Workshop on Holomorphic Dynamics

Holbæk, September 27-30, 2012

On Bill Thurston's binding entropy theory

This presentation is jointly prepared by

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I. A model for the parameter space

From Bill's talk at Topology Festival (May 4-7, 2012)

It turns out that the set of all "critical" degree d polynomials can be approximately described by collections of arcs of the disk whose endpoints have angle between them of the from k/d. It took me a while to realize that the set of all such arcs, along with the limiting cases where some endpoints coincide and there are additional implicit arcs, describe a spine for sets of d disjoint points in $\mathbb{C}$, that is, its fundamental group is the braid group and higher homotopy groups are trivial.

The space of primitive majors of degree d

Take a monic centered dynamical polynomial with locally connected Julia set. Take one external (union an eventual internal) landing ray per distinct critical value. Pullback this spider, record only the rays landing at the critical points. This is a critical portrait. It defines an equivalence relation on S^1, and a lamination called 'a primitive major'.

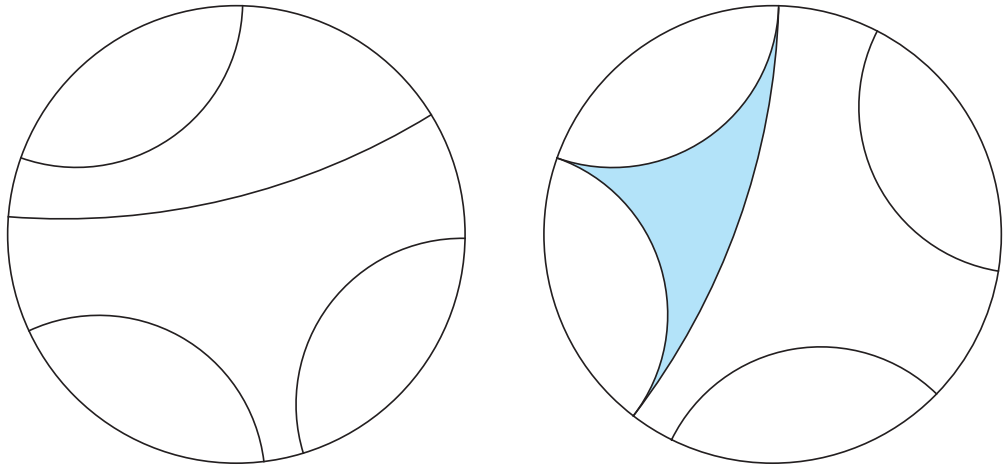
This motivates :

A primitive degree-d major is a collection of disjoint leaves and polygons in $\mathbb{D}$ each of whose vertices are identified under $z \mapsto z^d$, with the total criticality $d-1$ (a k-gon has *criticality* $k-1$.)

Their space is denoted by $PM(d)$. Each $m \in PM(d)$ induces a semi-metric $met(m)$ on S^1. One may equip $PM(d)$ a metric :

