8

Projective Geometry

8.1 Perspective

Perspective may be simply described as the realistic representation of spatial scenes on a plane. This of course has been a concern of painters since ancient times, and some Roman artists seem to have achieved correct perspective by the first century BCE; an impressive example is shown in Wright (1983), p. 38. However, this may have been a stroke of individual genius rather than the success of a theory, because the vast majority of ancient paintings show incorrect perspective. If indeed there was a classical theory of perspective, it was well and truly lost during the Dark Ages. Medieval artists made some charming attempts at perspective but always got it wrong, and errors persisted well into the fifteenth century. [Errors still survive in twentieth-century mathematics texts. Figure 8.1 shows a fifteenth-century artistic example from Wright (1983), p. 41, alongside a twentieth-century mathematical example from the exposé of Grünbaum (1985).]

The discovery of a method for correct perspective is usually attributed to the Florentine painter-architect Brunelleschi (1377–1446), around 1420. The first published method appears in the treatise *On Painting* by Alberti (1436). The latter method, which became known as *Alberti's veil*, used a piece of transparent cloth, stretched on a frame, and set in front of the scene to be painted. Then, viewing the scene with one eye, in a fixed position, one could trace the scene directly onto the veil. Figure 8.2 shows this method, with a peephole to maintain a fixed eye position, as depicted by Dürer (1525).

Figure 8.1: Errors in perspective

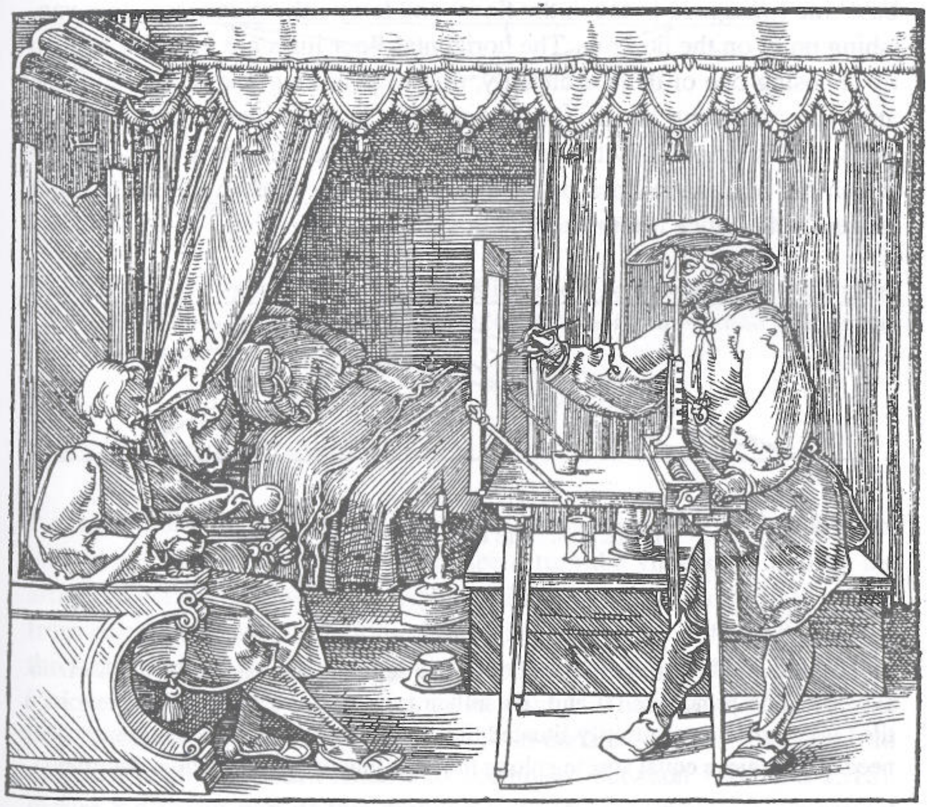

Figure 8.2: Dürer's depiction of Alberti's veil

Alberti's veil was fine for painting actual scenes, but to paint an imaginary scene in perspective some theory was required. The basic principles Renaissance artists used were the following:

(i) A straight line in perspective remains straight.

(ii) Parallel lines either remain parallel or converge to a single point (their *vanishing point*).

These principles suffice to solve a problem artists frequently encountered: the perspective depiction of a square-tiled floor. Alberti (1436) solved the special case of this problem in which one set of floor lines is horizontal, that is, parallel to the horizon. His method, which became known as the *costruzione legittima*, is indicated in simplified form in Figure 8.3. The nonhorizontal floor lines are determined by spacing them equally along the base line (imagined to touch the floor) and letting them converge to a vanishing point on the horizon. The horizontal floor lines are then determined by choosing one of them arbitrarily, thus determining one tile in the floor, and then producing the diagonal of this tile to the horizon. The intersections of this diagonal with the nonhorizontal lines are the points through which the horizontal lines pass. This is certainly true on the actual floor (Figure 8.4), hence it remains true in the perspective view.

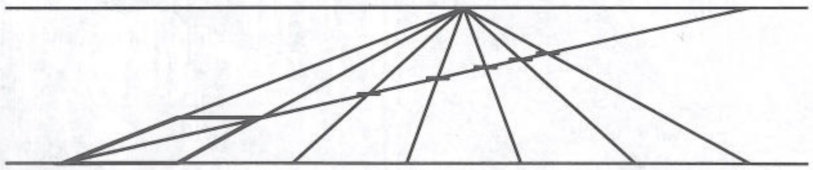

Figure 8.3: The *costruzione legittima*

EXERCISES

In almost all paintings of tiled floors, one set of lines is parallel to the horizon. However, the principles (i) and (ii) suffice to generate a perspective view of a tiled floor given an arbitrarily situated tile, and they show that no measurement is needed to achieve equal spacing along the base line in the *costruzione legittima*.

8.1.1 Use the lines shown in Figure 8.5 to determine all lines in a pavement generated by the given tile.

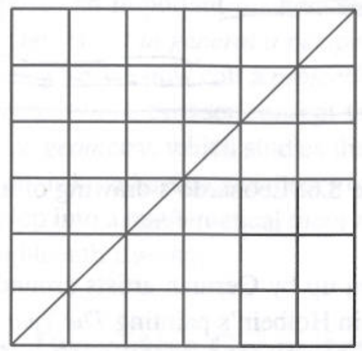

Figure 8.4: The actual floor

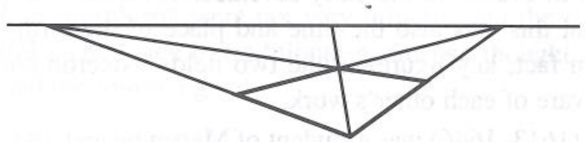

Figure 8.5: Tiled floor with arbitrary orientation

8.1.2 By using diagonals as in Exercise 8.1.1, show how to generate the lines in the tiling when the baseline is parallel to the horizon, without making any measurements.

8.2 Anamorphosis

It is clear from the Alberti veil construction that a perspective view will not look absolutely correct except when seen from the viewpoint used by the artist. Experience shows, however, that distortion is not noticeable except from extreme viewing positions. Following the mastery of perspective by the Italian artists, an interesting variation developed, in which the picture looks right from only one, extreme, viewpoint. The first known example of this style, known as *anamorphosis*, is an undated drawing by Leonardo da Vinci from the *Codex Atlanticus* (compiled between 1483 and 1518). Figure 8.6 shows part of this drawing, a child's face which looks correct when viewed with the eye near the right-hand edge of the page.

