An alternative proof of Mañé's theorem on non-expanding Julia sets

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Abstract

We give a proof of the following theorem of Mañé: A forward invariant compact set in the Julia set of a rational map is either expanding, contains parabolic points or critical points, or intersects the ω-limit set of a recurrent critical point. Moreover the boundary of a Siegel disc is contained in the ω-limit set of some recurrent critical point. We establish also a semi-local version of the result.

1 Statements

A classical theorem of Fatou says that for a given rational map f, if no critical points are in its Julia set $J(f)$, then f is uniformly expanding on a neighborhood of $J(f)$. Such a map is called *hyperbolic*. A local version of this says that if $x \in J(f)$ is not in the closure of the critical orbits, there is a neighborhood U of x such that the diameters of the components of $f^{-n}U$ shrink to zero as $n \to \infty$.

The next best category of rational maps are those with no parabolic orbits but with each critical point in the Julia set having a finite orbit. They are called *sub-hyperbolic* maps. Such maps expand also uniformly on a neighborhood of the Julia set, but with respect to a Riemannian metric with certain singularities.

If we now allow parabolic orbits, but still require critical points in the Julia set to have a finite orbit, we are into *geometrically finite* maps. These maps have also some sort of expansion on the Julia set, but only on sector neighborhoods of the parabolic points. See for example [TY].

A critical point in the Julia set having a finite orbit is a particular case of being *non-recurrent*, in the following sense: Denote by $\omega(c)$, the ω-limit set of c, to be $\{z \in \overline{\mathbb{C}} \mid \text{there exists } n_k \to \infty \text{ such that } z = \lim f^{n_k}(c)\}$. We say that c is *recurrent* if $c \in \omega(c)$. Mañé has a result that establishes expansion properties (or contraction properties of f^{-1}), at points or on compact invariant subsets of the Julia set away from the parabolic points and the ω-limit sets of recurrent critical points. This is an important result with many applications, for example combined with conditions on the critical orbits (such as Collet-Eckmann conditions), one may obtain metrical or geometrical information about the Julia set of more general type of rational maps.

In this paper we present an alternative proof of Mañé's result, along with the first application by Mañé, which is about the boundary of Siegel discs.

We then apply it to a semi-local setting, following ideas of H. Kriete (see [Kr]). There are many further applications in the literature, see for example [Yin], in this volume, [CJY] and [PR].

Mañé's result can be stated in the form of the following three theorems:

Denote by f a given rational map of degree at least two, by $J(f)$ its Julia set (which may or may not be $\overline{\mathbb{C}}$), and by N an integer depending only on f (to be made precise in §2).

Theorem 1.1. *(point version) If a point $x \in J(f)$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for all $\varepsilon > 0$ there exists a neighborhood U of x such that, for each $n \geq 0$ and each connected component V' of $f^{-n}(U)$,*

(a) the spherical diameter of V' is $\leq \varepsilon$ and $\deg(f^n : V' \to U) \leq N$.

(b) For all $\varepsilon_1 > 0$ there exists $n_0 > 0$ such that if $n \geq n_0$, the spherical diameter of V' is $\leq \varepsilon_1$.

Theorem 1.2. *(compact set version) Let $\Lambda \subset J(f)$ be a compact invariant set (i.e $f(\Lambda) \subset \Lambda$) not containing critical points or parabolic periodic points. If Λ is disjoint from the ω-limit set of every recurrent critical point, then it is expanding (see §3 for the precise definition). In particular a Cremer periodic point must be contained in $\omega(c)$ for some recurrent critical point c.*

Theorem 1.3. *(an application) Let Γ be either the orbit of a Cremer periodic point, the boundary of a Siegel disc, or a connected component of the boundary of a Herman ring. There exists a recurrent critical point c such that $\omega(c) \supset \Gamma$.*

A recent application of these results by Yin ([Yi]) shows that if $J(f)$ contains no recurrent critical points, then it is *shallow* (a notion introduced by C. McMullen), and therefore has Hausdorff dimension < 2 (a result first obtained by Urbanski).

Section 2 presents a proof of Theorem 1.1 due to M. Shishikura. It is edited by Tan Lei based on a hand-written manuscript of M. Shishikura, and is presented in a form that is also valid for the case $J(f) = \overline{\mathbb{C}}$, and for the generalization in Section 4. Appendix E is written by M. Shishikura. The rest is written by Tan Lei. Section 3 contains proofs of Theorems 1.2 and 1.3 following Mañé. Section 4 translates the above three theorems into a semi-local version, and gives a brief description of an interesting application of H. Kriete. The appendices contain some classical estimates on the Poincaré metric and alternative proofs of some of the lemmas.

Our proof of Theorem 1.1 is in spirit the same as Mañé's original one, but differs in presentation. It emphasizes on the use of Poincaré metric, and gives a direct argument rather than by contradiction. It also provides a bit of more quantitative information.

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2 Proof of Theorem 1.1

For $z \in \mathbb{C}$, denote by $D_r(z)$ the open disc centred at z of radius r, and by $\mathbb{D}$ the unit disc. We state first a general result:

Lemma 2.1. *For any $0 < r < 1$ and any positive integer p, there exists a constant $C(p,r) > 0$ such that for any holomorphic proper map $g : V \to \mathbb{D}$ of degree $\leq p$, with V simply connected, each connected component of $g^{-1}(\overline{D_r}(0))$ has diameter $\leq C(p,r)$ with respect to the Poincaré metric on V. Moreover* $\lim_{r\to 0} C(p,r) = 0$.

