Introduction

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Complex dynamical systems is a fascinating field. It has both the visual appeal of endlessly varying and beautiful fractals, and the intellectual appeal of sophisticated mathematical developments. Its theoretical challenges have attracted mathematicians from around the world.

A central object of study in complex dynamics is the *Mandelbrot set* M, a fractal shape that classifies the dynamics of the quadratic polynomials $f_c(z) = z^2 + c$. The Mandelbrot has a remarkably simple definition:

$$M = \{c \in \mathbb{C} \ : \ f_c^n(z) \text{ remains bounded as } n \to \infty.\}.$$

Nevertheless M exhibits a rich geometric and combinatorial structure, with many intriguing details and many remaining mysteries. Although it is defined in terms of quadratic polynomials, the Mandelbrot set reappears in virtually every other family of rational maps, as can be observed in computer experiments.

The mathematical theory of the Mandelbrot set and related objects has undergone rapid development during the last two decades, sparked by the pioneering work of Douady and Hubbard. This volume provides a coherent perspective on the present state of the field.

Its articles range from the systematic exposition of current knowledge about the Mandelbrot set, to the latest research in complex dynamics. In addition to presenting new work, this collection documents for the first time important results hitherto unpublished or difficult to find in the literature.

1 A detailed description of the contents of this volume

The Preface, by J. Hubbard, gives an intriguing first-hand account of developments in the field during the period 1976-1982. It recounts the discovery of the Mandelbrot set and the introduction of basic tools, such as the Riemann mapping to the exterior of M, Hubbard trees, quasiconformal mappings, polynomial-like mappings, holomorphic motions, Thurston's theory, etc.

Part one: Universality of the Mandelbrot set

With the advent of computers, it has been easy to experimentally observe copies of Mandelbrot set in many families of complex analytic dynamical systems, but it was a mathematical challenge to explain this universality.

The first results in this direction were obtained by Douady-Hubbard using polynomial-like mappings. This volume contains three articles about the subject:

[McMullen] shows that small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps. As a consequence, one obtains general estimates for the dimension of the bifurcation locus, using Shishikura's result that the Hausdorff dimension of the boundary of M is 2.

[Douady-Buff-Devaney-Sentenac] studies the birth of infinitely many baby Mandelbrot sets in parameter space near a rational map with a parabolic point. This paper shows each baby Mandelbrot set sits in the heart of a nest of imploded cauliflowers.

[Haïssinsky] explains why small copies of the Mandelbrot set appear within M itself.

Part two: Quadratic Julia sets and the Mandelbrot set

One of the central open questions concerning the Mandelbrot set is the following: is M a locally connected set? A positive answer would imply that there is a complete topological description of M, and in particular that hyperbolic dynamical systems are dense in the quadratic family. Similarly, one can completely answer questions about the topology of $J(f_c)$ when the Julia set is known to be locally connected.

A major breakthrough, due to Yoccoz, states that $J(f_c)$ is locally connected and M is locally connected at c, for certain special values of $c \in M$. These are the values where f_c has no indifferent points and is not renormalizable. Yoccoz's original proof is not published, but this volume includes two articles, [Milnor] and [Roesch], treating its central points:

[Milnor] shows that the Julia set of f_c is locally connected, using Yoccoz puzzles;

[Roesch] continues from [Milnor] and shows that M is locally connected at such parameters c. The proof given is a new argument, due to Shishikura, using holomorphic motions.

Turn to the case where f_c does have an indifferent point, [Tan] proves that M is locally connected when f_c has a parabolic point of multiplier 1. This paper also surveys more general results on the local connectivity of M when c has an indifferent point.

The landing behavior of external rays is often a starting point in the study of local connectivity. In this volume, [Petersen-Ryd] gives a new and elementary proof that external rays to M with rational angle do land, and describes the dynamics of f_c for the endpoint c.

Dynamical systems on the interval or the circle are closely related to dynamics in one complex variable. This volume includes two papers on real dynamics:

[Luzzatto] presents an annoted account of Jakobson's theorem, stating that the set of $c \in \mathbb{R}$ such that $f_c(z)$ has chaotic dynamics has positive Lebesgue measure. More precisely, $c = -2$ is a point of Lebesgue density of the set of $c \in \mathbb{R}$ such that f_c has an absolutely continuous invariant measure;

[Petersen] presents the Herman-Swiatek's theorem, stating that an analytic circle homeomorphism with a critical point and irrational rotation number θ is quasi-symmetrically conjugate to a rigid rotation if and only if θ is of bounded type. Using this theorem, [Petersen] also establishes the new result that there exist quadratic Siegel polynomials with a rotation numbers of *unbounded type* whose Julia sets are locally connected.

The delicate theory of bifurcations of indifferent period points is studied in two related papers:

[Jellouli1] shows that small perturbations of quadratic Siegel polynomials with Diophantine rotation number are conjugate to their linear parts up to an arbitrarily small error term. This work bears on the study of the Lebesgue measure of nearby filled Julia sets;

[Jellouli2] discusses various important quantities and their relations in the perturbation of a quadratic polynomial with a parabolic fixed point. These include the multiplier of the periodic orbit coming from the perturbation, its first and second derivatives with respect to the parameter, and the holomorphic indices.

