Geometry of planar harmonic polynomials

joint work with J. J. Loeb and M. El Amrani

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A planar harmonic polynomial takes the real form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}, \quad P, Q \text{ harmonic}$$

or the complex form

$$f(z) = p(z) - \overline{q(z)}, \quad p, q \text{ holomorphic} .$$

$$\deg f := \max\{\deg p, \deg q\} .$$

Questions:

- Number of roots of f ?
- $\#f^{-1}(c) =$?
- Fiber structure of $f : \mathbb{C} \to \mathbb{C}$?
- How does f map $\mathbb{C}$ into $\mathbb{C}$?
- Is f open? proper? surjective? Does $f(\mathbb{C})$ contain interior?
- dependence on parameter?

Why?

- real algebraic geometry
- singularity theory of planar maps
  (Whitney, Arnold et al. $z^2 \rightsquigarrow z^2 + \epsilon\bar{z}$)
- holomorphic dynamics
- gravitational lensing

The equivalence relation

$$f \sim \boxed{\underset{\text{bi-holomorphic}}{\xrightarrow{\psi}} \quad \xrightarrow{f} \quad \underset{\text{linear, invertible}}{\xrightarrow{\Phi}}}$$

keeps the above properties invariant.

Lemma (degree 1 normal forms): $z, \quad z - \overline{z} = \begin{pmatrix} 0 \\ 2y \end{pmatrix}$.

Theorem (quadratic normal forms + properties)

	open	proper	surjective	interior$(f(\mathbb{C})) \neq \emptyset$
holomorhpic z^2	yes	yes	yes	yes
$(z-\overline{z}) \circ (z^2)$	no	no	no	no
cone $\begin{pmatrix} x \\ xy \end{pmatrix}$	no	no	no	yes
fold $\approx \begin{pmatrix} x \\ y^2 \end{pmatrix}$	no	yes	no	yes
deltoid $z^2 - \overline{z}$	no	yes	yes	yes

Theorem Except $f \sim$ hol. or $(z-\overline{z})\circ$ hol.
(first and last columns true) f not open and $f(\mathbb{C}) \ni$ interior.
The last line is valid for $f \sim p - \overline{q}$, $\deg(p) \neq \deg(q)$.
proper $\not\Leftrightarrow$ surjective, proper $\Longrightarrow$ $int(f(\mathbb{C})) \neq \emptyset$.

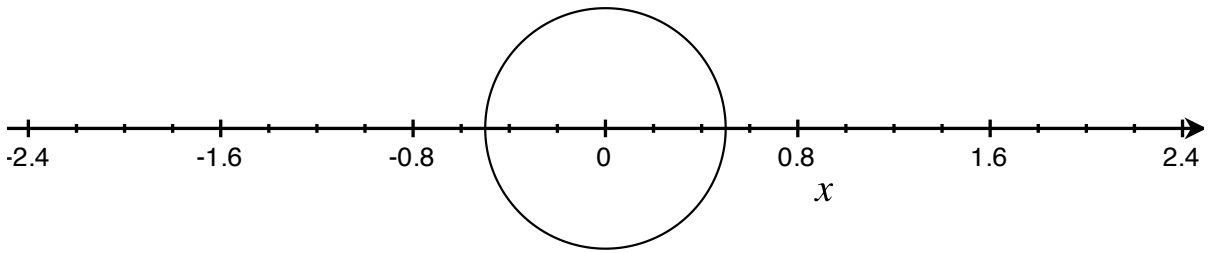
Critical set for $f = p - \overline{q} \not\sim$ hol.

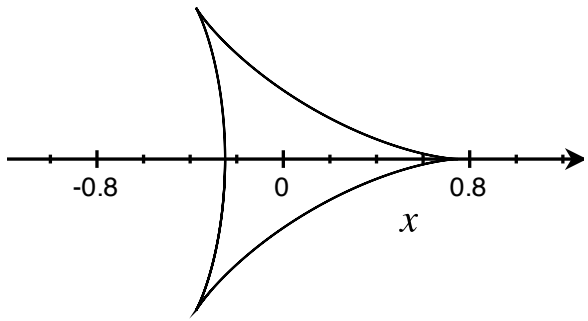
$\mathcal{C}_f$ = critical set = $\{z \mid Jac(f) = 0\} = \{z \mid |p'|^2 - |q'|^2 = 0\}$

$\mathcal{V}_f$ = critical value set = $f(C_f)$.

