Convergence of pinching deformations and matings of geometrically finite polynomials

Peter Haïssinsky & Tan Lei *

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Abstract. We give a thorough study of Cui's control of distortion technique in the analysis of convergence of simple pinching deformations, and extend his result from geometrically finite rational maps to some subset of geometrically infinite maps. We then combine this with mating techniques for pairs of polynomials to establish existence and continuity results for matings of polynomials with parabolic points. Consequently, if two hyperbolic quadratic polynomials tend to their respective root polynomials radially, and do not belong to conjugate limbs of the Mandelbrot set, their mating exists and deforms continuously to the mating of the two root polynomials.

1 Introduction

Throughout the paper, $d \geq 2$ will denote a fixed integer; the critical set of a polynomial f of degree d is defined as $C(f) = \{c,\, f'(c) = 0\}$, and the post-critical set is

$$\mathrm{Post}(f) = \cup_{c \in C(f)} \overline{\cup_{n \geq 1} f^n(c)}\,.$$

A rational map f of degree d is said to be *geometrically finite* (resp. *postcritically finite*) if its post-critical set has a finite accumulation set (resp. is a finite set). We will say that f is *sub-hyperbolic* (resp. *hyperbolic*) if it is geometrically finite with no parabolic points (resp. with no parabolic points and no critical point on the Julia set). The notion of *weakly hyperbolic* rational maps generalizes geometrically finite maps to some maps with an infinite postcritical accumulation set. The precise definition will be given later.

Our first task in this paper is to generalize an important result in [4] of Cui G. from geometrically finite maps to weakly hyperbolic maps:

Theorem A *Let f_0 be a weakly hyperbolic rational map with attracting points. Let (γ_i) be a collection of f_0-periodic cycles of star-like graphs linking a repelling central vertex to attracting vertices. Then there is a continuous path $(f_t)_{t\in[0,1)}$ of qc-deformations of f_0, called* SIMPLE PINCHING DEFORMATIONS, *converging uniformly to a limit rational map f_1, shrinking progressively each graph γ_i to a parabolic point for f_1. All the connected components of the preimages of these graphs are also shrunk, and these are the only changes in the limit.*

Detailed definitions and a more precise statement of Theorem A will be given in Section 2.1.

This result is actually more subtle than it appears to be. Although it is fairly easy to imagine the existence of a parabolic map with the right combinatorics, there might be many

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such maps, especially in the presence of several critical points in the basin. It is then quite surprising that the pinching path accumulates to a single parabolic map.

This theorem, combined with other techniques, create many interesting existence and continuity results. Cui's original work has used this to obtain a topological characterization of geometrically finite rational maps which generalizes Thurston's theorem on postcritically finite rational maps. See [4] and [27] for further details.

Here we combine it with the technique of matings of polynomials, and thus answer affirmatively a question raised by J. Milnor: Can the mating of two hyperbolic quadratic polynomials be deformed continuously to a mating of two parabolic polynomials?

A *marked mating* is, roughly speaking, a 4-tuple (f, g, q, R) with f and g two degree d monic polynomials having connected and locally connected filled Julia set K_f and K_g, with $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a rational map, and with q a homeomorphism from $K_f \sqcup K_g/(\gamma_f(t) \sim \gamma_g(-t))$ to $\overline{\mathbb{C}}$ conjugating the quotient dynamics induced by f and g to the dynamics of R, where $\gamma_f : \mathbb{R}/\mathbb{Z} \to \partial K_f$ (resp. γ_g) is the Carathéodory semiconjugacy. We establish:

Theorem B *Let (f_0, g_0, q_0, R_0) be a marked mating of geometrically finite polynomials with connected Julia sets and attracting points. Given a simple pinching path $(f_t)_{t\in[0,1)}$ of f_0 and a simple pinching path $(g_t)_{t\in[0,1)}$ of g_0, there is a simple pinching path $(R_t)_{t\in[0,1)}$ of R_0 together with a continuous path of maps q_t, such that*

1. *(f_t, g_t, q_t, R_t) is a geometric mating for every $t \in [0, 1)$ and depends continuously on t.*
2. *The three pinching paths (f_t), (g_t) and (R_t) converge to f_1, g_1 and R_1 respectively.*
3. *The maps q_t converge uniformly to a map q_1.*
4. *The quadruple (f_1, g_1, q_1, R_1) is a marked mating with parabolic points.*

Schematically, the following diagram is well defined and commutative:

$$\begin{array}{ccc} (f_0, g_0) & \overset{\text{pinch}}{\longrightarrow} & (f_1, g_1) \\ \text{mate}\downarrow & & \downarrow\text{mate} \\ R_0 & \overset{\text{pinch}}{\longrightarrow} & R_1 \end{array}$$

We will give more precise definition of marked matings in Section 3, especially we will make all q_t to be defined on a same space in order to talk about its continuity on t.

This theorem is to be compared with the examples constructed by Adam Epstein [8] showing that there are paths of marked matings (f_t, g_t, q_t, R_t) with f_t and g_t converge but R_t accumulates to a large set of limit parabolic maps. In these examples the multipliers of the corresponding attracting cycles tend to 1 along a horocycle, whereas pinching produces multipliers that converge to 1 radially, in the sense of McMullen (see [17]).

Applying these results to the quadratic polynomials $f_c : z \mapsto z^2 + c$, $c \in \mathbb{C}$, we get:

Corollary C *Two geometrically finite maps f_c and $f_{c'}$ are matable if and only if $\bar{c}$ and c' do not belong to the same limb of the Mandelbrot set.*

Combining Theorem B with techniques of perturbations of parabolics, we can actually obtain a more general parabolic⟷attracting-closing-lemma which we state as Theorem D and E: to each geometrically finite polynomial with connected Julia set can be associated a post-critically finite polynomial $\mathcal{T}(f)$ of the same degree with a homeomorphic Julia set and with conjugate dynamics on their Julia sets (see details below).

Theorem D *Two geometrically finite polynomials f and g with connected Julia sets and parabolic points are matable if and only if $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are matable.*

The following theorem gives more information:

Theorem E *Let f and g be two matable geometrically finite polynomials with connected Julia sets and parabolic points. Then there exist sub-hyperbolic perturbations $(f_t)_{t\in[0,1)}$ and $(g_t)_{t\in[0,1)}$ which converge to f and g respectively as t tends to 1 such that $J_{f_t} \approx J_f$ $J_{g_t} \approx J_g$, and such that their matings exist and converge to a mating of f and g.*

In § 2 we restate and prove Theorem A. For this we provide a thorough study of Cui's control of distortion technique. This technique is an important innovation to complex dynamics and will surely find wide applicability. The global structure of our proof is somewhat different from Cui's original one, and our definition of pinching is somewhat more general. We also use a different argument reducing the study around parabolic points to analysis of a simple model system. In § 3 we recall basic definitions and results about matings of polynomials and then apply Theorem A to study matings of geometrically finite polynomials. The paper also contains two appendices which may be of independent interest.

Background on complex dynamics can be found in [3, 19] and on quasiconformal maps in [1]. We also assume that the reader is familiar with basic quasiconformal surgeries as those explained in [3].

Acknowledgements. This work grew out from a question raised by J. Milnor to Tan Lei about the radial continuity of matings at root quadratic polynomials. We are grateful to him to have induced our project. We would like also to thank Cui. G. for enlightening explanations on his work, and K. Pilgrim and the anonymous referee for their valuable comments which have enabled us to improve the exposition.

2 Pinching deformation of rational maps, Theorem A

2.1 Definition and the pinching theorem

Let f be a rational map. Denote by J_f the Julia set and by $\mathcal{F}$ the Fatou set. A **simple pinching combinatorics** is a finite collection of γ_i satisfying:

• Each γ_i is a repelling star-like closed graph in the following sense:

– the central vertex β_i is a repelling periodic point, and is not in the ω-limit of recurrent critical points;

– every edge κ links β_i to an attracting periodic point α, and there are no other edges between these two vertices, further, for q the period of α, we have $f^q(\kappa) = \kappa$ and (f^q, B') is conformally conjugate to the translation by 1 on a horizontal strip (where B' is a neighborhood of $\kappa \smallsetminus \{\alpha, \beta_i\}$), for simplicity, we also require that κ intersects the boundary of any linearizable disk around α at only one point (examples of such (κ, B') are suitable straight lines and strips in the log-linearizing coordinates of α);

– $\gamma_i \smallsetminus \{\beta_i\} \subset \mathcal{F}$ and is disjoint from the orbits of the critical points.

- The γ_i's are mutually disjoint.
- $f : \gamma_i \to \gamma_j$ is a homeomorphism.

(The simplest example of a simple pinching combinatorics is the segment $[0, \frac{1}{2}]$ for $f(z) = z^2 + z/2$.)

This collection of γ_i can be decomposed into n_c cycles. Set $\widehat{R} = \bigcup_i \gamma_i$, $\{\widehat{R}_i\}_{1\le i\le n_c}$ be the set of cycles and $\mathcal{R} = \bigcup_n f^{-n}(\widehat{R})$. The definition above guarantees that each $\mathcal{R}$-component R is again star-like, with a unique Julia point $\beta(R)$ (the word 'simple' refers precisely to this fact). We make a (by no means canonical) choice of three distinct points a, b, c such that no pair belong to the same component of $\mathcal{R}$.

Definition of weak hyperbolicity. We say that f is weakly hyperbolic if there are constants $r > 0$ and $\delta < \infty$ such that, for all $z \in J_f \setminus \{\text{preparabolic points}\}$, there is a sub-sequence of iterates $(f^{n_k})_k$ such that

$$\deg(f^{n_k} : W_k(z) \longrightarrow D(f^{n_k}(z), r)) \le \delta$$

where $W_k(z)$ is the connected component of $f^{-n_k}(D(f^{n_k}(z), r))$ containing z. A simple uniform continuity argument implies that this definition is invariant under topological conjugacies.

Let us remark that if f is hyperbolic, then f is weakly hyperbolic too, and we can choose $\delta = 1$ and $r = (1/2)\mathrm{dist}(J_f, \mathrm{Post}\, f)$. It can be shown that geometrically finite rational maps are weakly hyperbolic.

The following is a restatement of Theorem A, in a more precise form:

Theorem 2.1 *Assume that f is a weakly hyperbolic rational map. Let (γ_i) be a simple pinching combinatorics for f. Then there is a convergent continuous path f_t of K_t-qc-deformations of f (with $K_t \to \infty$), shrinking progressively each γ_i to a parabolic point, and making no other changes.*

More precisely, there is a continuous path of complex structures σ_t, with σ_0 the standard complex structure, such that for h_t the integrating map of σ_t fixing a, b, c, and for $f_t = h_t \circ f \circ h_t^{-1}$, we have

(I) $h_t \rightrightarrows H$;   (II) The non trivial fibers of H coincide with the $\mathcal{R}$-components;

(III) $f_t \rightrightarrows F$;   (IV) $F \circ H = H \circ f$ and $H|_{J_f} : J_f \to J_F$ is a homeomorphism.

Let us note that h_0 is the identity since it integrates the standard complex structure and it fixes three points.

The sign $\rightrightarrows$ will mean throughout the paper uniform convergence.

This theorem is a generalization of a work of Cui G. [4] who proved the same result for geometrically finite maps. Our proof uses essentially one fundamental idea of Cui G. (see the Key Lemma below), but with a different presentation, and a somewhat more general definition of the pinching deformation.

Definition of σ_t. We will use different colors to design regions with special properties. Roughly speaking, the red set is the set to be pinched. The yellow set surrounds the red and contains of the support of σ_t. The green (resp. blue) set is a neighborhood of the yellow acting as protecting a neighborhood, in the sense that on green$\setminus$yellow and on blue$\setminus$green we are sure that there are no deformations. Note that some regions have several colorings.

Although these notions look very complicated, pinching is fundamentally a very simple operation to create parabolic points. But to guarantee the convergence, and that the limit has no other accidental changes in the dynamics, we need more precise information about where the deformation occurs and where not. The yellow set is assigned to locate the support of deformations. But we don't know how big its complement is, especially when we get close to a Julia point. The two consecutive protecting neighborhoods, green and blue, are assigned to guarantee some definite, un-deformed space around the yellow set. In most of the times the green neighborhood is enough. One appreciates best its importance in the Key Lemma below. The second neighborhood, blue, is however fundamental in the analysis around parabolic points, and in the proof that the limit dynamics is again weakly hyperbolic. For instance it will enable us to get bounds in the green set of the distortion of p-valent maps.

Here is a detailed description. We will first define appropriate quasiconformal deformations on some model strips then implement them into the dynamical plane. In the model, we have split the star along the red curve so it is doubled (cf. § 1 in [12] for a similar construction).

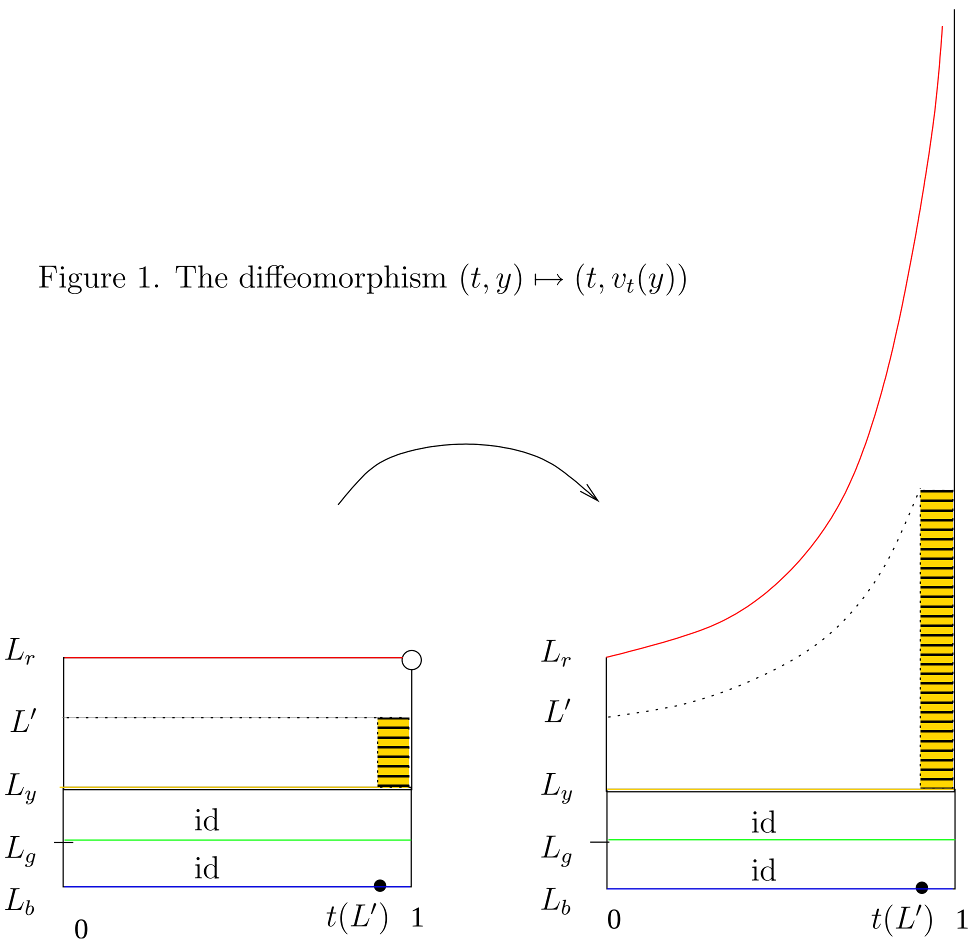
Our model spaces will be closed horizontal strips on upper or lower half-planes.

Choose a collection of numbers $0 < L_b < L_g < L_y < L_r$ (the indices b, g, y, r are colors blue, green, yellow and red respectively), and then an increasing C^1-function $\tau : [0,1[\to [L_r, +\infty[$. Let $M \subset \mathbb{R}^2$ be the closed subset bounded by

$$([0,1] \times \{L_b\}) \cup (\{0\} \times [L_b, L_r]) \cup (\{1\} \times [L_b, +\infty[) \cup (\{(t, \tau(t)), t \in [0,1[\}) \ .$$

Choose $v_t(y)$ so that $v_t(y) = y$ for $L_b \leq y \leq L_y$ and that $(t, y) \mapsto (t, v_t(y))$ is a C^1-diffeomorphism from $[0,1] \times [L_b, L_r] \smallsetminus \{(1, L_r)\} \to M$.

Figure 1. The diffeomorphism $(t, y) \mapsto (t, v_t(y))$



We make also the following technical assumption: For any $L' < L_r$, there is $t(L') \in]0,1[$ with $t(L') \to_{L'\to L_r} 1$, such that for any $(s,y) \in]t(L'),1] \times [L_b, L']$, we have $v_s(y) = v_{t(L')}(y)$. This assumption will be used only once in the proof of Lemma 2.8.

Now on the straight strip $\{L_b \le x \le L_r\}$, and for every $t \in [0,1]$, set

$$\widetilde{P}_t(x+iy) = x + i \cdot v_t(y) \ .$$

This map satisfies the following properties (cf. Fig. 1):

1. It commutes with the translation by 1 (and by any other real number).
2. It is the identity on the sub-strip $\{L_b \le y \le L_y\}$.
3. The coefficient of the Beltrami form
$$\left.\frac{\partial \widetilde{P}_t/\partial \bar{z}}{\partial \widetilde{P}_t/\partial z}\right|_{x+iy} = \frac{1-\frac{\partial}{\partial y}v_t(y)}{1+\frac{\partial}{\partial y}v_t(y)}$$
is continuous on $(t, x+iy) \in [0,1] \times \{L_b \le y \le L_r\}$, whose norm is locally uniformly bounded from 1 if $(t,y) \ne (1, L_r)$ and tends to 1 as $(t,y) \to (1, L_r)$.
4. The map $P_t(z) = -1/\widetilde{P}_t(-1/z)$ is continuous in (t,z). For $t<1$, P_t is injective.

Let us now define appropriate domains in the dynamical space which will support our deformation. Choose one edge in each edge orbit of $\widehat{R}$. Let κ be one such edge, of period q, and let B' be its invariant strip neighborhood (for f^q). Let B'^l denote the component of $B' \smallsetminus \kappa$ with κ as the left boundary (following the direction of the dynamics) and let B'^r be the other component.

By assumption, there are $0 < L_b < L_r$ such that (f^q, B'^l) is conjugate via a conformal map ψ to $(z \mapsto z+1, \{L_b < Imz < L_r\})$ so that $\psi(\kappa) = \{Imz = L_r\}$. Define as above a deformation $\widetilde{P}_t$ on the model strip $\{L_b < Imz < L_r\}$, and for $t \in [0,1[$, set $\sigma'_t = (\widetilde{P}_t \circ \psi)^*(\sigma_0)$ to be the pulled back complex structure on B'^l.

For the other half neighborhood B'^r of κ, use a complex conjugate model dynamics $(z \mapsto z+1, \{-L'_r < Imz < -L'_b < 0\})$ and complex conjugate deformations (so that κ corresponds always to the red boundary line). Note that the left and right model strips are not necessarily symmetric and we do not require symmetric deformations. We will use nevertheless the same letters for convenience.