Part three: Julia set of rational maps

A rational map f tends to be expanding on its Julia set $J(f)$, away from the orbits of recurrent critical points. This phenomenon, observed already by Fatou and Julia, was made precise in a strong form by Ricardo Mañé. In this volume:

[Shishikura-Tan] presents a new proof of Mañé's result, showing that any point $x \in J(f)$, which is neither a parabolic periodic point nor a limit point of a recurrent critical orbit, has a neighborhood which is contracted by the family $\{f^{-n}\}_{n \in \mathbb{N}}$.

[Yin] shows, as an application, that if all critical points in $J(f)$ are non-recurrent, then $J(f)$ is 'shallow' or 'porous'; consequently its Hausdorff dimension is less than two (as first shown by Urbanski).

The combinatorics of general rational maps, even of degree two, is still not well-understood. However many interesting rational maps can be constructed by *mating* pairs of polynomials of the same degree. If the polynomials are critically finite, then the existence of their mating can be reduced to a topological problem using Thurston's theory. In this volume:

[Shishikura1] provides a proof of a result, first developed by Mary Rees, showing that if a mating f exists combinatorially, then in fact f is topologically conjugate to a natural quotient of the dynamics of the two polynomials.

Part four: Foundational results

[Douady] gives a new proof of Ahlfors-Bers theorem on integrability of measurable complex structures. This theorem is of central importance for the surgery and deformation of holomorphic dynamical systems.

[Shishikura2] gives a thorough treatment of the theory of parabolic implosions (developed by Douady and Lavaurs from Ecalle-Voronin's theory). This powerful tool provides precise analytic information about the rational maps obtained by perturbing a parabolic cycle, and underlies Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension two. Other applications appear in [Douady-Buff-Devaney-Sentenac], [Jellouli2] and [Tan] in this volume. Refinements are included in the Appendix.

2 Techniques in complex dynamics

A wide range of modern methods of complex analysis and dynamics can be seen at work in the articles of this volume.

The most classical techniques, successfully used by Fatou and Julia, are the *Poincaré metric, Schwarz Lemma and Montel's theorem on normal families.* One can see these methods applied in [Petersen-Ryd], [Shishikura-Tan] and the appendix to [Haïssinsky].

The *Riemann mapping theorem, together with the Carathéodory theory*, is a central tool for understanding the topology of the Julia set and its complementary components. The rays coming from the Riemann representation often meet (land) at same points and cut the Julia set. They are preserved under iteration and transfer easily to the parameter spaces. Using external rays, one obtains a type of Markov partition called a *Yoccoz puzzle.* These methods appear in [Haïssinsky], [Milnor], [Roesch] and [Tan].

Quasiconformal deformations, the Beltrami equation, the Ahlfors-Bers theory and its refined version G. David's theorem. Starting with Sullivan's work, the integrability of *measurable* complex structures has found remarkable applications to complex dynamics. For example, it is essential to the Douady–Hubbard theory of *polynomial-like maps, Mandelbrot-like families*, which in turn forms the foundations of the theory of renormalization and the universality of the Mandelbrot set. These methods appear in [Douady], [McMullen], [Douady-Buff-Devaney-Sentenac] and [Haïssinsky].

Holomorphic motions. This tool for studying the variation of dynamics in families plays a crucial role in [Roesch]. See also [McMullen], [Douady-Buff-Devaney-Sentenac] and [Haïssinsky].

Grötzsch inequality about moduli of annuli. These conformal invariants play an essential role in work on local connectivity of Julia sets and the Mandelbrot set. See [Milnor] and [Roesch].

Yoccoz inequality. This application of the Grötzsch inequality is discussed in [Tan], Appendix D and in [Haïssinsky].

Parabolic implosion. This important modern theory is developed in detail in [Shishikura2], and is applied in [Tan], [Jellouli2] and [Douady-Buff-Devaney-Sentenac].

Expansion on subsets of the Julia set away from recurrent critical orbits. This is the content of Mañé's result and is the theme of [Shishikura-Tan]. See [Yin] for an example of application.

Jakobson's theorem. Collet-Eckmann maps. This is one of the key results in real dynamical systems having a complex flavor, and is exposed here in [Luzzatto].

Circle homeomorphisms, distortion of cross-ratios. They are important on their own right and have important applications in complex dynamics. Both features are exhibited in [Petersen].

Transferring results on the dynamical planes to the parameter space. This is the most common way to get information about the parameter space. See [McMullen], [Douady-Buff-Devaney-Sentenac], [Haïssinsky], [Roesch], [Tan], [Jellouli1] and [Luzzatto].

Thurston's theory, creating rational maps from combinatorial information. One example of this is mating of polynomials. And the fact that mating is an adequate way to describe topologically the dynamics is reported here in [Shishikura1].

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