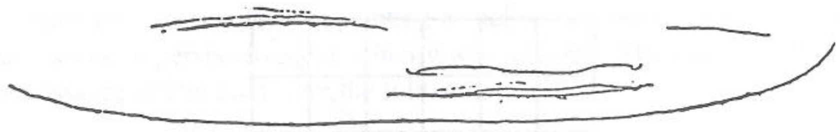

Figure 8.6: Leonardo's drawing of a face

The idea was taken up by German artists around 1530. The most famous example occurs in Holbein's painting *The Two Ambassadors* (1533). A mysterious streak across the bottom of the picture becomes a skull when viewed from near the picture's edge. For an excellent view of this picture and a history of anamorphosis, see Baltrušaitis (1977) and Wright (1983), pp. 146–156. The art of anamorphosis reached its technically most advanced form in France in the early seventeenth century. It seems no coincidence that this was also the time and place of the birth of projective geometry. In fact, key figures in the two fields, Niceron and Desargues, were well aware of each other's work.

Niceron (1613–1646) was a student of Mersenne and, like him, a monk in the order of Minims. He executed some extraordinary anamorphic wall paintings, up to 55 meters long, and also explained the theory in *La perspective curieuse* [Nicéron (1638)]. Figure 8.7 is his illustration of anamorphosis of a chair [from Baltrušaitis (1977), p. 44]. The anamorphosis, viewed normally, shows a chair like none ever seen, yet from a suitably extreme point one sees an ordinary chair in perspective.

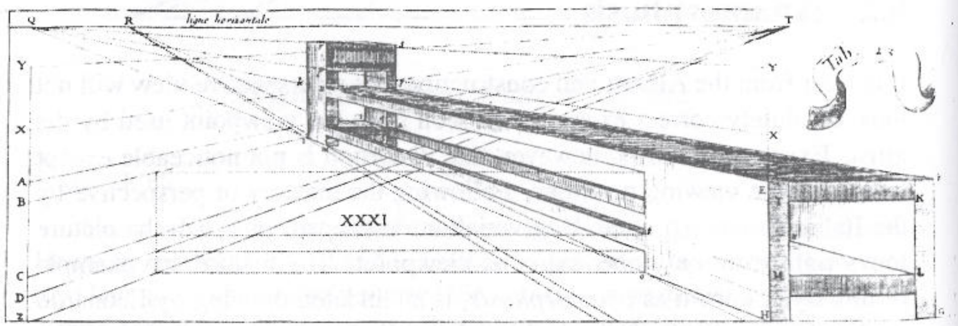

Figure 8.7: Niceron's chair

This example exposes an important mathematical fact: *a perspective view of a perspective view is not in general a perspective view*. Iteration of perspective views gives what we now call a *projective* view, and Niceron's chair shows that projectivity is a broader concept than perspectivity. As a consequence, *projective geometry*, which studies the properties that are invariant under projection, is broader than the theory of perspective. Perspective itself did not develop into a mathematical theory, *descriptive geometry*, until the end of the eighteenth century.

8.3 Desargues' Projective Geometry

The mathematical setting in which one can understand Alberti's veil is the family of lines ("light rays") through a point (the "eye"), together with a plane V (the "veil") (Figure 8.8). In this setting, the problems of perspective and anamorphosis were not very difficult, but the *concepts* were interesting and a challenge to traditional geometric thought. Contrary to Euclid, one had the following:

(i) Points at infinity ("vanishing points") where parallels met.

(ii) Transformations that changed lengths and angles (projections).

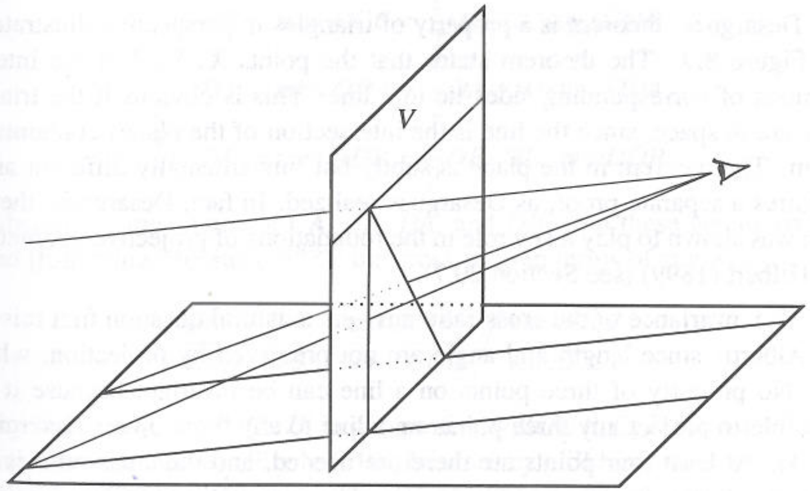

Figure 8.8: Seeing through Alberti's veil

The first to construct a mathematical theory incorporating these ideas was Desargues (1591–1661), although the idea of points at infinity had already been used by Kepler (1604), p. 93. The book of Desargues (1639), *Brouillon project d'une atteinte aux événemens des rencontres du cône avec un plan* (Schematic Sketch of What Happens When a Cone Meets a Plane), suffered an extreme case of delayed recognition, being completely lost for 200 years. Fortunately, his two most important theorems, the so-called Desargues' theorem and the invariance of the cross-ratio, were published in a book on perspective [Bosse (1648)]. The text of Desargues (1639) and a portion of Bosse (1648) containing Desargues' theorem may be found in Taton (1951). An English translation, with an extensive historical and mathematical analysis, is in Field and Gray (1987).

Kepler and Desargues both postulated one point at infinity on each line, closing the line to a "circle of infinite radius." All lines in a family of parallels share the same point at infinity. Nonparallel lines, having a finite point in common, do not have the same point at infinity. Thus any two distinct lines have exactly one point in common—a simpler axiom than Euclid's. Strangely enough, the line at infinity was only introduced into the theory by Poncelet (1822), even though it is the most obvious line in perspective drawing, the horizon. Desargues made extensive use of projections in the *Brouillon projet*; he was the first to use them to prove theorems about conic sections.

Desargues' theorem is a property of triangles in perspective illustrated by Figure 8.9. The theorem states that the points X, Y, Z at the intersections of corresponding sides lie in a line. This is obvious if the triangles are in space, since the line is the intersection of the planes containing them. The theorem in the plane is subtly but fundamentally different and requires a separate proof, as Desargues realized. In fact, Desargues' theorem was shown to play a key role in the foundations of projective geometry by Hilbert (1899) (see Section 20.7).