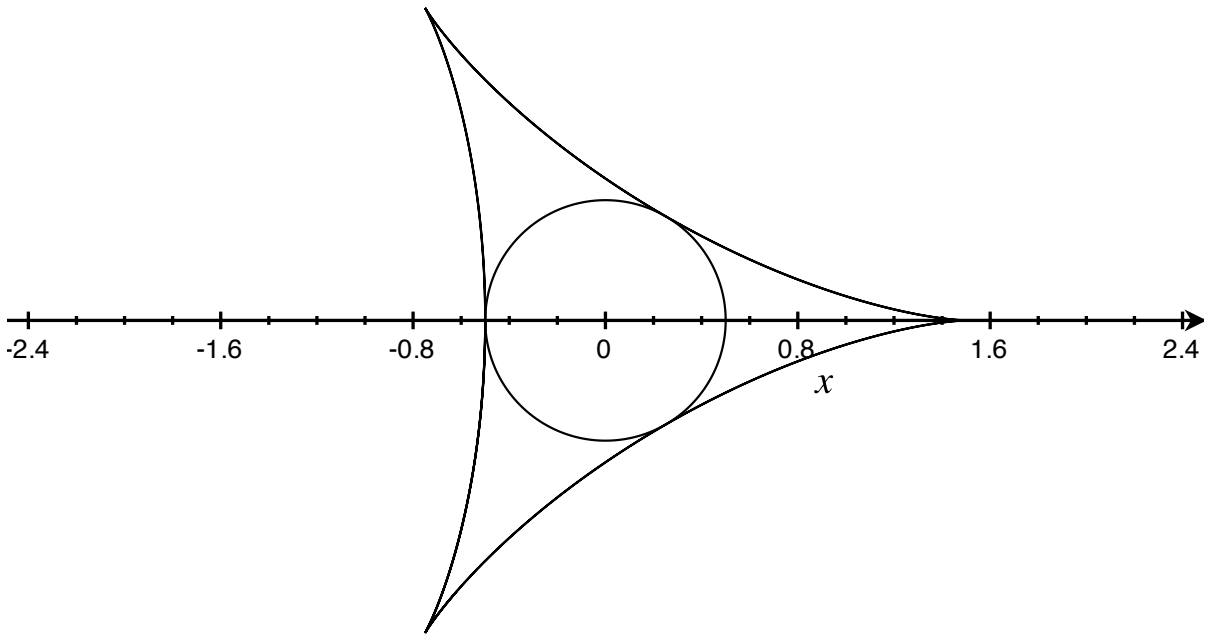
	$\mathcal{C}_f$	$\mathcal{V}_f$
z^2	$\{0\}$	$\{0\}$
$(z - \overline{z}) \circ z^2$	$\mathbb{C}$	$\mathbb{R}$
$\begin{pmatrix} x \\ xy \end{pmatrix}$	$i\mathbb{R}$	$\{0\}$
fold $\approx \begin{pmatrix} x \\ y^2 \end{pmatrix}$	a line	a parabola

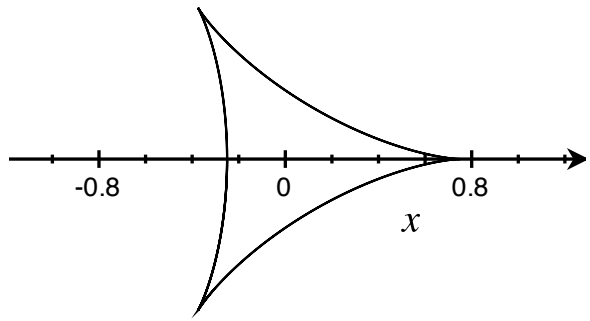
Case $z^2 - \overline{z}$, critical set and critical value set





Case $z^2 - \overline{z}$, co-critical set





Critical set for $f = p - \overline{q} \not\sim$ hol. , $\not\sim (z - \overline{z}) \circ$ hol.

Theorem

- $\mathcal{C}_f = \psi^{-1}(S^1) \cup \{\text{a finite set}\}$ for some rational map ψ, and conversely.
- $\mathcal{C}_f$ is a topologically finite (disconnected) graph;
- along an edge, the map f $\begin{cases} \text{either folds} \\ \text{or contracts to a constant} \end{cases}$

($\Longrightarrow$ f is not open.)