Proof. (For a more elementary proof due to K. Astala, see Appendix D.) Let A_i be the (finitely many) maximal concentric open annuli in $\mathbb{D}$ surrounding $\overline{D_r}(0)$ such that A_i does not contain critical values of g. For $E \subset V$ a component of $g^{-1}(\overline{D_r}(0))$ and each i, there is $A_i' \subset V$ such that A_i' surrounds E and $g : A_i' \to A_i$ is a covering. So

$$\text{mod } A_i' = \frac{1}{\deg(g|_{A_i'})} \text{ mod } A_i \geq \frac{1}{p} \text{ mod } A_i \quad \text{and}$$

$$\text{mod } (V \setminus E) \geq \sum_i \text{mod } A_i' \geq \frac{1}{p}\sum_i \text{mod } A_i = \frac{\log(1/r)}{2\pi p} .$$

The rest is classical (cf. Appendix B). ■

Next we define a universal constant.

Definition of N_0: There exist $z_1, \cdots, z_{N_0-1} \in \mathbb{D}$ such that $\{\frac{2}{3} \leq |z| \leq 1\} \subset \bigcup_{i=1}^{N_0-1} D_{\frac{1}{3}}(z_i)$.

Let f be the given rational map of degree at least two.

Definition of Ω_0 and J': Set $\Omega_0 = \overline{\mathbb{C}}$ if $J(f) = \overline{\mathbb{C}}$. Otherwise, by Sullivan's classification of the dynamics in the Fatou set we can find a non-empty compact set L as a disjoint union of finitely many closed Jordan domains and closed sub-annuli in the Herman rings and their preimages such that $f(L) \subset L$, L contains all critical points in the Fatou set and $L \cap J(f) = \{\text{parabolic periodic points}\}$. Set $\Omega_0 = \overline{\mathbb{C}} \setminus L$. In both cases we have $f^{-1}(\Omega_0) \subset \Omega_0$ and $\Omega_0 \cap J(f) = J(f) \setminus \{\text{parabolic periodic points}\}$.

In case $J(f) = \overline{\mathbb{C}}$ set $J' = \overline{\mathbb{C}}$. In case $J(f) \neq \overline{\mathbb{C}}$, i.e. $\overline{\Omega_0} \neq \overline{\mathbb{C}}$, we may then assume $\overline{\Omega_0} \subset \mathbb{C}$ (so it is bounded). We set $K' = \{z, f^n(z) \in \overline{\Omega_0} \text{ for all } n \geq 0\}$ and $J' = \partial K'$. Then $f^{-1}(J') \subset J'$ and the Hausdorff distance between J'

and $\partial f^{-n}(\Omega_0)$ tends to 0 as $n \to \infty$, with respect to the Euclidean metric (in fact both K' and J' are totally invariant but we will not need these properties). Notice that in this case $J(f) \subset J'$, moreover L can be chosen so that $J(f) \neq J'$ if and only if some Siegel disc or Herman ring of f contains postcritical points.

In both cases $\Omega_0 \cap J'$ contains no parabolic periodic points. The critical points in Ω_0 are all contained in J' and are of the following three types:
I. c is a recurrent critical point, i.e. $c \in \omega(c)$.
II. c is a non-recurrent critical point with $c \in \omega(c')$ for some recurrent critical point c'.
III. c is a non-recurrent critical point not contained in the ω-limit set of any recurrent critical point.

Definition of c_i, N and C_0: Denote by $c_1, \cdots, c_\nu$ the critical points of type III, $\deg(f, c)$ the local degree of f at c, and by N a positive integer greater than or equal to $\prod_{i=1}^{\nu} \deg(f, c_i)$ (this makes sense even when there are no critical points of type III)[1]. Set $C_0 = N \cdot N_0 \cdot C(N, 2/3)$, where $C(\cdot, \cdot)$ is the constant given by Lemma 2.1.

Denote by $d(z, E)$ the Euclidean distance between a point $z \in \mathbb{C}$ and a closed subset E of $\mathbb{C}$, and by $\operatorname{diam}_W(W')$ the diameter of W' with respect to the Poincaré metric of W (assuming that W is a hyperbolic Riemann surface and $W' \subset W$).

Lemma 2.2. *(1) Let $x \in \Omega_0 \cap J'$. There exists Ω_1 with $x \in \Omega_1$ and satisfying:*

$$(*) \begin{cases} \Omega_1 \subset \Omega_0 \text{ is open and hyperbolic, not necessarily connected,} \\ f^{-1}(\Omega_1) \subset \Omega_1,\ d_{\Omega_1}(f(c_i), f(c_j)) > C_0, \text{ if } f(c_i) \neq f(c_j) \\ d_{\Omega_1}(c_i, f^n(c_i)) > C_0, \text{ for } i = 1, \cdots, \nu \text{ and } n \geq 1\ , \end{cases}$$

where d_{Ω_1} is the Poincaré metric on Ω_1.

(2). Given Ω_1 satisfying $(*)$ above, for all $C > 0$ and $\varepsilon_1 > 0$, there exists n_0 such that if $n \geq n_0$ then for all $W' \subset W \subset f^{-n}(\Omega_1)$ such that W is simply connected, $W' \cap J' \neq \emptyset$ and $\operatorname{diam}_W(W') \leq C$, the set W' has spherical diameter $< \varepsilon_1$.

Proof. (For a different proof relating directly the Poincaré metric to the spherical metric, see Appendix E.)

