1 Branner-Hubbard conjecture

Theorem 1.1. *(Qiu-Yin, Kozlovski-van Strien) Let g be a polynomial with a disconnected its filled Julia set K_g. Assume that every critical component of K_g is aperiodic under the iteration of g. Then K_g is a Cantor set.*

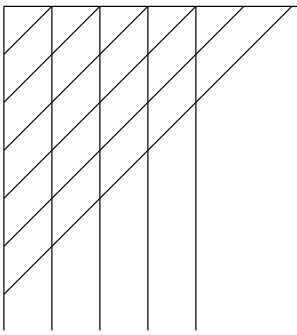
Set up. Let $f : \mathcal{E} \to \mathcal{L}$ be a proper holomorphic map such that $\mathcal{L} = \sqcup_{i\in I} L_i$ is the disjoint union of finitely many hyperbolic discs, $\mathcal{E} \subset\subset \mathcal{L}$, and $\mathcal{E} = \sqcup_{i,j\in I} \sqcup_{k\in\Lambda_{ij}} E_{ijk}$ is the union of finitely many discs with disjoint closures. Critical points are non-escaping and are in distinct L_i's.

$$\begin{array}{ccc} L_i & - & L_j \\ \cup & \nearrow_{f \text{ proper}} & \\ E_{ijk} & & \end{array}$$

$$\text{Set} \quad b = \#Crit(f) \quad \text{and} \quad \delta := \sup_{c\in Crit(f)} \deg_c(f) \ .$$

Theorem 1.2. *Assume that f* is persistenly recurrent *(see below). Then there is $C > 0$, such that for any $c \in Crit(f)$, there is a sequence of disjoint annuli A_n contained essentially in $P_0(c)\setminus\{c\}$ whose moduli $\mu(A_n)$ satisfy $\mu(A_n) \geq C > 0$. Moreover the constant $C = C(b, \delta)$ depends only on b and δ, and $A_n \cap \widehat{\mathcal{P}}_f = \emptyset$.*

Sketch of the proof. Starting from any critical piece K_{-1}, we define inductively a sequence $K_{n-1} \supset\supset K'_n \supset\supset K_n$ following Kozlovski-Shen-Strien and then a family of Kahn-Lyubich type maps with the same constants. We then prove that the moduli of $K'_n \setminus K_n$ are bounded from below, using the Kahn-Lyubich covering lemma. □



$\mathcal{T}(x) \subset \mathbb{R}^- \times \mathbb{R}$ a graph with

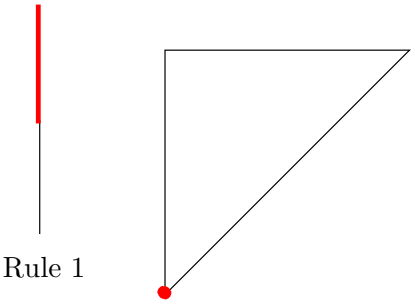
$\{vertices\} = -\mathbb{N} \times \mathbb{N}$, some are colored

3 types of edges: $-, |, \diagup$ (length=1)

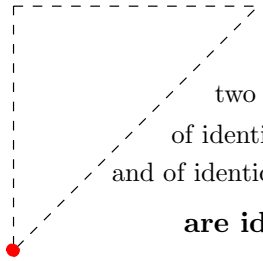
a modulus on each vertical segments (subadditive)

a degree on each diagonal segment starting with a given colored vertex (multiplicative)

degree($\diagup$ edge)=1 if the left vertex is incolored

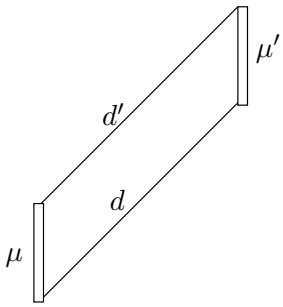


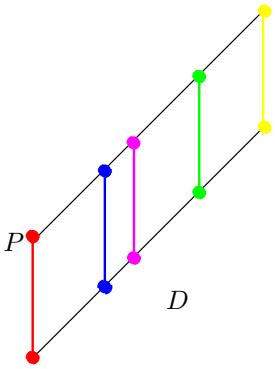
Rule 1



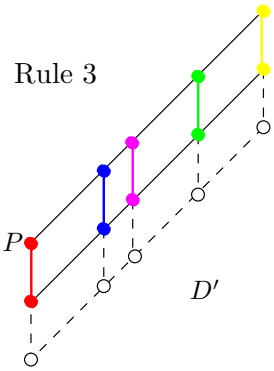
Rule2

two upper triangles of identical size and depth and of identically colored lower vertex **are identical**





