Jacobi is remembered for his contributions to many fields of mathematics, including differential geometry, mechanics, and number theory as well as elliptic functions. He was a great admirer of Euler and planned the edition of Euler's works that eventually began to appear, on a reduced scale, in 1911. In fact, in many ways Jacobi was a second, if lesser, Euler. He saw elliptic functions not so much as things in themselves, as Abel did, but as a source of dazzling formulas with implications in number theory. An astounding collection of formulas may be found in his major work on elliptic functions, the *Fundamenta nova* [Jacobi (1829)]. At the same time, he was deeply impressed by Abel's ideas and selflessly campaigned to make them better known. He introduced the terms "Abelian integral" and "Abelian function" for the generalizations of elliptic integrals and functions considered by Abel as well as "Abelian theorem" for Abel's theorem, which he described as "the greatest mathematical discovery of our time."

13 Mechanics

13.1 Mechanics before Calculus

The ambiguous title reflects the dual purpose of this section: to give a brief survey of the mechanics that came before calculus and to introduce the thesis that mechanics was psychologically, if not logically, a prerequisite for calculus itself. The remainder of the chapter expands on this thesis, demonstrating how several important fields in calculus (and beyond) originated in the study of mechanical problems. Lack of space, not to mention lack of expertise, prevents my venturing far into the history of mechanical concepts, so I shall assume some understanding of time, velocity, acceleration, force, and the like, and concentrate on the mathematics that emerged from reflection on these notions. These mathematical developments will be pursued as far as the nineteenth century. More details may be found in Dugas (1957, 1958) and Truesdell (1954, 1960). In the last century, mathematics seems to have been the motivation for mechanics rather than the other way round. The outstanding mechanical concepts of the twentieth century—relativity and quantum mechanics—would not have been conceivable without nineteenth-century advances in pure mathematics, some of which we discuss later.

It is mentioned in Section 4.5 that Archimedes made the only substantial contribution to mechanics in antiquity by introducing the basics of statics (balance of a lever requires equality of moments on the two sides) and hydrostatics (a body immersed in a fluid experiences an upward force equal to the weight of fluid displaced). Archimedes' famous results on areas and volumes were in fact discovered, as he revealed in his *Method*, by hypothetically balancing thin slices of different figures. Thus the earliest nontrivial results in calculus, if by calculus one means a method for discovering results about limits, relied on concepts from mechanics.

The medieval mathematician Oresme also was mentioned (Section 7.1) for his use of coordinates to give a geometric representation of functions. The relationship Oresme represented was in fact velocity v as a function of time t. He understood that displacement is then represented by the area under the curve, and hence in the case of constant acceleration (or "uniformly deformed velocity," as he called it) the displacement equals total time × velocity at the middle instant (Figure 13.1). This result is known as the "Merton acceleration theorem" [see, for example, Clagett (1959), p. 355] because it originated in the work of a group of mathematicians at Merton College, Oxford, in the 1330s. The first proofs were arithmetical and far less transparent than Oresme's figure.

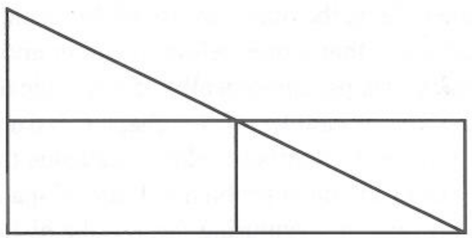

Figure 13.1: The Merton acceleration theorem

While constant acceleration was understood theoretically in the 1330s, it was not clear that it was actually a natural occurrence—namely, with falling bodies—until the time of Galileo (1564–1642). Galileo announced the equivalent result, that displacement of a body falling from rest at time $t = 0$ is proportional to t^2, in a letter [Galileo (1604)]. At first he was uncertain whether this derived from a velocity proportional to time $v = kt$ (that is, constant acceleration) or proportional to distance $v = ks$, but he resolved the question correctly in favor of $v = kt$ later [Galileo (1638)]. By composing the uniformly increasing vertical velocity with constant horizontal velocity, Galileo derived for the first time the correct trajectory of a projectile: the parabola.

The motion of projectiles was a matter of weighty importance in the Renaissance, and presumably observed often, yet the trajectories suggested before Galileo were quite preposterous (see, for example, Figure 6.3). The belief, deriving from Aristotle, that motion could be sustained only by continued application of a force led mathematicians to ignore the evidence and to draw trajectories in which the horizontal velocity dwindled to zero. Galileo overthrew this mistaken belief by affirming the *principle of inertia*: a body not subject to external forces travels with constant velocity.

The principle of inertia was Newton's starting point in mechanics; indeed, it is often called Newton's first law. It is a special case of his second law, that force is proportional to mass × acceleration [Newton (1687), p. 13]. Under this law, the motion of a body is determined by composition of the forces acting on it. The correct law for the composition of forces, that forces add vectorially, had been discovered in the case of perpendicular forces by Stevin (1586) and in the general case by Roberval [published in Mersenne (1636)]. The motion is thus determined by vector addition of the corresponding accelerations, the method Galileo used in the case of the projectile.

The determination of velocity and displacement from acceleration are of course problems of integration, so mechanics contributed a natural class of problems to calculus just at the time the subject was emerging. But more than this was true. The early practitioners of calculus believed that continuity was an essential attribute of functions, and the only way they were able to define continuity was ultimately by falling back on the dependence of a velocity or displacement on time. From this viewpoint, *all* problems of integration and differentiation were problems of mechanics, and Newton described them as such when explaining how his calculus of infinite series could be applied:

> It now remains, in illustration of this analytical art, to deliver some typical problems and such especially as the nature of curves will present. But first of all I would observe that difficulties of this sort may all be reduced to these two problems alone, which I may be permitted to propose with regard to the space traversed by any local motion however accelerated or retarded:
>
> 1. Given the length of space continuously (that is, at every time), to find the speed of motion at any time proposed.
> 2. Given the speed of motion continuously, to find the length of space described at any time proposed
>
> [Newton (1671), p. 71].

Of course we now know that the first problem requires differentiability rather than continuity for its solution, but the pioneers of calculus thought that differentiability was implied by continuity, and hence did not recognize it as a distinct notion. In fact it was a mechanical question—the problem of the vibrating string—whose investigation brought the distinction to light (see Section 13.4).