(1). For each $i = 1, \cdots, \nu$, there exists a repelling periodic point t_i arbitrarily close to c_i (and therefore with $f(t_i)$ close to $f(c_i)$) such that the orbit of t_i does not contain x. We choose these points so that for $Z_0 = \bigcup_i \operatorname{Orbit}(t_i)$, the set $\overline{\mathbb{C}} \smallsetminus Z_0$ is a hyperbolic surface, $d_{\overline{\mathbb{C}} \smallsetminus Z_0}(f(c_i), f(c_j)) > C_0$

[1]Note that N can be chosen depending only on the degree d of f, for example $= d^{2d-2}$.

if $f(c_i) \neq f(c_j)$ and $d_{\overline{\mathbb{C}} \smallsetminus Z_0}(c_i, f^n(c_i)) > C_0$ for $i = 1, \cdots, \nu$ and $n \geq 1$. Now set $\Omega_1 = \Omega_0 \smallsetminus Z_0 \subset \overline{\mathbb{C}} \smallsetminus Z_0$. It satisfies all the conditions.

(2) With the help of a Möbius transformation we may assume $0, \infty \notin \Omega_1$. We will need the following inequality. Let $a \in W' \subset W \subset \Omega \subset \overline{\mathbb{C}} \smallsetminus \{0, \infty\}$ with W a simply connected domain and $\mathrm{diam}_W(W') \leq C$; we have

$$(**) \quad \mathrm{diam}_{\mathrm{spherical}} W' \leq C' \cdot \inf \left\{ d(a, \partial\Omega), \frac{1}{|a|} \right\}$$

with C' a constant depending only on C. See Appendix A for a proof.

Case $J' \neq \overline{\mathbb{C}}$. We may then assume $J' \subset \overline{D_R}(0)$ for some $R > 0$. For W', W as in (2), choose $a \in W' \cap J'$ we get from $(**)$,

$$\mathrm{diam}_{\mathrm{spherical}} W' \leq C' \cdot d(a, \partial f^{-n}(\Omega_1)) \ .$$

Since the Hausdorff distance (with respect to the Euclidean metric) between J' and $\partial f^{-n}(\Omega_1)$ tends to 0 as $n \to \infty$, there is n_0 such that for $n \geq n_0$ and every $a \in J'$ we have $d(a, \partial f^{-n}(\Omega_1)) \leq \varepsilon_1 / C'$.

Case $J' = J(f) = \overline{\mathbb{C}}$. Let $S = C'/\varepsilon_1$. For a pair (W', W) as in (2), if there is $a \in W'$ with $|a| > S$, then by the inequality $(**)$ the spherical diameter of W' is less than ε_1. It remains to consider a pair W', W with $W' \subset \overline{D_S}(0)$. Choose $Z \subset \overline{\mathbb{C}} \smallsetminus \Omega_1$ a periodic orbit (which is surely non-exceptional). Now $J(f) \cap \overline{D_S}(0)$ is covered by finitely many discs $D_r(a_j)$ with $r = \varepsilon_1/(2C')$ and $a_j \in J(f) \cap \overline{D_S}(0)$. By properties of the Julia set, for each j, there is $N(j)$ such that $f^n(D_r(a_j)) \supset Z$ for $n \geq N(j)$. Hence $D_r(a_j) \cap f^{-n}(Z) \neq \emptyset$ for $n \geq N(j)$. Now let $n_0 = \max_j N(j)$. Fix $n \geq n_0$. Then for any $a \in J(f) \cap \overline{D_S}(0)$, there is j such that $a \in D_r(a_j)$. So $d(a, \partial f^{-n}(\Omega_1)) \leq d(a, f^{-n}(Z)) \leq 2r = \varepsilon_1/C'$. ∎

Lemma 2.3. *For any Ω_1 satisfying $(*)$ above, let $U_0 \subset \mathbb{C}$ be a round disc such that $U_0 \subset \Omega_1 \smallsetminus \bigcup_{c \text{ recurrent critical point}} \omega(c)$ and $\mathrm{diam}_{\Omega_1} U_0 \leq C_0$. Then, for every $n \geq 0$,*
deg(n). *for every $D_s(z) \subset U_0$ with $0 < s < d(z, \partial U_0)/2$, and every connected component V' of $f^{-n}(D_s(z))$, V' is simply connected and $\deg(f^n : V' \to D_s(z)) \leq N$;*
diam(n). *for every $D_r(w) \subset U_0$ with $0 < r < d(w, \partial U_0)/2$, and every connected component V of $f^{-n}(D_r(w))$, $\mathrm{diam}_{\Omega_1} V \leq C_0$.*

Proof. We claim at first that $f^{-n}(U_0)$ for any $n \geq 0$ contains no critical points other than $c_1, \cdots, c_\nu$. First note that $f(\omega(c)) = \omega(c)$ for a recurrent critical point c. So for $\Omega_2 = \Omega_1 \smallsetminus \bigcup_{c \text{ recurrent critical point}} \omega(c)$, we have $f^{-1}(\Omega_2) \subset \Omega_2$. In particular $f^{-n}(U_0) \subset \Omega_2$ for all $n \geq 0$. Secondly recall that the critical points in Ω_1 are in three different types, I, II and III, and those of types I and II are contained in $\bigcup_{c \text{ recurrent critical point}} \omega(c)$ (notice that for c a recurrent critical point, $\omega(c)$ coincides with the closure of the orbit $\{f^n(c),\ n > 0\}$). Therefore Ω_2, hence $f^{-n}(U_0)$, contains only critical points of type III, i.e. $c_1, \cdots, c_\nu$.

Now let us prove the assertion by induction on n. For $n = 0$, it is obvious. Suppose it is true up to $n-1$. We will prove **deg(n)** and **diam(n)** in two distinct steps.

Step 1. deg(n) *follows essentially from* **diam(0)**, $\cdots$, **diam(n-1)**.