$$d(m, m') = \sup_{x,y \in S^1} |met(m)(x,y) - met(m')(x,y)|.$$

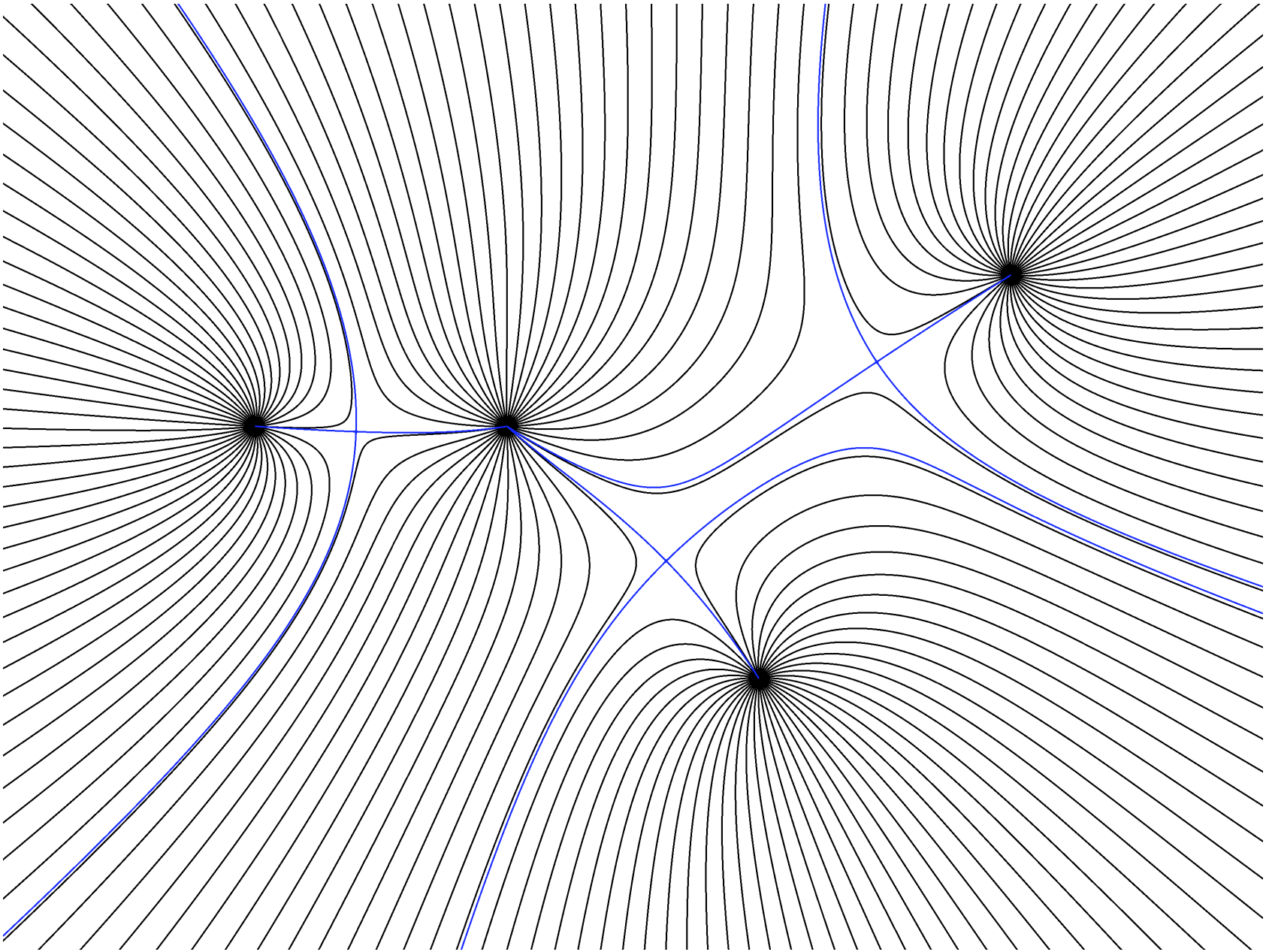
Two examples of degree 5 majors



Theorem (around 1 april 2011). The space $PM(d)$ *embeds in the space of monic centered degree d (non-dynamical) polynomials as a spine for the set of polynomials with distinct roots, that is, the complement of the discriminant locus. The spine consists of polynomials whose critical values are all on the unit circle.*

$\pi_1(PM(d)) =$ the d-strand braid group and all higher homotopy groups are trivial.

Proof of $\Longleftarrow$: Pullback the critical value radial spider.



They are also : separatrices of the gradient flow for $|P|$, $|P|^2$, $\log|P|$, or of the Newton flow $-\frac{P}{P'}$ (Newton tree)...

Proof of $\Longrightarrow$:

a primitive major –> a critical value radial spider + Hurwicz data –> reconstruction of P through cut-paste and Riemann Mapping Theorem. q.e.d.

A generic critical value radial spider –> d^{d-2} Hurwicz data. The counting is more complicated in the degenerate case.