Rule 3

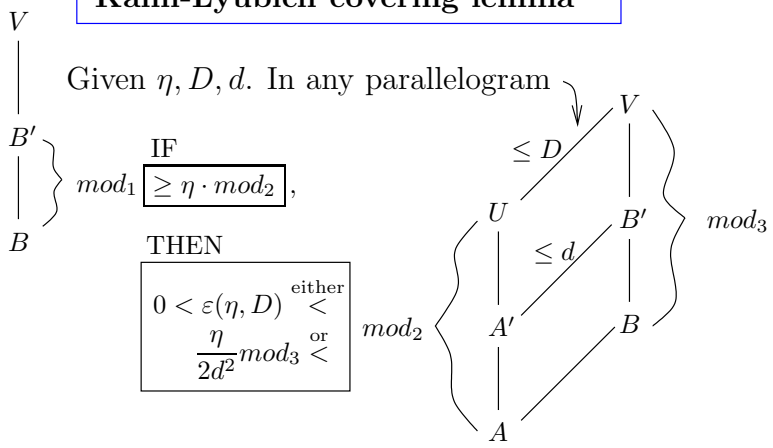


In a parallelogram, $d' \geq d$, $\mu \geq \frac{1}{d'}\mu$.

Some vertical segments are called **vacuous**, i.e $\cap \widehat{\mathcal{P}}_f = \emptyset$

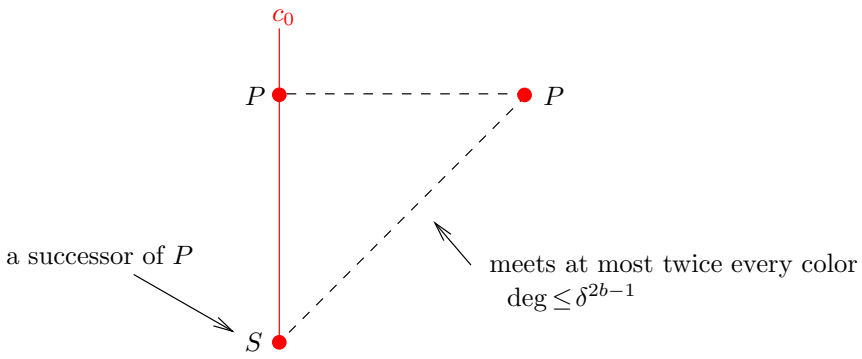
right side vacuous $\Longrightarrow$ left side vacuous, $d' = d$ and $\mu = \frac{1}{d}\mu'$

Kahn-Lyubich covering lemma



Given η, D, d. In any parallelogram

1.1 Successors and persistent recurrence



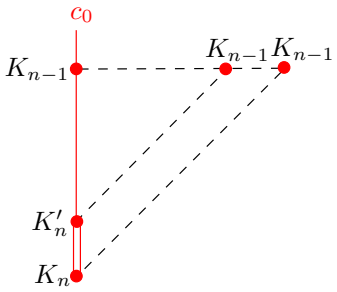
We say that f is **persistently recurrent** if:

