S^1$,
- $S = \overline{W} \sqcup A \sqcup \overline{W'} / \sim$,
  with $\infty \cdot e^{2\pi i\theta} \sim (-1, e^{2\pi i\theta})$ and $(+1, e^{2\pi i\theta}) \sim (\infty \cdot e^{-2\pi i\theta})'$,
- $\pi = id : \overline{W'} \to \overline{W}$.

f, g polynomials, degree$=d, d'$, postcritically finite,

$d = d'$, f, g monic mating(f, g) : M :	folding$(f,$ fold$)$: F : $F^2(\mathcal{C}^A) \subset K_f$, (pre)periodic
	

Problems

Use IMG/bimodule or other invariants to detect existence or absence of invariant multicurves, especially the Cantor multicurves and the equators.

Use IMG/bimodule or other invariants to classify indecomposable pieces.

Non-dynamical combinatorics, monodromy actions

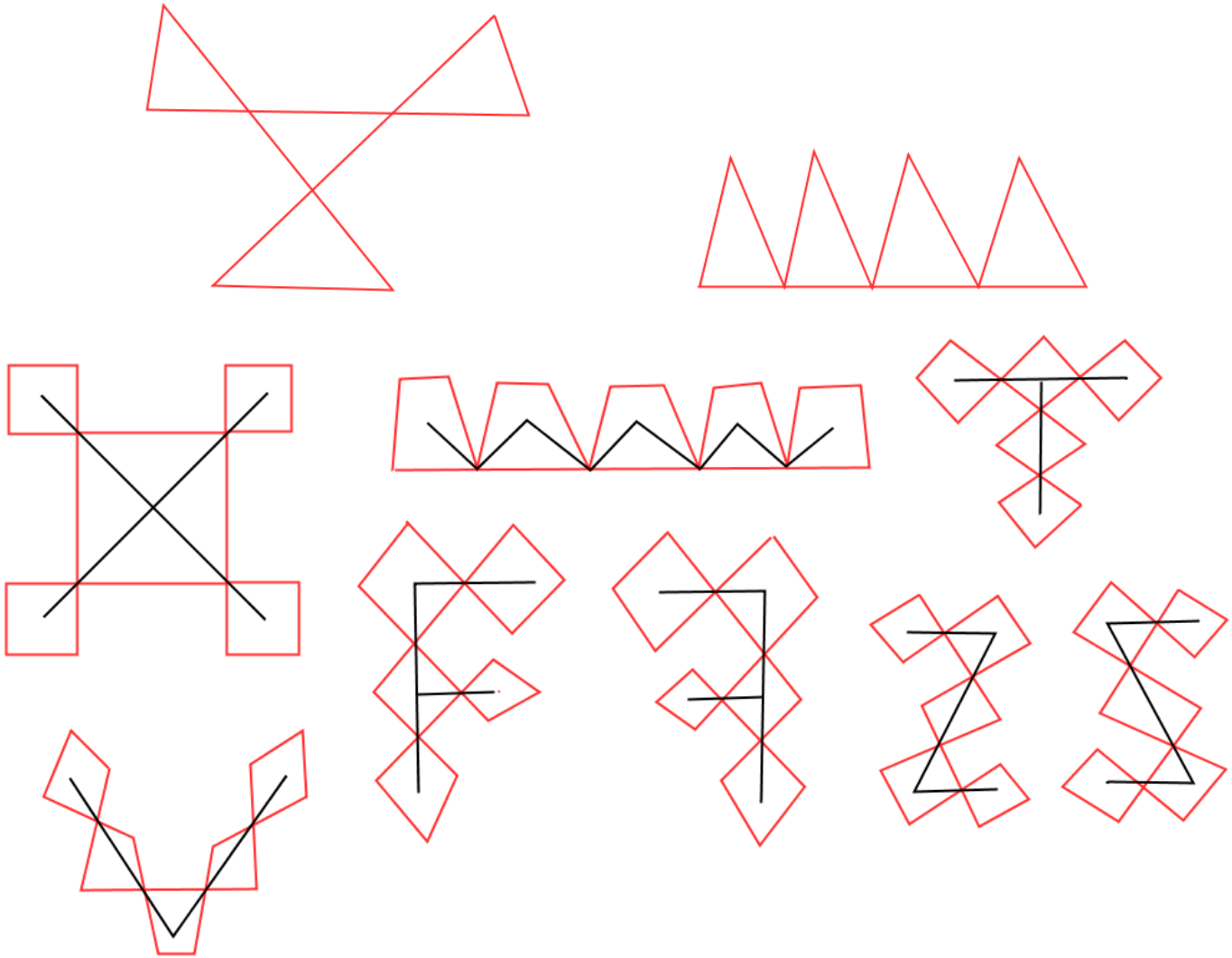
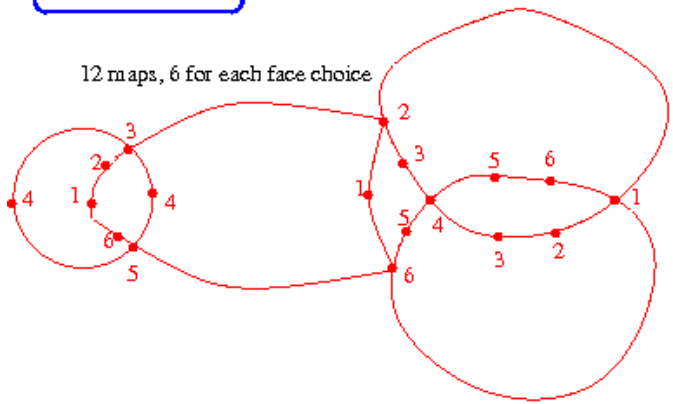