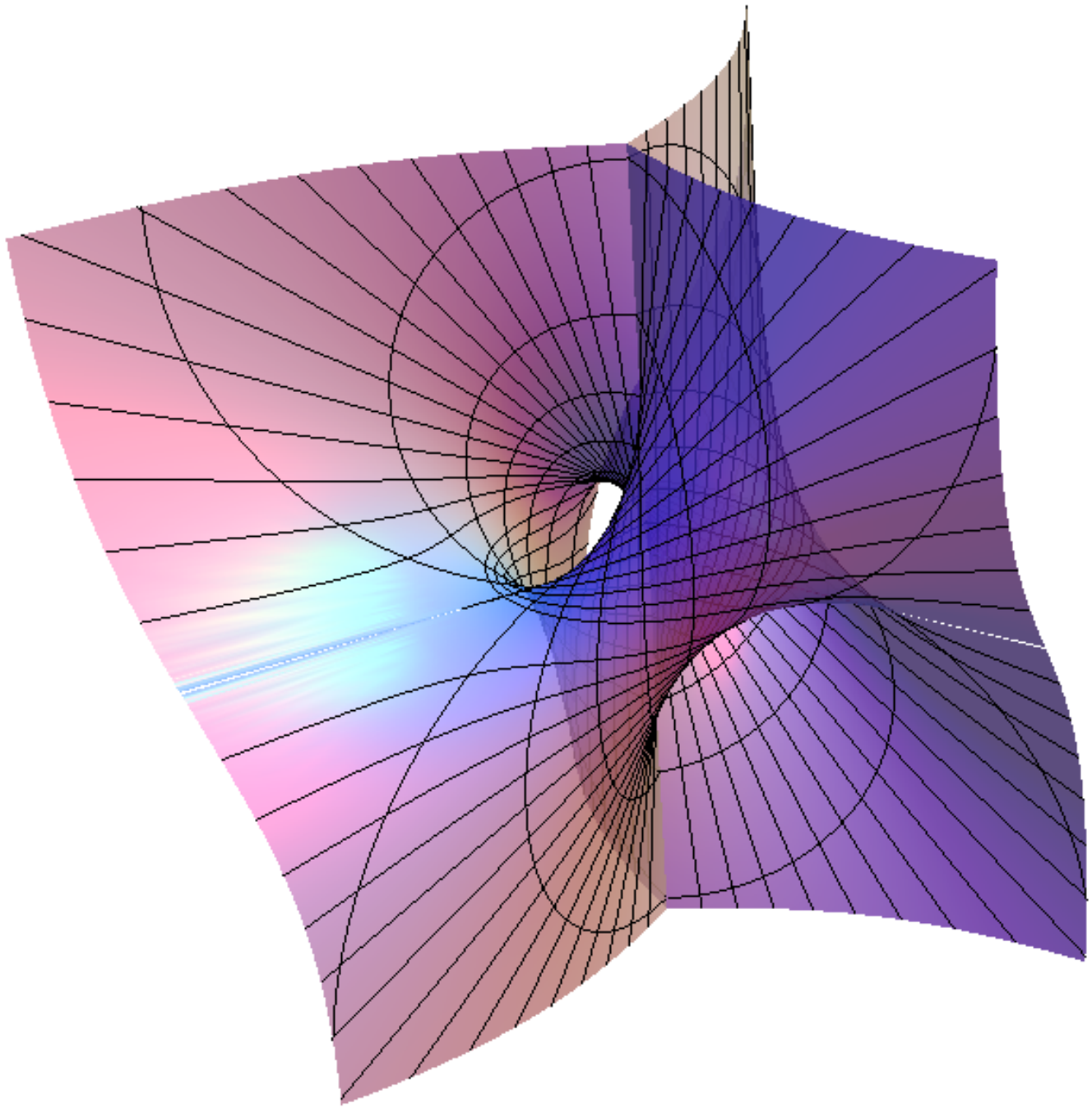
P has only simple roots $\leftrightarrow$ cr. values $\neq 0$. In this case one can slide the cr. values to the unit circle generating the same primitive major.

Show H.H. Rugh's scilab code...

From Bill's draft

A primitive quadratic major is just a diameter of the unit circle, so $PM(2)$ *is itself a circle. This corresponds to the fact that the 2-strand braid group is* $\mathbb{Z}$.

A primitive cubic major is either an equilateral triangle inscribed in the circle, or a pair of chords that each cut off a segment of angle-length $2\pi/3$. *There are a circle's worth of equilateral triangles. To each primitive cubic that consists of a pair of leaves, there is associated a unique diameter that bisects the central region. Thus, the space of two-leaf primitive cubic majors fibers over* S^1, *with fiber an interval. When the diameter is turned around by an angle of* π, *the interval maps to itself by reversing the orientation, so this is a Moebius band. The boundary of the Moebius band is wrapped attached to the circle of equilateral triangle configurations, wrapping around 3 times, since, given an equilateral triangle, you can remove any of its three edges to get a limit of two-leaf majors.* $[0, 1/2] \times]0, 1/6[/\{0, \epsilon\} \sim \{1/2, 1/6 - \epsilon\}$.



II. Binding entropy and the torus presentation

The **Binding Entropy** h for a postcritically finite polynomial map f of degree d is the topological entropy of the action of $(\theta, \eta) \mapsto (d\theta, d\eta)$ on pairs of external angles that end at the same point in the Hubbard tree H_f of f.

It is also the topological entropy of $f|_{H_f} : H_f \to H_f$, and $\lambda = e^h$ is the leading eigenvalue of the transition matrix of f on the edges of H_f.

For $\lambda = e^h$, the sequence λ^n approximates the growth of

the number of edges in $f^{-n}H_f \cap H_f$ (the lap number);
the maximal edge decomposition of an edge of H_f under f^{-n};
the maximal number of nth-preimages of a point in H_f;
the number of n-periodic ends in H_f;
the number of n-periodic angles landing on H_f;
the number of depth-n puzzle pieces cutting $H_f(??)$; $\cdots$

$\frac{1}{\lambda}$ = radius of conv. of the generating functions.

Torus presentations of laminations (around 21 April 2011)

Consider $Q : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $(\theta, \eta) \mapsto (2\theta, 2\eta)$ the angle doubling map.