The invariance of the cross-ratio answers a natural question first raised by Alberti: since length and angle are not preserved by projection, what is? No property of three points on a line can be invariant because it is possible to project any three points on a line to any three others (Exercise 8.3.1). At least four points are therefore needed, and the cross-ratio is in fact a projective invariant of four points. The cross-ratio $(ABCD)$ of points A,B,C,D on a line (in that order) is $\frac{CA}{CB}/\frac{DA}{DB}$. Its invariance is most simply seen by reexpressing it in terms of angles using Figure 8.10. Let O be any

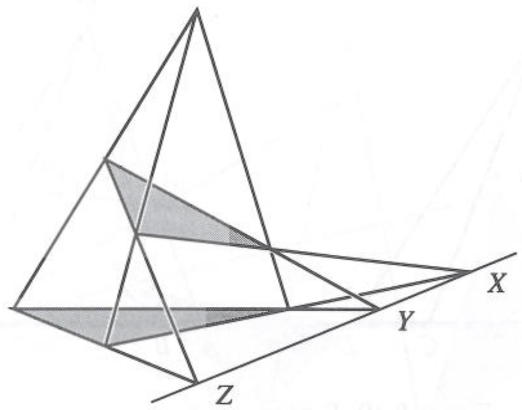

Figure 8.9: Desargues' theorem

point outside the line and consider the areas of the triangles OCA, OCB, ODA, and ODB. First compute them from bases on AB and height h, then recompute using OA and OB as bases and heights expressed in terms of the sines of angles at O:

$$\frac{1}{2}h\cdot CA = \text{area } OCA = \frac{1}{2}OA\cdot OC\sin\angle COA,$$
$$\frac{1}{2}h\cdot CB = \text{area } OCB = \frac{1}{2}OB\cdot OC\sin\angle COB,$$
$$\frac{1}{2}h\cdot DA = \text{area } ODA = \frac{1}{2}OA\cdot OD\sin\angle DOA,$$
$$\frac{1}{2}h\cdot DB = \text{area } ADB = \frac{1}{2}OB\cdot OD\sin\angle DOB.$$

Substituting the values of CA, CB, DA, and DB from these equations we find [following Möbius (1827)] the cross-ratio in terms of angles at O:

$$\frac{CA}{CB}\Big/\frac{DA}{DB} = \frac{\sin\angle COA}{\sin\angle COB}\Big/\frac{\sin\angle DOA}{\sin\angle DOB}.$$

Any four points A', B', C', D' in perspective with A, B, C, D from a point O have the same angles (Figure 8.10), hence they will have the same cross-ratio. But then so will any four points A'', B'', C'', D'' projectively related to A, B, C, D, since a projectivity is by definition the product of a sequence of perspectivities.

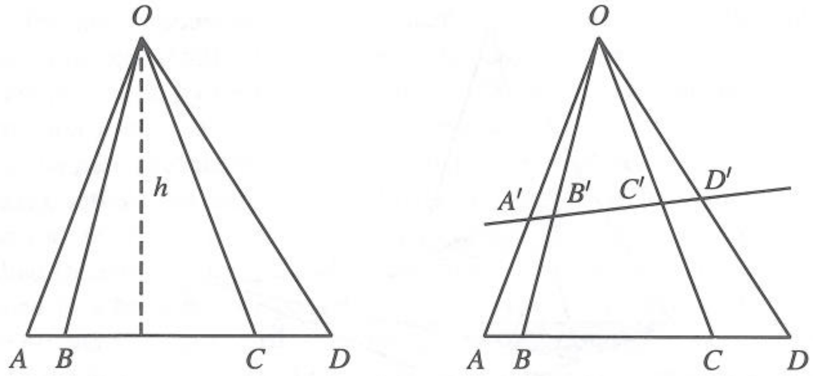

Figure 8.10: Evaluating the cross ratio

EXERCISES

As mentioned above, we cannot hope for an invariant that is simpler than the cross-ratio, because any three points in a line are projectively related to any other.

8.3.1 Show that any three points on a line can be sent to any other three points on a line by projection.

The case of Desargues' theorem where the two triangles lie in the same plane can be proved by viewing the plane in space. The setup for the proof is shown in Figure 8.11. The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in perspective from O in a plane Π, P is a point in space outside Π, and the line OD_1D_2 meets Π only at O.

8.3.2 Show that the triangles $A_1C_1D_1$ and $A_2C_2D_2$ are in different planes, and in perspective from O.

Thus it follows from the nonplanar version of Desargues' theorem that the intersections of the side pairs (A_1D_1, A_2D_2), (A_1C_1, A_2C_2), and (C_1D_1, C_2D_2) lie in a line.

8.3.3 Show that these intersections are projected from P to the intersections of the side pairs (A_1B_1, A_2B_2), (A_1C_1, A_2C_2), and (C_1B_1, C_2B_2), and hence deduce the planar Desargues' theorem.

8.3.4 Does this proof capture your intuitive idea of looking at the planar Desargues configuration (Figure 8.9) and interpreting it three-dimensionally? If so, what does the point P represent?

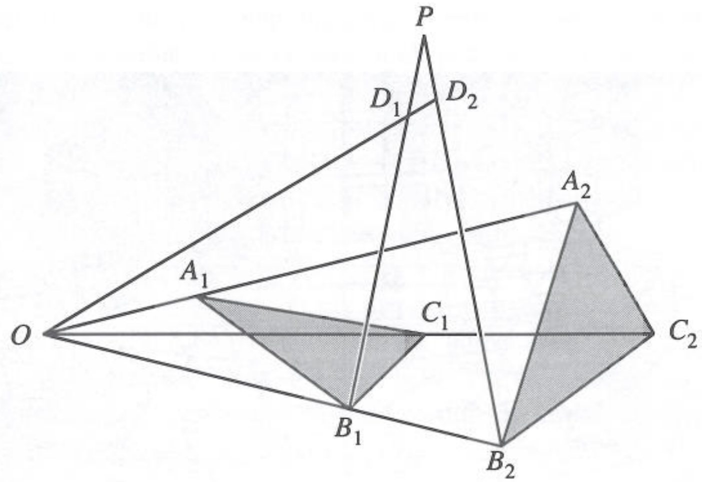

Figure 8.11: The planar Desargues' theorem

8.4 The Projective View of Curves

The problems of perspective drawing mainly involved the geometry of straight lines. There were, it is true, problems such as drawing ellipses to look like perspective views of circles, but artists were generally content to solve such problems by interpolating smooth-looking curves in a suitable straight-line framework. An example is the drawing of a chalice by Uccello (1397–1475) in Figure 8.12.

A mathematical theory of perspective for curves became possible with the advent of analytic geometry. When a curve is specified by an equation $f(x,y) = 0$, the equation of any perspective view is obtainable by suitably transforming x and y. However, this transformational viewpoint, even though quite simple algebraically, emerged only with Möbius (1827). The first works in projective geometry, by Desargues (1639) and Pascal (1640), used the language of classical geometry, even though the language of equations was available from Descartes (1637). This was understandable, not only because the analytic method was so obscure in Descartes, but also because the advantages of the projective method could be more clearly seen when it was used in a classical setting. Desargues and Pascal confined themselves to straight lines and conic sections, showing how projective geometry could easily reach and surpass the results obtained by the Greeks.