Let $D_s(z)$ be as in the Lemma. Let V' be a component of $f^{-n}(D_s(z))$. Then $V_j = f^j(V')$ are components of $f^{-(n-j)}(D_s(z))$ $(j = 0, 1, \cdots, n)$. So, by the hypothesis of induction, for $j = 1, \cdots, n$, the set V_j is simply connected and $\operatorname{diam}_{\Omega_1}(V_j) \le C_0$.

Now V' is also a component of $f^{-1}(V_1)$. Since $\operatorname{diam}_{\Omega_1}(V_1) \le C_0$, V_1 contains at most one critical value in $\{f(c_i), i = 1, \cdots, \nu\}$ (by Condition $(*)$), and contains no critical values of other types of critical points. Therefore V' is simply connected and $\deg(f : V_j \to V_{j+1}) = \deg(f, c_i)$ if $c_i \in V_j$ and $= 1$ otherwise.

Fix $i \in \{1, \cdots, \nu\}$. If $c_i \in V_j$ for some maximal j between 1 and n, then $f^m(c_i) \notin V_j$ for $m \ge 1$, by the induction assumption $\operatorname{diam}_{\Omega_1}(V_j) \le C_0$ and Condition $(*)$. Hence $c_i \notin V_l$ for $0 \le l < j$. This means that c_i appears at most once in $V_0 = V'$, V_1, $\cdots$, $V_n = D_s(z)$. Since no critical points other than $c_1, \cdots, c_\nu$ occur in these components, we have

$$\deg(f^n : V' \to D_s(z)) = \prod_{j=0}^{n-1} \deg(f : V_j \to V_{j+1}) \le \prod_i \deg(f, c_i) \le N \ .$$

Thus **deg(n)** is proved for any $D_s(z)$ as in the Lemma.

Step 2. *Given $D_r(w)$ as in the Lemma,* **diam(n)** *for $D_r(w)$ follows from* **deg(n)** *applied to several $D_s(z)$*'s.

Let V be a component of $f^{-n}(D_r(w))$. Apply **deg(n)** to $D_r(w)$ we know that V is simply connected and $\deg(f^n : V \to D_r(w)) \le N$.

Now choose $z_1, \cdots, z_{N_0-1} \in D_r(w)$ such that

$$D_r(w) \setminus D_{\frac{2}{3}r}(w) \subset \bigcup_{i=1}^{N_0-1} D_{\frac{1}{3}r}(z_i) \ .$$

We have, for $i = 1, \cdots, N_0 - 1$,

$$d(z_i, \partial U_0) \ge d(w, \partial U_0) - d(w, z_i) \ge d(w, \partial U_0) - r > 2r - r = r \ .$$

So we can apply **deg(n)** to $D_{\frac{r}{2}}(z_i)$ and V' a component of $f^{-n}(D_{\frac{r}{2}}(z_i))$ to conclude that V' is simply connected and $\deg(f^n : V' \to D_{\frac{r}{2}}(z_i)) \le N$.

Since $\frac{1}{3}r = \frac{2}{3}\cdot\frac{1}{2}r$, by Lemma 2.1 each connected component of $f^{-n}(D_{\frac{1}{3}r}(z_i))$ in V' has diameter $\leq C(N, \frac{2}{3})$ with respect to $d_{V'}$, and hence with respect to d_{Ω_1} since $V' \subset f^{-n}(\Omega_1) \subset \Omega_1$.

Similarly each connected component of $f^{-n}(D_{\frac{2}{3}r}(w))$ has diameter at most $C(N, \frac{2}{3})$ with respect to d_{Ω_1}.

Now $D_r(w)$ is the union of the following N_0 open connected sets: $D_{\frac{2}{3}r}(w)$ and $D_{\frac{1}{3}r}(z_i) \cap D_r(w)$ for $i = 1, \cdots, N_0 - 1$. Since $f^n : V \to D_r(w)$ is a proper map of degree $\leq N$, the preimage by f^{-n} of each of the above N_0 sets has at most N connected components in V, and V is covered by these components. Therefore V is covered by at most $N \cdot N_0$ sets of diameter $\leq C(N, \frac{2}{3})$. Hence $\mathrm{diam}_{\Omega_1} V \leq N \cdot N_0 \cdot C(N, \frac{2}{3}) = C_0$ since V is connected.

This completes the induction. ■

Proof of Theorem 1.1. Let $x \in \Omega_0 \cap J'$ not contained in the ω-limit set of a recurrent critical point. By Lemma 2.2(1), there is Ω_1 satisfying $(*)$ and

$$x \in (\Omega_1 \cap J') \setminus \bigcup_{c \text{ recurrent critical point}} \omega(c) \ .$$

We may assume $0, \infty \notin \Omega_1$. Fix $\varepsilon > 0$. We will prove Theorem 1.1 for all $x' \in (\Omega_1 \cap J') \setminus \bigcup_{c \text{ recurrent critical point}} \omega(c)$ with bounds depending only on Ω_1.

First note that there exists $\delta > 0$, if $V' \subset V \subset \Omega_1$ with V simply connected and $\mathrm{diam}_V(V') \leq \delta$ then the spherical diameter of V' is less than ε (see Appendix A). Take $0 < \rho < 1$ such that $C(N, \rho) \leq \delta$ (where $C(N, \rho)$ is given by Lemma 2.1).

Let $x' \in \Omega_1 \cap J' \setminus \bigcup_{c \text{ recurrent critical point}} \omega(c)$. Let U_0 be a round disc such that $x' \in U_0 \subset \Omega_1 \setminus \bigcup_{c \text{ recurrent critical point}} \omega(c)$ and $\mathrm{diam}_{\Omega_1} U_0 \leq C_0$ (as in Lemma 2.3).

Let $r > 0$ be such that $r < \frac{1}{2} d(x', \partial U_0)$. Define $U = D_{\rho r}(x') \subset D_r(x')$.