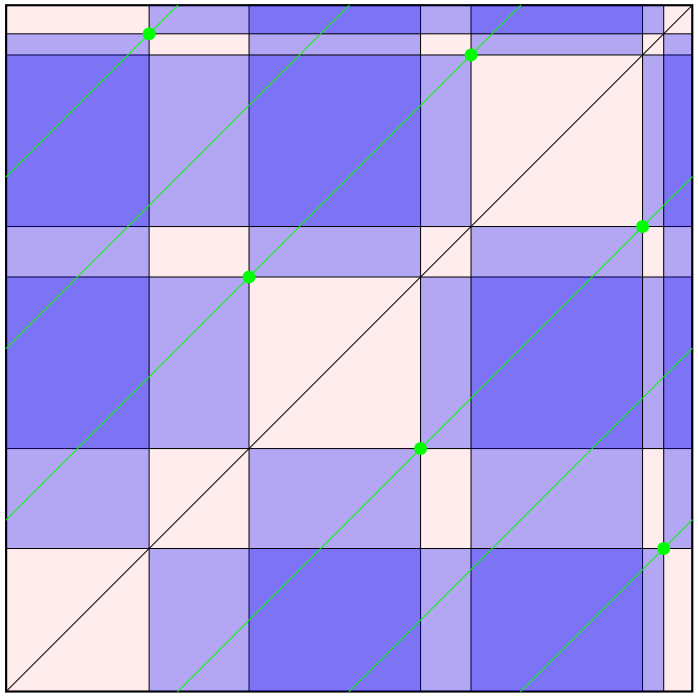
For any θ, let L_θ be the union of the two consecutive closed squares of size $\frac{1}{2}$ along the diagonal of $\mathbb{R}^2$, starting from $(\frac{\theta}{2}, \frac{\theta}{2})$.

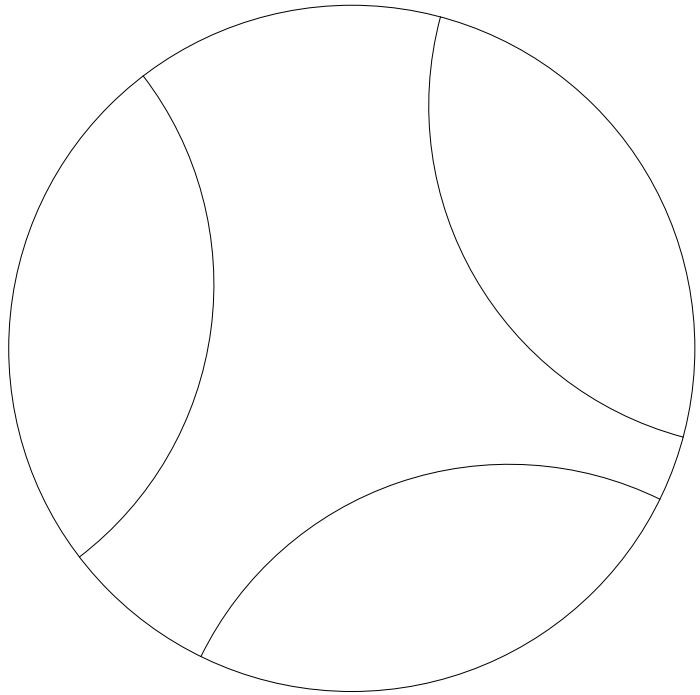
Let K_θ be the non escaping set of the map $Q|_{L_\theta} : L_\theta/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, and Δ be the diagonal.

Then : $K_\theta \setminus \Delta$ describes the binding set, or the set of pairs of angles having the same itinerary relative to the two closed half circles.

If θ is rational, $2^{\dim_H(K_\theta \setminus \Delta)} = \lambda(z \mapsto z^2 + c_\theta)$, *since* $K_\theta \setminus \Delta$ *has a measure that is locally multiplied by* λ *under* $\eta \mapsto 2\eta$ *which expands the metric of the circle by 2, so the measure is a conformal density of dimension* $\log \lambda / \log 2$.