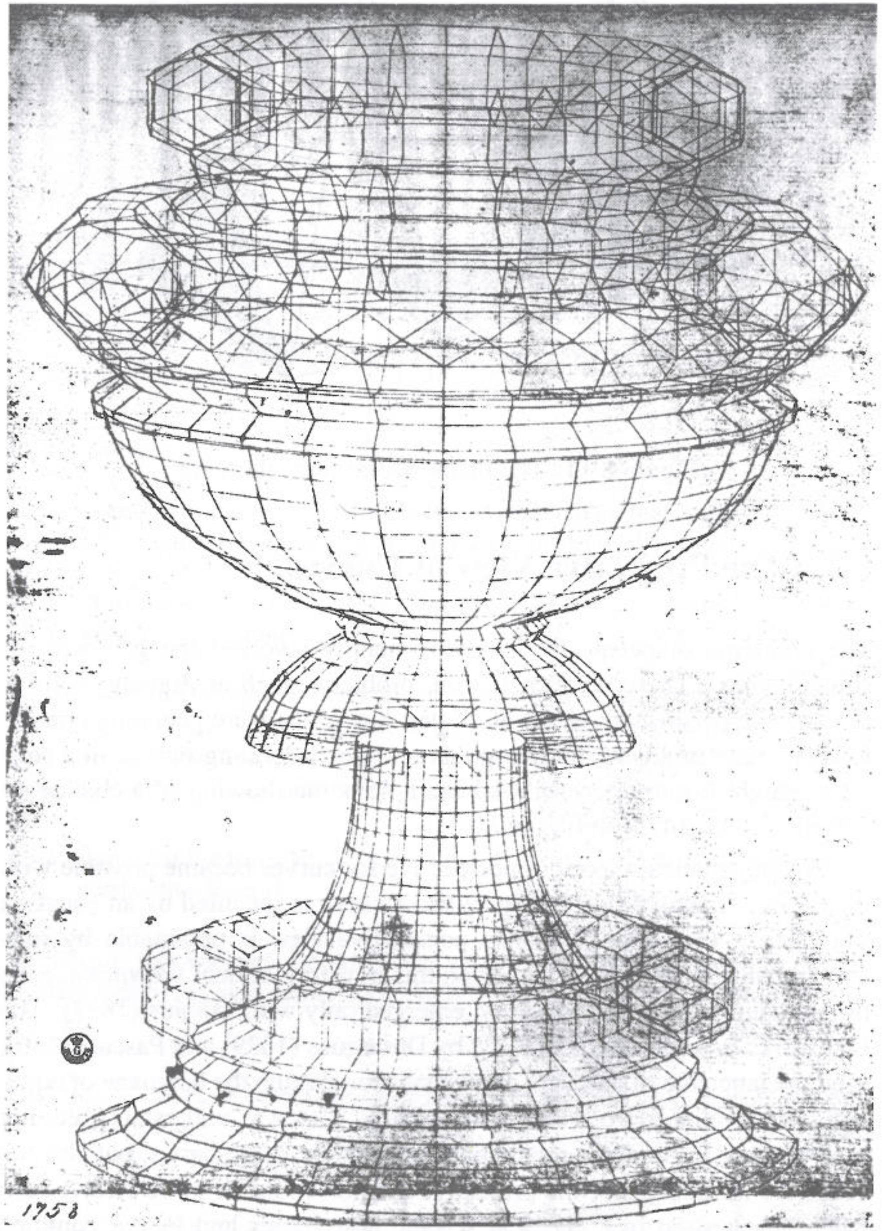

Figure 8.12: Drawing of a chalice by Uccello. (Uffizi, Florence)

Moreover, the projective viewpoint gave something else that would have been incomprehensible to the Greeks: a clear account of the behavior of curves at infinity.

For example, Desargues (1639) [in Taton (1951), p. 137] distinguished the ellipse, parabola, and hyperbola by their numbers of points at infinity, 0, 1, and 2, respectively. The points at infinity on the parabola and hyperbola can be seen quite plainly by tilting the ordinary views of them into perspective views (Figures 8.13 and 8.14). The parabola has just one point at infinity because it crosses each ray through 0, except the y-axis, at just one finite point. As for the hyperbola, its two points at infinity are where it touches its asymptotes, as seen in Figure 8.14. The continuation of the hyperbola above the horizon results from projecting the lower branch through the same center of projection (Figure 8.15).

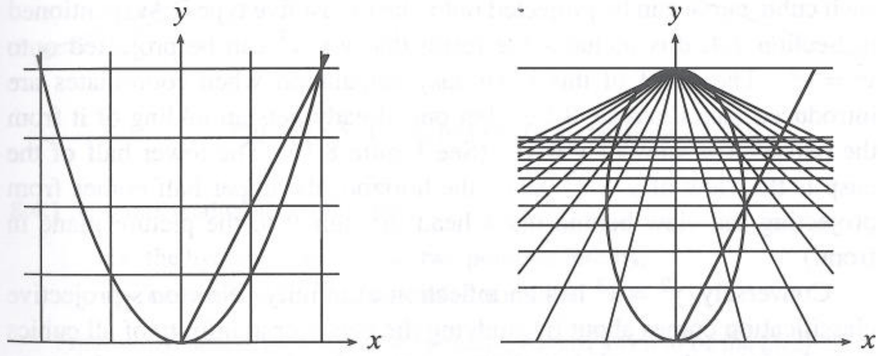

Figure 8.13: The parabola

Projective geometry goes beyond describing the behavior of curves at infinity. The line at infinity is no different from any other line and can be deprived of its special status. Then all projective views of a curve are equally valid, and one can say, for example, that all conic sections are ellipses when suitably viewed. This is no surprise if one remembers conic sections not as second-degree curves but as sections of the cone. Of course they all look the same from the vertex of the cone.

More surprisingly, a great simplification of cubic curves also occurs when they are viewed projectively. As mentioned in Section 7.4, Newton (1695) classified cubic curves into 72 types (and missed 6). However, in his Section 29, "On the Genesis of Curves by Shadows," Newton claimed that

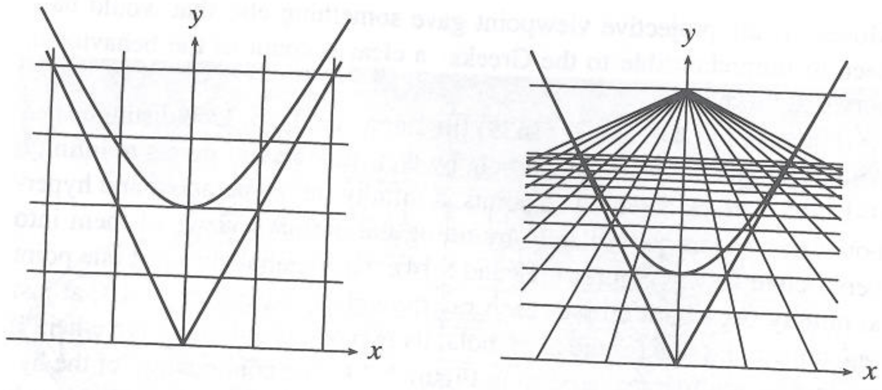

Figure 8.14: The hyperbola

each cubic curve can be projected onto one of just five types. As mentioned in Section 7.4, this includes the result that $y = x^3$ can be projected onto $y^2 = x^3$. The proof of this is an easy calculation when coordinates are introduced (see Exercise 8.5.3), but one already gets an inkling of it from the perspective view of $y = x^3$. (See Figure 8.16. The lower half of the cusp is the view of $y = x^3$ below the horizon; the upper half comes from projecting the view behind one's head through P to the picture plane in front.)

Conversely, $y^2 = x^3$ has an inflection at infinity. Newton's projective classification comes about by studying the behavior at infinity of all cubics and observing that each has characteristics already possessed, not necessarily at infinity, by curves of the form

$$y^2 = Ax^3 + Bx^2 + Cx + D.$$

Newton had already divided these into five types in his analytic classification (they are the five shown in Figure 7.3). Newton's result was improved only in the nineteenth century, when projective classification over the complex numbers reduced the number of types of cubics into just three. We discuss this later in connection with the development of complex numbers (Section 15.5).