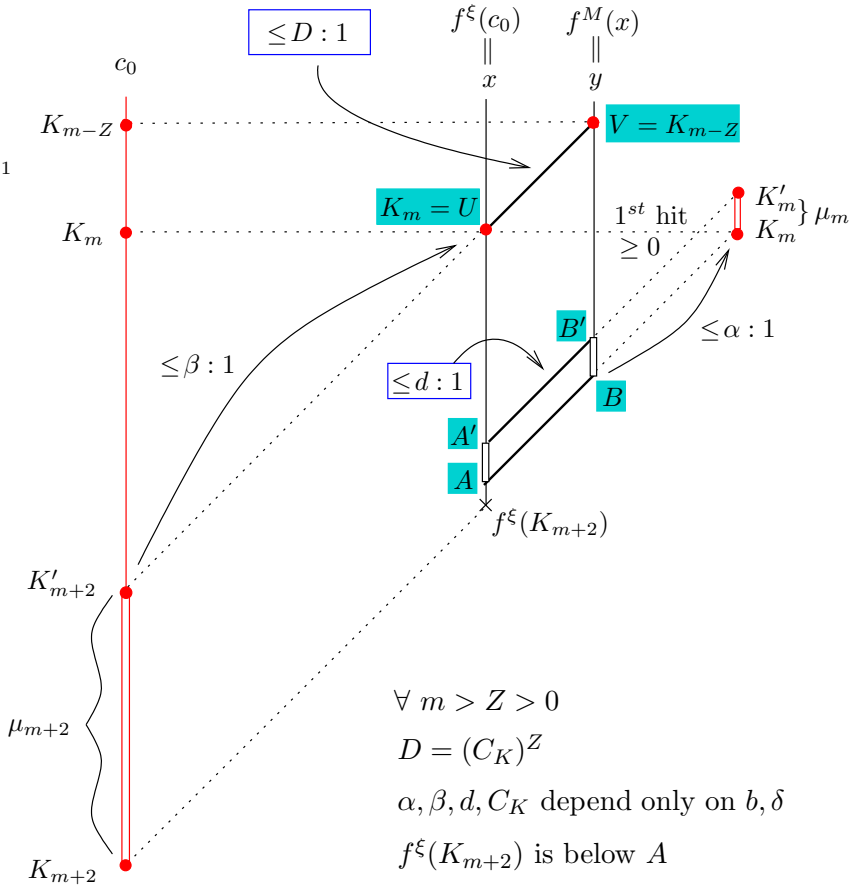
• given any couple $(c, c') \in Crit(f)^2$, marching horizontally to the right of any position in $\mathcal{T}(c)$, one will meet a position that has the color of c'; and $\mathcal{T}(c)$ contains no infinite long critical nests of c' (except of course on the 0th column);

• every critical piece of any color has $< +\infty$ successors.

Proposition A.

$\exists\ (K'_n, K_n)_n$ such that





$\forall\ m > Z > 0$

$D = (C_K)^Z$

α, β, d, C_K depend only on b, δ

$f^\xi(K_{m+2})$ is below A

Proof of Theorem 1.2.

Assume there is a sequence $k_n \to \infty$ such that $\mu_{m'} \overset{m'<k_n}{\geq} \mu_{k_n}$ (otherwise $\mu_n \geq \mu_{n_0} > 0$ for some n_0). Fix any n satisfying $k_n - 2 > Z$. Set $m = k_n - 2$. Then $m+2 = k_n$ and $\mu_{m'} \geq \mu_{m+2}$ for any $m' \leq m+2$. For this m, construct as above $(U, V, M, \xi, x, y, B', B, A', A)$. Then,

$$\mu_{m+2} \geq \frac{mod\begin{pmatrix} U \\ | \\ f^{\xi}(K_{m+2}) \end{pmatrix}}{\deg\left(K'_{m+2} \diagup U\right)} \underset{(d)}{\overset{(e)}{\geq}} \frac{mod\begin{pmatrix} U \\ | \\ A \end{pmatrix}}{\beta}; \tag{1}$$

$$mod\begin{pmatrix} B' \\ | \\ B \end{pmatrix} \overset{\text{no } \widehat{\mathcal{P}}_f}{=} \frac{\mu_m}{\deg\left(B' \diagup K'_m\right)} \underset{(b)}{\overset{\text{choice of } k_n}{\geq}} \frac{\mu_{m+2}}{\alpha} \overset{(1)}{\geq} \frac{mod\begin{pmatrix} U \\ | \\ A \end{pmatrix}}{\alpha\beta} =: \eta \cdot mod\begin{pmatrix} U \\ | \\ A \end{pmatrix}.$$

$$\beta \cdot \mu_{m+2} \overset{(1)}{\geq} mod\begin{pmatrix} U \\ | \\ A \end{pmatrix} \overset{\text{either}}{>} \varepsilon(\eta, D) > 0 \text{ ;} \quad \text{or} \quad > \frac{\eta}{2d^2} mod\begin{pmatrix} V \\ | \\ B \end{pmatrix} \overset{(*)}{\geq} \frac{\eta}{2d^2\alpha}(\mu_m + \cdots + \mu_{m-Z+1}) \overset{\text{choice of } k_n}{\geq} \frac{\eta Z}{2d^2\alpha}\mu_{m+2} \text{ ,}$$

where the inequality (*) is proved by using the pullback of the first ≥ 0 hit of y to K_j for $j = m, m-1, \cdots, m-Z+1$, and the property (K') that $(K'_j \smallsetminus K_j) \cap \widehat{\mathcal{P}}_f = \emptyset$.