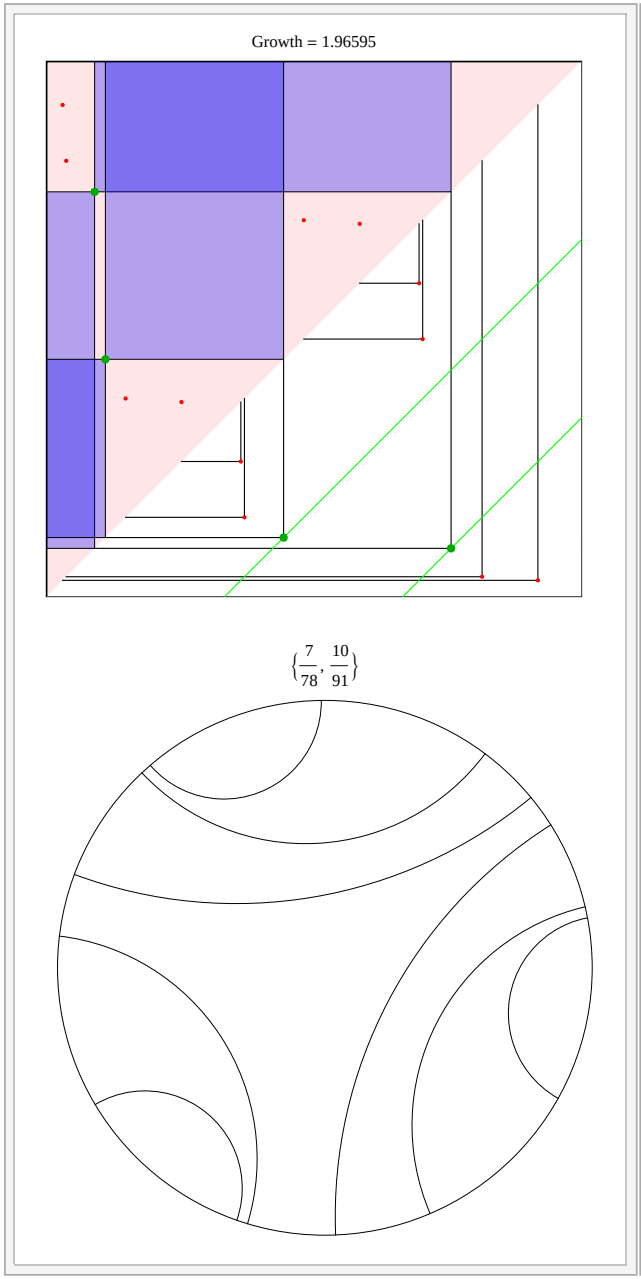
The set L for higher degree polynomials.

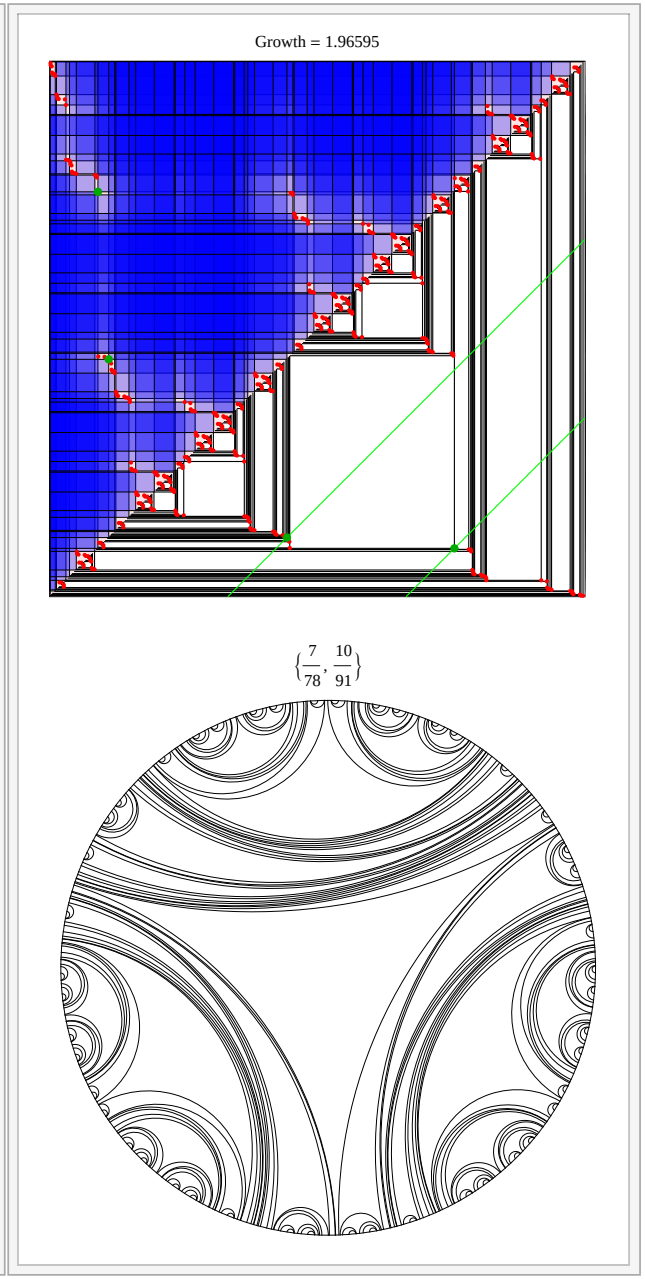




Given a primitive major $m \in PM(d)$, each region of $\mathbb{D} \setminus m$ touches S^1 in a union of one or more closed intervals $J_1 \cup \cdots \cup J_k$ of total length $1/d$. The product $(J_1 \cup \cdots \cup J_k) \times (J_1 \cup \cdots \cup J_k)$ is a union of k^2 rectangles whose total area is $1/d^2$ that maps under the degree d^2 covering map $(x, y) \mapsto (d \cdot x, d \cdot y)$ to the entire torus.