Exercises

As suggested above, the points at infinity of a curve may be counted by considering intersections of the curve with lines through the origin, and observing where they tend to infinity.

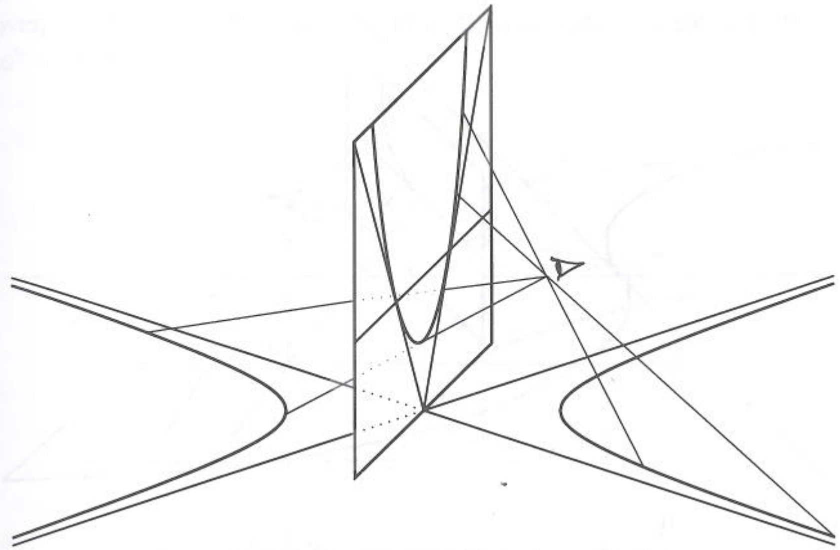

Figure 8.15: Branches of the hyperbola

8.4.1 Use this method to explain why

- the hyperbola $xy = 1$ has two points at infinity,
- the curve $y = x^3$ has one point at infinity.

Figures 8.13 and 8.14 were made by taking Alberti's veil to be the (x,z)-plane in (x,y,z)-space, with the "eye" at $(0,-4,4)$ viewing the (x,y)-plane.

8.4.2 Find the parametric equations of the line from $(0,-4,4)$ to $(x',y',0)$, and hence show that this line meets the veil where

$$x = \frac{4x'}{y'+4}, \quad z = \frac{4y'}{y'+4}.$$

8.4.3 Renaming the coordinates x, z in the veil as X, Y respectively, show that

$$x' = \frac{4X}{4-Y}, \quad y' = \frac{4Y}{4-Y}.$$

8.4.4 Deduce from Exercise 8.4.3 that the points (x',y') on the parabola $y = x^2$ have image on the veil

$$X^2 + \frac{(Y-2)^2}{4} = 1,$$

and check that this is the ellipse shown in Figure 8.13.

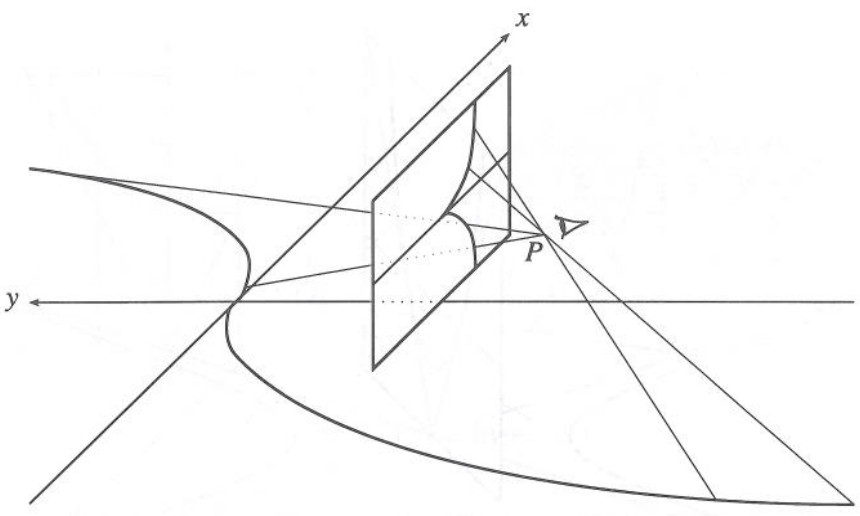

Figure 8.16: Perspective view of a cubic curve

8.5 Homogeneous Coordinates

The way in which projective geometry allows infinity to be put on the same footing as the finite points of the plane is intuitively clear when one thinks of the horizon in a picture, which is a line like any other. However, the most convenient way to formalize the idea is to introduce coordinates. This did not happen in Desargues' time, perhaps because of the resistance to coordinates in elementary geometry that was then prevalent (see Sections 7.4 and 7.5). Suitable coordinates, now known as *homogeneous coordinates*, were invented by Möbius (1827) and Plücker (1830). Homogeneous coordinates give a natural extension of the cartesian plane $\mathbb{R}^2$ by points at infinity by assigning new coordinates to the points already present and creating new points with the coordinates left over.

The homogeneous coordinates of a point $(X,Y) \in \mathbb{R}^2$ are all the real triples (Xz, Yz, z) with $z \neq 0$, that is, all real triples (x,y,z) with $x/z = X$, $y/z = Y$. If, following Klein (1925), we take X, Y to be the x, y coordinates in the plane $z = 1$, then these triples are just the coordinates of points $\neq O$ on the line in $\mathbb{R}^3$ from O to (X,Y) (Figure 8.17). Thus homogeneous coordinates give a one-to-one correspondence between points $(X,Y) \in \mathbb{R}^2$ and nonhorizontal lines through O in $\mathbb{R}^3$. The horizontal lines, whose points

have coordinates $(x,y,0)$, naturally correspond to points at infinity. Moreover, there is a natural way to determine which points at infinity "belong to" a given curve.

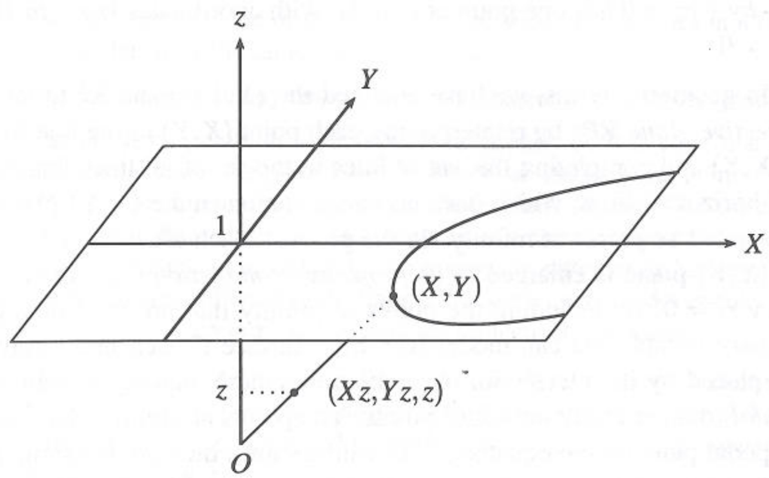

Figure 8.17: Constructing homogeneous coordinates

Each curve C in $\mathbb{R}^2$, expressed by an equation

$$p(X,Y) = 0 \tag{1}$$

say, can be reexpressed by the equation

$$p\left(\frac{x}{z}, \frac{y}{z}\right) = 0 \tag{2}$$

for $z \neq 0$. If p is a polynomial of degree n, we can extend (2) to all values of z by multiplying through by z^n, giving

$$z^n p\left(\frac{x}{z}, \frac{y}{z}\right) = \bar{p}(x,y,z) = 0, \tag{3}$$

where $\bar{p}$ is a *homogeneous polynomial* of degree n in x, y, z [that is, if (x,y,z) is a solution of (3), so is (tx, ty, tz)—as it should be, since these triples are coordinates of the same point]. For example, if the curve in $\mathbb{R}^2$ is the line $aX + bY + c = 0$, then the corresponding homogeneous equation (3) is $ax + by + cz = 0$.