- at the vertices $\exists$ local topological models (Whitney + Lyzzaik).
- If $\Gamma \overset{\psi}{\longrightarrow} S^1$ then $\Gamma \ni$ three cusps.

Bounding #roots for $f = p - \overline{q}$, $d = \deg(f)$

Assume f proper,
or more particularly $\deg(p) \neq \deg(q)$ ($\Longleftrightarrow \mathcal{C}_f$ bounded).

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$.

In case $p(z) - \overline{z}$, $d > 1$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$.

Bshouty-Lyzzaik (2004) The bound $3d - 2$ is sharp for $d = 4, 5, 6, 8$.

Geyer (2008) sharpness for $\forall\, d$ (see also Buff-Epstein-Koch).

Khavinson-Swiatek : $\#\{\overline{p(z)} - z = 0\} \leq 3d - 2$

Set $K(z) = \overline{p(z)}$, $f(z) = K(z) - z$.

$$\begin{array}{rcl} & & \{K(z) = z, |K'| > 1\} \qquad\qquad \{K(z) = z, |K'| < 1\} \\ & & \qquad\qquad \| \qquad\qquad\qquad\qquad\quad \| \\ \#\{f(z) = 0\} = & \#\{f(z) = 0, \deg_z = -1\} & + \#\{f(z) = 0, \deg_z = 1\} \\ & \overset{\text{degree}}{=} & d + \#\{\deg_z = 1\} \\ = & d + 2 \cdot \#\{\text{attracting fixed pts of } K\} & \end{array}$$

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 & = & d + 2 \cdot \#\{\text{attracting fixed pts of } K\} \\
 & \overset{\text{Fatou}}{\leq} & d + 2(d-1) = 3d - 2
\end{array}
$$

Proof of Geyer's theorem

Find $p \in \mathbb{R}[z]$ of degree d such that

$$p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \overline{c} \text{ (except possibly one)} \\ p(c) = \overline{c} \end{cases}$$

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2. Prove that $\not\exists$ obstructions (Levy).
3. Apply Thurston's theorem to obtain a polynomial P.
4. P is real.

Geometry and counting of Geyer polynomials

Geyer:

$\#\{\text{degree } d \text{ Geyer polynomials}\}/(\text{real conj.})$

$$= m - \text{th Catalan number} = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} \text{ if } \begin{cases} d = 2(m+1) \\ d = 2m+1 \end{cases}$$

Thurston's theorem $\longrightarrow$ Classification results.

Counting roots for $f = p - \overline{q}$, $d = \deg(f)$

Theorem (lower bound and surjectivity).

$$\deg p \neq \deg q \Longrightarrow \forall\, c \in \mathbb{C}, \quad \#f^{-1}(c) \geq d \ .$$

Geyer (again) $\forall\, d \geq 2$, $\exists$ $p(z) \in \mathbb{R}[z]$ with ≤ 1 real critical point s.t.

$$\exists\, c \in \mathbb{C} \quad \#\{p(z) - \overline{z} = c\} = 3d - 2 \ .$$

Theorem. Let $p(z) \in \mathbb{R}[z]$, $\deg(p) = d > 1$ and all critical points real. Then

$$\forall c \in \mathbb{R}, \quad \#\{z \in \mathbb{C} \mid p(z) \pm \overline{z} = c\} \leq \begin{cases} 2d & \text{if } d \text{ is even} \\ 2d+1 & \text{if } d \text{ is odd} \end{cases} .$$

And the bound is sharp for every d.

Theorem (cubic polynomials).

- $\boxed{f(z) = q(z) - \overline{z},\ \deg q = 3} \sim \boxed{\frac{z^3}{3} + pz - \overline{z}}$.

  - $\mathcal{C}_f = \begin{cases} \text{a Jordan curve} & \text{if } |p| < 1 \\ \text{a lemniscate} & \text{if } |p| = 1 \\ \partial(\Omega^1 \sqcup \Omega^2 \text{Jordan domains}) & \text{if } |p| > 1 \end{cases}$

- $|p| > 1 \Longrightarrow f|_{\overline{\Omega^k}}$ injective (behaves like $z^2 - \overline{z}$), and

  - $\#f^{-1}(c) = \begin{cases} 7 & | \ c \in f(\Omega^1) \cap f(\Omega^2) \\ 5 & | \ c \in f(\Omega^1) \triangle f(\Omega^2) \\ 3 & | \ c \in \overline{\mathbb{C}} \setminus f(\Omega^1) \cup f(\Omega^2) \end{cases}$