The binding set, whose dimension is to be measured





III. The entropy algorithm bypassing trees and dimensions (march 2011)

Start from any rational angle $\theta \neq 0$ and separate the circle into two halves by $\theta/2, \theta + 1/2$, each half is a closed segment in the circle (so the cutting points belong to both halves).

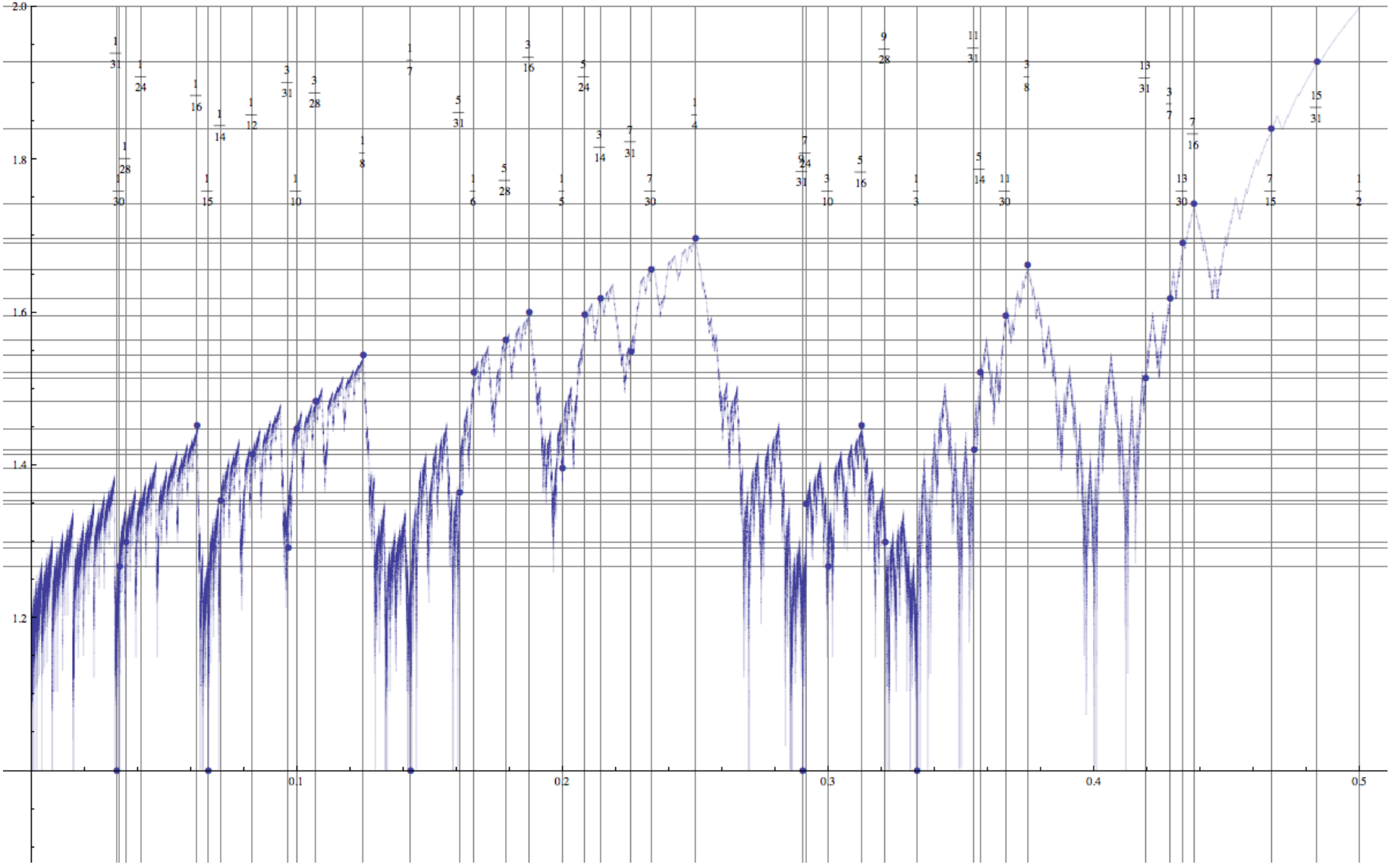
In the following all the $\theta/2$ should be replaced by $\theta + 1/2$ if the latter is periodic.