13.2 Celestial Mechanics

Astronomy has been a powerful stimulus to mathematics since ancient times. The epicyclic theory of Apollonius and Ptolemy introduced an interesting family of algebraic and transcendental curves, as we saw in Section 2.5, and the theory itself ruled Western astronomy until the seventeenth century. Even Copernicus (1472–1543), when he overthrew Ptolemy's earth-centered system with a sun-centered system in his *De revolutionibus orbium coelestium* [Copernicus (1543)], was unwilling to give up epicycles. Taking the sun as the center of the system simplifies the orbits of the planets but does not make them circular, so Copernicus, accepting the Ptolemaic philosophy that orbits must be generated by circular motions, modeled them by epicycles. In fact he used more epicycles than Ptolemy.

A more important advance, from the mathematical point of view, was Kepler's introduction of elliptical orbits in his *Astronomia nova* [Kepler (1609)]. When Newton explained these orbits as a consequence of the inverse square law of gravitation in the *Principia* [Newton (1687), p. 56] he showed that there was a deeper level of explanation—the infinitesimal level—where simplicity could be attained even when it was not possible at the global level. The force on a given body B_1 is simply the vector sum of the forces due to the other bodies $B_2, \ldots, B_n$ in the system, determined by their masses and distances from B_1 by the inverse square law and, by Newton's second law, this determines the acceleration of B_1. The accelerations of $B_2, \ldots, B_n$ are similarly determined, hence the behavior of the system is completely determined by the inverse square law, once initial positions and velocities are given. The inverse square law is an infinitesimal law in the sense that it describes the limiting behavior of a body—its acceleration—and not its global behavior such as the shape or period of its orbit.

As we now know, it is rarely possible to describe the global behavior of a dynamical system explicitly, so Newton found the only viable basis for dynamics in directing attention to infinitesimal behavior. Unfortunately, he communicated this insight poorly by expressing it in geometrical terms, in the belief that calculus, which he had used to discover his results, was inappropriate in a serious publication. By the eighteenth century this belief had been dispelled by Leibniz and his followers, and definitive formulations of dynamics in terms of calculus were given by Euler and Lagrange. They recognized that the infinitesimal behavior of a dynamical system was typically described by a system of *differential equations* and that the global behavior was derivable from these equations, in principle, by integration.

The question remained, however, whether the inverse square law did indeed account for the observed global behavior of the solar system. In a system with only two bodies, Newton showed [Newton (1687), p. 166] that each describes a conic section relative to the other, in normal cases an ellipse as stated by Kepler. With a three-body system, such as the earth–moon–sun, no simple global description was possible, and Newton could obtain only qualitative results through approximations. With the many bodies in the solar system, extremely complex behavior was possible, and for 100 years mathematicians were unable to account for some of the phenomena actually observed.

A famous example was the so-called secular variation of Jupiter and Saturn, which was detected by Halley in 1695 from the observations then available. For several centuries Jupiter had been speeding up (spiraling toward the sun) and Saturn had been slowing down (spiraling outward). The problem was to explain this behavior and to determine whether it would continue, with the eventual destruction of Jupiter and disappearance of Saturn. Euler and Lagrange worked on the problem without success; then, in the centenary year of *Principia*, Laplace (1787) succeeded in explaining the phenomenon. He showed that the secular variation was actually periodic, with Jupiter and Saturn returning to their initial positions every 929 years. Laplace viewed this as confirmation not only of the Newtonian theory but also of the stability of the solar system, though it seems that the latter is still an open question.

Laplace introduced the term "celestial mechanics" and left no doubt that the theory had arrived with his monumental *Mécanique céleste*, a work of five volumes that appeared between 1799 and 1825. In astronomy, the theory had its finest hour in 1846, with the discovery of Neptune, whose position had been computed by Adams and Leverrier from observed perturbations in the orbit of Uranus. The difficult question of stability was taken up again in the three volume *Les méthodes nouvelles de la mécanique* *céleste* of Poincaré (1892, 1893, 1899). In this work Poincaré directed attention toward asymptotic behavior, in a sense complementing Newton's infinitesimal view with a view toward infinity, and his methods have become highly influential in twentieth-century dynamics.

13.3 Mechanical Curves

When Descartes gave his reasons for restricting *La Géométrie* to algebraic curves (which he called "geometric"; see Section 7.3), he explicitly excluded certain classical curves on the rather vague grounds that they

> belong only to mechanics, and are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination.
>
> [Descartes (1637), p. 44]

The curves that Descartes relegated "to mechanics" were those the Greeks had defined by certain hypothetical mechanisms, for example, the epicycles (described by rolling one circle on another) and the spiral of Archimedes (described by a point moving at constant speed along a uniformly rotating line). He was probably aware that the spiral is transcendental by virtue of the fact that it meets a straight line in infinitely many points. This is contrary to the behavior of an algebraic curve $p(x,y)=0$, which meets a straight line $y=mx+c$ in only finitely many points, corresponding to the finitely many solutions of $p(x,mx+c)=0$. This proof that there are transcendental curves was given explicitly by Newton (1687), Lemma XXVIII.

We do not know whether Descartes distinguished, say, the algebraic epicycles from the transcendental ones; nevertheless, it is broadly true that his "mechanical" curves were transcendental. This remained true with the great expansion of mechanics and calculus in the seventeenth century, and indeed most of the new transcendental curves originated in mechanics. In this section we shall look at three of the most important of them: the catenary, the cycloid, and the elastica.

The *catenary* is the shape of a hanging cord, assumed to be perfectly flexible and with mass uniformly distributed along its length. In practice, the flexibility and uniformity of mass are realized better by a hanging chain,

hence the name "catenary," which comes from the Latin *catena* for chain. Hooke (1675) observed that the same curve occurs as the shape of an arch of infinitesimal stones. The catenary looks very much like a parabola and was at first conjectured to be one by Galileo. This was disproved by the 17-year-old Huygens (1646), though at the time he was unable to determine the correct curve. He did show, however, that the parabola was the shape assumed by a flexible cord loaded by weights that are uniformly distributed in the horizontal direction (as is approximately the case for the cable of a suspension bridge).