Fix $n \geq 0$. Let V be a connected component of $f^{-n}(D_r(x'))$ and V' be a connected component of $f^{-n}(U)$ in V. Then by Lemma 2.3 both V and V' are simply connected and $\deg(f^n : V \to D_r(x')) \leq N$. So $\mathrm{diam}_V V' \leq C(N, \rho) \leq \delta$ (Lemma 2.1). Hence V' has spherical diameter $< \varepsilon$. This proves (a). For (b), let $\varepsilon_1 > 0$ and $C = C(N, \rho)$. Let n_0 be the integer given in Lemma 2.2(2) (which depends only on Ω_1) and assume $n \geq n_0$. For V and V' as above (relative to f^{-n}) we have $V' \subset V \subset f^{-n}(\Omega_1)$, V is simply connected, $V' \cap J' \neq \emptyset$ (since $x' \in J' \cap U$ and $f^{-1}(J') \subset J'$) and $\mathrm{diam}_V(V') \leq C(N, \rho) = C$. So we can apply Lemma 2.2(2) to $W = V$ and $W' = V'$ to conclude that the spherical diameter of V' is less than ε_1.

This ends the proof of Theorem 1.1. ■

3 Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We say that a compact set Λ is *expanding* if there is N such that for any $n \geq N$, $\min_{z\in\Lambda} ||(f^n)'(z)|| > 1$, where $||\cdot||$ is with respect to the spherical metric.

Assume that Λ is a forward invariant compact set not containing critical points and parabolic points. We may assume $0, \infty \notin \Lambda$ and thus work with Euclidean metric rather than spherical metric.

Assume by contradiction that Λ is not expanding. In other words, there are $n_k \to \infty$, $z_k \in \Lambda$, such that $|(f^{n_k})'(z_k)| \leq 1$. We will show that any accumulation point of $\{f^{n_k}(z_k)\}_{k\in\mathbb{N}}$ is in the ω-limit set of some recurrent critical point.

Assume not. We may assume the entire sequence $f^{n_k}(z_k)$ converges to a point $x \in \Lambda$. Then x satisfies the conditions of Theorem 1.1. Take $\varepsilon > 0$ such that $d(\Lambda, c) > 2\varepsilon$ for every critical point c. Let U be a round disc centred at x associated to ε given by Theorem 1.1. For k large, $f^{n_k}(z_k)$ is in U. Let V_k be the component of $f^{-n_k}(U)$ containing z_k. By Theorem 1.1, $\text{diam}(f^j(V_k)) \leq \varepsilon$ for $0 \leq j \leq n_k$. Since $f^j(z_k) \in \Lambda$ for all j, we have $f^j(V_k)$ does not contain critical points of f. Therefore $f^{n_k} : V_k \to U$ is a bijection. Let $\varphi_k : U \to V_k$ be the inverse. Then the family $\{\varphi_k,\ k \in \mathbb{N}\}$ is normal and any limit function φ must be constant, since $\text{diam}V_k \to 0$ as $k \to \infty$ (Theorem 1.1). This contradicts the fact that

$$|\varphi'(x)| = \lim_{k\to\infty} |\varphi_k'(f^{n_k}(z_k))| = \lim_{k\to\infty} \frac{1}{|(f^{n_k})'(z_k)|} \geq 1 \ .$$

∎

Proof of Theorem 1.3. The orbit of a Cremer periodic point is in J', invariant and non-expanding. So by Theorem 1.2 it intersects $\omega(c)$ for some recurrent critical point c. It is therefore contained in $\omega(c)$.

Now let Γ denotes either the boundary of a Siegel disc or a component of the boundary of a Herman ring. We want to show that there is a recurrent critical point c such that $\Gamma \subset \omega(c)$. The proof consists of two steps.

1. $\Gamma \subset \bigcup_{c \text{ recurrent critical point}} \omega(c)$.

Since there are only finitely many (recurrent) critical points, the right hand set is closed. Note also that there are only finitely many critical points and parabolic periodic points.

Assume the assertion is not true. Then there is $x \in \Gamma$ satisfying the conditions of Theorem 1.1. There is therefore a connected open neighborhood V of x such that components of $f^{-nq}(V)$ intersecting Γ have diameter tending to 0. On the other hand, consider the "harmonic measure" μ of Γ, namely that induced by the boundary map of the conformal linearization map $\varphi : A \to B$, with B the Siegel disc with boundary Γ or respectively the Herman ring with one boundary component Γ, and with $A = \mathbb{D}$, or respectively an annulus. It is known that μ is non-atomic with support the whole Γ, f^q preserves μ and is ergodic with respect to it. From the properties of V and $f^{-nq}(V)$ we know that $V \cap \Gamma$ has positive harmonic measure and $f^{-nq}(V) \cap \Gamma$ has harmonic measure tending to 0 as n tends to ∞. This is a contradiction since f^q preserves the harmonic measure.

2. *There is one recurrent critical point c such that* $\Gamma \subset \omega(c)$.

For any recurrent critical point c, we have $f(\Gamma \cap \omega(c)) \subset (\Gamma \cap \omega(c))$. As $f^q : \Gamma \to \Gamma$ is ergodic with respect to μ, the set $\Gamma \cap \omega(c)$ has either 0 or full harmonic measure. But there are only finitely many such critical points. By **1** there is c recurrent such that $\Gamma \cap \omega(c)$ has full harmonic measure. Now since $\Gamma \cap \omega(c)$ is closed and the support of the harmonic measure is the whole set Γ, we must have $\Gamma \cap \omega(c) = \Gamma$. ∎

4 A semi-local version of the above results

Although we stated and proved Theorems 1.1, 1.2 and 1.3 for a rational map as a global dynamical systems of $\overline{\mathbb{C}}$, our proof actually works, with only minor modification, for restrictions of a rational map in a sub-domain of $\overline{\mathbb{C}}$, considered as a semi-local dynamical system, together with semi-local recurrent critical points and semi-local invariant compact sets. We may then conclude that only semi-local recurrent critical points are relevant to semi-local invariant subsets.