We may then define $Y(\kappa)$ (resp. $G'(\kappa)$, $B'(\kappa)$) the yellow (resp. green, blue) strip neighborhood, to be $\{\alpha\} \cup \psi(\{|\mathrm{Im}\, z| \ge L_y\})$ (resp. $\psi(\{|\mathrm{Im}\, z| > L_g\})$, $\psi(\{|\mathrm{Im}\, z| > L_b\}) = B'$). The green neighborhood $G(\kappa)$ will be of the form $G'(\kappa) \cup \Delta'_\alpha$ with Δ'_α a suitable neighborhood of the attracting end α, and the blue neighborhood $B(\kappa)$ will be of the form $B'(\kappa) \cup \Delta_\alpha \cup \Delta_\beta$, where Δ_α and Δ_β are suitable neighborhoods of the attracting end α and of the center β, such that $\overline{G(\kappa)} \subset B(\kappa)$.

Set $\sigma_t = \bigcup_n (f^n)^*(\sigma'_t)$. It is an f-invariant complex structure, and is conformal outside the grand orbit $\mathcal{Y}$ of $\bigcup_\kappa Y(\kappa)$.

Definition of a simple pinching deformation. A simple pinching deformation supported by the simple pinching combinatorics $\widehat{R}$ is given by the family of f-invariant complex structures

σ_t. We say that the deformation is convergent if the conclusions of Theorem 2.1 are satisfied for some homeomorphisms h_t integrating σ_t and maps $f_t = h_t \circ f \circ h_t^{-1}$.

Y= closed shaded strip$\setminus\{\beta\}$ (it contains the support of σ_t) and $Y^* = Y \cup \{\beta\}$
G'= first protecting strip of $Y \setminus \{\alpha\}$
$G = G' \cup \Delta'_\alpha$ (it is open and is bounded by the dotted curve)
$G^* = G \cup \{\beta\}$ (it is neither open nor closed)
B'= second protecting strip of $Y \setminus \{\alpha\}$
$B = B' \cup \Delta_\alpha \cup \Delta_\beta$ (bounded by the dashed curve)

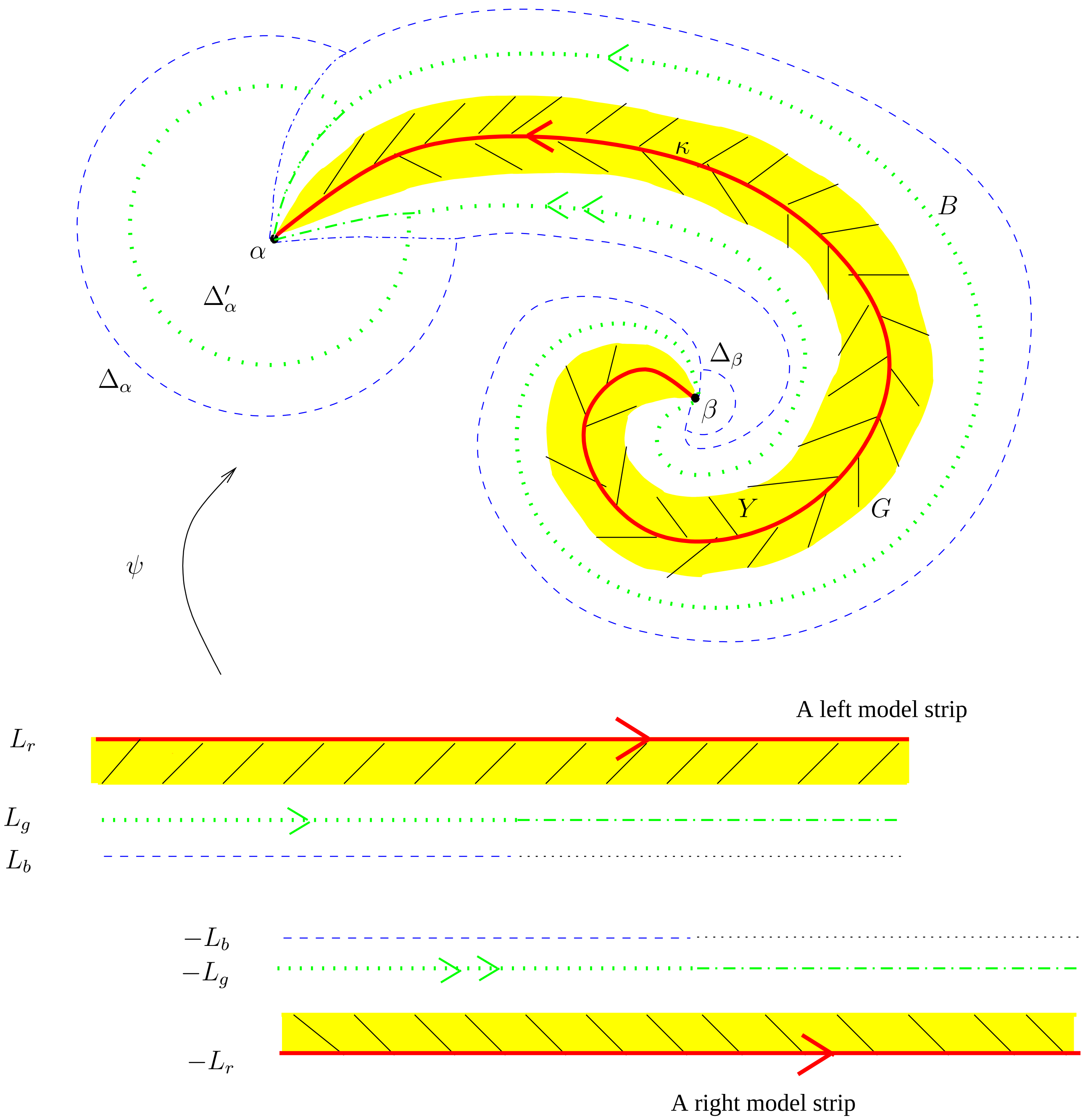


Figure 2. Deformation strip and protecting neighborhoods

Scheme of the proof. We will prove first that (h_t) is equicontinuous at every point z_0 (which would imply uniform equicontinuity). To do this, we will distinguish essentially three cases: $z_0 \notin J_f \cup \mathcal{R}$, $z_0 \in J_f \smallsetminus (\mathcal{R} \cup \{\text{preparabolics}\})$, and $z_0 \in \mathcal{R} \cup \{\text{preparabolics}\}$. While the equicontinuity in the first case is more or less automatic, in the last two cases it depends on estimates under deformation of the moduli of many annuli, that we will control thanks to a clever argument of Cui. G.

Once we know that (h_t) is an equicontinuous family, we will study the fiber structure of any limit map and show that it satisfies the conclusions of the theorem. This will enable us to prove that (f_t) is equicontinuous and that any limit map is again weakly hyperbolic. This will in turn imply that two limits of (f_t) have to be topologically conjugate, with a conjugacy that is conformal off the Julia sets. We can then use a rigidity result of P. Haïssinsky to conclude that this conjugacy is in fact the identity. This shows the convergence of the pinching deformation.

Notation. In this last introductory paragraph, we define families of sets associated to $\mathcal{R}$ (see Fig. 2). We use the assumption that the repelling ends are disjoint from the ω-limit set of recurrent critical points to show that these sets have bounded geometry (Lemma 2.2).

For each cycle $\widehat{R}_i$, $1 \le i \le n_c$ of $\widehat{R}$, we choose a connected component γ_i with center β_i. We also pick an attracting point $\alpha_{i,j}$ for each attracting cycle attached to β_i. If β_i is k_i-periodic then each $\alpha_{i,j}$ is $k_i c_i$-periodic for some $c_i \ge 1$. We will first define suitable linearizable disks for each of these points. Let us recall that when f is a holomorphic germ which fixes a point α such that $|f'(\alpha)| \notin \{0,1\}$, then there is a linearizing coordinate, that is, a univalent map ξ defined from a neighborhood of α onto a neighborhood N of 0 such that $\xi \circ f(z) = f'(\alpha)\cdot\xi(z)$. The preimage of any disk centered at the origin contained in N is by definition a linearizable disk for α.

Let us start with an attracting cycle. We let $\mathcal{C} = \{\alpha_{i,j}, \cdots, f^{c_i k_i}(\alpha_{i,j})\}$ and $\mathcal{C}' = f^{-1}(\mathcal{C})$. For each $\alpha \in \mathcal{C}'$, there is a minimal iterate $\ell = \ell(\alpha) \ge 0$ such that $f^{\ell}(\alpha) = \alpha_{i,j}$. There is also a neighborhood $N = N(\alpha)$ of α such that $f^{\ell}|_N$ is univalent and $f^{\ell}(N)$ is contained in a linearizable disk for $\alpha_{i,j}$. For $\alpha \in \mathcal{C}' \smallsetminus \mathcal{C}$, we may also assume that $N(\alpha)$ is disjoint from the postcritical set of f. We let $\Delta_{\alpha_{i,j}}$ be a linearizable disk contained in $\bigcap_{\alpha\in\mathcal{C}'\smallsetminus\{\alpha_{ij}\}} f^{\ell(\alpha)}(N(\alpha))$. For any preimage $\alpha \ne \alpha_{i,j}$ of $\alpha_{i,j}$ (including those in $\mathcal{C}$), we let ℓ be the minimal iterate such that $f^{\ell}(\alpha) = \alpha_{i,j}$ as above, and we define Δ_α to be the connected component of $f^{-\ell}(\Delta_{\alpha_{i,j}})$ which contains α.

Since β_i is disjoint from the ω-limit set of any recurrent critical point, it follows from R. Mañé's theorem [14, 23] that there are a linearizable disk Δ'_{β_i} and an integer $p > 0$ such that, for any iterate $n \ge 0$ and any connected component W of $f^{-n}(\Delta'_{\beta_i})$, the degree of $f^n|_W$ is at most p. We choose Δ'_{β_i} to be such a disk centered at β_i of a certain radius r'_{β_i}.

Definition of $\mathcal{Y}^*$, $\mathcal{G}$, $\mathcal{G}^*$. Recall that $\mathcal{Y}$ denotes the grand orbit of $\bigcup_\kappa Y(\kappa)$. It is fully invariant, contains the support of σ_t and $\mathcal{R} \smallsetminus J_f$. Set $\mathcal{Y}^* = \mathcal{Y} \cup \mathcal{R}$. Each component of $\mathcal{Y}^*$ is star-like, and consists of finitely many $\mathcal{Y}$-components together with a common boundary point. We define $\mathcal{G}$ (for green) an open neighborhood of $\mathcal{Y}$, with the property that on $\mathcal{G} \smallsetminus \mathcal{Y}$, σ_t is conformal, and $f(\mathcal{G}) \subset \mathcal{G}$. For each attracting point $\alpha_{i,j}$, we let $\kappa_{i,j}$ be the edge in $\widehat{R}$ which joins $\alpha_{i,j}$ to the center β_i. Let $G_{i,j} = G'(\kappa_{i,j}) \cup (1/2)\Delta_{\alpha_{i,j}}$. For any preimage $\kappa \ne \kappa_{i,j}$ of $\kappa_{i,j}$, let n be the first iterate such that $f^n(\kappa) = \kappa_{i,j}$. We define $G(\kappa)$ to be the connected component of $f^{-n}(G_{i,j})$ which contains κ. Similarly, we define $\mathcal{G}^* = \mathcal{G} \cup \mathcal{R}$ which is also the union of $\mathcal{G}$ with the grand orbits of all the centers of the stars γ_i. Each $\mathcal{G}^*$-component is again star-like. It is important for what follows that $\mathcal{G}$ is open, whereas $\mathcal{G}^*$ is neither open nor closed.

Definition of $\mathcal{B}$. We define $\mathcal{B}$ (for blue) as the collection of the following sets. We set

$$B(\gamma_i) = (\cup_\kappa (B'(\kappa) \cup \Delta_{\alpha_\kappa})) \cup \Delta_{\beta_i},$$

where Δ_{β_i} is a disk centered at β_i of radius $r_{\beta_i}/2 \le r'_{\beta_i}/6$, where κ ranges over the edges attached to β_i, and α_κ is the attracting point attached to κ. The constant r_{β_i} depends on f, and will be defined in the section 2.4 below in which we prove that any limit of (f_t) is weakly hyperbolic. The collection $\mathcal{B}$ is defined as follows: for any connected component R of $\mathcal{R}$, there is a minimal iterate such that $f^n(R) = \gamma$; we let $B(R)$ to be the connected component of $f^{-n}(B(\gamma))$ which contains R. Note that *deg* $f^n|_{B(R)} \le p$ by construction. Without loss of generality, we may assume that $p = \delta$.

We use G (resp. Y, R, G^*, Y^*) to denote a (general) $\mathcal{G}$ (resp. $\mathcal{Y}$, $\mathcal{R}$, $\mathcal{G}^*$, $\mathcal{Y}^*$)-component. Each point $\beta \in \mathcal{R} \cap J_f$ is the center of a unique R, Y^* and G^*, with $R \subset Y^* \subset G^*$ and they are compactly contained in a unique element B of $\mathcal{B}$. For each G, denote by $\beta(G)$ the unique Julia point on ∂G (it is a preimage of the central vertex β_i of some γ_i).

Normalization. The point $a = \infty$ is a critical point which belongs to a periodic Fatou component of f, which always exists since f has an attracting point. Its first return into the same Fatou component is $b = 0$. The point $c = 1$ is another point outside $\mathcal{G}^*$. Note that $a, b, c \notin \mathcal{G}^*$. This normalization has the advantage that we will be able to work with the Euclidean metric as well as with the spherical one, because then $\overline{J_f \cup \mathcal{B}} \subset \mathbb{C}$. If K is a 1-neighborhood of $\overline{J_f \cup \mathcal{B}}$ in the Euclidean metric, then both metrics are equivalent on K i.e., there is a constant $c_s > 1$ such that, for all $x, y \in K$, $|x-y|/c_s < d(x,y) < c_s|x-y|$.

Lemma 2.2 *1. For each component γ of $\mathcal{R}$, there is a $\mathcal{B}$-component $B(\gamma)$ containing γ and an iterate $n(\gamma)$ such that $f^{n(\gamma)}(\gamma)$ is the periodic star chosen above and such that the degree of $f^{n(\gamma)}|_{B(\gamma)}$ is at most δ.*

2. The diameter of any sequence of distinct stars G^*_k of $\mathcal{G}^*$ tends to 0.

PROOF. The first statement follows from the construction of $\mathcal{B}$. Set $B_k = B(G^*_k)$. There are (minimal) iterates $n_k \ge 0$ such that $f^{n_k}(B_k) = B(\gamma)$, with $n_k \to \infty$ as $k \to \infty$ Since *deg* $f^n|_{B_k} \le \delta$ for all k, the lemma is a consequence of the so-called shrinking lemma (see p. 86 of [13] for a proof). ■

2.2 Equicontinuity of (h_t) and fiber structure of limit maps

In this section we prove:

Proposition 2.3 *In the setting of Theorem 2.1, the maps (h_t) are equicontinuous. Furthermore, for any limit map H of (h_t), the nontrivial fibers of H are exactly the $\mathcal{R}$-components.*

For this we need

Lemma 2.4 *The family (h_t) is uniformly equicontinuous if and only if it is pointwise equicontinuous, i.e., for any $z_0 \in \overline{\mathbb{C}}$, any $\epsilon > 0$, there exist $\eta > 0$ and $t_0 < 1$, such that for any $t \in [t_0, 1)$ and any y with $d(y, z_0) \le \eta$, we have $d(h_t(y), h_t(z_0)) \le \epsilon$ (where d denotes the spherical metric).*

The proof is the same as for the statement that a pointwise continuous map on a compact set is uniformly continuous.

Now the proof of Proposition 2.3 is decomposed into 4 steps, which we proceed as follows.

Step 1. *Equicontinuity of (h_t) at any $z_0 \notin J_f \cup \mathcal{R}$.* This follows from the local uniform quasiconformality due to the construction of $\widetilde{P}_t$.

For the remaining cases the only information we have on the family (h_t) is on its Beltrami forms, and therefore how conformal invariants are modified after the application of h_t. So the equicontinuity will be proved by using the following lemma which enables us to translate conformal invariants estimates into metric estimates.

Lemma 2.5 (Equicontinuity criterion at a point) *Let $\mathcal{A} = \{h : \mathbb{D} \to \mathbb{C}\}$ be a family of continuous injective maps such that $\cup_{h \in \mathcal{A}} h(\mathbb{D})$ avoids at least 2 points in $\mathbb{C}$.*

1. Let $(U_n)_{n \geq 0}$ be a nested sequence of disk-like neighborhoods of the origin in the unit disk $\mathbb{D}$ such that $A'_n = \mathbb{D} \setminus \overline{U_n}$ is an annulus. If there exists a sequence $\eta_n \nearrow +\infty$ such that

$$\forall\, h \in \mathcal{A},\ \forall\, n \geq 0,\ \operatorname{mod} h(A'_n) \geq \eta_n,$$

then $\mathcal{A}$ is equicontinuous at the origin.

2. Let $A_n \subset \mathbb{D}$ be a nested sequence of annuli (i.e. for all n, A_{n+1} is contained in the component containing 0 of $\mathbb{C} \setminus \overline{A}_n$). If there is $M > 0$, such that

$$\forall\, h \in \mathcal{A},\ \forall\, n \geq 0,\ \operatorname{mod} h(A_n) \geq M,$$

then $\mathcal{A}$ is equicontinuous at the origin.

The proof of 1. relies on the fact that if an annulus A' in $\overline{\mathbb{C}}$ has modulus at least C, then one of the complementary component of A' has spherical diameter at most D, with D depends only on C. Please refer for example to Theorem 2.4 of [16] or to [23], appendices A and B. The part 2. follows by applying the Grøtzsch inequality.

Lemma 2.6 (Control of moduli of deformed annuli). *Let $A \subset \mathbb{C}$ be a bounded annulus such that $\partial A \cap \mathcal{G} = \emptyset$ (recall that $\mathcal{G}$ is open). Then there is $m > 0$ (depending on A but not on t), such that $\operatorname{mod} h_t(A) \geq m$ for all t.*

We postpone the proof of this lemma to § 2.3.

Lemma 2.7 (One good annulus around each Julia point). *Fix $r > 0$ (which will be the constant for f in the definition of weak hyperbolicity).*

For any $x \in J_f \setminus \mathcal{R}$, there are two open neighborhoods $N'(x)$ and $N(x)$ of x in $D(x, \frac{r}{4})$ and $m > 0$ such that $\operatorname{mod} h_t(N(x) \setminus \overline{N'(x)}) \geq m$ for all t.

For any $x = \beta_\gamma \in \mathcal{R}$, with γ an $\mathcal{R}$-component with repelling end β_γ and $B(\gamma)$ the corresponding $\mathcal{B}$-component, there are two open neighborhoods $N'(\gamma)$ and $N(\gamma)$ of γ in $(D(\beta_\gamma, \frac{r}{4}) \cup \mathcal{F}) \cap B(\gamma)$, labeled also by $N(\beta_\gamma)$ and $N'(\beta_\gamma)$, and there is $m > 0$, such that $\operatorname{mod} h_t(N(\beta_\gamma) \setminus \overline{N'(\beta_\gamma)}) \geq m$ for all t.