The problem of the catenary was finally solved independently by Johann Bernoulli (1691), Huygens (1691), and Leibniz (1691), in response to a challenge from Jakob Bernoulli in 1690. Johann Bernoulli showed that the curve satisfied the differential equation

$$\frac{dy}{dx}=\frac{s}{a},$$

where a is constant and $s=$ arc length OP (Figure 13.2). He derived this equation by replacing the portion OP of the chain, which is held in equilibrium by the tangential force F_1 at P and the horizontal force F_0, which is independent of P, by a point mass W equal to the weight of OP (hence proportional to s) held in equilibrium by the same forces. Comparing the directions and magnitudes of the forces gives

$$\frac{dy}{dx}=\frac{W}{F_0}=\frac{s}{a}.$$

By ingenious transformations Bernoulli reduced the equation to

$$dx=\frac{a\,dy}{\sqrt{y^2-a^2}},$$

in other words, to an integral. This solution was as simple as could be stated at the time, since x is a transcendental function of y and hence can be expressed, at best, as an integral. Today, of course, we recognize the function as one of the "standard" ones and abbreviate the solution as

$$y=a\cosh\frac{x}{a}-a.$$

The *cycloid* is the curve generated by a point on the circumference of a circle rolling on a straight line. Despite being a natural limiting case in

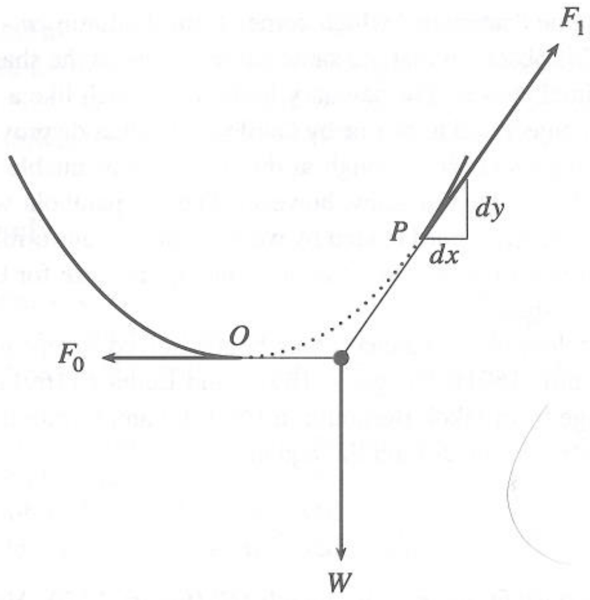

Figure 13.2: The catenary

the epicyclic family, the cycloid does not seem to have been investigated until the seventeenth century, when it became a favorite curve with mathematicians. It has many beautiful geometric properties, and even more remarkable mechanical properties. The first of these, discovered by Huygens (1659b), is that the cycloid is the *tautochrone* (equal-time curve). A particle constrained to slide along an inverted cycloid takes the same time to descend to the lowest point, regardless of its starting point.

Huygens (1673) made a classic application of this property to pendulum clocks, using a geometric property of the cycloid (Huygens, 1659c). If the pendulum, taken to be a weightless cord with a point mass at the end, is constrained to swing between two cycloidal "cheeks," as Huygens called them (Figure 13.3), then the point mass will travel along a cycloid. Consequently, the period of the cycloidal pendulum is independent of amplitude. This makes it theoretically superior to the ordinary pendulum whose period, though approximately constant for small amplitudes, actually involves an elliptic function. In practice, problems such as friction make the cycloidal pendulum no more accurate than the ordinary pendulum, but its theoretical superiority shut the ordinary pendulum out of mechanics for

some time. Newton's *Principia*, for example, often mentions the cycloidal pendulum but never the simple pendulum.

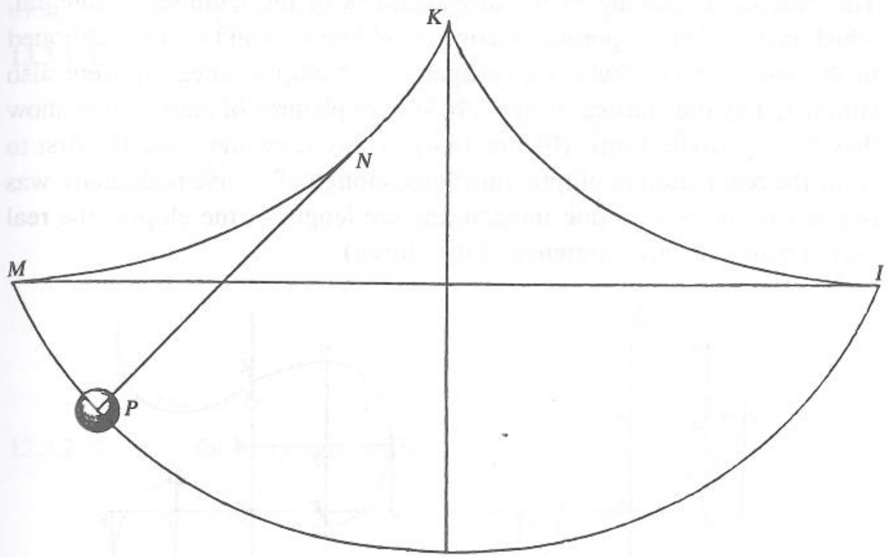

Figure 13.3: The cycloidal pendulum

The second remarkable property of the cycloid is that it is the *brachistochrone*, the curve of shortest time. Johann Bernoulli (1696) posed the problem of finding the curve, between given points A and B, along which a point mass descends in the shortest time. He already knew that the solution was a cycloid, and solutions were found independently by Jakob Bernoulli (1697), l'Hôpital (1697), Leibniz (1697), and Newton (1697). The problem is deeper than that of the tautochrone, because the cycloid has to be singled out from *all possible* curves between A and B. Jakob Bernoulli's solution was the most profound because it recognized the "variable curve" aspect of the problem, and it is now considered to be the first major step in the development of the calculus of variations.

The *elastica* was another of Jakob Bernoulli's discoveries, and likewise important in the development of another field—the theory of elliptic functions. The elastica is the curve assumed by a thin elastic rod compressed at the ends. Jakob Bernoulli (1694) showed that the curve satisfied a differential equation that he reduced to the form

$$ds = \frac{dx}{\sqrt{1-x^4}}.$$

To interpret this integral geometrically, he introduced the lemniscate and showed that its arc length was expressed by precisely the same integral. This was the beginning of the investigations of the lemniscatic integral, which included the important discoveries of Fagnano and Gauss mentioned in the last chapter. Euler's investigations of elliptic integrals were also stimulated by the elastica. Euler (1743) gave pictures of elastica that show they have periodic forms (Figure 13.4). These drawings were the first to show the real period of elliptic functions, though of course periodicity was implicit in the first elliptic integral, the arc length of the ellipse (the real period being the circumference of the ellipse).