More precisely, let $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. Let $Q' \subset Q$ be two open subsets of $\overline{\mathbb{C}}$ with $Q' \neq Q$ and $\overline{Q} \neq \overline{\mathbb{C}}$, such that $F|_{Q'} : Q' \to Q$ is a holomorphic proper map. Consider $f = F|_{\overline{Q'}} : \overline{Q'} \to \overline{Q}$ as a dynamical system that is only defined on $\overline{Q'}$. (This is a bit like polynomial-like mappings, although it is not required that Q' is relatively compact in Q, and no quasi-conformal technique is needed).

Now critical points of F in Q' are also critical points of f. We say that a critical point c *escapes* if there is $k \geq 0$ such that $c, f(c), \cdots, f^k(c) \in Q$ but $f^k(c) \notin Q'$. We call $\{c, f(c), \cdots, f^k(c)\}$ the f-orbit of c. Although such critical points may very well be recurrent under F, they will not play relevant roles to the f-invariant sets in Q'.

We modify the definition of Ω_0 as follows: Define L for the rational map F as before. Let L' be the union of L and the f-orbits of escaping critical points. Set $\Omega_0 = Q \setminus L'$. Since $f(L \cap Q') \subset F(L) \subset L$, we have $f(L' \cap Q') \subset L'$. Therefore $f^{-1}(\Omega_0) \subset \Omega_0$.

Now we can define $K' = \{z \in \overline{Q'}, f^n(z) \in \overline{f^{-1}(\Omega_0)} \text{ for all } n \geq 0\}$ and $J' = \partial K'$. Then again

(A) $f^{-1}(J') \subset J'$ and

(B) the Hausdorff distance between J' and $\partial f^{-n}(\Omega_0)$ tends to 0, as $n \to \infty$, with respect to the Euclidean metric.

We can now restate Theorems 1.1, 1.2 and 1.3 for f provided we make the following changes: In the statements of the theorems replace "$x \in J(f)$" by "$x \in \Omega_0 \cap J'$", "recurrent critical point" by "f-recurrent critical point", "$\Lambda \subset J(f)$" by "$\Lambda \subset \Omega_0 \cap J'$", and finally "$\Gamma$" by "$\Gamma \subset \Omega_0 \cap J'$ with $f^n(\Gamma) \subset \Omega_0$ for all n".

For the proof of these theorems, we make the following changes: In the proof of Lemma 2.2(1) choose the points t_i whose f-orbits escape, and set Z_0 to be the union of these f-orbits. Since there are such kind of points arbitrarily close to each c_i, Condition $(*)$ is easily achieved. In the proof of Lemma 2.2(2), the study of the case $J' = \overline{\mathbb{C}}$ can be deleted. Notice that the only properties of J' needed is (B) (in the proof of Lemma 2.2(2)) and (A) (at the very end of §2). This completes the proof of Theorems 1.1, 1.2 and 1.3 in this generalized setting.

We now describe briefly an application by H. Kriete (see [Kr]), of this generalized version of Mañé's result: Let F be a polynomial. Then Goldberg-Milnor's Fixed Point Portrait (cf. [GM]) provides regions in $\mathbb{C}$ bounded by periodic external rays that separate Cremer points and Siegel discs. Apply Theorem 1.3 to F restricted to each of these regions we obtain a critical point which is recurrent within the region and whose orbit accumulates to the Cremer point or Siegel boundary of the region.

A The inequality

Recall that $d(a, E)$ denotes the Euclidean distance between a point $a \in \mathbb{C}$ and a closed subset E of $\mathbb{C}$.

Lemma A.1. *Let $W \neq \mathbb{C}$ be an open and simply connected proper subset of $\mathbb{C}$. Let $a, b \in W$. Then*

$$|b - a| \leq (e^{2d_W(a,b)} - 1) \cdot d(a, \partial W) .$$

Proof. Let $F : \mathbb{D} \to W$ be a conformal map with $F(0) = a$. Set $r = |F^{-1}(b)|$. Then

$$d_W(a, b) = d_{\mathbb{D}}(0, F^{-1}(b)) = \log \frac{1 + r}{1 - r} .$$

Denote by $h(z) = (F(z) - a)/F'(0)$ the normalized univalent map. By Koebe 1/4 and distortion theorems we have

$$\frac{d(a, \partial W)}{|F'(0)|} = d(0, \partial h(W)) \geq \frac{1}{4} \quad \text{and} \quad \frac{|b - a|}{|F'(0)|} = |h(F^{-1}(b))| \leq \frac{r}{(1 - r)^2} .$$

Combining these we get

$$|b-a| \leq \frac{r}{(1-r)^2}|F'(0)| \leq \frac{4r}{(1-r)^2}d(a,\partial W) = \left(\frac{(1+r)^2}{(1-r)^2} - 1\right) d(a,\partial W) \ .$$

The right hand side is equal to $(e^{2d_W(a,b)} - 1) \cdot d(a, \partial W)$. ■

Lemma A.2. *Let $a \in W' \subset W \subset \Omega \subset \mathbb{C} \setminus \{0, \infty\}$ with W open and simply connected. Assume* $\operatorname{diam}_W(W') \leq C$. *Then*

$$\operatorname{diam}_{\text{spherical}} W' \leq C' \inf \left\{ d(a, \partial\Omega), \frac{1}{|a|} \right\}$$

where $C' = 2C''(e^{2C} - 1)$ with C'' a universal constant.