Equation (3) is satisfied by all points $(X,Y) = (x/z, y/z)$ of C, together with other possible coordinate triples with $z = 0$. The latter form horizontal lines approached by the lines from O to points of C as X or $Y \to \infty$, so it is natural to regard them as *the points at infinity of C*. In particular, each line $ax + by + cz = 0$ has one point at infinity, with coordinates $(tb, -ta, 0)$ for all $t \neq 0$.

In geometric terms, we have enlarged the (X,Y)-plane $\mathbb{R}^2$ to the *real projective plane* $\mathbb{R}P^2$ by reinterpreting each point (X,Y) as the line from O to (X,Y) and completing this set of lines to the set of all lines through O. The horizontal lines, which have no interpretation in the (X,Y)-plane, are interpreted as points at infinity. In the process, each algebraic curve C in the (X,Y)-plane is enlarged to its *projective completion* $\bar{C}$ [with equation $\bar{p}(x,y,z) = 0$] by including the points at infinity that are the limits of its ordinary points. We can model $\mathbb{R}P^2$ by a surface if each line through O is replaced by its intersection with the unit sphere, namely, a pair of *antipodal* (diametrically opposite) points. The points at infinity then become antipodal pairs on the equator $z = 0$, which shows they are the same as all other points. A line L in $\mathbb{R}^2$, given by a linear equation $aX + bY + c = 0$, has as completion the *projective line* $\bar{L}$ with homogeneous linear equation $ax + by + cz = 0$, which represents a plane through O. Thus the points of $\bar{L}$ lie in a plane through O and hence are modelled by the antipodal pairs on a great circle. The *line at infinity*, $z = 0$, consists simply of the antipodal pairs on the equator, and hence is the same as any other projective line.

A projective line can be visualized as a great semicircle (which contains one representative from each antipodal pair) with its ends identified. This is a closed curve, so Kepler and Desargues were not far wrong in thinking of a projective line as a circle. The projective plane, however, is not a sphere but something more peculiar, as was noticed by Klein (1874). On a sphere, any simple closed curve separates the surface into two parts. A "small" closed curve in the projective plane $\mathbb{R}P^2$, that is, one strictly contained in a hemisphere of the model, also separates $\mathbb{R}P^2$, but a "large" one does not. The equator, for instance, does not separate the upper hemisphere from the lower, because the hemispheres are the *same place* under the antipodal point identification! A less paradoxical view of this is seen by going back to the model of $\mathbb{R}P^2$ whose elements are lines through O. The lines through the equator do not separate the lines through the upper hemisphere from the lines through the lower hemisphere, because these are the same lines.

EXERCISES

The model of $\mathbb{R}P^2$ whose points are lines through O and whose lines are planes through O also helps in visualizing other basic properties of projective lines:

8.5.1 Use this interpretation of projective lines to show that all lines in a family of parallels have the same point at infinity.

8.5.2 Likewise, show that any two projective lines meet in exactly one point.

In passing from points in a plane to lines through O, it is clear that there is nothing special about the plane $z = 1$ in which we started—the points in *any* plane in $\mathbb{R}^3$ not containing O correspond to distinct lines through O. Conversely, we can pass from lines through O to points in any plane not containing O. In fact, it is often convenient to view projective curves in *different* such planes. This corresponds to taking different projections of the same curve, and it enables us to show, for example, that $y = x^3$ and $y^2 = x^3$ are projectively the same.

8.5.3 Let X, Y denote the x, y coordinates in the plane $z = 1$ (as before), and let X', Z' denote the x, z coordinates in the plane $y = 1$. Show that the curves $Y = X^3$, $(Z')^2 = (X')^3$ have the same equation in the homogeneous coordinates x, y, z.

8.5.4 Deduce that $Y = X^3$ is mapped onto $(Z')^2 = (X')^3$ by projection from O of the plane $z = 1$ onto the plane $y = 1$.

Now let us return to the interpretation of the projective plane $\mathbb{R}P^2$ as a surface, the sphere with antipodal points identified. The following result shows another way in which $\mathbb{R}P^2$ differs from a sphere.

8.5.5 Show that a strip of $\mathbb{R}P^2$ surrounding a projective line is a Möbius band (Figure 8.18.)

8.6 Bézout's Theorem Revisited

As we saw in Section 7.5, a precise account of points at infinity is needed to obtain Bézout's theorem that a curve of degree m meets a curve of degree n in mn points. The projective completion does this. The preceding exercises show that lines (curves of degree 1) meet in $1 \times 1 = 1$ point. In general, if C_m is a curve with homogeneous equation of degree m,

$$p_m(x,y,z) = 0 \tag{1}$$

and if C_n is a curve with homogeneous equation of degree n,

$$p_n(x,y,z) = 0 \tag{2}$$

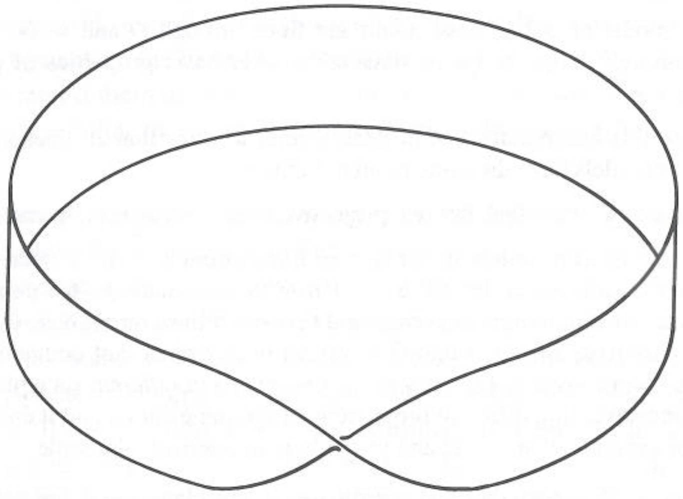

Figure 8.18: A Möbius band

one wishes to show that the equation

$$r_{mn}(x,y) = 0, \tag{3}$$

which results from eliminating z between (1) and (2), is homogeneous of degree mn. This is not hard to do (see exercises), but it seems that a homogeneous formulation of Bézout's theorem, with a rigorous proof that the resultant r_{mn} has degree mn, was not given until the late 1800s [according to Kline (1972), p. 553, the "proper count of multiplicities" was first made by Halphen in 1873].