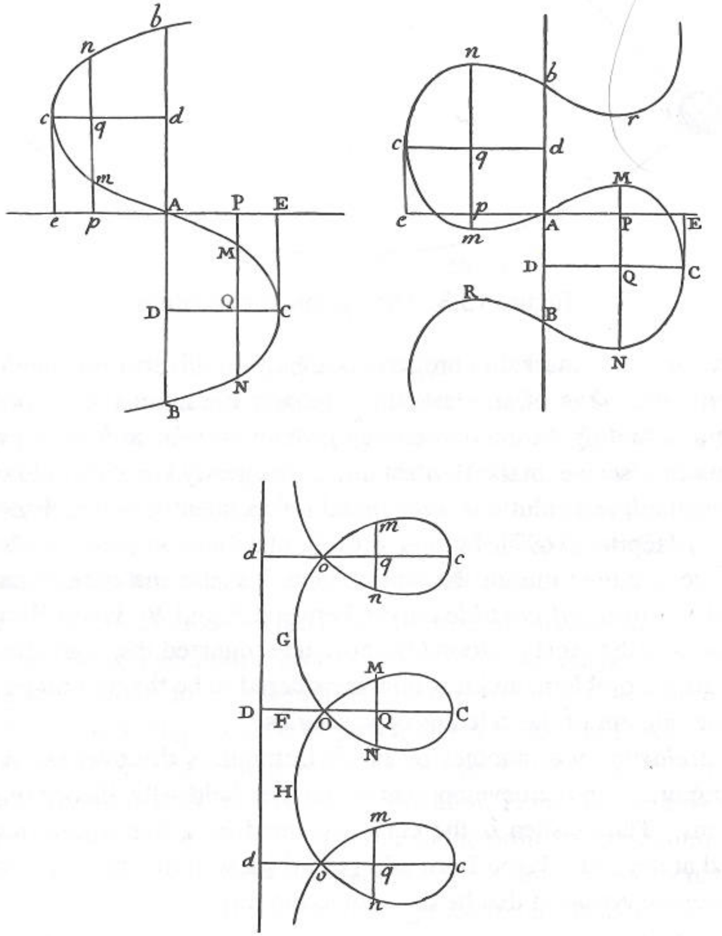

Figure 13.4: Forms of the elastica

EXERCISES

The derivation of the cosh function from the catenary equation is helped by a tricky formula for $\frac{d^2y}{dx^2}$, which you should verify first if it is not familiar to you.

13.3.1 Use

$$ds = \sqrt{dx^2+dy^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dy}\frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

to transform the differential equation

$$\frac{dy}{dx} = \frac{s}{a}$$

to

$$\frac{dx}{dz} = \frac{a}{\sqrt{1+z^2}}, \tag{1}$$

where $z = dy/dx$.

13.3.2 Solve (1) for x and hence show that the original equation has solution

$$y = a\cosh\frac{x}{a} + \text{const.}$$

It is considerably easier to solve the suspension bridge equation, which perhaps is why Huygens was able to do it at age 17, and before much calculus was known.

13.3.3 How should the formula $\frac{dy}{dx} = \frac{s}{a}$ be modified if the load is uniformly distributed in the horizontal direction (as in a suspension bridge)?

13.3.4 Solve the modified equation from Exercise 13.3.3, and hence show that the solution is a parabola.

Finally, we can verify that the catenary is indeed a transcendental curve.

13.3.5 Show that the functions sin and cos, and hence the functions sinh and cosh, are transcendental. *Hint*: You may need to use complex numbers.

13.4 The Vibrating String

The problem of the vibrating string is one of the most fertile in mathematics, being the source of such diverse fields as partial differential equations, Fourier series, and set theory. It is also remarkable in being perhaps the only setting in which the sense of hearing led to important mathematical discoveries. As we saw in Section 1.5, the Pythagoreans discovered the relationship between pitch and length by hearing the harmonious tones produced by two strings whose lengths were in a simple whole-number ratio. Thus in a sense it was possible to "hear the length of the string," and some later discoveries of mathematically significant properties of the strings—overtones, for example—were initially prompted by hearing [see Dostrovsky (1975)].

Various authors in ancient times suggested that the physical basis of pitch was frequency of vibration, but it was not until the seventeenth century that the precise relationship between frequency and length was discovered, by Descartes' mentor Isaac Beeckman. In 1615 Beeckman gave a simple geometric argument to show that frequency is inversely proportional to length; hence the Pythagorean ratios of lengths can also be interpreted as (reciprocal) ratios of frequencies. The latter interpretation is more fundamental because frequency alone determines pitch, whereas length determines pitch only when the material, cross section, and tension of the string are fixed. The relation between frequency ν, and tension T, cross-sectional area A, and length l was discovered experimentally by Mersenne (1625) to be

$$\nu \propto \frac{1}{l}\sqrt{\frac{T}{A}}.$$

The first derivation of Mersenne's law from mathematical assumptions was given by Taylor (1713), in a paper that marks the beginning of the modern theory of the vibrating string. In it he discovered the simplest possibility for the instantaneous shape of the string, the half sine wave

$$y = k\sin\frac{\pi x}{l}$$

and established generally that the force on an element was proportional to d^2y/dx^2.

