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Combinatorics of complex polynomials

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About polynomials

At c_i, $P'(c_i) = 0$. They are the **critical points** of P.

The values $v_i = P(c_i)$ are the **critical values** of P.

Main question : Can one use the c_i's or the v_i's to determine the polynomial?

Prescribing critical points

Given an (unordered) list X of $d-1$ points in $\mathbb{C}$ (might not be distinct), one can find a polynomial of degree d realizing X as the set of critical points :

$$P'(z) = \prod_{c \in X}(z-c),\ P(z) = \int_0^z P'(z)dz\ .$$

One can actually find all such polynomials :

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One can actually find all such polynomials :

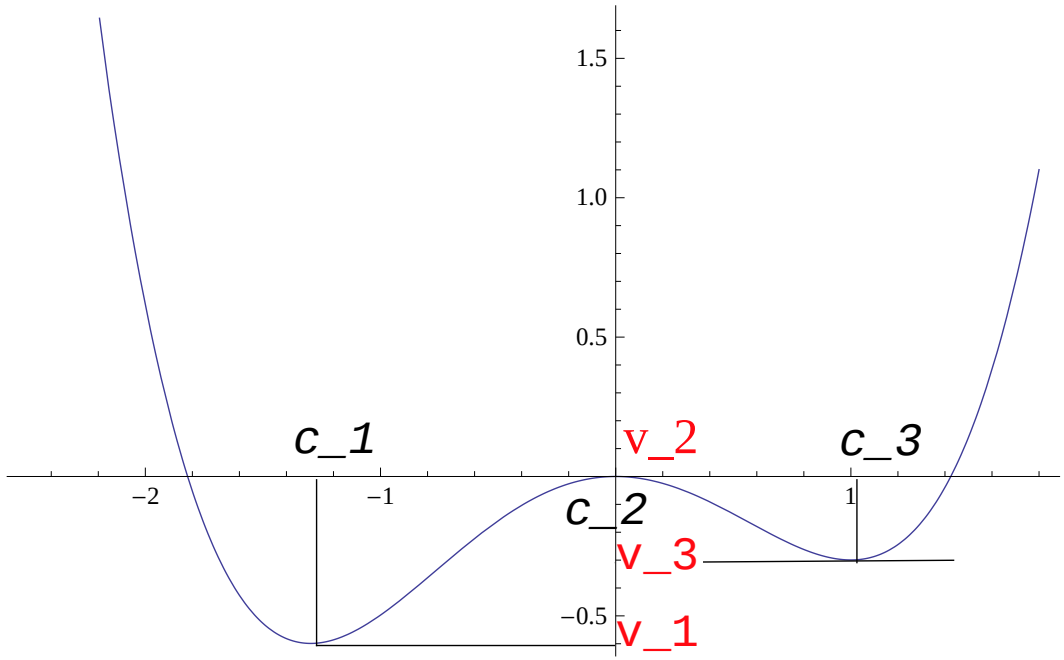
$$Q(z) = A\int_0^z P'(z)dz + B = \phi \circ P(z), \quad \phi(z) = Az + B$$

$A \in \mathbb{C}^*, B \in \mathbb{C}$.

$$\begin{array}{ccc} & \mathbb{C} & \\ P \swarrow & & \searrow Q \\ \mathbb{C} & \xrightarrow[\phi]{} & \mathbb{C} \end{array}$$

Prescribing critical values

Existence? How many? Explicit formulae?



Prescribing critical values

Existence? How many? Explicit formulae?

Two polynomials $f, g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are said to be isomorphic (or covering equivalent) if there is an affine map $M(z) = az + b$, $a \neq 0$, such that $\begin{array}{ccc} \overline{\mathbb{C}} & \stackrel{M}{\to} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array}$. Clearly f and g share the same degree and the same critical value set V.

So how many isomorphism classes are there ?

How many ? In the simple critical values case :

1. (Lyashko-Looijenga) For V a set of $d-1$ distinct points in $\mathbb{C}$, the number of isomorphism classes with V as the critical value set is equal to d^{d-3}:

$$1, 1, 1, 4, 25, 216, 2401, \cdots .$$

2. **Theorem LL** *For V a set of $d-1$ distinct points in $\mathbb{C}$, the number of monic centered polynomials with V as the critical value set is equal to d^{d-2}*:

$$1, 2, 3, 16, 125, 756, 16807, \cdots .$$

3. Real polynomials with real critical points and simple critical values : with generating function $\sec x + \tan x$:

$$1, 1, 1, 2, 5, 16, 61, 272, \cdots .$$

In particular any set of real points can be realized as critical values of a real polynomial.