Figure: Degree 4 and 5 polynomial combinatorics

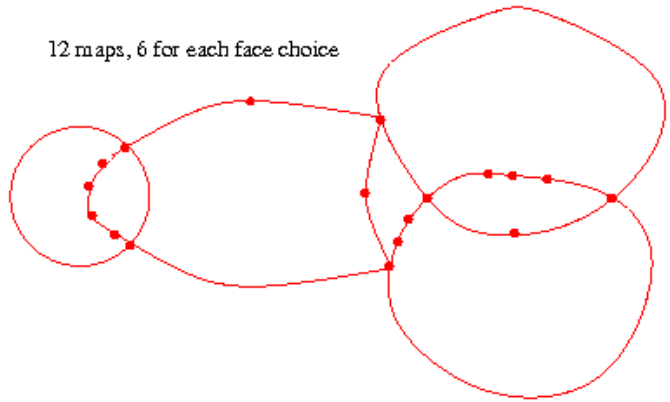
Quartic rational maps, samples of 120 isomorphism classes

60 maps

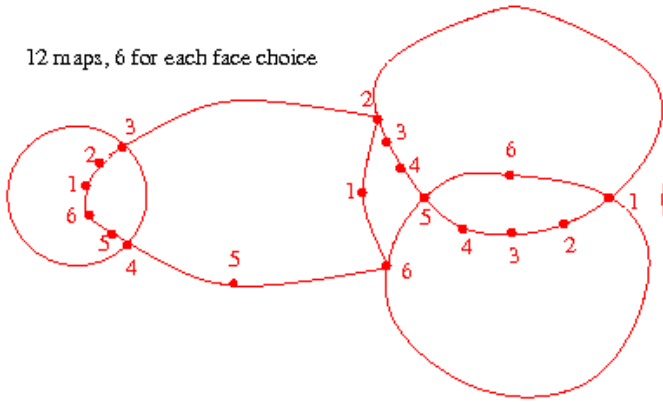
12 maps, 6 for each face choice



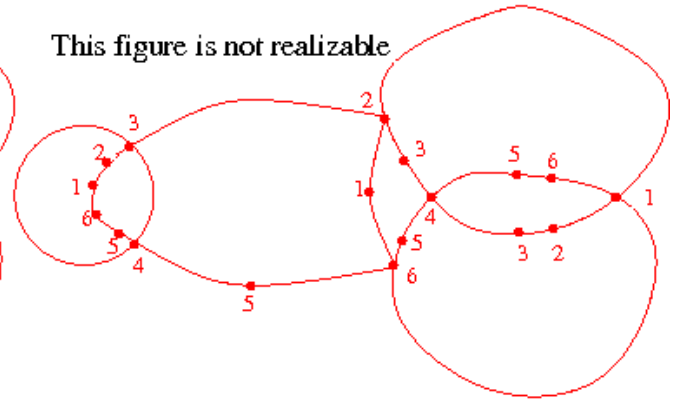
12 maps, 6 for each face choice



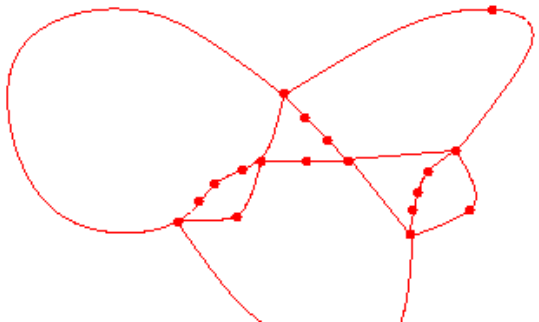
12 maps, 6 for each face choice



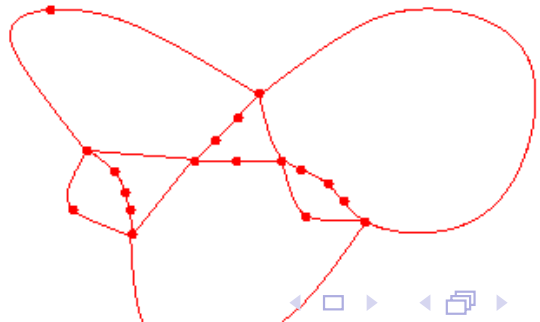
This figure is not realizable



12 maps, 6 for each face choice



12 maps, 6 for each face choice



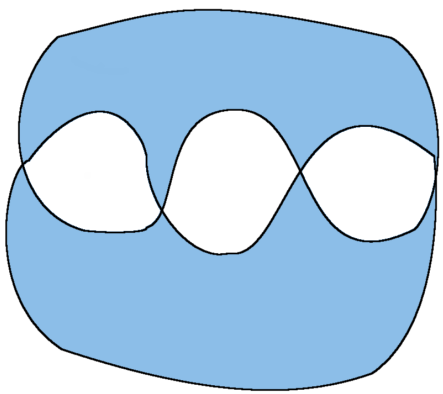
Underlying 4-valent planar graphs