The latter result was the starting point for a dramatic advance in the theory by d'Alembert (1747). Taking into account the dependence of y on time t as well as x, d'Alembert realized that acceleration should be expressed by $\partial^2 y/\partial t^2$ and the force found by Taylor by $\partial^2 y/\partial x^2$, hence *partial* derivatives are involved. Newton's second law then gives what is now called the *wave equation*,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$

writing the constant of proportionality as $1/c^2$. Undeterred by the novelty of this partial differential equation, d'Alembert forged ahead to a general

solution as follows. The equation may be simplified by a change of time scale $s = ct$ to

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial s^2}. \tag{1}$$

The chain rule gives

$$\begin{aligned} d\left(\frac{\partial y}{\partial x} \pm \frac{\partial y}{\partial s}\right) &= \frac{\partial^2 y}{\partial x^2}dx + \frac{\partial^2 y}{\partial x \partial s}(ds \pm dx) \pm \frac{\partial^2 y}{\partial s^2}ds \\ &= \left(\frac{\partial^2 y}{\partial s^2} \pm \frac{\partial^2 y}{\partial x \partial s}\right)(ds \pm dx) \end{aligned}$$

from which d'Alembert concluded that

$$\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial x \partial s}$$

is a function of $s + x$ and

$$\frac{\partial^2 y}{\partial s^2} - \frac{\partial^2 y}{\partial x \partial s}$$

is a function of $s - x$, whence, say

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial s} = \int \left(\frac{\partial^2 y}{\partial s^2} + \frac{\partial^2 y}{\partial x \partial s}\right) d(s+x) = f(s+x)$$

and similarly

$$\frac{\partial y}{\partial x} - \frac{\partial y}{\partial s} = g(s-x).$$

This gives

$$\frac{\partial y}{\partial x} = \frac{1}{2}(f(s+x) + g(s-x)), \quad \frac{\partial y}{\partial s} = \frac{1}{2}(f(s+x) - g(s-x)),$$

and finally

$$\begin{aligned} y &= \int \left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial s}ds\right) \\ &= \int \frac{1}{2}(f(s+x)(ds+dx) - g(s-x)(ds-dx)) \\ &= \Phi(s+x) + \Psi(s-x). \end{aligned}$$

Reversing the argument, we see that the functions Φ and Ψ can be arbitrary, at least as long as they admit the various differentiations involved.

But how arbitrary *is* an arbitrary function? Is it as arbitrary as an arbitrarily shaped string? The vibrating string problem caught eighteenth-century mathematicians unprepared to answer these questions. They had understood a function to be something expressed by a formula, possibly an infinite series, and this had been thought to guarantee differentiability. Yet the most natural shape of the vibrating string was one with a nondifferentiable point—the triangle of the plucked string as it is released—so nature seemed to demand an extension of the concept of function beyond the world of formulas.

The confusion was heightened when Daniel Bernoulli (1753) claimed, on physical grounds, that a general solution of the wave equation *could* be expressed by a formula, the infinite trigonometric series

$$y = a_1 \sin\frac{\pi x}{l}\cos\frac{\pi c t}{l} + a_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi c t}{l} + \cdots.$$

This amounts to claiming that any mode of vibration results from the superposition of simple modes, a fact he considered to be intuitively evident. The nth term in the series represents the nth mode, generalizing Taylor's formula for the fundamental mode and building in the time dependence; but Daniel Bernoulli gave no method for calculating the coefficient a_n.

We now know that his intuition was correct and that the triangular wave form, among others, is representable by a trigonometric series. However, it was well into the nineteenth century before anything like a clear understanding of trigonometric series was obtained. The fact that the triangular wave could be represented by a series made it a bona fide function by classical standards, hence mathematicians were brought to the realization that a series representation does not guarantee differentiability. Later, continuity was also called into question, and infinitely subtle problems concerning the convergence of trigonometric series led Cantor to develop the theory of sets (see Chapter 23).

These remarkably remote consequences of what seemed at first to be a purely physical question were of course not the only fruits of the vibrating string investigations. Trigonometric series proved to be valuable all over mathematics, from the theory of heat, where Fourier applied them with such success that they became known as *Fourier series*, to the theory of numbers. Their most famous application to number theory is probably the Dirichlet (1837) proof that any arithmetic progression $a, a+b, a+2b, \ldots$, where $\gcd(a,b) = 1$, contains infinitely many primes. Pythagoras would surely have approved!

EXERCISES

The simplest heat equation is the one-dimensional version,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

for the temperature T at time t and position x along an infinite straight wire. This equation may be derived from *Newton's law of cooling*, which asserts that the rate of heat flow between two points is proportional to their temperature difference.

Thus the approximate difference $\frac{\partial T}{\partial x}dx$ between T at x and $x+dx$ will induce heat to flow from $x+dx$ to x at a rate proportional to $\frac{\partial T}{\partial x}dx$. However, at the same time, heat will flow from $x-dx$ to x at approximately the same rate. To find the net flow toward x, and hence the rate $\frac{\partial T}{\partial t}$ of temperature increase, we need to take into account the rate of change of $\frac{\partial T}{\partial x}$, namely $\frac{\partial^2 T}{\partial x^2}$.

13.4.1 By pursuing this line of argument, give a plausible derivation of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}.$$

Sines and cosines arise from the heat equation when one solves it by the method of *separation of variables*.

13.4.2 Suppose the heat equation has a solution of the form $T(x,t) = X(x)Y(t)$, where X and Y are functions of the single variables x and t, respectively. Show that

$$\frac{1}{Y(t)}\frac{dY(t)}{dt} = \frac{\kappa}{X(x)}\frac{d^2X(x)}{dx^2} = \text{constant}.$$

13.4.3 Now explain how sines and cosines are involved in solving for $X(x)$.

13.5 Hydrodynamics

The properties of fluid flow have been investigated since ancient times, initially in connection with practical questions such as water supply and water-powered machinery. However, nothing like a mathematical theory was obtained before the Renaissance, and until the advent of calculus it was only possible to deal with fairly coarse macroscopic quantities such as the average speed of emission from an opening in a container. Newton (1687), Book II, introduced infinitesimal methods into the study of fluids, but much of his reasoning is incomplete, based on inappropriate mathematical models, or simply wrong. As late as 1738, when the field of hydrodynamics finally got its name in the classic *Hydrodynamica* of Daniel

Bernoulli, the basic infinitesimal laws of fluid motion had still not been discovered.

The first important law was discovered by Clairaut (1740), in a context that in fact was essentially static. Clairaut was interested in one of the burning questions of the time, the shape (or "figure") of the earth. Newton had argued that the earth must bulge somewhat at the equator as a result of its spin. Natural as this seems now (and indeed then, since the phenomenon was clearly observable in Jupiter and Saturn), it was opposed by the anti-Newtonian Cassini, who argued for a spindle-shaped earth, elongated toward the poles. Clairaut actually took part in an expedition to Lapland that confirmed Newton's conjecture by measurement, but he also attacked the problem theoretically by studying the conditions for the equilibrium of a fluid mass.