Set $Z = 2d^2\alpha^2\beta^2 + 1$. Then $Z > 2d^2\alpha^2\beta^2 = \frac{2d^2\alpha\beta}{\eta}$. Hence the second line above is impossible. So $\mu_{k_n} = \mu_{m+2} > \frac{1}{\beta} \cdot \varepsilon(\eta, D) > 0$. Therefore

$$\forall\, l \in \mathbb{N}, \quad \mu_l \geq \lim_{n\to\infty} \mu_{k_n} \geq \frac{1}{\beta} \cdot \varepsilon(\eta, D) =: C(b, \delta) > 0.$$

□

2 Construction of the KSS nest $(K'_n, K_n)_n$

We will define four pullback operators $\mathcal{Y}, \mathcal{A}, \mathcal{B}, \mathcal{C}$ from a critical nest into itself, and then set inductively $K_n = \mathcal{B} \circ \mathcal{A} \circ \mathcal{Y}(K_{n-1})$ and $K'_n = \mathcal{C} \circ \mathcal{Y}(K_{n-1})$.

2.1 First hit has bounded degree

$$\deg\left({}_{B'}\diagup^{K'_m} \right) = \deg\left({}_{B}\diagup^{K_m} \right) \quad \text{and} \quad \begin{array}{l} \overset{y}{\overline{\phantom{\times \overset{1^{st}\,\text{hit}}{\underset{\geq 0}{--}} K_m \text{ (red)}}}} \\ \times \overset{1^{st}\,\text{hit}}{\underset{\geq 0}{--}} K_m \text{ (red)} \\ \Big| \quad \diagup \longleftarrow \deg \leq \delta^{b-1} = \alpha \\ B \end{array} \tag{2}$$

Proof. $diag_{[B}\diagup^{K_m[}$ does not meet red color, and meets every other color at most once. Otherwise we get two upper triangles of different sizes, both with K_m as the right vertex, and both have a left side of the same coloring. Now moving the smaller triangle to the left and appling Rule 2 would imply an earlier hit of red spot on the horizontal segment containing K_m. □

2.2 Last successor operator Γ and the operator $\mathcal{Y} = \Gamma^{3b}$

For every critical puzzle piece P,

$\Gamma(P)$ = last successor of P,

$\mathcal{Y}(P) := \Gamma^{3b}(P)$ = the $3b$-th generation of last successors.

Lemma 2.1. *(a consequence of tableaux rules) Let P be a critical puzzle piece.*

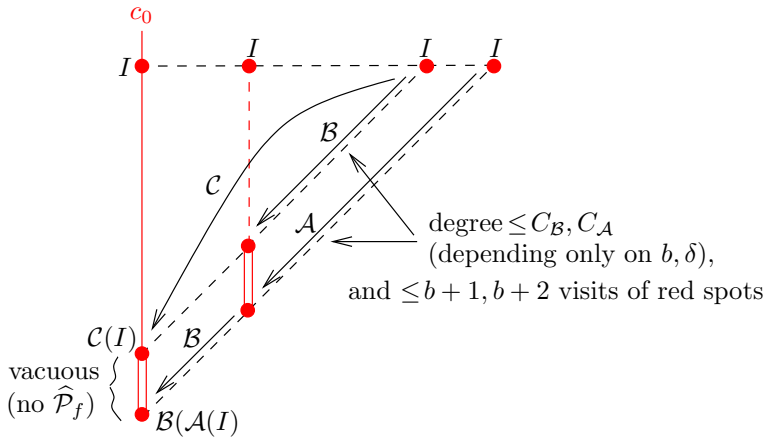
a) $2 \leq \#\{\text{successors of } P\} < +\infty$.

b) $\deg\left({}_{\mathcal{Y}(P)}\diagup^{P}\right) \leq C_{\mathcal{Y}} := \delta^{(2b-1)3b}$.

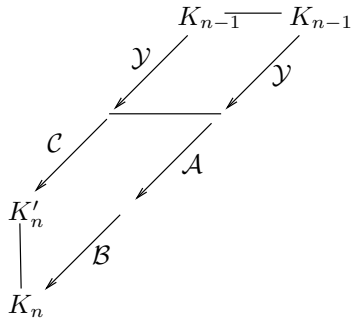
The only difficulty in this Lemma is the part that P has at least two successors.