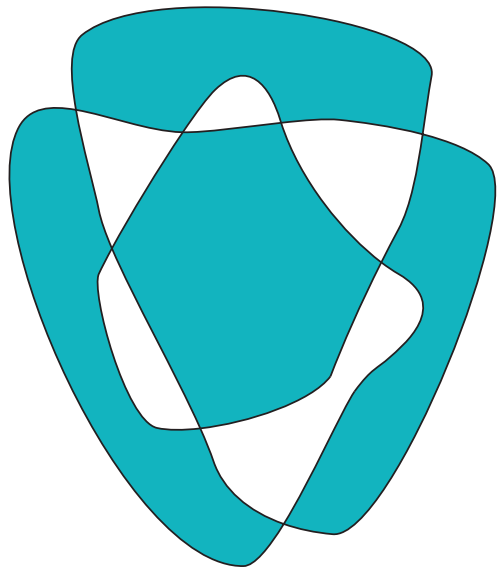
Theorem (W. Thurston, 2010) *A 4-valent connected planar graph* Γ *is homeomorphic to the pullback of a CV polygon of some rational map if and only if*

1. *(global balance) In an alternating coloring of the complementary faces, there are equal numbers of white and blue faces;*
2. *(local balance) For any oriented simple closed curve drawn in the graph that keeps blue faces on the left and white on the right (except at the corners), there are strictly more blue faces than white faces on the left side.*

A 4-valent planar graph with the above conditions will be said **balanced**.