How to distinguish?

1. No explicite formulae !!

2. For $d-1$ distinct real points $V = \{A_1 < A_2 < \cdots < A_{d-1}\}$, and any zig-zag permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & d-1 \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(d-1) \end{pmatrix}$$

such that $\sigma(d-1) < \sigma(d-2) > \sigma(d-3) < \sigma(d-4) \cdots$,
$\exists!$ degree d monic-centered real polynomial $f_{V,\sigma}$ with critical points $c_1 < c_2 < \cdots < c_{d-1}$ so that $v_i := f(c_i) = A_{\sigma(i)}$.

So the set of σ's provides a universal encoding of these real polynomials.

3. In degree 4, given $A < B < C$, $\exists$ two possibilities :
$v_1 = A < B = v_3$ and $v_2 = C$ or $v_3 = A < B = v_1$ and $v_2 = C$. In each case there is a unique real monic centered polynomial realizing the configuration.

Use parking functions (inspired by Tomasini)

Set $n = d - 1$. It is known :

Theorem $(n+1)^{n-1} = d^{d-2}$ *is the number of parking functions of n cars on n parking spots*
it is also the number of (non-embedded) rooted trees whose edges are labeled by $c_1, \cdots, c_n$.

A parking function is a function $p : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ such that $\#p^{-1}(\{1, 2, \cdots, i\}) \geq i, \quad \forall i = 1, \cdots, n$. Here is an example :

cars	1	2	3	4	5	6	7	8	9
favorite parking spots $p(i)$	5	5	2	1	5	1	7	3	9

This is a necessary and sufficient condition for the civilized parking algorithm to work. There is also a wild parking algorithm.

The round about argument of Pollack gives an easy proof about the number of these functions.

$\exists$ a bijection $p \leftrightarrow (\sigma_{civilized}, \sigma_{wild})$ (Poulalhon).

A bijection between parking functions and trees

A parking function can be represented fiber wisely as follows (each fiber should be displaced in increasing order from left to right) :

cars	$p^{-1}(1)$	$p^{-1}(2)$	$\cdots$	$p^{-1}(n)$
favorite parking spots	1	2	$\cdots$	n

Here is our example again:

cars	1	2	3	4	5	6	7	8	9
favorite parking spots $p(i)$	5	5	2	1	5	1	7	3	9

.

cars	4 6	3	8		1 2 5		7		9
favorite parking spots	1	2	3	4	5	6	7	8	9

.

For the cars to be all parked at least i cars should prefer to park the first i spots. That is :

$$\#p^{-1}(\{1, 2, \cdots, i\}) \geq i, \quad \forall i = 1, \cdots, n.$$

From a parking function to a tree

A rooted labeled tree is a non-embedded tree with marked vertex as root, and with edges labeled as $c_1, \cdots, c_n$.

Write down one by one the consecutive car numbers in the list $p^{-1}(1), p^{-1}(2), \cdots, p^{-1}(n)$, to obtain a list $q(1), q(2), \cdots, q(n)$. And then make a rooted tree so that the descendants edges of the root are $c_{p^{-1}(1)}$, and those of $c_{q(i-1)}$ are $c_{p^{-1}(i)}$ (no descendent if the set is empty).

the root and the edges	$*$	$q(1)$	$\cdots$	$q(n-1)$	$q(n)$
cars, descendant edges	$p^{-1}(1)$	$p^{-1}(2)$	$\cdots$	$p^{-1}(n)$	
favorite parking spots	1	2	$\cdots$	n	

.

In the above example,

root and edges	$*$	4	6	3	8	1	2	5	7	9
cars, descendants edges	4 6	3	8		1 2 5		7		9	
favorite parking spots	1	2	3	4	5	6	7	8	9	

From trees to parking functions

From a tree with root $*$ and with edges $c_1 \cdots, c_n$, we can make a breadth first search starting from $*$: we build a queue, each time putting the descendant edges of the new queue-head to the end of the queue, in increasing order if there are several descendant edges. The favorite parking spot corresponds to the entering order of the group of descendant edges to the queue.

heads of search	$*$	4	6	3	8	1	2	5	7	9
cars, descendant edges	4 6	3	8		1 2 5		7		9	
entering order of desc.	1	2	3	4	5	6	7	8	9	
favorite parking spots										

Proof of LL, bijection between trees to monic centered polynomials

Embed the tree in the plane by increasing order around each vertex.