2.3 The operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$

Lemma 2.2. *For any critical puzzle piece I, there are $\mathcal{A}(I), \mathcal{B}(I), \mathcal{C}(I)$ such that*



2.4 The KSS nest



Fix $c_0 \in Crit(f)$. Let K_{-1} be a critical puzzle piece of c_0. We define inductively

$K_n = \mathcal{B}(\mathcal{A}(\mathcal{Y}(K_{n-1})))\quad K'_n = \mathcal{C}(\mathcal{Y}(K_{n-1}))$.

They satisfy (following Lemmas 2.1, 2.2):

(K) $\begin{cases} \deg\left({}_{K_n}\!\diagup^{K_{n-1}}\right) \le C_K := C_{\mathcal{B}} \cdot C_{\mathcal{A}} \cdot C_{\mathcal{Y}} \ \text{ (depending only on } b, \delta); \\ 3b+2 \le \#(\{\text{red spots}\} \cap \ diag_{[K_n}\!\diagup^{K_{n-1}[}) \le b+b+1+3b = 5b+1. \end{cases}$

(K') $(K'_n \smallsetminus K_n) \cap \widehat{\mathcal{P}}_f = \emptyset$, $\deg\left({}_{K'_n}\!\diagup^{K_{n-1}}\right) \le C_{\mathcal{B}}^2 \cdot C_{\mathcal{Y}}$.

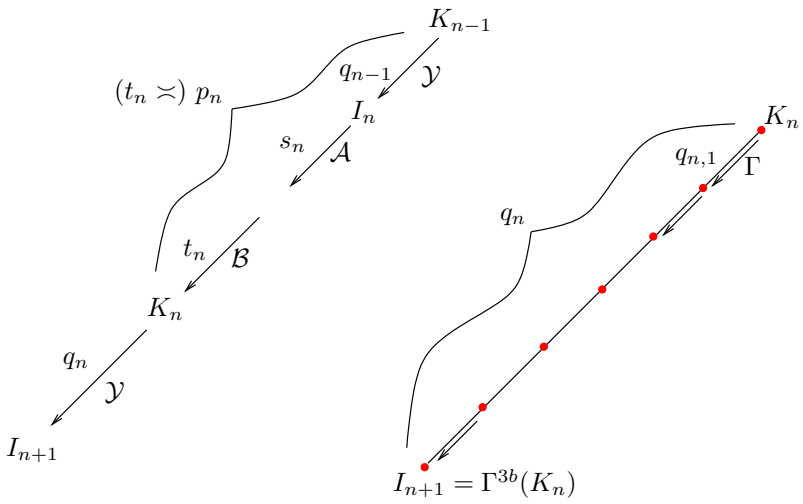
Therefore:

$$\deg\left({}_{K_m}\!\diagup^{K_{m-Z}}\right) \le (C_K)^Z =: D \ ; \tag{3}$$

$$\deg\left({}_{K'_{m+2}}\!\diagup^{K_m}\right) \le \deg\left({}_{K'_{m+2}}\!\diagup^{K_{m+1}}\right) \cdot \deg\left({}_{K_{m+1}}\!\diagup^{K_m}\right) \le C_{\mathcal{B}}^2 \cdot C_{\mathcal{Y}} \cdot C_K =: \beta \ . \tag{4}$$

2.5 Proof of Proposition A

It remains to prove $\deg\left({}_{A'}\diagup^{B'}\right) = \deg\left({}_{A}\diagup^{B}\right) \leq d$ and that $f^{\xi}(K_{m+2})$ is below A.



For J colored piece, denote by $r(J) = \min\{|J - J|,\ \text{among all tableaux}\}$.