1. Form the set of all possible unordered pairs $\{2^n\theta, 2^l\theta\}$ so that both l, n are ≥ -1 and the two angles are unequal. Consider them as a basis of an abstract linear space.
2. Compute the leading eigenvalue of the linear map :

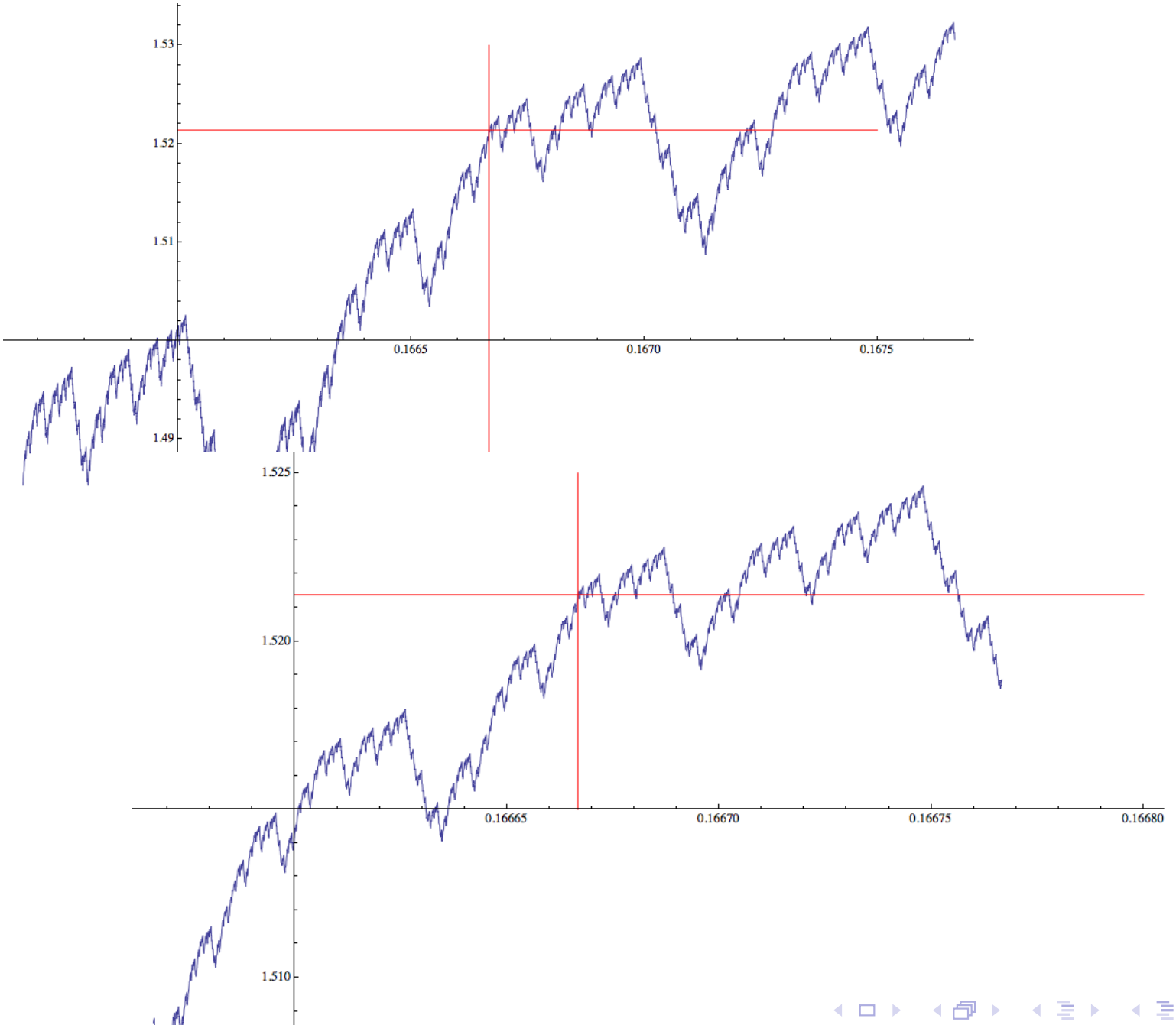
$$\{2^n\theta, 2^l\theta\} \mapsto \begin{cases} \{2^{n+1}\theta, 2^{l+1}\theta\} & \text{if } 2^n\theta, 2^l\theta \text{ are in the same half-circle} \\ \{2^{n+1}\theta, \theta\} + \{\theta, 2^{l+1}\theta\} & \text{otherwise} \end{cases}$$