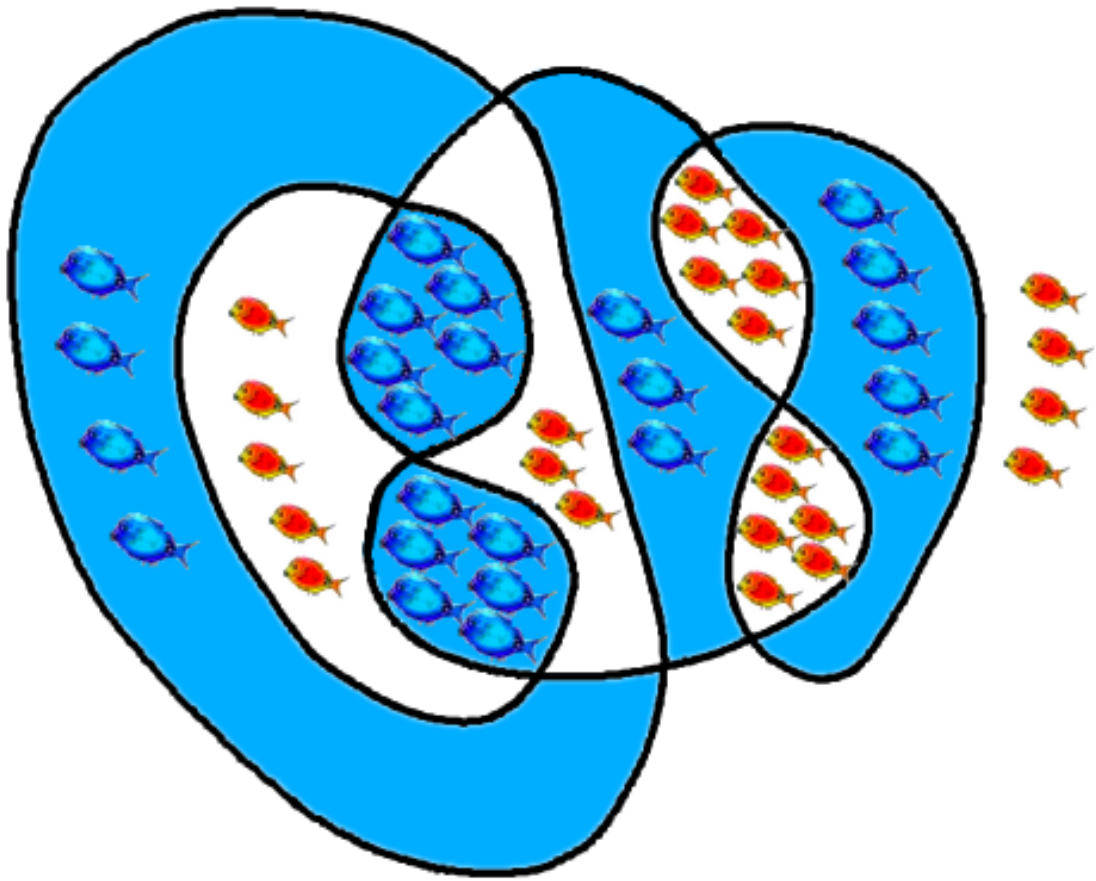
Counter examples, globally imbalanced





4 blue-green regions
6 white regions
Two regions same-color regions
share at most 3 vertices