An obvious condition must be included in the hypothesis of Bézout's theorem: that the curves C_m and C_n have no common component. The algebraic equivalent of this condition is that the polynomials p_m, p_n have no nonconstant common factor. Then the form of Bézout's theorem that can be proved with the help of homogeneous coordinates is *curves C_m, C_n with homogeneous equations $p_m(x,y,z) = 0$, $p_n(x,y,z) = 0$ of degrees m, n and no common component have intersections given by the solutions of a homogeneous equation $r_{mn}(x,y) = 0$ of degree mn.*

A useful consequence of Bézout's theorem is that curves C_m, C_n of degrees m, n with *more* than mn intersections have a common component.

EXERCISES

As the Chinese discovered (see Section 6.2), the problem of elimination belongs to linear algebra. In the case of Bézout's theorem, this includes the determinant criterion for a set of homogeneous equations to have a nonzero solution, and it leads to an expression for the resultant r_{mn} as a determinant.

8.6.1 Suppose that

$$p_m(x,y,z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m,$$
$$p_n(x,y,z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n$$

are homogeneous polynomials of degrees m, n. Thus $a_i(x,y)$ is homogeneous of degree i, $b_j(x,y)$ is homogeneous of degree j. By multiplying p_m and p_n by suitable powers of z, show that the equations

$$p_m = 0 \quad \text{and} \quad p_n = 0$$

are equivalent to a system of $m+n$ homogeneous linear equations in the variables $z^{m+n-1}, \ldots, z^2, z^1, z^0$, which in turn is equivalent to

$$r_{mn}(x,y) \equiv \begin{vmatrix} a_0 & a_1 & \cdots & a_m & 0 & & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_m & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \ddots & \ddots & \\ & & & & & & & 0 \\ 0 & \cdots & 0 & a_0 & & \cdots & & a_m \\ b_0 & b_1 & \cdots & & b_n & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & & b_n & & \vdots \\ \vdots & & & & & & \ddots & 0 \\ 0 & \cdots & 0 & b_0 & & \cdots & & b_n \end{vmatrix} = 0.$$

8.6.2 Show that a polynomial $p(x,y)$ is homogeneous of degree $k \Leftrightarrow p(tx,ty) = t^k p(x,y)$.

8.6.3 Show $r_{mn}(tx,ty) = t^{mn} r_{mn}(x,y)$. *Hint*: Multiply the rows of $r_{mn}(tx,ty)$ by suitable powers of t to arrange that each element in any column contains the same power of t. Then remove these factors from the columns so that $r_{mn}(x,y)$ remains.

8.7 Pascal's Theorem

Pascal's *Essay on Conics* [Pascal (1640)] was written in late 1639, when Pascal was 16. He probably had heard about projective geometry from his father, who was a friend of Desargues. The *Essay* contained the first statement of a famous result that became known as Pascal's theorem or the *mystic hexagram*. The theorem states that the pairs of opposite sides of a hexagon inscribed in a conic section meet in three collinear points. (The vertices of the hexagon can occur in any order on the curve. In Figure 8.19 the order was chosen to enable the three intersections to lie inside the curve.) Pascal's proof is not known, but he probably established the theorem for the circle first, then trivially extended it to arbitrary conics by projection.

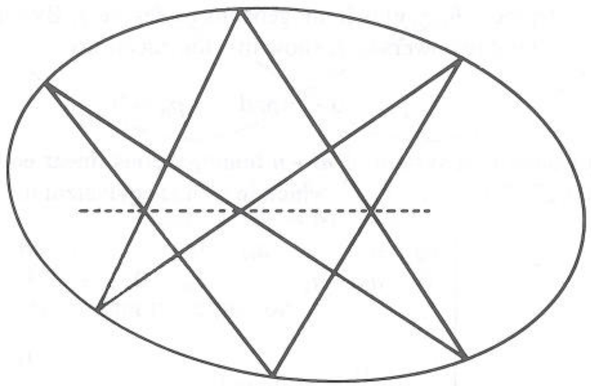

Figure 8.19: Pascal's theorem

Plücker (1847) threw new light on Pascal's theorem by showing it to be an easy consequence of Bézout's theorem. Plücker used an auxiliary theorem about cubics which can be bypassed, giving the following direct deduction from Bézout's theorem.

Let $L_1, L_2, \ldots, L_6$ be the successive sides of the hexagon. The unions of alternate sides, $L_1 \cup L_3 \cup L_5$ and $L_2 \cup L_4 \cup L_6$, can be regarded as cubic curves

$$l_{135}(x,y,z) = 0, \qquad l_{246}(x,y,z) = 0,$$

where each l is a product of three linear factors. These two curves meet in nine points: the six vertices of the hexagon and the three intersections of opposite sides. Let

$$c(x,y,z) = 0 \tag{1}$$

be the equation of the conic that contains the six vertices.

We can choose constants α, β so that the cubic curve

$$\alpha l_{135}(x,y,z) + \beta l_{246}(x,y,z) = 0 \tag{2}$$

passes through any given point P. Let P be a point on the conic, unequal to the six vertices. Then the curves (1), (2) of degrees 2, 3, have $7 > 2 \times 3$ points in common, and hence a common component by Bézout's theorem. Since c has no nonconstant factor, by hypothesis, this common component must be c itself. Hence

$$\alpha l_{135} + \beta l_{246} = cp \tag{3}$$

for some polynomial p, which must be linear since the left-hand side of (3) has degree 3 and c has degree 2. Since the curve $\alpha l_{135} + \beta l_{246} = 0$ passes through the nine points common to $l_{135} = 0$ and $l_{246} = 0$, while $c = 0$ passes through only six of them, the remaining three (the intersections of opposite sides) must be on the line $p = 0$.

Exercises

8.7.1 Generalize the preceding argument to show that if two degree n curves meet in n^2 points, nm of which lie on a curve of degree m, then the remaining $n(n-m)$ points lie on a curve of degree $n-m$.

An important special case of Pascal's theorem was discovered around 300 CE by Pappus, and it is called the *theorem of Pappus*. In this theorem, the conic is a "degenerate" conic section, consisting of two straight lines.

The usual statement of Pappus' theorem, like that of Pascal's theorem, says that the intersections of opposite sides of the hexagon are in a straight line. However, if we avail ourselves of the freedom to take this line to be at infinity, then Pappus' theorem takes a form that is easier to visualize and prove.

8.7.2 Interpret Figure 8.20 as an illustration of Pappus' theorem.

8.7.3 Write down a statement of the theorem corresponding to Figure 8.20, the conclusion of which is that P_1Q_3 and P_2Q_2 are parallel. (Equivalently, $OP_1/OP_2 = OQ_3/OQ_2$.)

8.7.4 Deduce the required equation from two other equations that express parallelism in Figure 8.20.

8.7.5 Also draw the figure and prove the theorem in the case where the two lines P_1P_2 and Q_1Q_2 do *not* meet at O, that is, when they too are parallel.