He considered the vector field of force acting on the fluid and observed that it must be what we now call a *conservative*, or *potential* field. That is, the integral of the force around any closed path must be zero; otherwise the fluid would circulate. The condition he actually formulated was the equivalent one that the integral between any two points be independent of the path. In the special two-dimensional case where there are components P, Q of force in the x and y directions, the quantity to be integrated is

$$P\,dx + Q\,dy.$$

Clairaut argued that for the integral to be path-independent, this quantity must be a complete differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Consequently, $P = \partial f/\partial x$, $Q = \partial f/\partial y$ and P, Q satisfy the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \qquad (1)$$

This condition is indeed necessary, but the existence of the potential f involved more mathematical subtleties than could have been foreseen at the time. Clairaut derived the corresponding equations for the components P, Q, R in the physically more natural three-dimensional case and went as far as studying the equipotential surfaces $f =$ constant. He also found a satisfying solution to the problem of the figure of the earth. When the

force at a point is the resultant of gravity and the rotational force, then an ellipsoid of revolution is an equilibrium figure, with the axis of rotation being the shorter axis of the ellipse [Clairaut (1743), p. 194].

The two-dimensional equation (1), despite being physically special if not unnatural, turned out to have a deep mathematical significance. This was discovered in the dynamic situation, with P, Q taken to be components of velocity rather than force. In this case, (1) still holds when the flow is independent and irrotational as d'Alembert (1752) showed by an argument similar to Clairaut's. The crucial additional fact that now emerges is that P, Q satisfy a second relation

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \qquad (2)$$

derived by d'Alembert as a consequence of the incompressibility of the fluid. He considered an infinitesimal rectangle of fluid with corners at the points (x,y), $(x+dx,y)$, $(x,y+dy)$, $(x+dx,y+dy)$, and the parallelogram into which it is carried in an infinitesimal time interval by the known velocities (P,Q), $(P+(\partial P/\partial x)dx, Q+(\partial Q/\partial x)dx)$, ... Equating the areas of these two parallelograms leads to (2). In the three-dimensional case one similarly gets

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0,$$

but the significance of (1) and (2), as d'Alembert discovered, is that they can be combined into a single fact about the complex function $P + iQ$. This flash of inspiration became the basis for the theory of complex functions developed in the nineteenth century by Cauchy and Riemann (see Section 16.1).

Exercises

To understand the concept of irrotational flow more directly, it helps to consider a flow that is clearly *rotational*, for example a rigid rotation of the plane about the origin at constant angular velocity ω.

13.5.1 For this flow, show that the velocity components at the point (x,y) are

$$P = -\omega y, \quad Q = \omega x,$$

and deduce that $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = -2\omega$.

Thus the quantity $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ is a measure of the amount of rotation of the flow. It is, in fact, sometimes called the "rotation" but it is more often called the *curl*, a term James Clerk Maxwell introduced in 1870.

The quantity $\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$ is called the *divergence* because it measures the amount of "expansion"of the fluid. As one would expect, the divergence is zero for the rigid flow above.

13.5.2 Check that the divergence is zero for the rigid rotation about the origin.

A more direct way to see that divergence is zero for any incompressible flow in the plane is to consider a *fixed* rectangle, with fluid flowing through it.

Consider the rectangle with corners fixed in the plane at (x,y), $(x+dx,y)$, $(x,y+dy)$, $(x+dx,y+dy)$, and consider the instantaneous flux of fluid through it. Fluid flows in the x end at speed P, hence the influx is proportional to $P\,dy$, and it flows out the $x+dx$ end at speed $P+(\partial P/\partial x)\,dx$, etc.

13.5.3 Show that the net influx of fluid is

$$-\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dxdy,$$

and hence that the divergence is zero for incompressible flow.

13.5.4 Show similarly that

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0$$

for an incompressible flow in three dimensions.

13.6 Biographical Notes: The Bernoullis

Undoubtedly the most outstanding family in the history of mathematics was the Bernoulli family of Basel, which included at least eight excellent mathematicians between 1650 and 1800. Three of these, the brothers Jakob (1654–1705) and Johann (1667–1754) and Johann's son Daniel (1700–1782), were among the great mathematicians of all time, as one may guess from their contributions already mentioned in this chapter. In fact, all the mathematicians Bernoulli were important in the history of mechanics. One can trace their influence in this field in Szabó (1977), which also contains portraits of most of them, and in Truesdell (1954, 1960). However, Jakob, Johann, and Daniel are of interest from a wider point of view, in mathematics, as well as in their personal lives. The Bernoulli family, with all its mathematical talent, also had more than its share of arrogance and jealousy, which turned brother against brother and father against son. In three successive generations, fathers tried to steer their sons into nonmathematical careers, only to see them gravitate back to mathematics. The fiercest conflict occurred among Jakob, Johann, and Daniel.