Proof. Let C'' be the Lipschitz constant between the spherical metric and the Euclidean metric. We have

$$\begin{aligned} \operatorname{diam}_{\text{spherical}} W' &\leq C'' \operatorname{diam}_{\text{Euclidean}} W' \leq 2C'' \sup_{b \in W'} |b-a| \\ &\leq 2C'' \sup_{b \in W'} (e^{2d_W(a,b)} - 1) d(a, \partial W) \\ &\leq 2C''(e^{2C} - 1) d(a, \partial W) = C' d(a, \partial\Omega) \ . \end{aligned}$$

For a suitable choice of the spherical metric, the map $H(z) = 1/z$ is an isometry. Apply the above inequality to $H(a) \subset H(W') \subset H(W) \subset H(\Omega)$ we get

$$\operatorname{diam}_{\text{spherical}} W' = \operatorname{diam}_{\text{spherical}} H(W') \leq C' d(H(a), \partial H(\Omega)) \leq \frac{C'}{|a|} \ ,$$

where the last inequality is due to the facts that $H(a) = \frac{1}{a}$ and $0 \notin H(\Omega)$. ■

From this one deduces easily that for any $\varepsilon > 0$, there is δ such that if $C \leq \delta$ then $\operatorname{diam}_{\text{spherical}} W' \leq \varepsilon$. This fact was needed at the end of §2 for $\Omega = \Omega_1$. ■

B Control between modulus and diameter

Lemma B.1. *For $\delta \in (0, \infty)$, there are two strictly decreasing continuous functions $u(\delta)$ and $v(\delta)$ with*

$$\lim_{\delta \to 0^+} u(\delta) = \lim_{\delta \to 0^+} v(\delta) = +\infty; \ \lim_{\delta \to +\infty} u(\delta) = \lim_{\delta \to +\infty} v(\delta) = 0 \ ,$$

satisfying the following property: Given any pair (V, E) with $V \subset \overline{\mathbb{C}}$ a hyperbolic open simply connected subset (i.e. $\#(\overline{\mathbb{C}} \setminus V) > 2$) and $E \subset V$ compact so that $V \setminus E$ is an open annulus, then for m the modulus of $V \setminus E$ and δ the diameter of E with respect to the Poincaré metric on V, we have

$$u(\delta) \leq m \leq v(\delta) \ .$$

Proof. We may assume $V = \mathbb{D}$, and E contains 0 and s with $s \in (0,1)$ and $d_{\mathbb{D}}(0,s) = \delta$. We have

$$\log \frac{1+s}{1-s} = \delta \text{ therefore } s = s(\delta) = \frac{e^\delta - 1}{e^\delta + 1} \ .$$

Note that $E \subset \overline{D}_s(0)$ so $A \supset \mathbb{D} \smallsetminus \overline{D}_s(0)$. As a consequence

$$m \geq \text{ mod } (\mathbb{D} \smallsetminus \overline{D}_s(0)) = \frac{1}{2\pi} \log \frac{1}{s} = \frac{1}{2\pi} \log \frac{e^\delta + 1}{e^\delta - 1} =: u(\delta) \ .$$

To get $v(\delta)$, we use the solution of Grötzsch extremal problem (cf. [Al1], Chapter III) that $m \leq \text{mod } (\mathbb{D} \smallsetminus [0, s(\delta)]) =: v(\delta)$. ■

In the case of Lemma 2.1, we have $v(\delta) \geq m \geq (\log(1/r))/(2\pi p)$ so

$$\delta \leq v^{-1}(\frac{\log(1/r)}{2\pi p}) =: C(p,r) \ .$$

The function $C(p,r)$ as a function of r is strictly increasing and continuous, with $\lim_{r \to 0^+} C(p,r) = 0$ and $\lim_{r \to 1^-} C(p,r) = +\infty$.

C Bounded distortion

We use Lemma 2.1 to recover a distortion estimate in [CJY] (although we don't need it in our paper). In the same setting as Lemma 2.1, denote by $B_V(z,s)$ the Poincaré disc in V centred at z with Poincaré radius s. Define $B_{\mathbb{D}}(w,t)$ similarly. Then there is a constant C_1 depending only on s and p such that, for any $z \in V$,

$$B_{\mathbb{D}}(g(z), C_1) \subset g(B_V(z,s)) \subset B_{\mathbb{D}}(g(z), s) \ .$$

Proof. The right hand inclusion is due to Schwarz Lemma. As for the left hand side, we may assume $g(z) = 0$. Let $B_{\mathbb{D}}(0,t)$ be the largest disc contained in $g(B_V(z,s))$. Let E be the connected component of $g^{-1}(B_V(0,t))$ containing z. Obviously $\overline{E} \cap \partial B_V(z,s) \neq \emptyset$. Therefore the Poincaré diameter δ of E satisfies $\delta \geq s$. By Lemma 2.1 and Lemma B.1, $s \leq \delta \leq C(p, r(t)) =: C_p(t)$, where

$$r(t) = \frac{e^t - 1}{e^t + 1} \ .$$

Clearly $C_p(t)$ is a continuous strictly increasing function of t. As a consequence $t \geq (C_p)^{-1}(s) =: C_1(p,s)$.

D An elementary proof of Lemma 2.1

This proof is due to K. Astala, who kindly allowed us to include it here.

In the setting of Lemma 2.1 we may assume $V = \mathbb{D}$. Therefore $g : \mathbb{D} \to \mathbb{D}$ is a Blaschke product of degree at most p, more precisely

$$g(z) = \prod_{i=1}^{l} \frac{z - a_i}{1 - \bar{a}_i z}$$

with $a_i \in \mathbb{D}$ and $l \leq p$.