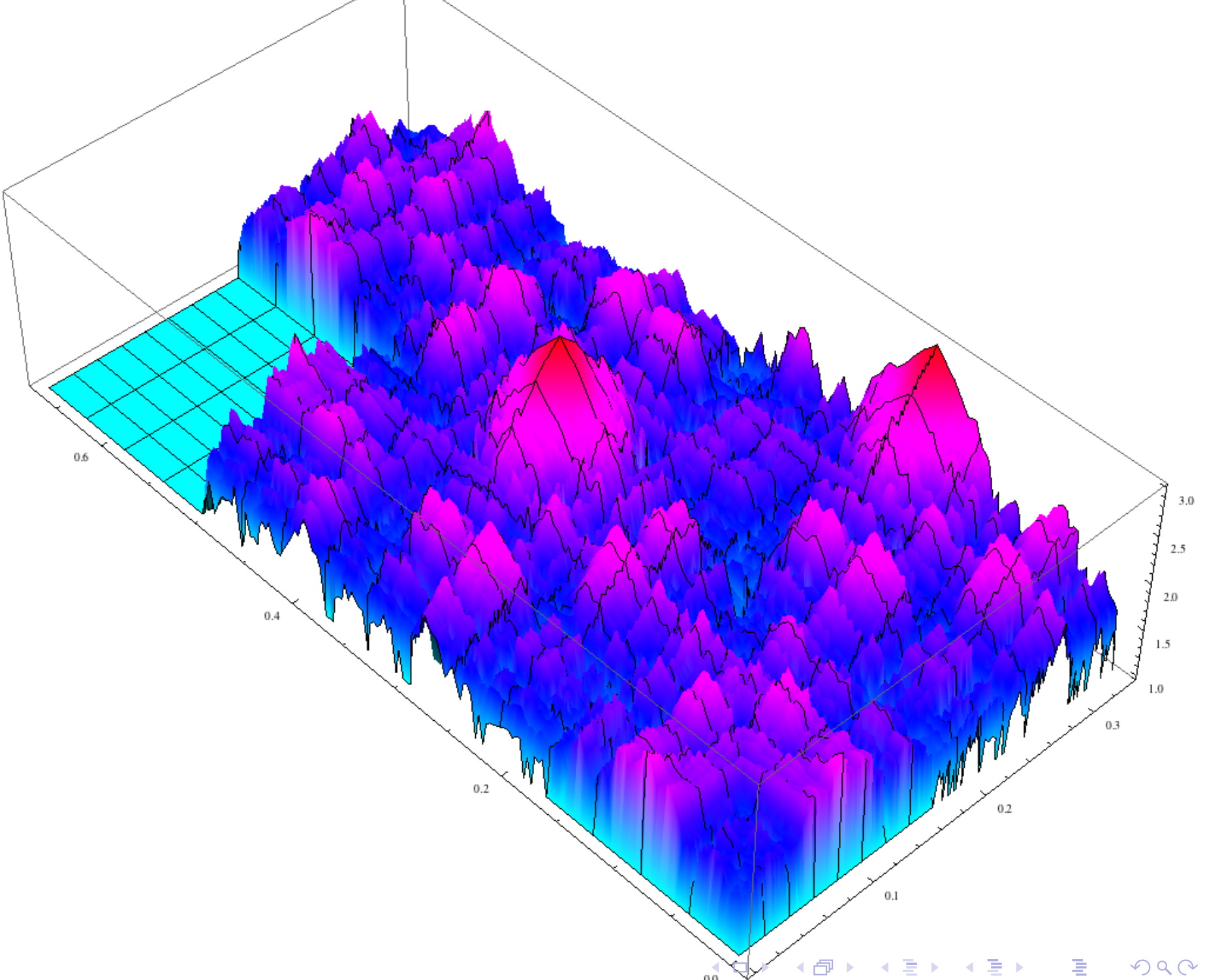
3. You get e^h. A similar algorithm works for any degree.

The plots (on March 13, 2011)



and two zooms at 1/6 (the same day)




0.6
0.4
0.2
0.0
0.1
0.2
0.3
1.0
1.5
2.0
2.5
3.0

IV. Continuity (? ?)

From a message of Bill on March 19 2012 : *Hubbard now thinks he can prove the continuity of entropy, as does Dierk (...). I think I know how to give a complete proof as well, but I'll wait and see what Hamal and Dierk come up with.*

The first issue is to define the binding entropy for any angle...

See work of Chris Penrose, Branner-Hubbard, Kiwi, Poirier, Tao Li, Dujardin-Favre, Bruin-Schleicher, Bardoldi-Dudko-Nefrashevych, Tiozzo, among others...