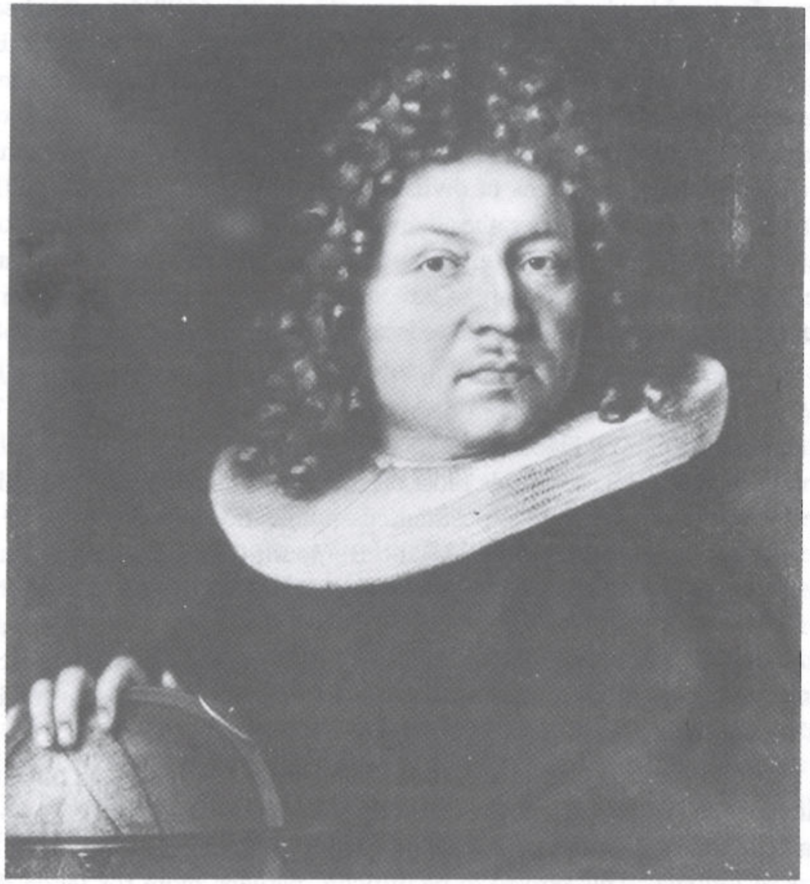

Figure 13.5: Portrait of Jakob Bernoulli by Nicholas Bernoulli

Jakob, the first mathematician in the family, was the oldest son of Nicholas Bernoulli, a successful pharmacist and civic leader in Basel, and Margaretha Schönauer, the daughter of another wealthy pharmacist. There were three other sons: Nicholas, who became an artist and in 1686 painted the portrait of James seen here (Figure 13.5); Johann; and Hieronymus, who took over the family business. Their father's wish was that Jakob should study theology, which he initially did, obtaining his licentiate in 1676. However, Jakob also began to teach himself mathematics and astronomy, and he traveled to France in 1677 to study with the followers of Descartes. In 1681 his astronomy brought him into conflict with the theologians. Inspired by the appearance of a great comet in 1680, he published a pamphlet that proposed laws governing the behavior of comets and claiming that their appearances could be predicted. His theory was not actually correct (this was six years before *Principia*), but it certainly clashed with the theology of the time, which exploited the unexpectedness of comets in claiming they were signs of divine displeasure. Jakob decided that his future was in mathematics rather than theology, and he adopted the motto *Invito Patre, Sidera verso* (Against my father's will, I will turn to the stars). He made a second study tour, to the Netherlands and England, where he met Hooke and Boyle, and began to lecture on mechanics in Basel in 1683.

He married Judith Stepanus in 1684, and they eventually had a son and daughter, neither of whom became a mathematician. In a sense, the mathematical heir of Jakob was his nephew Nicholas (son of the painter), who carried on one of Jakob's most original lines of research, probability theory. He arranged for the posthumous publication of Jakob's book on the subject, the *Ars conjectandi* [Jakob Bernoulli (1713)], which contains the first proof of a law of large numbers. Jakob Bernoulli's law described the behavior of long sequences of trials for which a positive outcome has a fixed probability p (such trials are now called Bernoulli trials). In a precise sense, the proportion of successful trials will be "close" to p for "almost all" sequences.

In 1687 Jakob became professor of mathematics in Basel and, together with Johann (whom he had been secretly teaching mathematics), set about mastering the new methods of calculus that were then appearing in the papers of Leibniz. This proved to be difficult, perhaps more for Jakob than Johann, but by the 1690s the brothers equaled Leibniz himself in the brilliance of their discoveries. Jakob, the self-taught mathematician, was the slower but more penetrating of the two. He sought to get to the bottom of every problem, whereas Johann was content with any solution, the quicker the better.

Johann was the tenth child of the family, and his father intended him to have a business career. When his lack of aptitude for business became clear, he was allowed to enter the University of Basel in 1683 and became a master of arts in 1685. During this time he also attended his brother's lectures and, as mentioned earlier, learned mathematics from him privately. Their rivalry did not come to the surface until the catenary contest of 1690, but Jakob may have felt uneasy about his younger brother's talent as early as 1685. In that year he persuaded Johann to take up the study of medicine,

making the highly optimistic forecast that it offered great opportunities for the application of mathematics. Johann went into medicine quite seriously, obtaining a licentiate in 1690 and a doctorate in 1695, but by that time he was more famous as a mathematician. With the help of Huygens he gained the chair of mathematics in Groningen, and thus became free to concentrate on his true calling.

The great applications of mathematics to medicine did not eventuate, though Johann Bernoulli did make an amusing application of geometric series which still circulates today as a piece of physiological trivia. In his *De nutritione* [Johann Bernoulli (1699)] he used the assumption that a fixed proportion of bodily substance, homogeneously distributed, is lost each day and replaced by nutrition, to calculate that almost all the material in the body would be renewed in three years. This result provoked a serious theological dispute at the time, since it implied the impossibility of resurrecting the body from all its past substance.

Johann Bernoulli made several important contributions to calculus in the 1690s, outside mechanics. One was the first textbook in the subject, the *Analyse des infiniment petits*. This was published under the name of his student, the Marquis l'Hôpital (1696), apparently in return for generous financial compensation. Another contribution, made jointly with Leibniz, was the technique of partial differentiation. The two kept this discovery secret for 20 years in order to use it as a "secret weapon" in problems about families of curves [see Engelsman (1984)]. Other discoveries still remain outside the territory usually explored in calculus, for example,

$$\int_0^1 x^x\,dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots.$$

This startling result of Johann Bernoulli (1697) can be proved using a suitable series expansion of x^x and integration by parts (see exercises).

The rivalry between Jakob and Johann turned to open hostility in 1697 over the *isoperimetric problem*, the problem of finding the curve of given length which encloses the greatest area. Jakob correctly recognized that this was a calculus of variations problem but withheld his solution, whereas Johann persisted in publicizing an incorrect solution and claiming that Jakob had no solution at all. Jakob presented his solution to the Paris Academy in 1701, but it somehow remained in a sealed envelope until after his death. Even when the solution was made public in 1706, Johann refused to admit his own error or the superiority of Jakob's analysis.

Johann was married to Dorothea Falkner, the daughter of a parliamentary deputy in Basel, and through his father-in-law's influence was awarded the chair of Greek in Basel in 1705. This enabled him to return to Basel from Groningen, but his real goal was the chair of mathematics, not Greek. Jakob was then in ill health, and his last days were embittered by the belief that Johann was plotting to take his place, using the Greek offer as a stepping stone. This is precisely what happened, for when Jakob died in 1705 Johann became the professor of mathematics.