Then add extra edges at each vertex so that each vertex has n edges and they are increasingly ordered from 1 to n.

This tree is now the union of d stars joint at $n = d - 1$ points $c_v, v \in \{1, \cdots, n\}$. In the star centered at the root there is a unique edge from the root to the edge 1. Orient this edge from the root.

A set V of n distinct points on the unit circle, defines a star $\mathcal{S}$ in $\overline{\mathcal{D}}$ centered at 0, with n radial lines as branches, naturally ordered from 1 to n, following their angles in $[0, 2\pi[$.

Now construct a topological polynomial sending each star to $\mathcal{S}$ preserving the cyclic order. Uniformize it so that a ray landing on the right of the marked edge is sent to $[1, +\infty[$. This determines a unique monic centered polynomial with critical value set V.

Conversely

Any monic centered polynomial P with critical value set $V \subset S^1$ pulls back $\mathcal{S}$ to a planar tree T. We may call c_v the critical point that is mapped to the v-th critical value. Then T is d copies of $\mathcal{S}$ joint at $c_1, \cdots, c_n$. The ray $[0, +\infty[$ has a unique preimage that is asymptotic to the positive real axis at ∞. Assign the star-center on the left of its landing point to be the root of T.

This bijection gives a combinatorial proof of Theorem LL.

Conversely

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This bijection gives a combinatorial proof of Theorem LL.

The proof is NOT constructive, no formulae for the coefficients !

The set of parking functions gives a non universal encoding of the polynomials.

Algebraic proof of Theorem LL, computing the coefficients

Set

$$\mathcal{LL} : f \mapsto \mathcal{LL}_f, \quad \mathcal{LL}_f(t) = \prod_{v \text{ critical values of } f} (t - v).$$

Note that $\mathcal{LL}$ sends a degree d monic centered polynomial in z to a degree d-1 monic polynomial in t.

Notice that if we replace f by $f - w$, we will move all critical values by a translation by $-w$. So

$$\mathcal{LL}_{f-w}(t) = \prod_{u \text{ critical value of } f-w} (t - u) =$$

$$\prod_{v \text{ critical value of } f} (t - (v - w)) = \mathcal{LL}_f(t + w).$$

So for a given vector $\mathbf{c} \in \mathbb{C}^{d-1}$, centered, we may assign a unique polynomial $P_{\mathbf{c}}$ with critical point set $\mathbf{c}$ and with critical value set centered. We set $\widehat{\mathcal{LL}}(\mathbf{c}) = \mathbf{v}$ to be the corresponding critical value vector of this unique $P_{\mathbf{c}}$.

Theorem $\widehat{\mathcal{LL}}(\lambda \mathbf{c}) = \lambda^d \widehat{\mathcal{LL}}(\mathbf{c})$ *(it is a homogenous polynomial of degree d per each coordinate), it is proper, in particular open and surjective. And it is of multiplicity* d^{d-2}.

An example : in degree 4 :

$$\widehat{\mathcal{LL}} : \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \mapsto \frac{1}{3} \begin{pmatrix} -c_1^4 + 2c_1^3(c_2+c_3) - 6c_1(c_1c_2c_3) \\ -c_2^4 + 2c_2^3(c_1+c_3) - 6c_2(c_1c_2c_3) \\ -c_3^4 + 2c_3^3(c_1+c_2) - 6c_3(c_1c_2c_3) \end{pmatrix} - \text{center}$$

$$\underset{\sum c_i = 0}{=} - \begin{pmatrix} c_1^4 + 2c_1(c_1c_2c_3) \\ c_2^4 + 2c_2(c_1c_2c_3) \\ c_3^4 + 2c_3(c_1c_2c_3) \end{pmatrix} + \frac{1}{3} \begin{pmatrix} c_1^4 + c_2^4 + c_3^4 \\ c_1^4 + c_2^4 + c_3^4 \\ c_1^4 + c_2^4 + c_3^4 \end{pmatrix}$$

The proof (inspired by LL and Douady-Sentenac)

Set $c_i^* = \lambda c_i$. In degree 4, set $g^*(x) = 4(x - c_1^*)(x - c_2^*)(x - c_3^*)$.
So $g^*(\lambda x) = \lambda^3 4(x - c_1)(x - c_2)(x - c_3) = \lambda^3 g(x)$ and
$g^*(\lambda x) = \lambda^{d-1} g(x)$ in general.