Proof. Let $x \in J_f \setminus \mathcal{R}$. Choose $\overline{N'} \subset N \subset D(x, \frac{r}{4})$ Jordan neighborhoods of x avoiding at least two marked points such that no $\mathcal{G}^*$-component would have a closure that intersects both boundaries. Then

$$N' \cup \left(\bigcup_{\substack{G \cap \partial N' \neq \emptyset \\ G \,\mathrm{a}\, \mathcal{G}\text{-comp.}}} \overline{G} \cup \partial N' \right) \subset N \setminus \left(\bigcup_{\substack{G \cap \partial N \neq \emptyset \\ G \,\mathrm{a}\, \mathcal{G}\text{-comp.}}} \overline{G} \cup \partial N \right).$$

Moreover the right hand set is open and the left hand set is compact connected. So their difference has an annular component A, satisfying $\partial A \cap \mathcal{G} = \emptyset$.

There is $m > 0$, such that

$$\bmod h_t(N \setminus \overline{N'}) \overset{\text{Grötzsch}}{\geq} \bmod h_t(A) \overset{\text{Lemma 2.6}}{\geq} m\ .$$

Let $x = \beta \in J_f \cap \mathcal{R}$. We will choose N and N' similarly, but as neighborhoods of $\overline{G^*}(\beta)$ and as subsets of $(\mathcal{B}(x) \cap \mathcal{F}) \cup D(x, \frac{r}{4})$. ∎

Now we can prove

Step 2. *Equicontinuity of* (h_t) *at* $z_0 \in J_f \setminus ((\mathcal{R} \cup \{\textit{preparabolics}\}))$.

Proof. We will use a standard pullback argument. When x ranges over J_f, the sets $N'(x)$ define an open cover of $J_f \cup \mathcal{R}$, which is compact. We extract a finite sub-covering $N'(x_i), i = 1, \cdots, l$. Note that Lemma 2.7 implies the existence of $m > 0$ such that, for any t and any $i \in \{1, \cdots, l\}$, $\bmod h_t(N(x_i) \setminus \overline{N'(x_i)}) \geq m$.

Assume $z_0 \in J_f \setminus (\mathcal{R} \cup \{\text{preparabolics}\})$. By weak hyperbolicity, there are infinitely many n (the good iterates), such that f^n blows up a neighborhood of z_0 to $D(f^n(z_0), r)$ with degree at most δ. There is $i(n)$ such that $f^n(z_0) \in N'(x_{i(n)})$. Taking a subsequence if necessary we may assume $i(n) \equiv i$.

We distinguish two cases. Either $x_i = \beta_\gamma$ for some γ, then we use the fact $N(x_i)$ is contained in a $\mathcal{B}$-component so that we may apply Lemma 2.2. Or

$$f^n(z_0) \in N'(x_i) \subset N(x_i) \subset D\left(x_i, \frac{r}{4}\right) \subset D\left(f^n(z_0), \frac{r}{2}\right) \subset D(f^n(z_0), r)$$

for infinitely many n. Let E, U, with $E \subset U$, be the respective components of $f^{-n}(\overline{N'_i})$ and of $f^{-n}(N_i)$ containing z_0. Set $A_n = U \setminus E$. Then

$$\bmod h_t(A_n) \geq \frac{1}{\delta} \cdot \bmod h_t(N(x_i) \setminus \overline{N'(x_i)}) \geq \frac{m}{\delta}$$

(for the first inequality, see for example [23], the proof of Lemma 2.1). Taking again a subsequence if necessary, we may assume that the annuli A_n are disjoint, nesting down to z_0 (this follows from the shrinking lemma, due to the “Koebe space” $D(r) \setminus D(\frac{r}{2})$). This shows that (h_t) is equicontinuous at z_0. ∎

Step 3. *Equicontinuity of* (h_t) *at* $z_0 \in (\mathcal{R} \cup \{\textit{preparabolics}\})$. This part is postponed to §2.5.

In order to study the fiber structure of limit maps of (h_t), We will make use of the following lemma which is proved in §2.3 :

Lemma 2.8 (4 points). *Let (z_1, z_2, z_3, z_4) be four distinct points such that no pair belong to a same $\mathcal{R}$-component. Then, for Γ the set of Jordan curves which separate (z_1, z_2) from (z_3, z_4) ; there is $m > 0$ such that $\Lambda(h_t(\Gamma)) \geq m$ (for all t), (where $\Lambda(\Gamma)$ denotes the extremal length of the curve family Γ, cf. [2])).*

Step 4. *For any limit map of (h_t), the nontrivial fibers are $\mathcal{R}$-components.*

Proof. Recall that (h_t) are normalized to fix three points a, b, c in the complement of $\mathcal{G}^*$.

Assume $h_{t_n} \rightrightarrows H$. By Step 3 and its proof, H maps each $\mathcal{R}$-component to a point. We will show that they are the only fibers of H.

Choose $z \neq w$, so that z, w are not in the same $\mathcal{R}$-component. Let us assume that $H(z) = H(w)$. We may assume that $H(z) \notin \{a, b\}$ and that $\{z, w, a, b\}$ are in different $\mathcal{R}$-components by relabelling the points a, b, c if necessary. Let $\Gamma_{(z,w),(a,b)}$ be the set of curves separating $\{z, w\}$ to $\{a, b\}$. The assumptions of the Lemma 2.8 are satisfied, so $\Lambda(h_t(\Gamma_{(z,w),(a,b)})) \geq m > 0$ for all t. Consequently $H(z) \neq H(w)$ (see Corollary B.2 in the appendix). ∎

This ends the proof of Proposition 2.3, modulo Lemma 2.6, Lemma 2.8 and Step 3.

2.3 Estimates of conformal invariants

The technical Lemma 2.6 and Lemma 2.8 used above are proved in this section. The main idea is to formalize the non-influence of the deformation in appropriate cases by "forgetting" its support.

We start with the key Lemma due to Cui G.

Key Lemma. *There is a uniform constant $0 < c \leq 1$ with the following properties. Let $\eta : [0, 1] \to \mathbb{C}$ (resp. $\eta : [0, 1] \to \overline{\mathbb{C}}$) be a rectifiable curve with end points outside $\mathcal{G}$. Then*

$$l_{\rho_e}(\eta) \geq c \cdot d_e(\eta(0), \eta(1)), \quad (\textit{resp. } l_\rho(\eta) \geq c \cdot d(\eta(0), \eta(1))\ ) \ ,$$

where d_e is the Euclidean metric (resp. d the spherical metric), ρ_e (resp. ρ) is the same metric but with zero density in $\mathcal{Y}$ (the support of σ_t), i.e.

$$\rho_e(z)|dz| = (1 - \chi_{\mathcal{Y}}(z))|dz| \ , \qquad \rho(z)|dz| = \frac{1 - \chi_{\mathcal{Y}}(z)}{1 + |z|^2} \cdot |dz| \ ,$$

where $\chi_{\mathcal{Y}}$ denotes the indicatrix function associated to $\mathcal{Y}$.

Proof. We will modify η in the following way that will also be used later on: Let I be a maximal open subinterval such that $\eta(I) \subset \mathcal{G}$ (so that $\eta(I) \subset G$ and $\eta(\partial I) \subset \partial G$ for some G). Define $\eta'(I)$ to be the Euclidean (spherical) geodesic linking the two ends of $\eta(I)$. We claim then: for some $c > 0$,

$$l_{\rho_e}(\eta(I)) \geq c \cdot l_e(\eta'(I)), \quad (\text{resp. } l_\rho(\eta(I)) \geq c \cdot l(\eta'(I))\ ) \ .$$

Proof of the claim: If $\eta(I) \cap \mathcal{Y} = \emptyset$, then $l_{\rho_e}(\eta(I)) = l_e(\eta(I)) \geq l_e(\eta'(I))$. If $\eta(I) \cap \mathcal{Y} \neq \emptyset$, set $I =]s_1, s_2[$, then there are two subintervals, one on each end, $I_1 =]s_1, s_1'[$ and $I_2 =]s_2', s_2[$, such that $\eta(s_i) \in \partial G$ and $\eta(s_i') \in \partial(\mathcal{Y} \cap G)$. We claim that there are constants $c_G, c > 0$, with c independent of G, such that

$$l_{\rho_e}(\eta(I)) \geq \sum_{i=1,2} l_{\rho_e}(\eta(I_i)) = \sum_{i=1,2} l_e(\eta(I_i)) \geq \sum_{i=1,2} |\eta(s_i) - \eta(s_i')|$$

$$\stackrel{*}{\geq} c_G \cdot \left(\sum_{i=1,2} |\eta(s_i) - \beta(G)| \right) \geq c_G \cdot |\eta(s_1) - \eta(s_2)| = c_G \cdot l_e(\eta'(I)) \stackrel{*}{\geq} c \cdot l_e(\eta'(I)) \ .$$

Here the only non trivial inequalities are the ones marked with $*$. They are proved as follows: Lemma 2.9 proves the first $*$-inequality.

Lemma 2.9 *If $g : \mathbb{D} \to \mathbb{C}$ be a univalent function such that $g(z) = \lambda \cdot z + O(z)$ at the origin with $|\lambda| > 1$, and if γ_1 and γ_2 are two disjoint invariant arcs in $\mathbb{D} \setminus \{0\}$ landing at 0, then $\gamma_1 \cup \gamma_2$ forms a "quasi-arc" in the following sense: there is a constant $c > 0$, depending only on the germ g, such that for any $z \in \gamma_2$, $d_e(z, \gamma_1) \geq c \cdot d_e(z, 0)$.*

Proof. Let $\phi : \overline{U} \to \overline{D_r}$ be a linearization mapping (with $\overline{U} \subset \mathbb{D}$ so that $\phi(\overline{U})$ is a round disc). As $|\phi'(z)|$ is bounded from above and from below, we may estimate the distance in the $\phi(z)$ coordinate. For $\lambda = g'(0)$, $\phi(\gamma_1)$ and $\phi(\gamma_2)$ are both invariant by $z \mapsto \lambda \cdot z$. In other words, they are self-similar. There is a constant $c_0 > 0$ such that for $z \in \phi(\gamma_2) \cap \{\frac{r}{|\lambda|} \leq |w| \leq r\}$, $d(z, \phi(\gamma_1)) \geq c_0 \cdot |z|$, due to compactness and the fact that the two arcs are disjoint. The rest follows by self-similarity and bounded distortion of the univalent map ϕ. ■

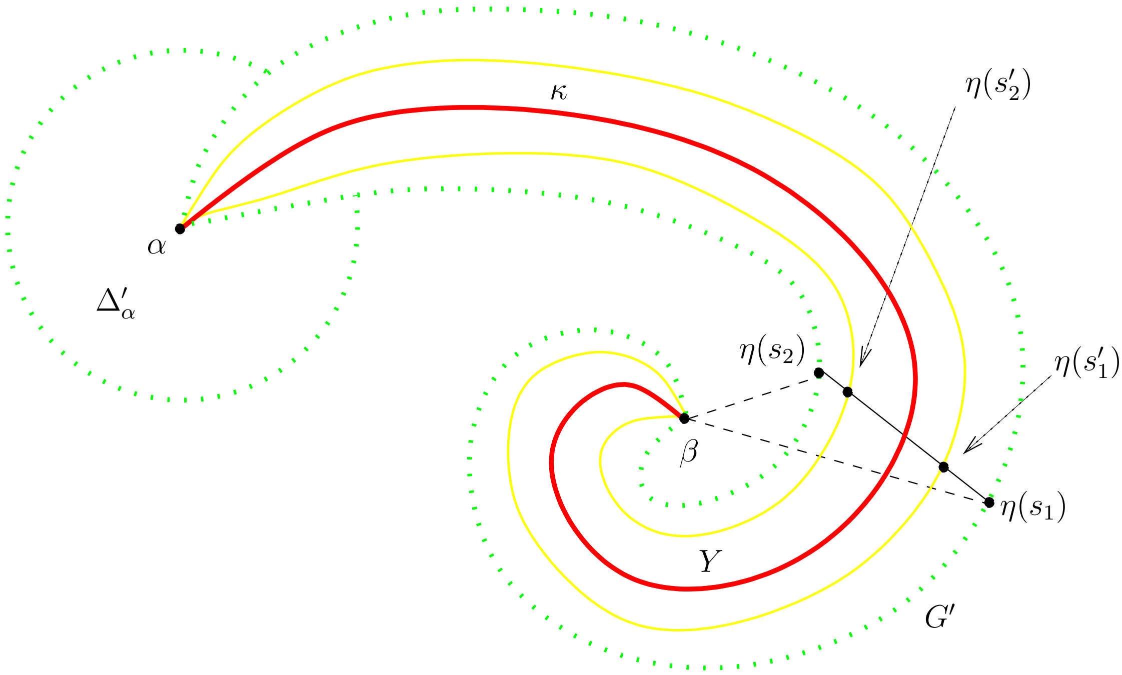


Figure 3. Cui's inequality

For each $\mathcal{G}$-component G there is a minimal n such that $f^n(\beta(G))$ is q-periodic and repelling, and f^n is either locally injective at $\beta(G)$ or has an isolated critical point at $\beta(G)$. We may then apply the above lemma to the (well-defined) map $F = f^{-n} f^q f^n$ in a neighborhood U_G of $\beta(G)$. Combining this with a compactness argument for $\overline{G} \setminus U_G$, we obtain the constant c_G.

It remains to show the second starred inequality, that is $c = \inf_G c_G > 0$.

Assume by contradiction that $\inf_G c_G = 0$. Then for some periodic γ and its corresponding $\mathcal{B}$-component B, there is $n_k \to \infty$, $B_k \in \mathcal{B}$, G_k a $\mathcal{G}$-component in B_k with attaching point β_k, and $z_k \in \partial G_k \setminus \{\beta_k\}$, $z'_k \in G_k \cap \partial \mathcal{Y}$ such that $|z_k - z'_k| / |z_k - \beta_k| \to 0$, as $k \to \infty$.

We let $\widehat{G}$ denote the union of the $\mathcal{G}$-components attached to $\beta = \beta(\gamma)$ (i.e. $\widehat{G} = G^*(\beta) \smallsetminus \{\beta\}$). Set $y_k = f^{n_k}(z_k)$, $y'_k = f^{n_k}(z'_k)$. We have $\beta = f^{n_k}(\beta_k)$ and it follows from above that $|y_k - y'_k|/|y_k - \beta| \geq C' > 0$. We will uniformize the domains in order to deal with proper maps ψ_k of the unit disc of degree at most δ :

$$\begin{array}{ccc} (\beta_k, z_k, z'_k),\ B_k & \xrightarrow{f^{n_k}} & B,\ (\beta, y_k, y'_k) \\ h_k \downarrow & & \downarrow h \\ (0, w_k, w'_k),\ \mathbb{D} & \xrightarrow{\psi_k} & \mathbb{D},\ (0, x_k, x'_k) \end{array}$$

As h has bounded distortion on $\widehat{G}$ which contains y_k, y'_k, we have $|x_k - x'_k|/|x_k| \geq C > 0$ and $h(\widehat{G}) \subset \overline{D}(0, r)$ for some $r < 1$. As $w_k, w_k, 0$ are in the same component of $\psi_k^{-1}(h(\widehat{G}))$, we know that $|w_k|, |w'_k| \leq s < 1$ (see for example [23], Lemma 2.1). Therefore h_k^{-1} has uniform bounded distortion, so that $|w_k - w'_k|/|w_k|$ is comparable to $|z_k - z'_k|/|z_k - \beta_k|$ and tends to 0.

Case 1. $|x_k| \geq r_1 > 0$. This implies that $|x_k - x'_k| \geq C r_1 > 0$. Switch to the hyperbolic metric $\rho(\cdot, \cdot)$ for convenience, and use the fact that ψ_k contracts ρ, we have

$$\frac{\rho(w_k, w'_k)}{\rho(w_k, 0)} \geq \frac{\rho(x_k, x'_k)}{C_1} \geq C_2 > 0 \ ,$$

which is a contradiction.

Case 2. $|x_k| \to 0$. One can check that w_k and 0 are in the same component of $\psi_k^{-1}(D(0, 2|x_k)|)$. This implies that $\rho(w_k, 0) \to 0$ (see for example [23], Lemma 2.1), (therefore $\rho(w_k, w'_k) \to 0$ and $\rho(x_k, x'_k) \to 0$).

We claim that there is $a > 0$ such that $|x'_k| \leq a|x_k|$. For otherwise for infinitely many k, $\psi_k^{-1}(\{z, |x_k| < |z| < |x'_k|\})$ would contain a round annulus centered at 0, separating w_k to w'_k, and having definite modulus. This contradicts the fact that $|w_k - w'_k|/|w_k| \to 0$.

For all large k, set $W_k = D(0, 2a|x_k|) \subset \mathbb{D}$ and let W'_k be the component of $\psi^{-1}(W_k)$ containing 0. One checks again that $w_k, w'_k \in W'_k$. Now uniformizing W_k, W'_k again we are more or less back to the situation of Case 1 (by checking as well that $w_k, w'_k, 0$ are always in the same pulled-back components). Therefore $\inf_G c_G > 0$.

This ends the proof of the claim for l_e. The spherical case is similar. We then get the Key Lemma by replacing $\eta(I)$ by $\eta'(I)$ for every possible I. ■

Proof of Lemma 2.6 (control of moduli of deformed annuli). Denote by δ the Euclidean distance between the two components of ∂A, and let Γ be the family of rectifiable curves joining them. It follows from the Key Lemma that for any $\eta \in \Gamma$, $l_{\rho_e}(\eta) \geq c\delta$. Set $A_t = h_t(A)$, and Γ_t the family of arcs joining the two boundary components of A_t, and $\rho_t = h_{t*}(\rho)$ (with $\rho_t = 0$ on $h_t(\mathcal{Y})$). Then

$$\operatorname{mod} A_t = \Lambda(\Gamma_t) = \sup_{\rho'} \frac{L_{\rho'}(\Gamma_t)^2}{Area_{\rho'} A_t} \geq \frac{L_{\rho_t}(\Gamma_t)^2}{Area_{\rho_t} A_t} = \frac{L_\rho(\Gamma)^2}{Area_\rho A} \geq \frac{(c \cdot \delta)^2}{Area_\rho A} \geq \frac{(c \cdot \delta)^2}{Area(A)} \overset{def}{\equiv} m(A) > 0 \,,$$

where the supremum is taken over all measurable conformal metrics ρ' on A_t, and $L_{\rho'}(\Gamma_t)$ denotes the infimum of the ρ'-lengths of curves in Γ_t. ■

Before proving Lemma 2.8, we first establish a lemma that gives a uniform control. This estimate will be also used to prove the weak hyperbolicity of limit maps of (f_t).

Let $z \in \overline{\mathbb{C}}$. We assign a compact subset $K(z)$ in the following way. If $z \in \mathcal{G}^*$, then there is a unique $\mathcal{G}^*$-component G^* and a unique $\mathcal{Y}^*$-component Y^* such that $z \in G^*$ and $Y^* \subset G^*$. We set $K(z) = Y^*$. If $z \notin \mathcal{G}^*$, we set $K(z) = \{z\}$.

Let $\mathcal{Q}$ be the set of distinct quadruples $q = (z_1, z_2, z_3, z_4)$ ($z_i \neq z_j$ for $i \neq j$). If $q \in \mathcal{Q}$, we let Γ_q be the set of rectifiable curves which separate (z_1, z_2) from (z_3, z_4).