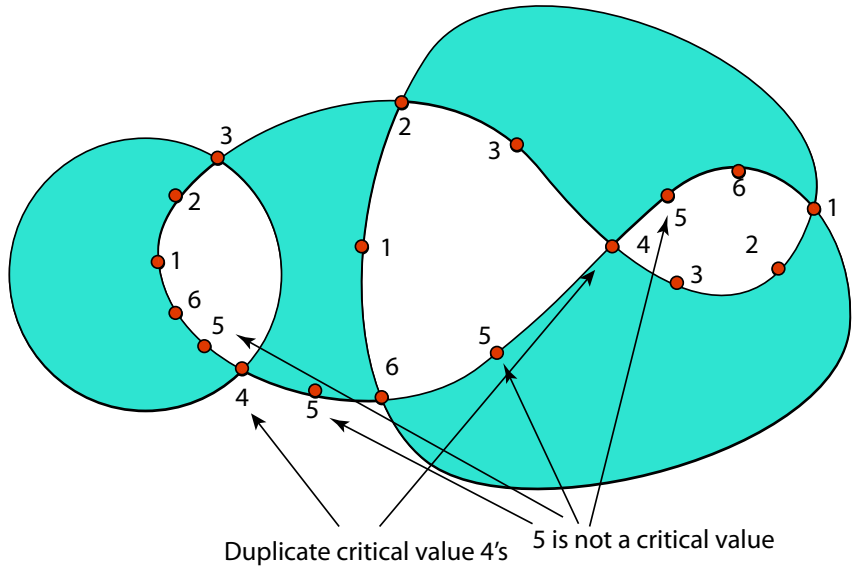
Counter examples, locally imbalanced



In each face of this diagram, the number of shown fish plus the number of corners equal to 8 (the degree). The graph is globally balanced. But the right half of the diagram violates the local balance condition.

Proof.

1. This condition is equivalent to the possibility of distributing the 2-valent dotted vertices, which is a marriage problem or graph flow problem in graph theory.
2. The dots can be consistently labelled, due to a cohomology argument with coefficients in $\mathbb{Z}/(|V| \cdot \mathbb{Z})$.
3. Perturb to get rid of duplicate critical points.



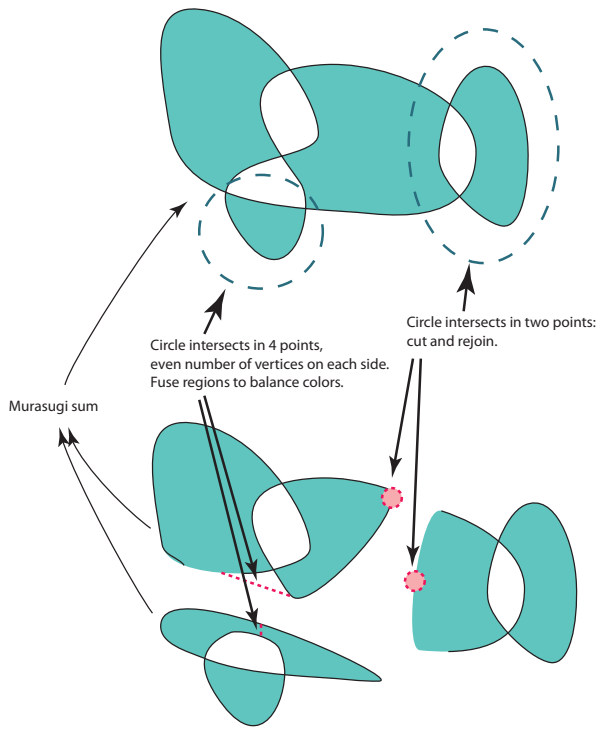
Note how the numbering goes clockwise around each white face and counterclockwise around each blue face. There's a duplicate critical value: the two vertices labeled 4 go to the same point, and none of the vertices labeled 5 are at a crossing point, so 5 is not a critical value.

With such a diagram, one can always perturb it to make the duplicate critical values distinct (in two different ways to turn one of label 4 into 5, or more if there are more duplications), and eliminate all dots with a label that is not on a critical point, to get valid labelings.