Let $z' \in \mathbb{D}$ such that $|g(z')| \leq r$. Then there is at least one i such that

$$\left| \frac{z' - a_i}{1 - \bar{a}_i z'} \right| \leq r^{1/p} \ .$$

As $p_i(z) := \frac{z-a_i}{1-\bar{a}_i z}$ is an isometry for the Poincaré metric, we have $d_{\mathbb{D}}(z', a_i) = d_{\mathbb{D}}(p_i(z'), 0) \leq \log \frac{1+r^{1/p}}{1-r^{1/p}} =: M$. As a consequence $z' \in \bigcup_{i=1}^{l} B_{\mathbb{D}}(a_i, M)$.

Therefore $g^{-1}(\overline{D}_r) \subset \bigcup_{i=1}^{l} B_{\mathbb{D}}(a_i, M)$. Now any connected component of $g^{-1}(\overline{D}_r)$ would have diameter with respect to $d_{\mathbb{D}}$ at most $p \times 2M =: C'(p, r)$.

From the definition of M one can see clearly that $C'(p, r) \to 0$ and $r \to 0$.

E An alternative proof of Lemma 2.2

The Poincaré metric $\lambda(z)|dz|$ of $\mathbb{C} \smallsetminus \{0, 1\}$ has an estimate

$$\log \lambda(z) = \log \frac{1}{|z| \log \frac{1}{|z|}} + O(1) \text{ as } z \to 0 \ .$$

See [Al2, p.18, (1-24)] (or also [McM, p.13, Theorem 2.3]). An easy calculation shows that for any $C > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in \mathbb{C}$ and $0 < |y| < \delta$ then

$$\{z |\ d_{\mathbb{C} \smallsetminus \{0,1\}}(z, y) \leq C\} \subset D_\epsilon(0) \ .$$

Using affine transformations sending 0, 1, ∞ to 1, w, ∞, one can show that for any $0 < r < R < \infty$, $C > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $r < |w| < R$, $d_{\text{spherical}}(w, y) < \delta$ and $y \in \mathbb{C} \smallsetminus \{0, 1, w\}$, then

$$\text{diam}_{\text{spherical}}\{z |\ d_{\mathbb{C} \smallsetminus \{0,w\}}(z, y) \leq C\} < \epsilon.$$

The same argument can be used with $\{0, \infty\}$ replaced by $\{0, 1\}$ or $\{1, \infty\}$. Therefore we conclude that for any $C > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $d_{\text{spherical}}(w, y) < \delta$ and $y \in \mathbb{C} \smallsetminus \{0, 1, w\}$ then

$$\text{diam}_{\text{spherical}}\{z |\ d_{\mathbb{C} \smallsetminus \{0,1,w\}}(z, y) \leq C\} < \epsilon \ .$$

Now let us define Ω_1. Let t be a repelling periodic point which is not in the orbits of x and c_i's. Let $Z_0 = \cup_{n=0}^{M} f^{-n}(t)$, where M is to be determined later and $\Omega_1 = \Omega_0 - Z_0$. By the choice of t, we know that x and the orbits of c_i's belong to Ω_1. We may suppose that $0, 1, \infty \in Z_0$.

To prove (1), set $C = C_0$ and

$$\epsilon = \min\{d_{\text{spherical}}(f(c_i), f(c_j)) | f(c_i) \neq f(c_j)\} \cup$$
$$\{d_{\text{spherical}}(c_i, f^n(c_j)) | i = 1, \ldots, \nu,\ n \geq 1\}\ ,$$

and we obtain δ satisfying the above. Since the inverse orbit of t is dense in the Julia set, we can choose large M so that for any $z \in J(f)$ there exists an element $w \in Z_0 = \cup_{n=0}^{M} f^{-n}(t)$ with $d_{\text{spherical}}(z, w) < \delta$. In particular, if $y = f(c_i)$, $w \in Z_0$ and $d_{\text{spherical}}(w, y) < \delta$ then

$$\text{diam}_{\text{spherical}}\{z |\ d_{\Omega_1}(z, y) \leq C_0\} \leq \text{diam}_{\text{spherical}}\{z |\ d_{\mathbb{C}\smallsetminus\{0,1,w\}}(z, y) \leq C_0\} < \epsilon\ .$$

Therefore $d_{\text{spherical}}(f(c_i), f(c_j)) > C_0$ if $f(c_i) \neq f(c_j)$. The same argument applies to $d_{\text{spherical}}(c_i, f^n(c_j))$.

To prove (2), set $C = C$ and $\epsilon = \epsilon_1$ and we obtain δ satisfying the above property. Choose n_0 so that every point of the Julia set is within spherical distance δ from $f^{-n_0}(Z_0)$. Then for any $y \in J' \cap \Omega_0 \subset J(f)$, there exists $w \in Z_0$ such that $d_{\text{spherical}}(w, y) < \delta$ and hence

$$\text{diam}_{\text{spherical}}\{z |\ d_{f^{-n}(\Omega_1)}(z, y) \leq C\}$$
$$\leq \text{diam}_{\text{spherical}}\{z |\ d_{\mathbb{C}\smallsetminus\{0,1,w\}}(z, y) \leq C\} < \epsilon_1\ .$$

It follows that if $n \geq n_0$, $W' \subset f^{-n}(\Omega_1)$ and $\text{diam}_{f^{-n}(\Omega_1)}(W') \leq C$ then W' has a spherical diameter less than ϵ_1. ■

Remark. [McM, p.39, Theorem 3.6] uses a similar argument.

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