For $r > 0$, we define $\mathcal{Q}_r \subset \mathcal{Q}$ as the set of quadruples such that $d(K(z_1), K(z_2)) \geq r$, $d(K(z_3), K(z_4)) \geq r$, $d(z_1, z_2) \geq r$ and $d(z_3, z_4) \geq r$.

Lemma 2.10 (Uniform control of lengths) *For all $r > 0$, there is a constant $\ell = \ell(r) > 0$ such that, for any $q \in \mathcal{Q}_r$ and any $\gamma \in \Gamma_q$, we have $l_\rho(\gamma) \geq \ell$ (where l_ρ is given in the Key Lemma).*

Proof. If not, let $q_n \in \mathcal{Q}_r$, $\gamma_n \in \Gamma_{q_n}$ such that $l_\rho(\gamma_n) \to 0$. Taking subsequences if necessary, we may assume that q_n tends towards $(z_1, z_2, z_3, z_4) \in \overline{\mathbb{C}}^4$, that γ_n tends in the Hausdorff topology towards a compact subset γ of diameter at least r.

Let $\gamma_n' = \gamma_n \setminus \mathcal{G}$. If there is a subsequence such that $diam(\gamma_{n_k}') \geq \delta > 0$ for all $k \geq 0$, then it follows from the Key lemma that

$$l_\rho(\gamma_{n_k}) \geq 2c \cdot diam(\gamma_{n_k}') \geq 2c\delta > 0$$

which is a contradiction.

Therefore, $diam(\gamma_n') \to 0$. Taking a further subsequence we may also assume that $\gamma_n' \to \{a\}$ in the Hausdorff topology. As $\mathcal{G}$ is open, we have $a \notin \mathcal{G}$. Therefore, for any $s, 0 < s < r$, and for all large n, we have

$$\gamma_n' \subset D(a, s)\ ,\ \gamma_n \subset D(a, s) \cup \mathcal{G}^*\ .$$

Hence γ_n is contained in the connected component D_s' of $D(a, s) \cup \mathcal{G}^*$ containing a.

Case 1. $a \notin \overline{G^*}$ for any $\mathcal{G}^*$-component G^*. Then $diam\, D_s' \to_{s\to 0} 0$. This contradicts $diam\, \gamma_n \geq r$.

Case 2. $a \in \partial G^* \setminus \{\beta(G^*)\}$ for some $\mathcal{G}^*$-component G^*. Then for s small enough, and n large, $\gamma_n \subset D_s' = D(a, s) \cup G^*$. Hence a pair of the quadruple q_n, say $\{z_{1n}, z_{2n}\}$ are in $D(a, s) \cup G^*$. But $q_n \in \mathcal{Q}_r$. This implies that $\{z_{1n}, z_{2n}\} \not\subset D(a, s)$, and $\{z_{1n}, z_{2n}\} \not\subset G^*$. Therefore, say, $z_{1n} \in G^*$ and $z_{2n} \in D(a, s)$, and consequently $z_1 \in \overline{G^*} \setminus D(a, s)$ and $z_2 = a$. It follows that $l_\rho(\gamma) \geq \min\{d(a, \mathcal{Y}), d(a, z_1)\} > 0$ which is a contradiction.

Case 3. $a = \beta(G^*)$ for some $\mathcal{G}^*$-component G^*. In this case $diam(D_s' \setminus \overline{G^*}) \to_{s\to 0} 0$. It is then easy to see that $d(K(z_{1n}), K(z_{2n})) \to 0$. This gives us a contradiction. ■

Proof of Lemma 2.8 (4 points). We proceed as in Lemma 2.6 but with the spherical metric. If neither sets $\{z_1, z_2\}$, $\{z_3, z_4\}$ is a subset of a green star of $\mathcal{G}^*$, then $q = (z_1, z_2, z_3, z_4)$ belongs to some $\mathcal{Q}_r$. Therefore, Lemma 2.10 implies that $l_\rho(\gamma) \geq \ell > 0$ for all $\gamma \in \Gamma_q$ and for some $\ell = \ell(r)$, and we obtain

$$\Lambda(h_t(\Gamma)) \geq \frac{\ell^2}{4\pi} > 0\,.$$

Otherwise, z_1 and z_2, say, belong to the same star G^* of $\mathcal{G}^*$. By assumption, at least one point in each group is not in $\mathcal{R}$. We consider now a new, smaller pair $(\hat{\mathcal{Y}}^*, \hat{\mathcal{G}}^*)$ of neighborhoods of $\mathcal{R} \setminus J_f$ such that neither $\{z_1, z_2\}$ nor $\{z_3, z_4\}$ is a subset of a $\hat{\mathcal{G}}^*$-component. Now we will need our technical assumption on the complex structure $\sigma_t(z)$. It implies that there is $t_0 > 0, t_0 < 1$, such that $\sigma_s = \sigma_{t_0}$ off $\hat{\mathcal{Y}}^*$, for all $t_0 < s < 1$. Therefore $h_s \circ h_{t_0}^{-1}$ is conformal off $h_{t_0}(\hat{\mathcal{Y}}^*)$. On the other hand, we may check that the proofs of Key Lemma and Lemma 2.10 are still valid on $h_{t_0}(\overline{\mathbb{C}})$, for the dynamics of f_{t_0}, and for the pair $(h_{t_0}(\hat{\mathcal{Y}}^*),\, h_{t_0}(\hat{\mathcal{G}}^*))$ (but probably with another constant c). Therefore, for all $t_0 < s < 1$,

$$\Lambda(h_s h_{t_0}^{-1}(h_{t_0}(\Gamma))) \geq \frac{\hat{\ell}^2}{4\pi} > 0\,.$$

This implies that $\Lambda(h_s(\Gamma)) \geq C > 0$ for $t_0 < s < 1$, with C independent of s. But $(h_t)_{0\leq t\leq t_0}$ is uniformly quasiconformal, so $\Lambda(h_t(\Gamma)) \geq m > 0$ for all $t \in [0,1[$, with m independent of t. ■

2.4 Equicontinuity of (f_t) and the proof of Theorem 2.1

In this section, we prove the equicontinuity of $(f_t)_t$ together with the weak hyperbolicity of any of its limits, and then Theorem 2.1.

The equicontinuity of (f_t) is due to Lemma A.4 in the appendix: in our case, assume $h_{t_n} \rightrightarrows H$. Then each $\mathcal{R}$-component is a fiber of H and all the other fibers are points. Therefore f maps any fiber of H into a fiber of H. Replacing both F_t and G_t by h_t, and replacing g by f in Lemma A.4, we conclude that $f_{t_n} = h_{t_n} \circ f \circ h_{t_n}^{-1}$ converges uniformly to a limit map F. It is automatically a rational map of the same degree as f.

Let us prove that F is weakly hyperbolic. Let $C = \inf_t dist_e(0, J_{f_t})$. Due to the facts that 0 is fixed in the Fatou set and disjoint from $J_f \cup \mathcal{R}$, and that $\{h_t\}_t$ is equicontinuous, we have $C > 0$.

Lemma 2.11 *Let $r < C/2$ then there is some constant $r' > 0$ such that, for all $x \in J_f \setminus (\mathcal{R} \cup \{preparabolics\})$ and any $y \notin \mathcal{G}^* \cup D(x,r)$, we have $|H(x) - H(y)| \geq r'$.*

Proof. We use the notation of §2.3. Let $x \in J_f \setminus (\mathcal{R} \cup \{\text{preparabolics}\})$, $y \in \mathbb{C} \setminus \mathcal{G}^*$ with $|y - x| \geq r$. Then $K(x) = \{x\}$ and $K(y) = \{y\}$. Let Γ be the set of rectifiable curves which separate $\{x, y\}$ from $\{0, \infty\}$. The quadruple belongs to $\mathcal{Q}_r$. The Lemma 2.10 implies that, for all $t < 1$, $\Lambda(h_t(\Gamma)) \geq C'_r$. We claim that $|h_t(x) - h_t(y)| \geq C \cdot \exp(-2\pi/C'_r)$. This would imply the lemma.

If $|h_t(x) - h_t(y)| \geq C$ then the claim is proved. Otherwise, $|h_t(x) - h_t(y)| < |h_t(x)|$, and therefore Lemma B.1 in the appendix implies that

$$|h_t(x) - h_t(y)| \geq |h_t(x)| \exp(-2\pi/C'_r) \geq C \cdot \exp(-2\pi/C'_r)\,,$$

which is the claim. ■

Proof that F is weakly hyperbolic. We assume that f is weakly hyperbolic with constants (δ, r_0). The weak hyperbolicity will come from two different arguments. The first will follow from Lemma 2.11 when the point considered is far from $\mathcal{G}^*$-components with large diameter, the second from Lemma 2.2. We will use the notation introduced in §2.1.

Given $n_0 \in \mathbb{N}$, we let $\mathcal{G}(n_0)^*$ be the union of the $\mathcal{G}^*$-components G^* such that there is an iterate $n \leq n_0$ such that $f^n(G^*)$ is the periodic G_i chosen in the cycle. Since the diameters of the connected component of $\mathcal{G}^*$ shrink to 0, we may choose n_0 large enough so that the diameter of any $\mathcal{G}^* \setminus \mathcal{G}^*(n_0)$-component does not exceed $\min\{r_0, r'_{\beta_i}\}/3$.

Let us define $r_{\beta_i} = \min\{\frac{r_0}{3}, d(\beta_i, \mathcal{G}^*(n_0) \setminus G^*(\beta_i)), r'_{\beta_i}/3\}$. Recall that $B_{\beta_i} = B'_{\beta_i} \cup D(\beta_i, r_{\beta_i}/2)$. We let $r_1 = \min d(G^*, \partial B)$ where G^* ranges over $\mathcal{G}^*(n_0)$-components and $B = B_{G^*} \in \mathcal{B}$ satisfies $G^* \subset B$. We set $A(\beta_i) = \Delta_{\beta_i} \setminus f^{-k_i}(\Delta_{\beta_i})$. There is an $r_2 > 0$ such that, for any $z \in J_f \cap A(\beta_i)$, the disk $D(z, r_2)$ is disjoint from $\mathcal{G}^*(n_0)$. Let us note that $r_2 \leq |z - \beta_i| \leq r_{\beta_i}/2$.

Let $r < \min\{\frac{C}{2}, \frac{r_0}{3}, \frac{r_1}{2}, r_2\}$, and let us consider a point $x \in J_f \setminus (\mathcal{R} \cup \{\text{preparabolics}\})$. There is a sequence (n_k) such that $deg(W_k(x) \xrightarrow{f^{n_k}} D(f^{n_k}(x), r_0)) \leq \delta$. We construct inductively a sequence (n'_p) such that $deg(W'_p(H(x)) \xrightarrow{F^{n'_p}} D(F^{n_p}(H(x)), r')) \leq \delta$ where r' is associated to r by Lemma 2.11.

We assume that we already constructed $n'_1, \ldots, n'_{p-1}$. Let k be the smallest index so that $n_k > n'_{p-1}$. We distinguish two cases.

If $D(f^{n_k}(x), r) \cap \mathcal{G}^*(n_0) = \emptyset$, then we write D'_k to be the union of $D(f^{n_k}(x), r)$ with all $\mathcal{G}^*$-components G^* such that $G^* \cap D(f^{n_k}(x), r) \neq \emptyset$. Since $r < r_0/3$, it follows that $D'_k \subset D(f^{n_k}(x), r_0)$ so it is also the case for the fill-in D_k of D'_k. It follows from Lemma 2.11 that $H(D_k) \supset D(H(f^{n_k}(x), r'))$. Let $W_k(H(x))$ be the connected component of $F^{-n_k}(D(F^{n_k}(H(x)), r')$ which contains $H(x)$. Then the degree is at most δ since $D_k \subset D(f^{n_k}(x), r_0)$. Therefore, we set $n'_p = n_k$.

If $D(f^{n_k}(x), r) \cap \mathcal{G}^*(n_0) \neq \emptyset$, then there is a $\mathcal{G}^*(n_0)$-component G^* such that $f^{n_k}(x) \in B = B_{G^*}$ because $r < r_1$. Therefore, there is a minimal iterate j such that $f^{n_k+j}(x) \subset A(\beta)$. We set $n'_p = n_k + j$. Since $r < r_2$, it follows that $D(f^{n'_p}(x), r) \cap \mathcal{G}^*(n_0) = \emptyset$; we let D' be the union of $D(f^{n'_p}(x), r)$ with all $\mathcal{G}^*$-components which intersect that disk. Let $w \in D(f^{n'_p}(x), r)$, then

$$|w - \beta| \leq |w - f^{n'_p}(x)| + |f^{n'_p}(x) - \beta_i| \leq \left(r + \frac{r'_\beta}{3}\right) + \frac{r_\beta}{2} \leq \left(r_2 + \frac{r'_\beta}{3}\right) + \frac{r'_\beta}{6} \leq 2\frac{r'_\beta}{3},$$

since $r_2 \leq r_\beta/2$. It follows that $D'_k \subset \Delta'_\beta$ and the fill-in D_k of D'_k is also contained in Δ'_β. Therefore $H(D(f^{n'_p}(x), r)) \supset D(F^{n'_p}(H(x)), r')$, and the degree of the restriction of $F^{n'_p}$ to any connected component of $F^{-n'_p}(D(F^{n_k+j}(H(x)), r'))$ is at most δ (cf. the definition of r'_β).

This implies that F is weakly hyperbolic. ■

Proof of Theorem 2.1. We first assume that we are under the normalization fixing $(0, 1, \infty)$. It follows from Proposition 2.3 that (H_t) is an equicontinuous family, and that (f_t) is also. To prove the convergence of the deformation, it suffices to prove the uniqueness of the limits as $t \to 1$.

Therefore, we may now assume that there are sequence (t_n) and (s_n) tending to 1 such that $h_{t_n} \rightrightarrows H_1$, $h_{s_n} \rightrightarrows H_2$, $f_{t_n} \rightrightarrows F_1$, $f_{s_n} \rightrightarrows F_2$. As $H_{1,2}$ have the same fiber systems, there is a homeomorphism ϕ making the following diagram commute:

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{Id} & \overline{\mathbb{C}} \\ H_1 \downarrow & & \downarrow H_2 \\ \overline{\mathbb{C}} & \xrightarrow{\phi} & \overline{\mathbb{C}} \end{array} .$$

We claim that $H_i(J_f \cup \mathcal{R}) = J_{F_i}$ for the left hand set is F_i-fully invariant, has no isolated points, and periodic points are dense inside. This implies that ϕ is a topological conjugacy from F_1 to F_2. Moreover ϕ restricted to the Fatou set $\mathcal{F}$ of F_1 is equal to $H_2 \circ H_1^{-1}$. But due to our technical assumption on the complex structure σ_t, on any compact set of $H_1^{-1}(\mathcal{F})$ the two maps H_1 and H_2 integrate the same complex structure (this is still true even without the technical assumption, but the proof is more delicate). By local uniqueness the map ϕ is conformal on $\mathcal{F}$. The key point here is to apply a rigidity result of P. Haïssinsky, restated in Theorem 2.12 below, to conclude that in this case ϕ is globally conformal. But it fixes $0, 1, \infty$ due to normalization. So $\phi = id$. This gives the convergence of (h_t), with the same fiber system, i.e. properties I and II. The other properties follow. ■

Theorem 2.12 *Let f be a weakly hyperbolic rational map which is not a Lattès example. Assume that φ is a homeomorphism of $\overline{\mathbb{C}}$, conformal on $\overline{\mathbb{C}} \setminus J_f$, such that $\varphi \circ f \circ \varphi^{-1}$ is again a rational map. Then φ is a Moebius transformation.*

See [11] for a proof.

2.5 Equicontinuity of (h_t) at $z_0 \in \mathcal{R} \cup \{$preparabolic points$\}$ (Step 3)

The idea is to compare deformations of local dynamics with that of model parabolic dynamics.

For each $\nu \in \mathbb{N}$, we define the *model parabolic maps* $g_\nu(z)$ and the standard projection π_ν to be:

$$g_1(z) = g(z) = \frac{z}{1-z}\ ,\ g_\nu(z) = \left(\frac{g(\nu z^\nu)}{\nu}\right)^{\frac{1}{\nu}} = z\left(\frac{1}{1-\nu z^\nu}\right)^{\frac{1}{\nu}} = z(1+z^\nu+\cdots)\ ,\ \pi_\nu(z) = \nu z^\nu\ .$$

The map g_ν maps univalently $\mathbb{C} \setminus \pi_\nu^{-1}([1,\infty[)$ onto $\mathbb{C} \setminus \pi_\nu^{-1}(]-\infty,-1[)$, and has a parabolic fixed point at 0 with ν attracting petals. Note that $\pi(z) = -1/z$ conjugates $g(z)$ to the standard translation $T : z \mapsto z+1$. Therefore $\pi\pi_\nu$ is the Fatou coordinate for g_ν.

A *left sepal* (resp. *right sepal*) for g_ν is an invariant region corresponding to an upper (resp. lower) half-plane H in the $\pi \circ \pi_\nu(z)$ coordinates with $TH = H$. In case that the boundary is a straight line, it becomes a round disk centered in $i\mathbb{R}$ tangent to 0 for the map g.

One defines similarly *attracting (resp. repelling) petals* corresponding to right (resp. left) half-planes L with $TL \subset L$ (resp. $T^{-1}L \subset L$), and then *attracting and repelling invariant sectors* corresponding to right and left half strips. A *flower neighborhood* corresponds to the union of left and a right sepal, together with an attracting and a repelling petal.

When the boundaries of these regions are horizontal or vertical straight lines (in the $\pi\pi_\nu$ coordinate), we call them straight sepals, petals, etc.

Lemma 2.13 *Given any parabolic germ F with ν attracting fixed petals at 0, there is a neighborhood V of 0 for F, a straight flower neighborhood U of 0 for g_ν, and a qc-conjugacy φ from $(g_\nu(z), U)$ to $(F(z), V)$. Moreover φ can be chosen to be conformal on the sepals and on the repelling petals.*

Proof. We shall construct $\chi = \varphi^{-1}$.

We let $\{\mathcal{P}_{j,+}(F)\}_{j\in\mathbb{Z}/\nu\mathbb{Z}}$ (resp. $\{\mathcal{P}_{j,-}(F)\}_{j\in\mathbb{Z}/\nu\mathbb{Z}}$) be ν disjoint repelling (resp. attracting) petals of F numbered in cyclic order such that $\mathcal{P}_{j,+}(F)$ lies in between $\mathcal{P}_{j,-}(F)$ and $\mathcal{P}_{j+1,-}(F)$.

Let $\mathcal{S}_{l,j}(F)$ be a sepal which intersects $\mathcal{P}_{j,+}(F)$ and $\mathcal{P}_{j+1,-}(F)$ and $\mathcal{S}_{r,j}(F)$ be a sepal which intersects $\mathcal{P}_{j,+}(F)$ and $\mathcal{P}_{j,-}(F)$. We let $v_j(F)$ be the repelling axis of $\mathcal{P}_{j,+}(F)$. Finally, we let $\Phi_{j,\pm} : \mathcal{P}_{j,\pm}(F) \to \mathbb{C}$ be Fatou coordinates.