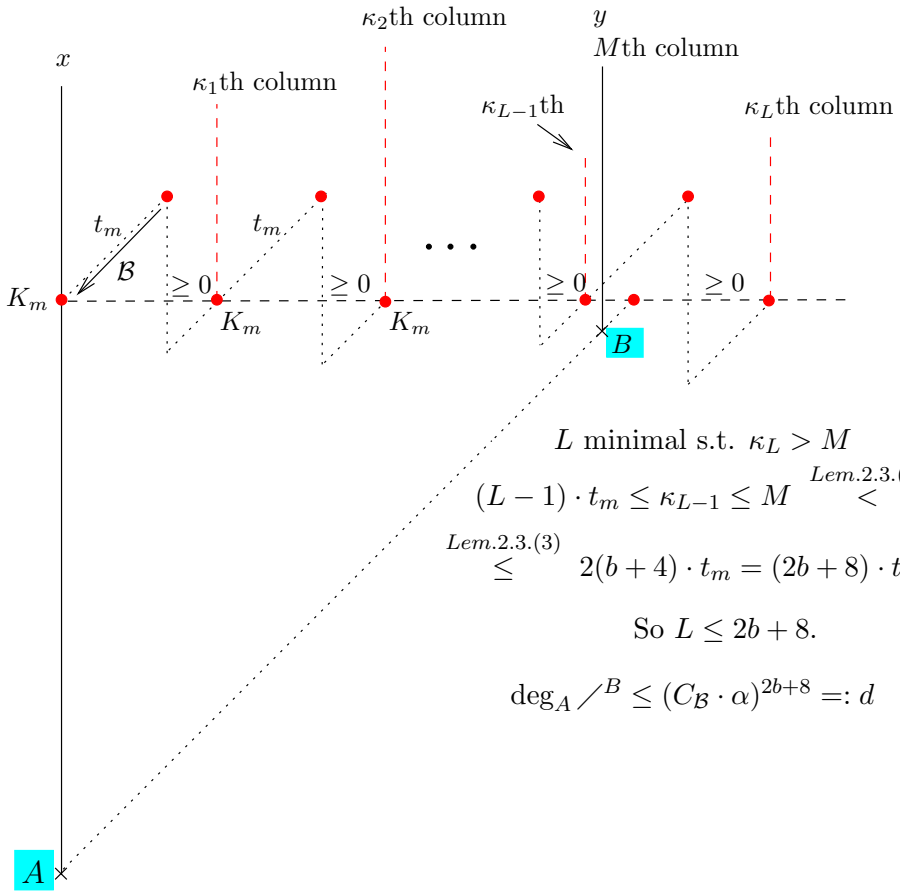
Lemma 2.3. *We have*

(1) $2 \cdot r(K_n) \leq q_{n,1} \leq \frac{1}{2^{3b-1}} r(I_{n+1})$;

(2) *any two consecutive red spots on* $row(K_n)$ *of* $\mathcal{T}(c_0)$ *have a column difference* $\leq q_{n,1}$;

(3) $t_n \leq p_n \leq (b+4)t_n$, *and* $p_n \geq 2p_{n-1}$;

(4) *For any* $m > m^-$, *and for* M *such that* $f^M(K_m) = K_{m^-}$, *we have* $M < 2p_m$.



L minimal s.t. $\kappa_L > M$

$$(L-1)\cdot t_m \leq \kappa_{L-1} \leq M \overset{Lem.2.3.(4)}{<} 2p_m \overset{Lem.2.3.(3)}{\leq} 2(b+4)\cdot t_m = (2b+8)\cdot t_m.$$

So $L \leq 2b+8$.

$$\deg_A \nearrow^B \leq (C_{\mathcal{B}} \cdot \alpha)^{2b+8} =: d$$

Proof of that $f^{\xi}(K_{m+2})$ is below A.

For this we will estimate the number of red spots on the top edge of the two upper triangles T' and T, where T' has a left side $\begin{matrix} K_m \\ | \\ f^{\xi}(K_{m+2}) \end{matrix}$ on $column(y)$ of $\mathcal{T}(c_0)$, and T has a left side $\begin{matrix} K_m \\ | \\ A \end{matrix}$ on the same column. Set $r = |T'|$. Then

$$q_{m,1}\cdot(\#\{\text{reds} \in top(T')\} - 1) \overset{Lem.2.3.(2)}{\geq} r \overset{*}{\geq} r(I_{m+2}) \overset{Lem.2.3.(1)}{\geq} \frac{r(I_{m+2})}{r(I_{m+1})} 2^{3b-1} q_{m,1} \overset{**}{\geq} 2^{6b-1} q_{m,1}. \tag{5}$$

where the $*$ inequality is due to the fact that r is also equal to the column difference of two red spots on $row(I_{m+2})$ (see Figure 2.2), and the $**$ inequality is due to $r(I_{n+1}) \geq 2^{3b} r(K_n) \geq 2^{3b} r(I_n)$ for all $n \geq 0$ (by Lemma 2.3.(1)).

On the other hand,

$$\#\{\text{reds} \in top(T)\} - 1 \underset{def.\ of\ L}{\overset{\text{Property of } \mathcal{B}}{\leq}} L \cdot (b+1) \leq (2b+8)(b+1) \underset{b \geq 1}{\leq} 2^{6b-1}\ . \tag{6}$$

Now (5) and (6) together imply that $f^{\xi}(K_{m+2})$ is below A. □