$$
\begin{aligned}
v_i^* &= \left(\int_0^{c_i^*} - \frac{1}{d-1} \Big[\sum_{j=1}^{d-1} \int_0^{c_j^*} \Big] \right) g^*(u) du \\
&= \left(\int_0^{\lambda c_i} - \frac{1}{d-1} \Big[\sum_{j=1}^{d-1} \int_0^{\lambda c_j} \Big] \right) g^*(u) du \\
&= \left(\int_0^{c_i} - \frac{1}{d-1} \Big[\sum_{j=1}^{d-1} \int_0^{c_j} \Big] \right) g^*(\lambda x)\, \lambda\, dx = \\
&= \lambda^d \left(\int_0^{c_i} - \frac{1}{d-1} \Big[\sum_{j=1}^{d-1} \int_0^{c_j} \Big] \right) g(x)\, dx = \lambda^d v_i \ .
\end{aligned}
$$

Properness

Now we need to determine $\widehat{\mathcal{LL}}^{-1}(\mathbf{0})$. Notice that all critical values of f are at 0 then f is necessarily $(z-c)^d$. Making f centered means that $c=0$ and $f(z)=z^d$. So all critical points of f are also at 0. Therefore $\widehat{\mathcal{LL}}^{-1}(\mathbf{0})=\{\mathbf{0}\}$ and $\widehat{\mathcal{LL}}$ is proper.

Now one can apply Bezout's theorem to conclude that $\widehat{\mathcal{LL}}$ is of multiplicity d^{d-2} from the $d-2$-dimensional hypersurface $\sum c_i=0$ to the $d-2$-dimensional hypersurface $\sum v_i=0$. □

Given now an ordered list V of $d-1$ points in $\mathbb{C}$, here is how to get all monic centered polynomials realizing V as critical value set:

$$V \overset{\text{give an order}}{\longrightarrow} \mathbf{v} \overset{\text{centering}}{\longrightarrow} \mathbf{v}-C \underset{d^{d-2}:1}{\overset{\widehat{\mathcal{LL}}}{\longleftarrow}} \mathbf{c} \overset{\text{forget order}}{\longrightarrow} P_{\mathbf{c}} \longrightarrow P_{\mathbf{c}}+C=f_{\mathbf{a}}.$$

This proof is more constructive, but gives no encoding. We will see that $\begin{pmatrix}\mathbf{a}\\ \mathbf{c}\end{pmatrix} \to \mathbf{v}$ is a covering over the set of $\mathbf{v}$ with pairwise distinct coordinates.

Analytic continuation, the scilab code of H.H. Rugh.

For a vector $\mathbf{a} = \begin{pmatrix} a_d \\ \vdots \\ a_2 \end{pmatrix} \in \mathbb{C}^{d-1}$, consider a monic centered polynomial $f_{\mathbf{a}}(z) = a_d + a_{d-1}z + \cdots + a_2 z^{d-2} + z^d$.

Theorem. *The set*

$$W = \left\{ \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{v} \end{pmatrix} \in \mathbb{C}^{3(d-1)} \;\middle|\; \begin{array}{l} \mathbf{c} = \textit{critical points of } f_{\mathbf{a}} \\ \mathbf{v} = f_{\mathbf{a}}(\mathbf{c}) \end{array} \right\}$$

is an algebraic set. The projection $\pi_{\mathbf{v}} : W \to \mathbb{C}^{d-1}$ is a finite degree covering over the subset $\mathcal{V}$ of $\mathbf{v}$ with pairwise distinct coordinates.

(We have used a short hand notation that for a function $\phi : \mathbb{C} \to \mathbb{C}$, the formula $\phi(\mathbf{c})$ denotes the column vector $(\phi(c_1), \cdots, \phi(c_{d-1}))^t$.)

One can then expect to lift paths realizing prescribed $\mathbf{v}$.

The algebraic set and the non-autonomous ODE

Proof. Write in inner product form
$f_{\mathbf{a}}(z) = \begin{pmatrix} 1 & z & \cdots & z^{d-2} \end{pmatrix} \cdot \mathbf{a} + z^d = B_0(z) \cdot \mathbf{a} + z^d$
$g_{\mathbf{a}}(z) := f'_{\mathbf{a}}(z) = \begin{pmatrix} 0 & 1 & \cdots & (d-2)z^{d-3} \end{pmatrix} \cdot \mathbf{a} + dz^{d-1} =$
$B_1(z) \cdot \mathbf{a} + dz^{d-1}$
where $B_0(z)$, $B_1(z)$ denote the horizontal vectors in the formulae.