We define $\{v_j(g_\nu)\}_{j\in\mathbb{Z}/\nu\mathbb{Z}}$ to be the ν repelling axes of g_ν numbered in cyclic order.

Let $\chi : \mathcal{P}_{j,+}(F) \to \mathbb{C}$ be defined by $\chi(z) = (\pi\pi_\nu)^{-1} \circ \Phi_{j,+}(z)$ with the inverse branch of $(\pi\pi_\nu)$ chosen so that, for all $j \in \mathbb{Z}/\nu\mathbb{Z}$, χ maps $v_j(F)$ to $v_j(g_\nu)$. Restricting $\mathcal{P}_{j,+}(F)$ if necessary, we may assume that its image is contained in $\mathbb{D}$. It follows that

$$\chi \circ F = g_\nu \circ \chi$$

holds on $F^{-1}(\cup_j \mathcal{P}_{j,+}(F))$.

Therefore, we may use this functional equation to extend χ univalently to the sepals, shortening them if necessary. It remains to extend χ quasiconformally to invariant attracting sectors.

For each $j \in \mathbb{Z}/\nu\mathbb{Z}$, we let $\mathcal{P}'_j \subset \mathcal{P}_{j,-}$ be disjoint attracting invariant sectors which cover the set on which χ is not defined. On the attracting quotient cylinder $\mathbb{C}/\mathbb{Z}$, $\mathcal{P}'_j$ is an annulus. We assume that its modulus is finite and that its boundary is composed of two disjoint horizontal curves.

It follows from the uniqueness of Fatou coordinates up to an additive constant that the expression of χ on those two curves differ by a constant. Therefore, a quasiconformal extension of χ exists which conjugates the dynamics. The map $\varphi = \chi^{-1}$ satisfies the requirement of the lemma. ■

Let us return to our rational map f. Let $\gamma = \gamma_i$ be one of the repelling star-like graphs for f. Assume that it has ν edges $(\kappa_j)_{j\in\mathbb{Z}/\nu\mathbb{Z}}$ (numbered in cyclic order). Denote by q the (common) period of κ_j. Choose a smooth Jordan neighborhood $\Delta(\beta)$ of β so that its boundary intersects (transversally) each κ_j at only one point. A flower neighborhood of γ is in the form $\Delta(\beta) \cup_j (B'(\kappa_j) \cup \Delta(\alpha(\kappa_j)))$, that is, like a blue set.

Lemma 2.14 *There is a flower neighborhood V of γ, a flower neighborhood U of 0 for the model map g_ν, a collection R of closed sepals, one in each sepal of U, and a qc-conjugacy φ from $((U \setminus R), g_\nu)$ to $((V \setminus \gamma), f^q)$, such that φ maps conformally (sepals$(U) \setminus R$ onto $\bigcup_j B'(\kappa_j) \setminus \gamma$, and φ is conformal on a union C of invariant repelling sectors with $\varphi(C) \supset (J_f \cup \mathcal{G}) \cap (\Delta(\beta) \setminus \bigcup_j B'(\kappa_j))$.*

Proof. We proceed as in the previous lemma and define the inverse map $\chi = \varphi^{-1}$. Let V be a flower neighborhood of f as above, and let $\widehat{\psi} : \Delta(\beta) \setminus \{\beta\} \to \mathbb{C}$ be a multivalued, locally univalent map which satisfies $\widehat{\psi} \circ f^q(z) = \widehat{\psi}(z) + 1$.

1. For each κ_j, recall that $B'(\kappa_j)$ is the invariant strip neighborhood chosen in the set up of simple pinching.

Let $\{\mathcal{P}_{j,+}\}_{j\in\mathbb{Z}/\nu\mathbb{Z}}$ be the connected components of $\Delta(\beta) \setminus \gamma$ numbered in cyclic order.

2. For each j, choose an invariant subsector with piecewise smooth boundary $\mathcal{P}'_{j,+}$ such that

$$(J_f \cup \mathcal{G}) \cap (\Delta(\beta) \setminus \bigcup_i B'(\kappa_i)) \ \subset\ \mathcal{P}'_{j,+} \ \subset\ \mathcal{P}_{j,+} \setminus \bigcup_i \overline{B'}(\kappa_i)$$

(we assume that the boundaries intersect just at β and $\partial\Delta(\beta)$).

3. Around each repelling axis $v_j(g_\nu)$, choose a straight invariant sector neighborhood C_j such that the quotient Riemann surfaces have the same moduli :

$$\mathrm{mod}\ C_j/g_\nu = \quad \mathrm{mod}\ \mathcal{P}'_{j,+}/f^q\ .$$

Define $\chi : (\mathcal{P}'_{j,+}, f^q) \to (C_j, g_\nu)$ to be a conformal conjugacy (adjusting the outer boundary of C_j if necessary).

4. For each attracting axis of g_ν, choose one left sepal $\mathcal{S}_{j,l}$ and one right sepal $\mathcal{S}_{j,r}$ disjoint from $\bigcup_i C_i$.

5. Inside each sepal $\mathcal{S}_{j,l}$, define $R_{j,l}$ the uniquely determined sub-sepal so that

$$\mathrm{mod}\ (\mathcal{S}_{j,l} \smallsetminus R_{j,l})/g_\nu = \quad \mathrm{mod}\ B'^l(\kappa_j)/f^q$$

(where $B'^l(\kappa_j)$ is the left component of $B'(\kappa_j) \smallsetminus \kappa_j$).

Define $\chi : (B'^l(\kappa_j), f^q) \to (\mathcal{S}_{j,l} \smallsetminus R_{j,l}, g_\nu)$ to be a conformal conjugacy.

Do the same for each right sepal.

6. Extend χ quasiconformally to the domains $\Delta(\alpha_j)\setminus B'(\kappa_j)$ and $\Delta(\beta)\smallsetminus\bigcup_i(B'(\kappa_i)\cup\mathcal{P}'_{i,+})$. In both cases, the boundary are periodic half-strips in the log-linearizing coordinates for $\Delta(\alpha_j)\setminus B'(\kappa_j)$ and in $\widehat{\psi}$-coordinates for $\Delta(\beta)\smallsetminus\bigcup_i(B'(\kappa_i)\cup\mathcal{P}'_{i,+})$, and the restriction of χ to the boundary commutes with the translation by 1. This ensures the existence of a quasiconformal extension which conjugates the dynamics. The map $\varphi = \chi^{-1}$ satisfies the requirements of the lemma. ■

Proposition 2.15 (equicontinuity of (h_t) at $z_0 \in \mathcal{R}$). *Fix $z_0 \in \gamma$. For any $\varepsilon > 0$, there is $\delta > 0$, for any $|z - z_0| < \delta$ and any t, $|h_t(z) - h_t(z_0)| < \varepsilon$.*

Proof. For simplicity we will only treat the case that γ has a unique edge κ.

Let φ, U, R be as in the above lemma. Note that $\pi = \pi^{-1}$. Denote by B^l the left sepal of U. There is a translation T_σ by a pure imaginary constant, mapping $\pi(B^l)$ to $\{y > L_b\}$ and $\pi(B^l \smallsetminus R)$ to $\{L_b < y < L_r\}$. Define $P_t(z)$ on B^l to be $\pi T_\sigma^{-1}\widetilde{P}_t T_\sigma \pi(z)$. Define $P_t(z)$ on the right sepal B^r similarly. It extends by identity to the remaining part of U.

Set $\mu_t = \varphi^*\sigma_t = \mu_t^b \wedge \mu_t^c \wedge \mu^s$, where we have decompose μ_t into three structures with disjoint support $B(= B^l \cup B^r)$, C and S respectively. Note that $\mu_t^s = \mu^s$ is independent of t.

We will integrate them one by one: at first P_t integrates μ_t^b, conjugates g to g, and is the identity on $S \cup C$. The next step C_t integrates μ_t^c, conjugates g to a germ X_t, finally S_t integrates $(C_t)_*\mu^s$, it is a K-qc map with K independent of t, and conjugates X_t to a parabolic germ Y_t.

We have

$$h_t|_V = H^{-1} \circ S_t \circ C_t \circ P_t \circ \varphi^{-1}|_V$$

where H is a suitable univalent map.

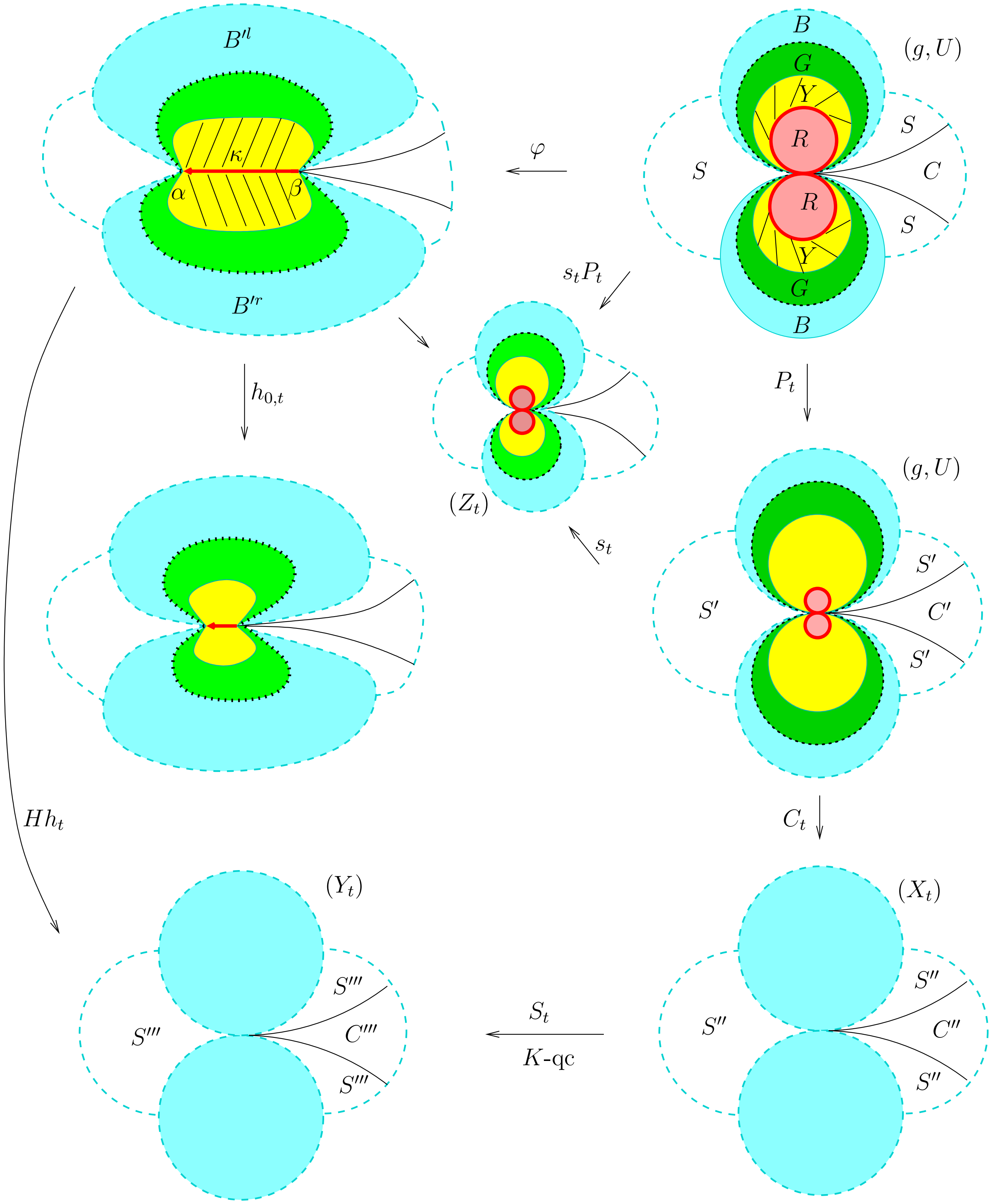


Figure 4. Relations with various models

Let us end up the proof assuming the following lemma.

Lemma 2.16 *For any $M > 0$, there are $t_0 < 1$ and a closed neighborhood $\widehat{E}$ of γ, such that for any $t > t_0$,* $\operatorname{mod} h_t(V \setminus \widehat{E}) > M$.

Due to the normalization, Lemma 2.16 implies that

Given any $\varepsilon > 0$*, there are* $t_0 < 1$ *and a closed neighborhood* $\widehat{E}$ *of* γ*, such that for any* $t > t_0$*,* $\operatorname{diam} h_t(\widehat{E}) < \varepsilon$*.*

This, together with the continuity of $h_t(z)$ on $(t, z) \in [0, t_0] \times \overline{\mathbb{C}}$, gives the proposition. ■

In order to prove Lemma 2.16, we study at first properties of P_t and C_t:

Property of P_t**:** *For any* $r > 0$*, there are* $t_0 < 1$ *and* E *a full continuum neighborhood of* R*, such that for any* $t \geq t_0$*,* $P_t(E) \subset \overline{D}_r$*.*

This is due to the continuity of $P_t(z)$ on (t, z) with $P_1(R) = \{0\}$.

Property of C_t**:** *For some sequence* $r_k \searrow 0$*,* $\operatorname{mod} C_t(U \setminus \overline{D}_{r_k}) \to_{r_k \to 0} \infty$ *uniformly on* $t \in [0, 1[$*.*

Proof. Note that $\pi(C)$ is a left half strip in the form $\{x \leq a' < 0, |y| \leq b\}$. Set $\mathcal{G}' = \pi(\mathcal{G} \cap C)$. Denote by G' a general $\mathcal{G}'$-component. Define, for $a \in \mathbb{R}$, $a < a'$,

$$\Sigma_a = \{x = a, \, |y| \leq b\} \cup \bigcup_{G' \cap \{x \equiv a\} \neq \emptyset} \overline{G'} \, .$$

All discs $D(r_i)$ below are centered at 0. Choose one by one $R_0^- > 0$, $x_0 < 0$, $R_0^+ > 0$ and $N \in \mathbb{N}$ such that

$$\partial D(R_0^-) \subset \pi(U) \, , \; \Sigma_{x_0} \cap \overline{D}(R_0^-) = \emptyset \, , \; \Sigma_{x_0} \subset \overline{D}(R_0^+) \, , T^{-N} \Sigma_{x_0} \cap \overline{D}(R_0^+) = \emptyset \, .$$

Fix $k \in \mathbb{N}$. Set $x_k = x_0 - kN$. Let R_k^- be the radius of the largest disk not intersecting the interior of Σ_{x_k}. Define R_k^+ such that $-R_k^+ - x_k = -R_0^+ - x_0 =: C_0$. Set $r_k = 1/R_k^+$ and let Q be the rectangle $\{\log R_0^- < x < \log R_k^+; \; |y| \leq \pi\}$. Then

$$\operatorname{mod} C_t(U \setminus \overline{D}_{r_k}) \geq \operatorname{mod} C_t(\pi(\{R_0^- < |z| < R_k^+\})) = \operatorname{mod} C_t \pi(-e^Q) \, .$$

Define now ρ to be the density of the Euclidean metric on Q minus the yellow set, (more precisely on $Q \setminus \log(-\pi(\varphi^{-1}(\mathcal{Y} \cap \varphi(C))))$), and to be zero elsewhere. Let Γ be the set of arcs in Q connecting the two vertical segments. Then

$$\operatorname{mod} C_t(U \setminus \overline{D}_{r_k}) \geq \frac{L_\rho(\Gamma)^2}{Area_\rho(Q)} \overset{*}{\geq} \frac{(c(\log R_k^- - \log R_0^+))^2}{2\pi(\log R_k^+ - \log R_0^-)} \overset{k \to \infty}{\longrightarrow} \infty \, .$$

where $*$ is due to the following two facts:

1. Each arc in Γ contains a sub-arc with end points outside $-\log \mathcal{G}'$, of Euclidean length at least $\log R_k^- - \log R_0^+$, due to the construction of $R_k^\pm$.

2. There is a uniform Koebe space around each green component in Q so that the estimates in the Key Lemma still hold, probably with a different constant.

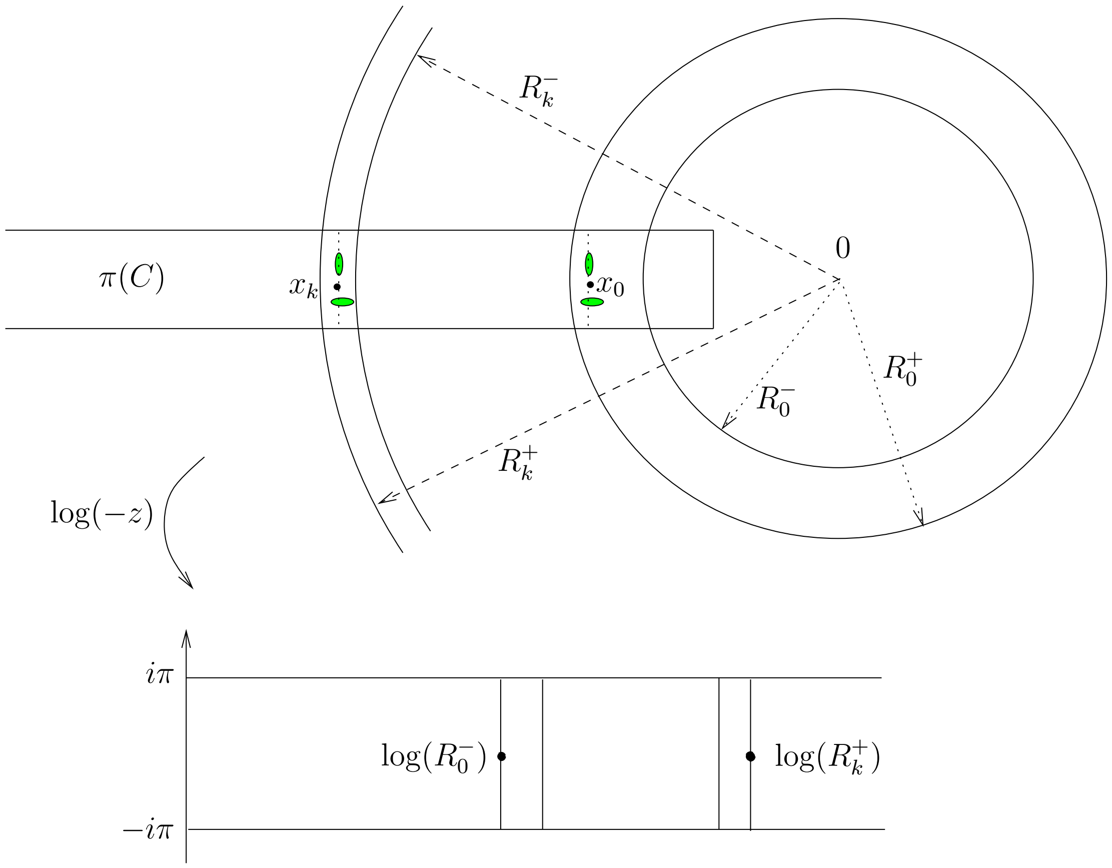


Figure 5. Estimates in the log-coordinate

Now the limit follows from $R_k^+ - R_k^- \le R_k^+ - (-x_k) + diam\,\Sigma_{x_k} = C_0 + diam\,\Sigma_{x_0}$ as $T_{kN}\Sigma_{x_k} = \Sigma_{x_0}$. This ends the proof of Property of C_t.