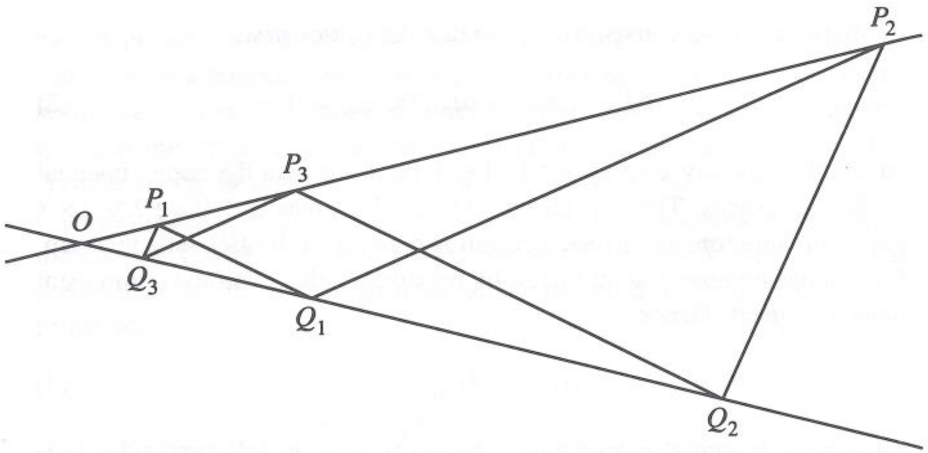

Figure 8.20: Illustration of Pappus' theorem

8.8 Biographical Notes: Desargues and Pascal

Girard Desargues was born in Lyons in 1591 and died in 1661. He was one of nine children of Girard Desargues, a tithe collector, and Jeanne Croppet. He was evidently brought up in Lyons, but information about his early life is lacking. By 1626 he was working as an engineer in Paris and may have used his expertise in the famous siege of La Rochelle in 1628, during which a dike was built across the harbor to prevent English ships from relieving the city.

In the 1630s he joined the circle of Marin Mersenne, which met regularly in Paris to discuss scientific topics, and in 1636 contributed a chapter to a book of Mersenne on music theory. In the same year he published a 12-page booklet on perspective, the first hint of his ideas in projective geometry. The *Brouillon projet* [Desargues (1639)] was published in an edition of only 50 copies and won very little support. In fact, its reception was generally hostile, and Desargues was engaged in a pamphleteering battle for years with his detractors [see Taton (1951), pp. 36–45]. At first his only supporters were Pascal, most of whose work on projective geometry is also lost, and the engraver Abraham Bosse, who expounded Desargues' perspective method [Bosse (1648)]. Desargues became discouraged by the attacks on his work and left the dissemination of his ideas up to Bosse, who was not really mathematically equipped for the task. Projective geometry secured a place in mathematics only with the publication of a book

by Phillipe de la Hire [de la Hire (1673)], whose father, Laurent, had been a student of Desargues. It seems quite likely that la Hire's book influenced Newton. For this and more on Desargues' mathematical legacy, see Field and Gray (1987), Ch. 3.

Around 1645 Desargues turned his talents to architecture, perhaps to demonstrate to his critics the practicality of his graphical methods. He was responsible for various houses and public buildings in Paris and Lyons, excelling in complex structures such as staircases. His best-known achievement in engineering, a system for raising water at the château of Beaulieu, near Paris, is also interesting from the geometrical viewpoint. It makes the first use of epicyclic curves (Section 2.5) in cogwheels, as was noted by Huygens (1671). Huygens visited the château at the time when it was owned by Charles Perrault, the author of *Cinderella* and *Puss in Boots*.

Desargues apparently returned to scientific circles in Paris toward the end of his life—Huygens heard him give a talk on the existence of geometric points on November 9, 1660—but information about this period is scanty. His will was read in Lyons on October 8, 1661, but the date and place of his death are unknown.

Blaise Pascal (Figure 8.21) was born in Clermont-Ferrand in 1623 and died in Paris in 1662. His mother, Antoinette Bagon, died when he was three, and Blaise was brought up by his father, Etienne. Etienne Pascal was a lawyer with an interest in mathematics who belonged to Mersenne's circle and, as mentioned earlier, was a friend of Desargues. He has a curve named after him, the *limaçon of Pascal*. In 1631 Etienne took Blaise and his two sisters to Paris and gave up all official duties to devote himself to their education. Thus Blaise Pascal never went to school or university, but by the age of 16 he was learned in Latin, Greek, mathematics, and science. And of course he had written his *Essay on Conics* and discovered Pascal's theorem.

The *Essay on Conics* [Pascal (1640)] is a short pamphlet containing an outline of the great treatise on conics he had begun to prepare, and which is now lost. It includes a statement of Pascal's theorem for the circle. Pascal worked on his treatise until 1654, when it was nearly complete, but he never mentioned it thereafter. Leibniz saw the manuscript when he was in Paris in 1676, but no further sightings are known.

In 1640 Pascal and his sisters joined their father in Rouen, where he had become a tax official. Pascal got the idea of constructing a calculating machine to help his father in his work. He found a theoretical solution around

Figure 8.21: Pascal

the end of 1642, based on toothed wheels, but difficulties in the production of accurate parts delayed the appearance of the machine until 1645. This was the first working computer. The gear mechanism for addition seems rather obvious to us now, but in Pascal's day it already raised questions of the "Can a machine think?" kind. Pascal himself was sufficiently amazed by the mechanism to say that "the arithmetical machine produces effects which approach nearer to thought than all the actions of the animal. But it does nothing which would enable us to attribute will to it, as to the animals" (Pascal, *Pensées*, 340). The machine greatly impressed the French chancellor, and Pascal was granted exclusive rights to manufacture and sell it. Whether it was a commercial success is not known, but for a time, at least, Pascal was diverted by the opportunity to cash in on his ideas.

The direction of Pascal's life began to shift away from such worldly concerns in 1646, when his father was treated for a leg injury by two local bonesetters. The bonesetters were Jansenists, then a fast-growing sect within the Catholic church. Their influence resulted in the conversion of the whole family to Jansenism, and Pascal began to devote more time to religious thought. For some years, though, he continued with scientific work.

In 1647 he investigated the variation of barometric pressure with altitude, resulting in his *New Experiments Concerning the Vacuum*, published the same year; in 1651 he did pioneering work in hydrostatics, resulting in his *Great Experiment Concerning the Equilibrium of Fluids*, published in 1663; and in 1654 he investigated the so-called Pascal's triangle, making fundamental contributions to number theory, combinatorics, and probability theory (for more on this, see Chapter 11). In 1654 Pascal experienced a "second conversion," which led to his almost complete withdrawal from the world and science and his increasing commitment to the Jansenist cause. Only in 1658 and 1659 did he concentrate at times on mathematics (on one occasion, so the story goes, to take his mind off the pain of a toothache). His favorite topic at this stage was the cycloid, the curve generated by a point on the circumference of a circle that rolls on a straight line. Later in the seventeenth century the cycloid became important in the development of mechanics and differential geometry (see Chapters 13 and 17).

Mathematicians are of course very sorry about Pascal's withdrawal from mathematics at an early age; however, it was not just religion that gained from Pascal's conversion. The *Provincial Letters*, which he wrote to promote Jansenist ideas, and his *Pensées*, which were edited by the Jansenists after his death, became classics of French literature. Undoubtedly Pascal is the only great mathematician whose standing is equally great among writers. Moreover, his devotion to the Jansenist ideal of serving the needy had one enduring practical consequence: his idea of a public transport system. Shortly before his death in 1662, Pascal saw the inauguration of the world's first omnibus service. Coaches could be taken from the Porte Sainte-Antoine to the Luxembourg in Paris for 5 sous, with profits being directed to the relief of the poor.