For any $\mathbf{c} \in \mathbb{C}^{d-1}$ consider $F : \mathbb{C}^{2(d-1)} \to \mathbb{C}^{2(d-1)}$:

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} \mapsto \begin{pmatrix} f_{\mathbf{a}}(\mathbf{c}) \\ g_{\mathbf{a}}(\mathbf{c}) \end{pmatrix} = \begin{pmatrix} B_0(\mathbf{c}) \\ B_1(\mathbf{c}) \end{pmatrix} \cdot \mathbf{a} + \begin{pmatrix} \mathbf{c}^d \\ d\,\mathbf{c}^{d-1} \end{pmatrix}.$$

$$\text{Then } W = \left\{ \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{v} \end{pmatrix} \in \mathbb{C}^{3(d-1)} \,\middle|\, F \begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right\}.$$

The jacobian, and the Implicit Function Theorem

At a point $\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \\ \mathbf{v} \end{pmatrix} \in W$, the jacobian matrix of

$F\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix} - \begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} f_{\mathbf{a}}(\mathbf{c}) - \mathbf{v} \\ g_{\mathbf{a}}(\mathbf{c}) \end{pmatrix}$ takes the form $\begin{pmatrix} A & O & -Id \\ C & D & O \end{pmatrix}$ with O the $(d-1) \times (d-1)$ zero matrix, D the diagonal matrix with diagonal entries $f''_{\mathbf{a}}(c_1), \cdots, f''_{\mathbf{a}}(c_{d-1})$ and

$$A = \begin{pmatrix} 1 & c_1 & \cdots & c_1^{d-2} \\ 1 & c_2 & \cdots & c_2^{d-2} \\ \vdots & \vdots & & \vdots \\ 1 & c_{d-1} & \cdots & c_{d-1}^{d-2} \end{pmatrix}, \; D = \begin{pmatrix} f''_{\mathbf{a}}(c_1) & 0 & \cdots & 0 \\ 0 & f''_{\mathbf{a}}(c_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f''_{\mathbf{a}}(c_{d-1}) \end{pmatrix}.$$

If the c_i's are pairwise distinct, on one hand the Van der Monde matrix A is invertible, on the other hand all critical points of $f_{\mathbf{a}}$ are simple so $f''_{\mathbf{a}}(c_i) \neq 0$. Therefore A, D and $\begin{pmatrix} A & O \\ C & D \end{pmatrix}$ are invertible.

The scilab program CV-05.sci lifts any paths within a coordinate v_i of the covering over $\mathcal{V}$, and projects the path to the $\mathbf{c}$-coordinates.

Given a smooth path $\mathbf{v}(t) \subset \mathcal{V}$ and $\begin{pmatrix} \mathbf{a}(0) \\ \mathbf{c}(0) \\ \mathbf{v}(0) \end{pmatrix} \in W$, the corresponding path $\begin{pmatrix} \mathbf{a}(t) \\ \mathbf{c}(t) \end{pmatrix}$ is the solution of the following non-autonomous ODE :

$$\begin{pmatrix} A(\mathbf{a},\mathbf{c}) & O \\ C(\mathbf{a},\mathbf{c}) & D(\mathbf{a},\mathbf{c}) \end{pmatrix} \begin{pmatrix} \dot{\mathbf{a}} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{v}}(t) \\ \mathbf{0} \end{pmatrix}$$

or

$$\begin{pmatrix} \dot{\mathbf{a}} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} A(\mathbf{a},\mathbf{c}) & O \\ C(\mathbf{a},\mathbf{c}) & D(\mathbf{a},\mathbf{c}) \end{pmatrix}^{-1} \begin{pmatrix} \dot{\mathbf{v}}(t) \\ \mathbf{0} \end{pmatrix}.$$

The scilab program shows $\mathbf{c}(t)$ when the mouse describes $\mathbf{v}(t)$.

Pairing critical points and critical values

Lemma *The set of pairs (critical point, its critical value) of a rational map f determines uniquely f. In other words two maps sharing the same set are identical.*

For polynomials, the set of pairs (critical point, its critical value) can be represented by one straight segment in $\mathbb{C}$ linking each pair of points. This is what CV-05.sci and CV-web-02.sci show.

The program CV-web-02.sci shows the monodromy action, when moving the critical values, of the tree encoding.

Relation with holomorphic dynamics

Implementing Thurston's algorithm...