Proof of Lemma 2.16. For any $M > 0$, there is $r > 0$, for any t, $\mathrm{mod}\,(C_t(U \smallsetminus \overline{D}_r)) \ge KM$; and there are t_0, E such that for any $t > t_0$, $P_t(E) \subset \overline{D}_r$; set $\widehat{E} = \varphi(E)$; for any $t > t_0$,

$$\mathrm{mod}\,h_t(V \smallsetminus \widehat{E}) = \mathrm{mod}\,Hh_t(V \smallsetminus \widehat{E}) = \mathrm{mod}\,S_tC_tP_t(U \smallsetminus E)$$

$$\overset{S_t\text{ is K-qc}}{\ge} \frac{1}{K}\mathrm{mod}\,C_tP_t(U \smallsetminus E) \overset{\text{Grötzsch}}{\ge} \frac{1}{K}\mathrm{mod}\,C_t(U \smallsetminus \overline{D}_r) \ge \frac{1}{K}KM = M\ .$$

■

There are two variants of the above proof, one uses only the properties of P_t (which is used in Cui's original manuscript), and the other only the properties of C_t.

Denote by $h_{0,t} : V \to \overline{\mathbb{C}}$ that integrates the complex structure μ_t that is σ_t on $B'(\kappa)$ and σ_0 elsewhere, normalized so that $h_{0,t}(0) = 0$ and $h_{0,t}(z) = z + O(1)$ at ∞.

Now $(P_t \circ \varphi^{-1})_*(\mu_t)$ is g-invariant with dilatation K on S and conformal elsewhere. Denote by s_t the integrating map normalized as $h_{0,t}$. Then s_t conjugates g to a parabolic germ Z_t.

Then the argument above proves that $h_{0,t}$ is equicontinuous at γ and $h_{0,t}(\gamma) \to \{0\}$ as $t \to 1$. (As this does not rely on the estimates in the Key Lemma, it works for non-simple pinching as well, see [4].)

The other variant is the following:

Lemma 2.17 *Let F be a parabolic germ with r petals, defined and univalent on a neighborhood V of 0. Assume that $\mathcal{Y} \subset \mathcal{G} \subset V$ satisfies :*

a) $\mathcal{G}$ is open, and $\overline{G}_i \to \{point\}$ for any convergent sequence of distinct $\mathcal{G}$-components G_i,

b) $F^{-1}(\mathcal{G}) \subset \mathcal{G}$,

c) the Key Lemma holds for paths in V.

Then given any family of F-invariant structures σ_t with support contained in $\overline{\mathcal{Y}}$, the integrating map H_t (normalized as above) is equicontinuous at 0.

For this we use φ as in Lemma 2.13, and then C_t and S_t as above.

Apply this lemma to f^l near an l-periodic parabolic point, we get the equicontinuity of h_t at these points.

Finally in order to prove equicontinuity of h_t at backward components of γ_i or at preparabolic points, we use the same technique as the one to find c_G for preperiodic G. Details are omitted.

3 Matings of polynomials

3.1 Definition, existence and unicity

There are many equivalent ways to define matings of polynomials. We have chosen here the one presented by J. Milnor [18]:

Definition (the sphere). Following the terminology of J. Milnor, we consider $\mathbb{S}^2$ as the subspace of $\mathbb{C}\times\mathbb{R}$ defined by $\mathbb{S}^2 = H_+ \cup H_- \cup \{(z,0) \in \mathbb{C}\times\mathbb{R},\ |z| = 1\}$, where $H_+ = \{(z,r) \in \mathbb{C}\times\mathbb{R} | |z|^2 + r^2 = 1,\ r > 0\}$ and $H_- = \{(z,r) \in \mathbb{C}\times\mathbb{R} | |z|^2 + r^2 = 1,\ r < 0\}$. Let $\tau_\pm : \mathbb{C} \to H_\pm$ be the *gnomonic* projections defined by:

$$\tau_+(z) = (z,1)/\sqrt{|z|^2+1}, \quad \tau_-(z) = (\bar{z},-1)/\sqrt{|z|^2+1}\ .$$

The hemispheres $H_\pm$ are equipped with conformal structures by $\tau_\pm$.

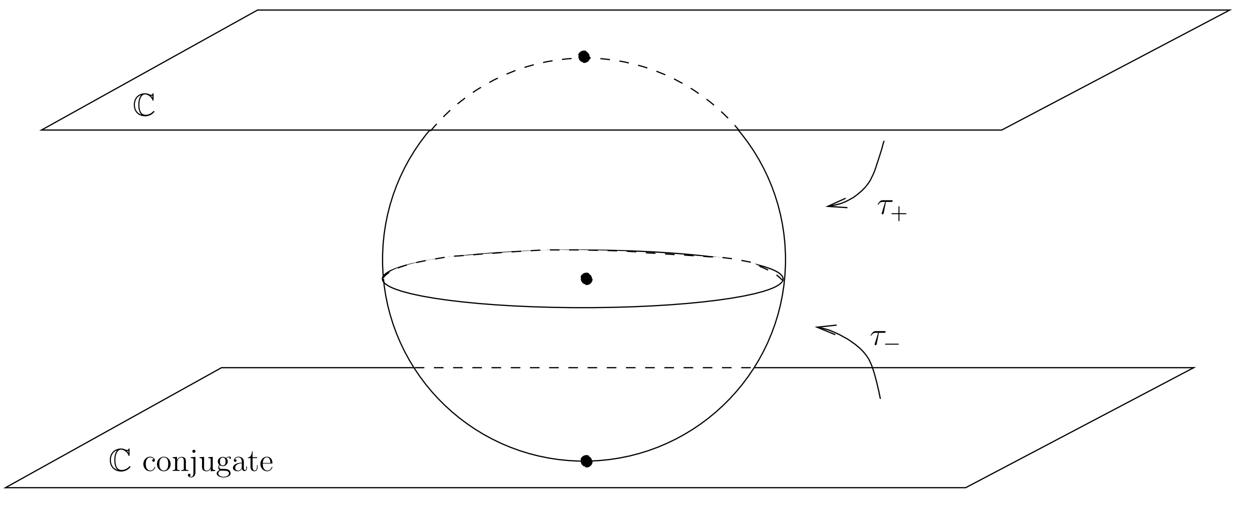


Figure 6. Matings and gnomonic projections

Definition (topological mating). Let $F, G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be two degree d ramified coverings. Here we allow d to be 1. Assume that ∞ is totally invariant for both F and G, and F and G are holomorphic in a neighborhood of ∞ with local expansion $z^d(1 + O(\frac{1}{z}))$ (i.e. they have the same leading term in their local expansion at ∞). We define the *topological mating* $F \perp\!\!\!\perp G : \mathbb{S}^2 \to \mathbb{S}^2$ to be the unique extension on $\mathbb{S}^2$ of $\tau_+ \circ F \circ \tau_+^{-1}|_{H_+}$ and $\tau_- \circ G \circ \tau_-^{-1}|_{H_-}$. By abuse of notation we will not distinguish F and $\tau_+ \circ F \circ \tau_+^{-1}|_{H_+}$ (resp. G and $\tau_- \circ G \circ \tau_-^{-1}|_{H_-}$). An easy calculation shows that the map $F \perp\!\!\!\perp G$ is a well defined degree d branched covering. In particular, if $d = 1$ then we get a homeomorphism of the sphere.

Assume now that f and g are two monic geometrically finite polynomials of degree $d \geq 2$ with connected Julia sets. The sphere $\mathbb{S}^2$ on which $f \perp\!\!\!\perp g$ is defined is equipped with the ray equivalence relation, which is defined to be the smallest equivalence relation generated by $x \sim y$ if x, y belong to the closure in $\mathbb{S}^2$ of an external ray of f or an external ray of g. The map $f \perp\!\!\!\perp g$ preserves this relation.

Definition. We say that (f, g, q, R) is a *marked mating*, if f, g are monic polynomials of degree d, R is a degree d rational map and $q : \mathbb{S}^2 \to \overline{\mathbb{C}}$ is a continuous map such that

1. (semi-conjugacy) $q \circ (f \perp\!\!\!\perp g) = R \circ q$;
2. (identification) $q(x) = q(y)$ if and only if x and y are ray-equivalent ;
3. (maximal conformality) q is conformal in $int(K_f) \cup int(K_g)$, $q(int(K_f) \cup int(K_g)) = \overline{\mathbb{C}} \setminus J_R$ and $q^{-1}(\overline{\mathbb{C}} \setminus J_R) = int(K_f) \cup int(K_g)$.

We will say that two monic polynomials f and g are *matable* if there exist a continuous map q and a rational map R such that (f, g, q, R) is a marked mating.

Unicity of R and q. K. Pilgrim provided examples of non-unicity of the conformal conjugacy class of R in a marked mating (f, g, q, R) in case that R is a Lattès example. For details, see [18], Appendix B.9. However, we have:

Proposition 3.1 *Let (f, g, q, R) be a marked mating. If f, g, R are weakly hyperbolic and R is not a Lattès example, then R is unique up to conformal conjugacy, and for a given R, the map q is unique up to postcomposition of an automorphism of R.*

Proof. Consider two marked matings (f, g, q, R) and (f, g, q', R'). By Lemma A.2 below, the map $q' \circ q^{-1}$ is a homeomorphism conformal off the Julia set which conjugates R to R'. We conclude by the rigidity Theorem 2.12. ■

Remark that it is not always easy to check that R is weakly hyperbolic, knowing that f and g are weakly hyperbolic. But if f and g are geometrically finite, so is R. Moreover, without any normalization on f and g, there are generally $d - 1$ possible topological matings.

Existence. There are many known results regarding the existence and non-existence of matings of postcritically finite polynomials, especially in degree 2 and 3. Interested readers can go to [18] and [24] for surveys of these results. The central tool is Thurston's theory for postcritically finite branched coverings. We mention here only the quadratic case: combining results of S. Levy, M. Rees, M. Shishikura and Tan L., we have:

Proposition 3.2 *Two postcritically finite quadratic polynomials f_c and $f_{c'}$ are matable if and only if c and c' do not belong to conjugate limbs of the Mandelbrot set.*

See [22, 25] for more details. One of our tasks here is to extend this result to geometrically finite quadratic polynomials (Corollary C).

Existence of marked matings of some postcritically infinite polynomials can now follow from surgeries. We state this in the next proposition, which in particular provides us with an extension of Proposition 3.2 to hyperbolic polynomials with infinite critical orbits.

Proposition 3.3 *Let (f, g, q, R) be a marked mating of polynomials f and g. Let us assume that $\hat{f}$ (resp. $\hat{g}$) is a quasi-regular map which coincides with f (resp. g) on its basin of infinity, and that μ (resp. ν) is a $\hat{f}$-invariant Beltrami form (resp. $\hat{g}$-invariant) supported on its filled-in Julia set. If we let φ (resp. ψ) be a quasiconformal homeomorphism which integrates μ (resp. ν) normalized at infinity to be tangent to the identity, then the polynomials $f_1 = \varphi \circ \hat{f} \circ \varphi^{-1}$ and $g_1 = \psi \circ \hat{g} \circ \psi^{-1}$ are matable.*

Proof. Define

$$\varphi \perp\!\!\!\perp \psi : \mathbb{S}^2 \to \mathbb{S}^2 \text{ by: } \varphi \perp\!\!\!\perp \psi = \begin{cases} \tau_+ \circ \varphi \circ \tau_+^{-1} & \text{on } H_+ \\ id & \text{on the equator} \\ \tau_- \circ \psi \circ \tau_-^{-1} & \text{on } H_- \end{cases}$$

The map φ restricted to the basin of ∞ of f realizes a conformal conjugacy from f to f_1. The normalizations guarantee that φ maps the θ external ray of f to the ray with the same angle of f_1. Therefore the map $\varphi \perp\!\!\!\perp \psi$ is a homeomorphism of $\mathbb{S}^2$, and a conjugacy from $\hat{f} \perp\!\!\!\perp \hat{g}$ to $f_1 \perp\!\!\!\perp g_1$ which preserves the external rays of the same angle.

Now we define a 'q-pushed forward' Beltrami form on the Riemann sphere ξ as follows:

$$\xi = \begin{cases} q_*\mu & \text{on } q(intK_f) \\ q_*\nu & \text{on } q(intK_g) \\ 0 \cdot d\bar{z}/dz & \text{elsewhere.} \end{cases}$$

Since q is conformal on $int(K_f) \cup int(K_g)$, the form ξ is R-invariant. Therefore, the measurable Riemann mapping theorem provides us with a quasiconformal homeomorphism $\chi : (\overline{\mathbb{C}}, a, b, c) \to (\overline{\mathbb{C}}, a, b, c)$ which solves the Beltrami equation associated to ξ.

Set $R_1 = \chi \circ R \circ \chi^{-1}$ and $q_1 = \chi \circ q \circ (\varphi \perp\!\!\!\perp \psi)^{-1} : \mathbb{S}^2 \to \overline{\mathbb{C}}$.

We can then check easily that (f_1, g_1, q_1, R_1) is a marked mating. ■

Remark. The proof of Theorem D will follow the same lines as Proposition 3.3, but instead of using the measurable Riemann mapping theorem, we shall use a generalization due to G. David [5].

3.2 Continuous paths of matings

Definition. Let $(f_t)_{t \in T}$ and $(g_t)_{t \in T}$ be two continuous family of matable polynomials. Let us say that they define a continuous family of matings if there exist continuous families $(q_t)_{t \in T}$ and $(R_t)_{t \in T}$ such that $(f_t, g_t, q_t, R_t)_{t \in T}$ defines a family of marked matings. If $T = [0, 1)$, we say that a continuous path of matings $(f_t, g_t, q_t, R_t)_{t \in T}$ is convergent if there exist limits f, g, q and R of f_t, g_t, q_t and R_t as t tends to 1 such that (f, g, q, R) is a marked mating.

We need at first a preliminary result which is a version with parameter of Proposition 3.3.

Let (f_0, g_0, q_0, R_0) be a marked mating of polynomials with connected Julia sets.

Let $(\hat{f}_t)_{t\in T}$ be a continuous family of quasi-regular maps, each coincides with f_0 on the basin of ∞ and maps the interior of K_{f_0} onto itself.

Let μ_t be $\hat{f}_t$-invariant Beltrami forms, continuous on $t \in T$, and with support contained in K_{f_0}. The measurable Riemann mapping theorem provides us with a continuous family of quasiconformal maps φ_t, integrating μ_t, and being normalized to be tangent to the identity at infinity. Clearly φ_t is conformal on the basin of ∞ of f_0.

Define $f_t = \varphi_t \circ \hat{f}_t \circ \varphi_t^{-1}$. They are again monic polynomials with connected Julia set, and depend continuously on t.

Assume that $\hat{g}_t, \nu_t, \psi_t, g_t$ are similar deformations of g_0.

Proposition 3.4 *In the above set up the marked matings $(f_t, g_t, q_t, R_t)_{t\in T}$ exist and depend continuously on t. In particular, if $(f_t)_{t\in[0,1)}$ and $(g_t)_{t\in[0,1)}$ are simple pinching deformations of f_0 and g_0 respectively (in this case $\hat{f}_t \equiv f_0$ and $\hat{g}_t \equiv g_0$), then (R_t) is a simple pinching deformation of R_0 and $(f_t, g_t, q_t, R_t)_{t\in[0,1)}$ is a continuous path of marked matings.*

Proof. We repeat the proof of Proposition 3.3 adding the subscript t. So we define subsequently the maps $\varphi_t \amalg \psi_t$, the q_0-pushed forward forms ξ_t and the quasi-regular dynamics $\hat{R}_t$. Since q_0 is conformal on $int(K_{f_0}) \cup int(K_{g_0})$, the forms ξ_t are $\hat{R}_t$-invariant and depend continuously on t. Therefore, the measurable Riemann mapping theorem provides us with a continuous family of quasiconformal homeomorphisms $\chi_t : (\overline{\mathbb{C}}, a, b, c) \to (\overline{\mathbb{C}}, a, b, c)$ which solves the Beltrami equation associated to ξ_t.

Set $R_t = \chi_t \circ \hat{R}_t \circ \chi_t^{-1}$ and $q_t = \chi_t \circ q_0 \circ (\varphi_t \amalg \psi_t)^{-1} : \mathbb{S}^2 \to \overline{\mathbb{C}}$.

We can then check easily that $(f_t, g_t, q_t, R_t)_{t\in T}$ are marked matings and are continuous with respect to $t \in T$.

If $(f_t)_t$ and $(g_t)_t$ are simple pinchings, we check that the q-pushed forward of the Beltrami forms defines a simple pinching deformation of R_0. ■

Now we can turn to the limit of simple pinchings. Consider simple pinchings (f_t, φ_t) and (g_t, ψ_t). We apply Proposition 3.4 and wish to prove Theorem B, which can be summed up by the following diagram:

$$\begin{array}{ccccc}
(\mathbb{S}^2, f_0 \amalg g_0) & \xrightarrow{\varphi_t \amalg \psi_t} & (\mathbb{S}^2, f_t \amalg g_t) & \xrightarrow{t\to 1} & (\mathbb{S}^2, f_1 \amalg g_1) \\
\downarrow q_0 & & \downarrow q_t & & \downarrow q_1 \\
(\overline{\mathbb{C}}, R_0) & \xrightarrow{\chi_t} & (\overline{\mathbb{C}}, R_t) & \xrightarrow{t\to 1} & (\overline{\mathbb{C}}, R_1)
\end{array}$$

Proof of Theorem B. It follows from Proposition 3.4 that (f_t, g_t, q_t, R_t) exists and is a marked mating for any $0 \le t < 1$, and that it depends continuously on t.

We can then apply Theorem 2.1 to all three deformations for f_0, g_0 and R_0 respectively and establish the convergences $f_t \rightrightarrows f_1$, $g_t \rightrightarrows g_1$, $\varphi_t \rightrightarrows \varphi_1$, $\psi_t \rightrightarrows \psi_1$, $R_t \rightrightarrows R_1$ and $\chi_t \rightrightarrows \chi_1$. One checks easily that $\varphi_1 \amalg \psi_1$ is well defined and semi-conjugates $f_0 \amalg g_0$ to $f_1 \amalg g_1$. It remains to prove the uniform convergence of q_t to a map q_1 as $t \to 1$, and that the limit 4-tuple (f_1, g_1, q_1, R_1) is also a marked mating.

By our definition of pushed forward deformation, the map q maps each equivalence class of $\varphi_1 \amalg \psi_1$ (a red star in K_{f_0} or in K_{g_0}) into an equivalence class of χ_1.

Define now $q_1 = \chi_1 \circ q \circ (\varphi_1 \amalg \psi_1)^{-1}$. By Lemma A.4, it satisfies the following property:

1. q_1 is well defined and continuous, q_t converges uniformly to q_1 and $q_1 \circ (\varphi_1 \amalg \psi_1) = \chi_1 \circ q$.

We have furthermore

2. $q_1 \circ (f_1 \amalg g_1) = R_1 \circ q_1$, since $q_t \circ (f_t \amalg g_t) = R_t \circ q_t$ and every map in the equation converges uniformly to the corresponding limit map.