With the death of Jakob and the virtual retirement of Leibniz and Newton, Johann enjoyed about 20 years as the leading mathematician in the world. He was particularly proud of his successful defense of Leibniz against the supporters of Newton:

> When in England war was declared against M. Leibniz for the honour of the first invention of the new calculus of the infinitely small, I was despite my wishes involved in it; I was pressed to take part. After the death of M. Leibniz the contest fell to me alone. A crowd of English antagonists fell upon my body. It was my lot to meet the attacks of Messrs Keil, Taylor, Pemberton, Robins and others. In short I alone like the famous Horatio Cocles kept at bay at the bridge the entire English army. [Translation by Pearson (1978), p. 235]

His portrait from this era shows the Bernoulli arrogance at its peak (Figure 13.6).

Johann Bernoulli finally met his match at the hands of his own pupil Euler in 1727. There was no open warfare, just a polite exchange of correspondence on the logarithms of negative numbers, but it revealed that Johann Bernoulli understood some of his own results less well than Euler did. Johann Bernoulli persisted in his stubborn misunderstanding for another 20 years, while Euler went on to develop his brilliant theory of complex logarithms and exponentials (see Section 16.1). Johann Bernoulli seems not to have minded his pupil's success at all; instead, he became consumed with jealousy over the success of his son Daniel.

Daniel Bernoulli (Figure 13.7) was the middle of Johann's three sons, all of whom became mathematicians. The oldest, Nicholas (called Nicholas II by historians to distinguish him from the first mathematician Nicholas), died of a fever in St. Petersburg in 1725 at the age of 30. The youngest,

Johann II, was the least distinguished of the three, but he fathered the next generation of Bernoulli mathematicians, Jakob II and Johann III.

Figure 13.6: Johann Bernoulli

Daniel's path to mathematics was very similar to his father's. During his teens he was tutored by his older brother; his father wanted him to go into business, but when that career failed Daniel was permitted to study medicine.

He gained his doctorate in 1721 and made several attempts to win the chair of anatomy and botany in Basel, finally succeeding in 1733. By that time, however, he had drifted into mathematics, with such success that he had been called to the St. Petersburg Academy. During his years there (1725–1733) he conceived his ideas on modes of vibration and produced the first draft of his *Hydrodynamica*. Although he missed finding the basic partial differential equations of hydrodynamics, the *Hydrodynamica* made other important advances. One was the systematic use of a principle of conservation of energy; another was the kinetic theory of gases, including the derivation of Boyle's law that is now standard.

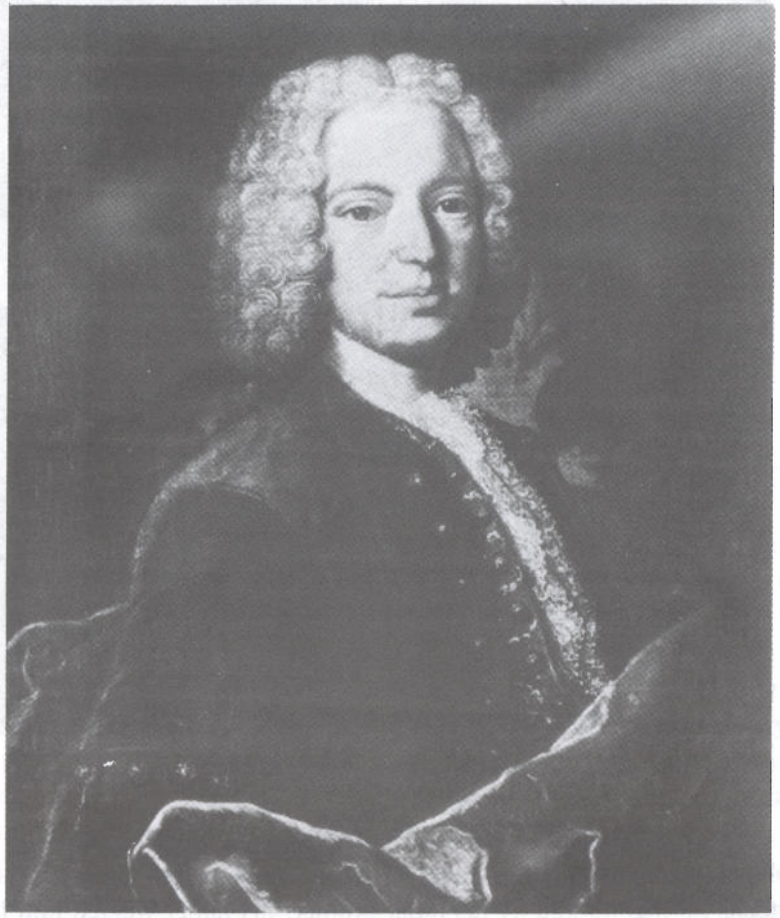

Figure 13.7: Daniel Bernoulli

Unfortunately, publication of the *Hydrodynamica* was delayed until 1738. This left Daniel's priority open to attack, and the one to take advantage of him was his own father. The self-styled Horatius of the priority dispute between Leibniz and Newton attempted the most blatant priority theft in the history of mathematics by publishing a book on hydrodynamics in 1743 and dating it 1732. Daniel was devastated, and wrote to Euler:

> Of my entire *Hydrodynamics*, not one iota of which do in fact I owe to my father, I am all at once robbed completely and lose thus in one moment the fruits of the work of ten years. All propositions are taken from my *Hydrodynamics*, and then my father calls his writings *Hydraulics, now for the first time disclosed*, 1732, since my *Hydrodynamics* was printed only in 1738.
>
> [Daniel Bernoulli (1743), in the Truesdell (1960) translation]

The situation was not quite as clear-cut as Daniel claimed [a detailed assessment is in Truesdell (1960)], but at any rate Johann Bernoulli's move backfired. His reputation was so tarnished by the episode that he did not even receive credit for parts of his work that *were* original. Daniel went on to enjoy fame and a long career, becoming professor of physics in 1750 and lecturing to enthusiastic audiences until 1776.

Exercises

13.6.1 Use integration by parts to show that

$$\int_0^1 x^n (\log x)^n \, dx = \frac{(-1)^n n!}{(n+1)^{n+1}}.$$

13.6.2 Deduce that

$$\int_0^1 x^x \, dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots$$

using a series expansion of $x^x = e^{x \log x}$.