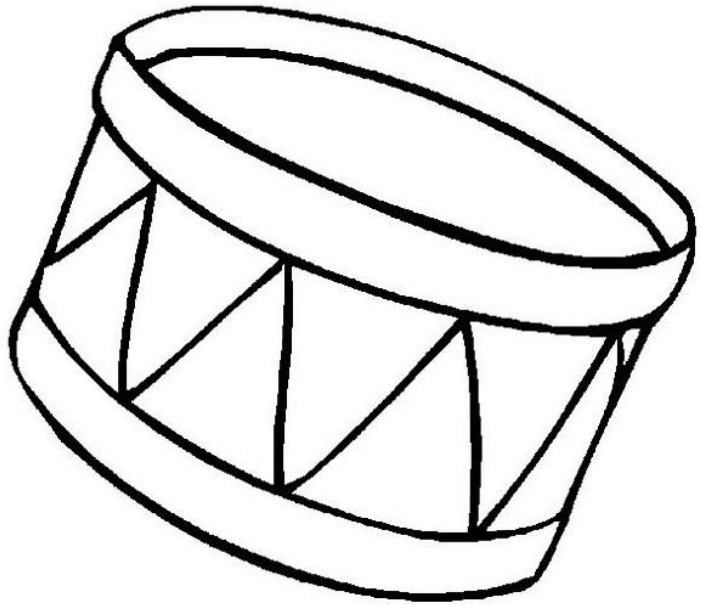
Structures and decompositions

Following ideas of W.Thurston, in parallel to Martin Bridgeman's link projection classification, Tomasini obtains:

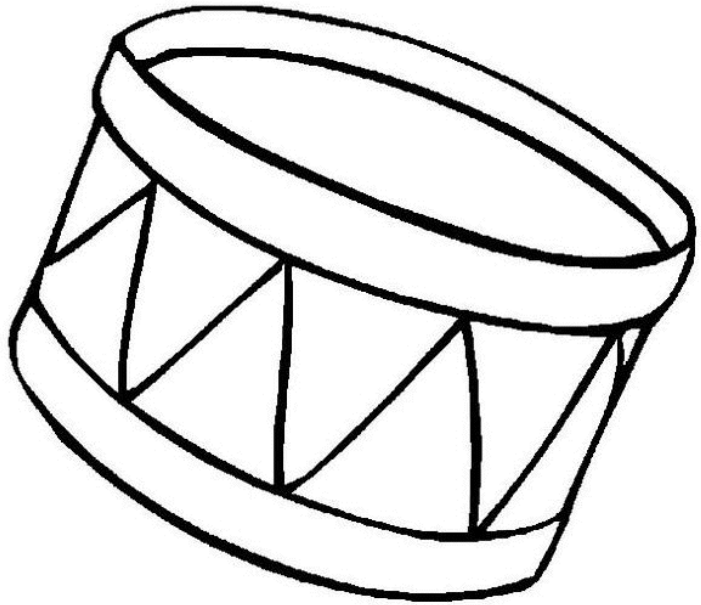
Theorem. *Every balanced graph can be decomposed, after cutting along essential Jordan curves intersecting the graph at 2 or 4 points, simultaneously unpinch in opposite colored components, into Turksheads or union of two circles.*



Turksheads

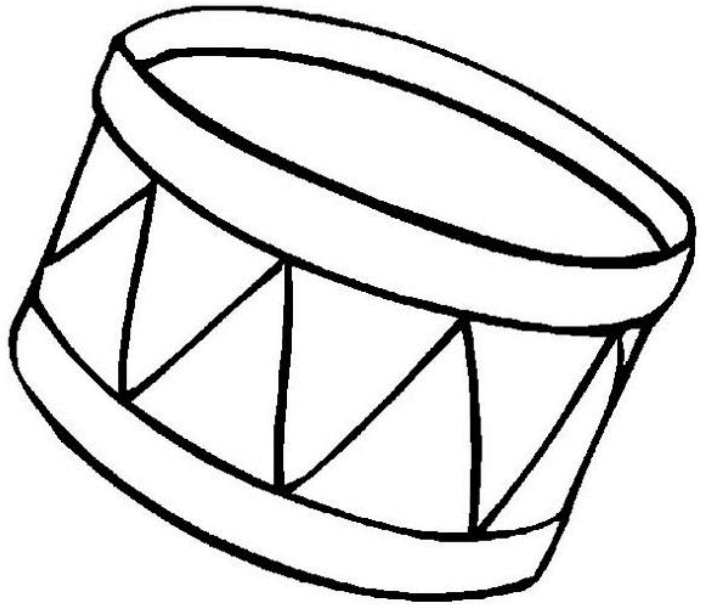


Turksheads



Thank you very much for your attention

Turksheads



Thank you very much for your attention

and your eventual solutions!