3. $q_1(x) = q_1(y)$ iff x and y are ray-equivalent.

By Point 1 and Lemma A.3 we just need to show that $x \sim_q x'$ implies $\varphi_1 \amalg \psi_1(x) \sim_{q_1} \varphi_1 \amalg \psi_1(x')$. The map $\varphi_1 \amalg \psi_1$, as the uniform limit of $\varphi_t \amalg \psi_t$, maps external rays of $f_0 \amalg g_0$ bijectively onto external rays of $f_1 \amalg g_1$ with the same angle. This is exactly what we wanted.

4. q_1 is univalent in $int(K_{f_1}) \cup int(K_{g_1})$. Fix any open set L compactly contained in $int(K_{f_1}) \cup int(K_{g_1})$. For t close to 1 the maps q_t are univalent on L. So q_1 is either univalent or constant on L. The injectivity follows from Points 1 and 3 above. ∎

Remark. The only assumptions that we have really used were the existence of the marked mating (f_0, g_0, q_0, R_0) and the convergence of the three pinching deformations (f_t, φ_t), (g_t, ψ_t) and (R_t, χ_t).

As a consequence, we get a positive answer of J. Milnor's original question (in a more general form):

Corollary 3.5 *Let f and g be two quadratic hyperbolic polynomials in non-conjugate limbs of the Mandelbrot set. Then they are matable. Furthermore, assume that both maps have an attracting but non-superattracting periodic cycle. Then there exist a simple pinching path of f and a simple pinching path of g. The mating of the two paths yields a continuous path of marked matings, and converges to a marked mating of parabolic polynomials. In the particular case that the multipliers of the attractors are real and positive, the pinching paths can be chosen so that the multipliers remain real and tend to* 1 *(this is stronger than radial convergence in the sense of McMullen).*

Proof. In the hyperbolic component of f there is a unique postcritically finite map f_0 (one can obtain this by surgery). Similarly for g_0. They do not belong to conjugate limbs of the Mandelbrot set. So Proposition 3.2 ensures that f_0 and g_0 are matable, and that f and g are matable as well.

To find a simple pinching combinatorics for f and for g, we just need to lift a suitable simple closed curve in the quotient torus to the attractor. See for example [26].

Now we can apply Theorem B to conclude.

In the particular case with real positive multipliers for the attractors, we define an adapted pinching deformation. Let us first consider a conformal mapping which maps the Fatou component which contains the critical point to the unit disk. If it is properly normalized then it conjugates the first return map to

$$B(z) = z\frac{z + \lambda}{1 + \lambda z}$$

on the unit disc with $\lambda \in (0,1)$. We will define the deformation for B and then pull it back to our original map. We can check that the critical point of B is on the segment $(-1,0)$, choose the pinching combinatorics to be the real segment $[0,1]$, and the Beltrami forms in the pinching deformations to be symmetric with respect to the real axis for B. These choices will guarantee that the deformed maps f_t and g_t continue to have a real positive multiplier for the attractor, and these multipliers tend to 1 at the limit of pinching. ■

Proof of Corollary C. Let c and c' be geometrically finite polynomials. If they belong to conjugate limbs of the Mandelbrot set their respective α-fixed point have opposite external angles and therefore $\mathbb{S}^2/\sim_{ray}$ is not a sphere and the marked mating can not exist. See [25] for further details.

On the other hand, if they are not in conjugate limbs, then it is the same for $\mathcal{T}(c)$ and $\mathcal{T}(c')$, the center of the hyperbolic component having c, resp. c' as root. Therefore the marked mating of f_c and $f_{c'}$ exists by Corollary 3.5. ■

3.3 Theorems D and E

In this section, we prove Theorems D and E.

Postcritically finite polynomials associated to geometrically finite polynomials. Let f be a monic and centered geometrically finite polynomial with connected Julia set of degree $d \geq 2$. Let us recall that there is a unique conformal map $B_f : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_f$ which is tangent to the identity at infinity such that $B_f(z^d) = f \circ B_f(z)$. Since the Julia set is locally connected this map extends continuously to the closure and induces the Carathéodory loop $\gamma_f : \mathbb{S}^1 \to J_f$. We wish to associate a canonical postcritically finite polynomial $\mathcal{T}(f)$. We will proceed in two steps. The first step follows from the statement proved in [10] :

Proposition 3.6 *Let f be a monic centered geometrically finite polynomial with connected Julia set. Then there exist continuous families of sub-hyperbolic polynomials $(f_t)_{0\leq t<1}$ and of orientation preserving homeomorphisms $(h_t : J_f \to J_{f_t})_{0\leq t<1}$ such that (f_t, h_t) tends to (f, id) and such that $h_t \circ f = f_t \circ h_t$ on J_f. The maps h_t are obtained as continuous extensions of the composition $B_{f_t} \circ B_f^{-1}$ of the Böttcher coordinates of f and f_t, so $h_t = \gamma_{f_t} \circ \gamma_f^{-1}$ formally.*

Remarks. The construction of this perturbation is made in two steps: in the first step, we perturb f in the space of polynomial-like mappings in order to destroy all parabolic points and to get a stable family; in the second step, the straightening theorem is used to get back to the polynomial world. The stability and the hand-made peturbation help showing that the mappings h_t can be defined as continuous extensions of the Böttcher coordinates of f and f_t. They have no particular regularity besides continuity in this statement. It follows that K_f and K_{f_t} are described by the same pinched disk model (cf. [6]) so that h_t admits a homeomorphic extension to the plane which coincides with $B_{f_t} \circ B_f^{-1}$. It could be possible to prove that they admit $W^{1,2}_{loc}$ extensions to the plane, but this will not be needed in the sequel (see however Corollary 3.10).

We would like also to emphasize that, at this point, the perturbation which is given by this proposition has nothing to do with pinching deformations. But later on, we will use Proposition 3.6 to prove that we can actually choose it that way (see Proposition 3.11).

Given a monic and centered geometrically finite polynomial f with connected Julia set, the proposition provides us with a (monic and centered) sub-hyperbolic polynomial map $\mathcal{S}(f) = f_0$ with a conjugacy $h : J_f \to J_{f_0}$. If f is sub-hyperbolic, then we set $\mathcal{S}(f) = f$. By a simple surgery, we can then associate a canonical postcritically finite polynomial $\mathcal{T}(f)$ to $\mathcal{S}(f)$ (see for instance Theorem VI.5.1 of [3] or [15]).

We cut Theorem D into the next two propositions:

Proposition 3.7 *Let f, g be two monic and centered geometrically finite polynomials with connected Julia sets. If $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are matable, then f and g are matable too.*

Proposition 3.8 *Let f, g be two monic and centered geometrically finite polynomials with connected Julia sets. If f and g are matable, then $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are matable too.*

Our proof of Proposition 3.7 will rely on the following proposition [9].

Proposition 3.9 (parabolic surgery) *Let f be a sub-hyperbolic rational map with connected Julia set. We assume that f has an attracting cycle α and a repelling cycle β on the boundary of its immediate basin with period less or equal to α's. Then there are another rational map g and a μ-homeomorphism φ, locally quasiconformal in the Fatou set, univalent in the basin of ∞ and tangent to the identity at ∞ such that:*
(i) $\varphi(J_f) = \varphi(J_g)$, $\varphi(\beta)$ is parabolic and the immediate basin of α becomes $\varphi(\beta)$'s;
(ii) outside α's immediate basin, $\varphi \circ f = g \circ \varphi$; in particular, $\varphi : J_f \to J_g$ is a homeomorphism which conjugates the dynamics.

The class of μ-homeomorphisms is a generalization of quasiconformal maps [5]. It is stable under composition by quasiconformal maps and the associated Beltrami equation admits a unique normalized solution as the measurable Riemann mapping theorem. For more details on μ-homeomorphisms, please refer to [5]. Here also, the construction of φ has nothing to do with pinchings, but this statement remains true for a much wider class than sub-hyperbolic maps.

Corollary 3.10 *Let f be a geometrically finite polynomial with connected Julia set. There is a μ-homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ \mathcal{T}(f) = f \circ \varphi$ on $J_{\mathcal{T}(f)}$.*

Proof. Since $\mathcal{T}(f)$ and $\mathcal{S}(f)$ are quasiconformally conjugate on their Julia sets, we may replace $\mathcal{T}(f)$ by $\mathcal{S}(f)$. Let us apply Proposition 3.9 to $\mathcal{S}(f)$ at points which used to be parabolic for f. One obtains a polynomial F with a homeomorphism $h : J(f) \to J(F)$ such that $h \circ f = F \circ h$. This yields a correspondence between the Fatou components of f and F which has the property of matching the periodic points and components of the same nature together, and the restrictions of f and F to these domains have the same degree. It follows from C.T. McMullen's Proposition 6.7 and 6.8, and Theorem 6.1 d) from [15] that we may extend h as a plane homeomorphism which will be quasiconformal off the Julia sets. The rigidity Theorem 2.12 then implies that it is globally quasiconformal so that f and F are quasiconformally conjugate. Since the class of μ-homeomorphisms is preserved under the composition of quasiconformal maps, we have established the corollary. ∎

We are now ready to prove Proposition 3.7.

Proof of Proposition 3.7. We proceed as in Proposition 3.3. Let us apply Corollary 3.10 to f and g and let us denote by φ and ψ the μ-homeomorphisms that are given.

We denote by $(\mathcal{T}(f), \mathcal{T}(g), q, R)$ a marked mating. We let μ and ν be the Beltrami forms associated to φ and ψ. We push them forward by q on $\overline{\mathbb{C}}$ and denote the result by ξ. Let us define

$$H = q \circ (\varphi^{-1} \circ f \circ \varphi) \perp\!\!\!\perp (\psi^{-1} \circ g \circ \psi)) \circ q^{-1}.$$

This map is well defined and we can check that it is ACL. Moreover, ξ is H-invariant, and we may complete the proof analogously as the proof of Proposition 3.3 using G. David's generalization of the measurable Riemann mapping theorem (see [9] for further details). ■

Remark. We have used Proposition 3.9 because this strategy can be applied to a larger variety of matings for which we wish to create parabolic points. We could have used instead pinching techniques.

We propose an alternative proof of the Corollary C which does not use pinching deformations.

PROOF OF COROLLARY C. Let c and c' be geometrically finite polynomials. If they belong to conjugate limbs of the Mandelbrot set, we argue as in the first proof.

On the other hand, if they are not in conjugate limbs, then it is the same for $\mathcal{T}(c)$ and $\mathcal{T}(c')$, the center of the hyperbolic component having c, resp. c' as its root. Therefore the marked mating of f_c and $f_{c'}$ exists by Proposition 3.7. ■

It remains to prove Proposition 3.8 and Theorem E. The following will be used several times.

Proposition 3.11 *Let f be a geometrically finite polynomial with connected Julia set. There exist a sub-hyperbolic map f_0 and a simple pinching deformation of f_0 which converges to f.*

Remark. Cui G.Z. has proved this result for geometrically finite rational maps, regardless whether the Julia set is connected or not [4]. But his proof is more involved since it relies on Thurston's characterization of rational maps.

PROOF. We assume that f is monic and centered. We denote by A_{par} the union over all the parabolic points of their basin of attraction, and by A^*_{par} their immediate basins.

The Proposition 3.6 enables us to define a first continuous family $(\hat{f}_t)_{0\leq t<1}$ of monic, centered, sub-hyperbolic polynomials with homeomorphims $h_t : J_f \longrightarrow J_{\hat{f}_t}$ which conjugates the dynamics. Furthermore, for each $t \in [0, 1)$, the map $h_t \circ h_0^{-1} : J_{\hat{f}_0} \longrightarrow J_{\hat{f}_t}$ extends as a K_t-quasiconformal homeomorphism $\widehat{\varphi}_t$ where K_t is a non-decreasing function of t and $\widehat{\varphi}_0 = id$ because they lie in a stable analytic family of polynomials (cf. Prop. 4.2 in [10]). These maps coincide with the composition of the Böttcher coordinates off the filled-in Julia sets.

Actually, we may assume that $\widehat{\varphi}_t$ is uniformly quasiconformal off the closure of the new attracting domains. To see this, it is enough to restrict ourselves to the periodic attracting cycles of f. Conjugating the first return maps $\hat{f}_t^k$ to finite Blaschke products of the unit disk by Riemann maps, we see that we get a family of rational maps which are uniformly hyperbolic.

Let us note that this perturbation comes with a simple pinching combinatorics $\widehat{R}_t$ which link together all the new attracting and repelling points created by the desingularization of the parabolic points (cf. Prop. 2.1 in [10]).

Of course, there are no reasons why this deformation could be interpreted as a pinching deformation. The rest of the proof will be broken into three steps. They consist in "correcting" this path of polynomials to make it a pinching path. The first step will fix the right dynamics in the Fatou components which we wish to keep intact, the second step will study the limit of the simple pinching associated to the combinatorics coming from above, and the third will provide us with the sought pinching deformation.

Step 1. If f has no attracting point, then we define $\widehat{F}_t = \hat{f}_t$, and $\psi = id$, and we may directly proceed to Step 2. Otherwise, let $\alpha \in \mathbb{C}$ be an attracting periodic point of f of period k. Let U be the Fatou component containing α. For all t, we denote by $\hat{\alpha}_t$ the perturbation of α and $\widehat{U}_t$ the Fatou component bounded by $h_t(\partial U)$.

Since the Julia sets are locally connected, both components U and $\widehat{U}_0$ are Jordan domains. Furthermore, the restrictions of f^k and $\hat{f}^k$ to these domains have the same degree and contain an attracting point by continuity. Therefore, Proposition 6.7 in [15] implies that there is a quasiconformal homeomorphism $\hat{h}_0 : U \to \widehat{U}_0$ which coincides with h_0 on the boundary. We proceed similarly for all bounded Fatou components disjoint from A_{par}. We note that the quasiconformal distortion can be chosen to be uniformly bounded. For each t, we may define $h_t = \widehat{\varphi}_t \circ h_0$.

Let us define a new family of maps g_t by setting $g_t = h_t \circ f \circ h_t^{-1}$ where h_t^{-1} is defined, and $g_t = \hat{f}_t$ elsewhere. Each map g_t is continuous. Since g_t is quasi-regular on the complement of where it coincides with $\hat{f}_t$, Rickman's removability theorem implies that g_t is a well-defined quasi-regular map (see [21] or Lemma I.2 in [7]). Furthermore, we may define a g_t-invariant ellipse field E_t by setting $E_0 = (h_t)_* \mathbb{S}^1$ on the domain of h_t^{-1} and $E_t = \mathbb{S}^1$ elsewhere, where $\mathbb{S}^1$ denotes the conformal structure induced by the standard complex structure. Thus, the measurable Riemann mapping theorem provides us with a quasiconformal homeomorphism ψ_t and a monic centered polynomial $\widehat{F}_t = \psi_t \circ g_t \circ \psi_t^{-1}$. It follows from the construction that (ψ_t) is a normalized family of uniformly quasiconformal homeomorphisms, so that there is a quasiconformal map ψ which is the limit of a convergent sequence $(\psi_{t_n})_n$ with $t_n \to 1$.

Therefore, one also gets a limit $\hat{F}$ of $(\hat{F}_{t_n})$. The maps $\psi_t \circ h_t$ are conformal on $\mathbb{C} \setminus \overline{A_{par}}$, so we get a conformal conjugacy between f and $\hat{F}$ on this set. Furthermore, for any $z \in J_f$, $\psi_{t_n} \circ h_{t_n}(z)$ tends to $\psi(z)$. On the other hand, on any compact subset of A_{par}, $g_t = \hat{f}_t$ for t close enough to 1, so that $\hat{F} = \psi \circ f \circ \psi^{-1}$. Therefore f and $\hat{F}$ are conjugate by a homeomorphism which is conformal off J_f. It follows from Theorem 2.12 and the normalization that $\hat{F} = f$. Furthermore, the whole path $(\hat{F}_t)_t$ converges to f since it has only one accumulation point.

Step 2. We may define a simple pinching deformation (F_t, ϕ_t) of $F_0 = \widehat{F}_0$ supported by $\psi_0(\widehat{R}_0)$ with $\phi_t(z) = z + o(1)$ at infinity. We let $\mathcal{Y}$ be the yellow set for F_0. It follows from Theorem 2.1 that this deformation converges to some (F, ϕ). The map $H = \phi \circ \psi_0 \circ h_0$ defined off A_{par} defines a conjugacy between f and F which is conformal off $\overline{A_{par}}$.

We end this step by proving that H admits a quasiconformal extension to $\mathbb{C}$ which will extend the conjugacy. We will need more knowledge on $\hat{f}_t$ to proceed. We will refer to [10] when this is needed. The conjugacy between the parabolic basins will be first defined on the level of Fatou coordinates and then lifted to the dynamical planes.

Let β_f be a parabolic point for f, and let us denote by $A(\beta_f)$ its basin of attraction and by $\Phi_f : A(\beta) \to \mathbb{C}$ a Fatou coordinate that will be normalized later on. We define $\beta_F = \psi(\beta_f)$ (ψ is has been defined in Step 1), $A(\beta_F)$ and Φ_F similarly.

Let $c_1, \ldots, c_\nu$ be the critical points of f in $A(\beta)$. It follows from the construction of $\hat{f}_t$ that we may follow continuously the critical points with respect to t (cf. Prop. 3.1 in [10]). We may assume that c_1 is on the boundary of a petal of β_f defined as a preimage of a right-half plane and that $\Phi_f(c_1) = 0$.

The perturbation $(\hat{f}_t)_t$ enables us to follow continuously each critical point of $C(f) \cap A(\beta_f)$ (cf. Prop. 3.1 in [10]). This correspondence extends to $(\hat{F}_t)_t$. Hence we obtain a correspondence between $C(f) \cap A(\beta_f)$ and $C(F) \cap A(\beta_F)$ (cf Prop. 3.1). We denote by $H(c_j)$ the point corresponding to c_j. We normalize Φ_F so that $\Phi_F(H(c_1)) = 0$.

We let $\{\widehat{\gamma}_j, 1 \le j \le \nu\}$ be pairwise disjoint invariant curves of the plane by the map $T : z \mapsto z + 1$ which go through $\Phi_F(c_j)$. These curves define an order on the critical points since each splits the plane in two components. Let also $\widehat{\gamma}_j{}^+$ be the closure of the forward T-invariant component of $\widehat{\gamma}_j \backslash \{\Phi_f(c_j)\}$ (so it contains $\Phi_f(c_j)$), and γ_j be the connected component of $\Phi_f^{-1}(\widehat{\gamma}_j{}^+)$ which contains c_j. This curve joins c_j to finitely many preimages of β_f. The perturbation which defines $(\hat{f}_t)_t$ enables us also to follow continuously γ_j as a curve joining the critical point to preimages of the new attracting points (cf. Prop. 2.1 of [10]). Applying ϕ defines curves which join $H(c_j)$ to preimages of β_F which are the images by H of the corresponding preimages of β_f. We note that we may choose the pinching deformation (F_t, ϕ_t) so that the yellow set $\mathcal{Y}$ be disjoint of these curves. The Fatou coordinate Φ_F defines an order on $H(c_j)$ which is the same as the previous one. This means that we may define curves $\delta_j : [0,1] \to \mathbb{C}$ joining $\Phi_f(c_j)$ to $\Phi_F(c_j)$ such that their quotient $\pi(\delta_j)$ in $\mathbb{C}/\mathbb{Z}$ are pairwise disjoint, where $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ is the canonical projection. We let $D_j \subset \mathbb{C}/\mathbb{Z}$ be pairwise disjoint neighborhoods of $\pi(\delta_j)$. We choose them so that these domains are disjoint from $\pi \circ \Phi_F \circ \phi(\mathcal{Y})$. Therefore, there is a quasiconformal isotopy $(\overline{\omega}_s)_{0 \le s \le 1}$ defined on $\mathbb{C}/\mathbb{Z}$, supported on $\cup_j D_j$ such that $\overline{\omega}_0 = id$ and $\overline{\omega}_s(\pi \Phi_f(c_j)) = \pi \delta_j(s)$. Hence, there is a lift $(\omega_s)_s$ of $(\overline{\omega_s})_s$ to $\mathbb{C}$ such that $\omega_0 = id$ and $\omega_1(\Phi_f(c_j)) = \Phi_F(H(c_j))$.

We claim that there is a map $\tilde{\omega} : A(\beta_f) \to A(\beta_F)$ such that $\Phi_F \circ \tilde{\omega} = \omega_1 \circ \Phi_f$ and $\tilde{\omega}(c_j) = H(c_j)$. This follows from the facts that the critical points of Fatou coordinates are the precritical points of the polynomials and that their critical points are associated to the corresponding preimages of the parabolic points. Furthermore, $\tilde{\omega}$ maps petals to petals, so $\tilde{\omega}(\beta_f) = \beta_F$, and the extension to the boundary coincides with H. We denote by H this homeomorphism. Another application of Theorem 2.12 implies that H is a global quasiconformal map which conjugates f to F.

Step 3. We first define an F_0-invariant ellipse field E. On $\mathbb{C} \backslash \phi(\mathcal{Y})$, we let $E = (H \circ \phi)^* \mathbb{S}^1$, and on $\phi(\mathcal{Y})$, we let $E = \mathbb{S}^1$ be the field of circles, representing the standard complex structure. Let χ be given by the measurable Riemann mapping theorem with $\chi(z) = z + o(1)$ at infinity so that $f_0 = \chi \circ F_0 \circ \chi^{-1}$ is also monic and centered. Since χ is conformal on $\mathcal{Y}$, we may transport the simple pinching deformation defined for F_0 to f_0. We let (f_t, φ_t) be this new deformation and (f_1, φ) be the limit of this deformation provided by Theorem 2.1. We note that, since the fibers of ϕ and of $\varphi \circ \chi$ are the same, we may define a homeomorphism $\varphi \circ \chi \circ (H \circ \phi)^{-1}$ of the sphere which is conformal off J_F and which conjugates f to f_1. It follows from Theorem 2.12 that this homeomorphism is an affine map which has to be the identity thanks to the normalization. ■

We assume the reader is familiar with Thurston obstructions, Levy cycles ... as in [24].

Proof of Proposition 3.8. Suppose that (f, g, q, R) is a geometric mating of geometrically finite polynomials. We wish to prove that $\mathcal{T}(f)$ and $\mathcal{T}(g)$ are matable.

It follows from the Proposition 3.6 that there is a homeomorphism $h_f : J_{\mathcal{T}(f)} \to J_f$ which conjugates $\mathcal{T}(f)$ to f. We let $H_f : \mathbb{C} \to \mathbb{C}$ be a homeomorphic extension of h_f which is conformal off $K_{\mathcal{T}(f)}$ (cf. the Remark following the statement of Prop. 3.6 or Cor. 3.10). We define similarly h_g and H_g for g.

The map

$$H = q \circ (H_f \perp\!\!\!\perp H_g) : K_{\mathcal{T}(f)} \sqcup K_{\mathcal{T}(g)} / \sim_{ray} \to \overline{\mathbb{C}}$$

is a homeomorphism which maps $J_{\mathcal{T}(f)} \sqcup J_{\mathcal{T}(g)} / \sim_{ray}$ onto J_R. Let us define $T = H \circ (\mathcal{T}(f) \perp\!\!\!\perp \mathcal{T}(g)) \circ H^{-1}$ on the Riemann sphere. It follows that T is a postcritically finite ramified covering which agrees with R on J_R.

Claim 1. T has no Thurston obstruction.

Let us assume that T has an obstruction Γ. For each parabolic point of R and in each component of its immediate basin, we consider a hyperbolic geodesic ray which joins the point in $Post(T)$ to the parabolic point. The union of these curves produces a finite set of starlike graphs $\mathcal{S}$ as a simple pinching combinatorics. Let us remark that, for each parabolic point, we can include the critical orbits of these immediate basins (and only those) into a Jordan domain which can be contracted onto the starlike graph attached to this parabolic point. Let Ω be the union of these domains.

If Γ can be homotoped rel. $Post(T)$ so that it does not intersect $\mathcal{S}$, then we may assume that Γ is disjoint from Ω. Therefore, Γ is also an obstruction for R, which is impossible by Theorem B.4 in [16]. Whence Γ intersects $\mathcal{S}$. By the intersection theory of [24], this in turn implies that Γ is a Levy cycle. Therefore, there is a curve γ cutting $\mathcal{S}$ and a preimage γ' by some iterate $(\mathcal{T}(f) \perp\!\!\!\perp \mathcal{T}(g))^k$ isotopic to γ rel. the post-critical set such that $(\mathcal{T}(f) \perp\!\!\!\perp \mathcal{T}(g))^k : \gamma' \to \gamma$ is a homeomorphism.

The application of Proposition 3.11 produces simple pinchings (F_t, φ_t) and (G_t, ψ_t) of $F = F_0 = \mathcal{S}(f)$ and $G = G_0 = \mathcal{S}(g)$ which converge to $F_1 = f$ and $G_1 = g$. Let us consider maps h_F, h_G, H_F, H_G and

$$H_0 = q \circ (H_F \perp\!\!\!\perp H_F) : K_F \sqcup K_G / \sim_{ray} \to \overline{\mathbb{C}}$$

as above. Let also $S = H_0 \circ (F \perp\!\!\!\perp G) \circ H_0^{-1}$.

It follows that Γ is also a Levy cycle for S which intersects the pushforward L of the simple pinching combinatorics.

By pushing γ and γ' with $q \circ (\varphi_1 \perp\!\!\!\perp \psi_1) \circ H_0^{-1}$, one can extract curves ℓ and ℓ' in $\overline{\mathbb{C}}$ such that :
- ℓ and ℓ' join parabolic points of R in their repelling directions, or join a parabolic point in its repelling direction to a repelling postcritical point,
- ℓ' is a preimage of ℓ by an iterate $R^{k'}$,
- ℓ' is isotopic to ℓ rel. the post-critical set of R,
- all curves in their isotopy class has definite diameter and
- $R^{k'} : \ell' \to \ell$ is a homeomorphism.

This yields a contradiction because of Fatou's shrinking lemma (this is to be compared to the notion of degenerate combinatorial equivalence of Cui G. in [4], and was inspired by the proof of Theorem A therein). This proves the claim.

Since T has no obstruction, Thurston's characterization of rational maps implies that T is combinatorially equivalent to a rational map R_T. It follows from Theorem 2.1 in [22] and its proof that there is a continuous map $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, obtained as a uniform limit of homeomorphisms, such that $h \circ T = R_T \circ h$ and its restriction

$$h|_{\overline{\mathbb{C}} \setminus J_R} : \overline{\mathbb{C}} \setminus J_R \to \overline{\mathbb{C}} \setminus J_{R_T}$$

is a homeomorphism.

Claim 2. h is a homeomorphism of $\overline{\mathbb{C}}$.

It remains to prove that h is injective on J_R. Let us recall that $h \circ R = R_T \circ h$ holds on J_R.

Since h is a limit of homeomorphisms, it follows that preimages of connected sets are connected and full. Furthermore, since R and R_T are both degree d mappings, if K' is a connected component of $R_T^{-n}(K)$ for some continuum $K \subset J_R$ and the degree of $R_T|_{K'}$ is δ, then the restriction of R to $h^{-1}(K')$ is also a degree δ mapping onto $h^{-1}(K)$.

We work with the spherical metric.

Since R_T is sub-hyperbolic, a maximal degree δ and a radius $r > 0$ exist such that, for any $z \in J_{R_T}$, the degree of the restriction of R_T^n to any connected component of $R_T^{-n}(D(z, r))$ is bounded by δ.

Since R is geometrically finite, a weaker statement remains true: there is an $r' > 0$ such that, for any $\varepsilon > 0$ and any continuum $K \subset J_R$ of diameter less than r', there is an iterate n_0 such that for any $n \geq n_0$, any connected component L of $R^{-n}(K)$ has diameter at most ε (see for instance Prop. 3.1 in [11]).

We note that the diameter of $h^{-1}(\{z\})$ is bounded uniformly in z, so, using the (uniform) local connectivity of J_R, there is some integer N independent of z such that we may cover $h^{-1}(\{z\})$ in J_R by at most N continua of diameter less than r'. Let us fix $z \in J_{R_T}$ and set $K_n = h^{-1}(R_T^n(z))$. For $\varepsilon > 0$, let n be such that the diameter of any connected component of $R^{-n}(K)$ of any continuum $K \subset J_R$ of diameter less than r' is less than $\varepsilon/(\delta N)$. Let us cover K_n by N continua of diameter at most r'. It follows that $h^{-1}(\{z\})$ is covered by at most $N\delta$ continua of diameter at most $\varepsilon/(N\delta)$, so that the diameter of $h^{-1}(\{z\})$ is less than ε. This implies that $h^{-1}(\{z\})$ is a point and it establishes the claim that h is a homeomorphism.

Therefore, $(\mathcal{T}(f), \mathcal{T}(g), h \circ H, R_T)$ defines a geometric marked mating and this establishes Theorem D. ■

We may now prove Theorem E, which we recall here in a more precise form:

Corollary 3.12 *If (f, g, q, R) is geometrically finite marked mating with at least one parabolic point, then there exist sub-hyperbolic perturbations $(f_t)_{t\in[0,1)}$ and $(g_t)_{t\in[0,1)}$ which converge to f and g respectively as t tends to 1 such that $J_{f_t} \approx J_f$ $J_{g_t} \approx J_g$, and such that their matings (f_t, g_t, q_t, R_t) exist and converge to (f, g, q, R).*

Proof. Let us apply Proposition 3.11 to f and g so that we obtain two pinching deformations $(f_t, \varphi_t)_{t\in[0,1)}$ and $(g_t, \psi_t)_{t\in[0,1)}$. By Theorem D, there are q_0, R_0 so that (f_0, g_0, q_0, R_0) is a marked mating. Now Theorem B ensures that the path of marked matings (f_t, g_t, q_t, R_t) exists and converges to a marked mating (f, g, q_1, R_1). Now R_1 is again geometrically finite and has parabolic points, so is not a Lattès example. We can then apply the unicity result Proposition 3.1 to conclude that $R = HR_1H^{-1}$ with H a Möbius transformation. If $\hat{q}_t = H \circ q_t$ and

$\hat{R}_t = HR_tH^{-1}$, then $(f_t, g_t, \hat{q}_t, \hat{R}_t)_t$ is again a path of marked matings and it converges to $(f, g, H \circ q_1, R)$. Now $H \circ q_1$ and q differ by a Möbius map G with $GR = RG$. We may then re-adjust the mating path as above to get it converging to (f, g, q, R). ∎

A Quotient topology

Definition. Let A, B be two compact Hausdorff topological spaces and $F : A \to B$ be a continuous surjective map. We define the equivalence relation $\sim_F$ on A by: $x \sim_F y$ if and only if $F(x) = F(y)$.

We give several topological lemmas, admitting the first two.

Lemma A.1 *The equivalence relation $\sim_F$ is closed (i.e. if $x_n \sim_F y_n$, $x_n \to x$, $y_n \to y$ then $x \sim_F y$), and $[F] : [x]_F \to F(x)$ is a homeomorphism from $A/\sim_F$ to B.*

Lemma A.2 *Let $(X_i, \sim_i)$ $i = 1, 2$ be two topological spaces equipped with an equivalence relation each. Let $h : X_1 \to X_2$ be continuous such that if $x \sim_1 x'$ then $h(x) \sim_2 h(x')$. Let $\pi_i : X_i \to X_i/\sim_i$ be the quotient projections. Then the quotient map $[h] : X_1/\sim_1 \to X_2/\sim_2$ is well defined and continuous, and is surjective if h is.*

Lemma A.3 *Let A, B, C, D be compact Hausdorff topological spaces, $l : A \to C$, $r : A \to D$ and $s : C \to B$ be continuous surjective maps. In particular $s \circ l$ generates an equivalence relation in A. Assume that $x \sim_r x' \Rightarrow l(x) \sim_s l(x')$. Then there is a continuous surjective map $v : D \to B$ such that $v \circ r = s \circ l$ and $\sim_v = r_*(\sim_{s \circ l})$ (where $r_*(\sim)$ denotes the pushed forward equivalence relation, that is, $x \sim_{r_*(\sim)} x'$ if and only if $r^{-1}(x) \cup r^{-1}(x')$ belongs to a single equivalence class of $\sim$).*

$$\begin{tikzcd} & A \arrow[dl, "l"'] \arrow[dr, "r"] & \\ C \arrow[dr, "s"'] & & D \arrow[dl, "v"] \\ & B & \end{tikzcd}$$

Proof. By Lemma A.1, s and r can be considered as quotient maps. The map l satisfies the condition of h in Lemma A.2. Therefore v is well defined and continuous by Lemma A.2 with $v \circ r = s \circ l$. Since $s \circ l$ is surjective, v is surjective.

Let $x, x' \in D$. Now $r^{-1}(x) \sim_{sol} r^{-1}(x')$ if and only if $s \circ l(r^{-1}(x)) = s \circ l(r^{-1}(x'))$ if and only if $v(x) = v(x')$. ∎

Lemma A.4 *Let $g : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous surjective map. For $t \in [0, 1)$, let $F_t, G_t : \mathbb{S}^2 \to \mathbb{S}^2$ be two families of homeomorphisms of $\mathbb{S}^2$. Assume that, as $t \to 1$, F_t and G_t converge uniformly to continuous maps F_1, G_1 respectively, and g maps each fiber of F_1 into a fiber of G_1. Then $g_t = G_t \circ g \circ (F_t)^{-1} : \mathbb{S}^2 \to \mathbb{S}^2$ converges uniformly to a continuous map g_1, and $g_1 \circ F_1 = G_1 \circ g$.*

Proof. Define $g_1 = G_1 \circ g \circ (F_1)^{-1}$.

1. g_1 **is well defined and continuous.** For this we apply Lemma A.3. Let $l = g$, $r = F_1$ and $s = G_1$. We can check that all the conditions of Lemma A.3 are satisfied and therefore $g_1 = v$ is well defined, continuous, surjective and with fibers $F_{1*}(\sim_{G_1 \circ g})$.

2. g_t **converges uniformly to** g_1**.** In other words, for any $\varepsilon > 0$, there is t_0, for any $1 > t > t_0$ and any y, $|g_t(y) - g_1(y)| < \varepsilon$.

Assume by contradiction that g_t does not converge uniformly to g_1. That is there is $\varepsilon_0 > 0$, $t_n \to 1$ and $y_n \in \mathbb{S}^2$ such that $|g_{t_n}(y_n) - g_1(y_n)| > 2\varepsilon_0$.

We may assume $y_n \to y$ (by taking subsequences). Since g_1 is continuous, we have $g_1(y_n) \to g_1(y)$ and $|g_{t_n}(y_n) - g_1(y)| > \varepsilon_0$.

Let $x_n = (F_{t_n})^{-1}(y_n)$. We may assume $x_n \to x$ (by taking subsequences). We claim that $F_{t_n}(x_n) \to F_1(x)$ and $g_{t_n} \circ F_{t_n}(x_n) \to g_1 \circ F_1(x)$. The first limit is due to

$$|F_{t_n}(x_n) - F_1(x)| \leq |F_{t_n}(x_n) - F_1(x_n)| + |F_1(x_n) - F_1(x)| \to 0 \text{ as } n \to \infty$$

by uniform convergence of F_t. The second limit can be proved similarly, by uniform convergence of $g_t \circ F_t$ to $g_1 \circ F_1$, which is a consequence of the two equalities $g_t \circ F_t = G_t \circ g$, $G_1 \circ g = g_1 \circ F_1$ and the fact that $G_t \circ g$ converges uniformly to $G_1 \circ g$.

Therefore $F_1(x) = y$ and $g_{t_n}(y_n) = g_{t_n} \circ F_{t_n}(x_n) \to g_1 \circ F_1(x) = g_1(y)$. This leads to a contradiction.

3. $g_1 \circ F_1 = G_1 \circ g$, since $g_t \circ F_t = G_t \circ g$ for $t \in [0,1)$ and, as $t \to 1$, all maps in the equation converge uniformly to the corresponding maps. ∎

B An inequality

Lemma B.1 *Let $z, w \in \mathbb{C} \smallsetminus \{0\}$ be points in the plane such that $|z - w| < |w|$, and let Γ be the family of rectifiable curves which separate $\{0, \infty\}$ from $\{z, w\}$. Then*

$$|z - w| > |w| \exp \frac{-2\pi}{\Lambda(\Gamma)} \cdot$$

Proof. Let

$$h(\zeta) = \frac{\zeta - w}{\zeta + w}$$

be the Moebius transformation which maps $(0, \infty, w)$ to $(-1, 1, 0)$. Since

$$|z + w| = |2w + (z - w)| \geq 2|w| - |z - w| > |w|$$

and $\left|\frac{z-w}{z+w}\right| < 1$, the annulus

$$A = \left\{ \zeta \in \mathbb{C}, \left|\frac{z - w}{z + w}\right| < |\zeta| < 1 \right\}$$

is well defined and non-degenerate. Therefore,

$$\frac{1}{2\pi} \log \left|\frac{z + w}{z - w}\right| = \operatorname{mod} A \leq \frac{1}{\Lambda(h(\Gamma))} = \frac{1}{\Lambda(\Gamma)} \cdot$$

Hence
$$|z-w| \geq |z+w| \exp \frac{-2\pi}{\Lambda(\Gamma)} \geq |w| \exp \frac{-2\pi}{\Lambda(\Gamma)} \cdot$$
■

Corollary B.2 *Let z_t, w_t, a, b be four distinct points with z_t, w_t depending on a parameter t. Assume that $d(w_t, \{a,b\}) \geq C > 0$ for all t. If $d(z_t, w_t) \to 0$, then $\Lambda(\Gamma_{(z_t,w_t),(a,b)}) \to 0$. Consequently, if $\Lambda(\Gamma_{(z_t,w_t),(a,b)}) \geq C' > 0$, then $d(z_t, w_t) \geq C'' > 0$.*

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Peter Haïssinsky, LATP/CMI, Université de Provence, 39, rue Frédéric Joliot-Curie, 13453 Marseille cedex 13, France.
E-mail: phaissin@cmi.univ-mrs.fr

Tan Lei, Unité CNRS-UMR 8088, Département de Mathématiques, Université de Cergy-Pontoise, 2, avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France.
E-mail: tanlei@math.u-cergy.fr