7

Analytic Geometry

7.1 Steps toward Analytic Geometry

The basic idea of analytic geometry is the representation of curves by equations, but this is not the whole idea. If it were, then the Greeks would be considered the first analytic geometers. Menaechmus was perhaps the first to discover equations of curves, along with his discovery of the conic sections, and we have seen how he used equations to obtain $\sqrt[3]{2}$ as the intersection of a parabola and a hyperbola (Section 2.4). Apollonius' study of conics used equations obtained as by-products of geometric arguments.

What was lacking in Greek mathematics was both the inclination and the technique to manipulate equations to obtain information about curves. The Greeks used curves to study algebra rather than the other way around. Menaechmus' construction of $\sqrt[3]{2}$ is an excellent example of this: extraction of roots was not a given operation but one that had to be secured by geometric construction. Similarly, an equation was not an entity in its own right but a property of a curve that could be discovered after the curve had been constructed geometrically. This was a natural state of affairs as long as equations were written out in words. When, as in Apollonius, an equation takes half a page to write out, it is difficult to form a general concept of equation, function, or curve. Hence the lack of a general concept of curve in Greek mathematics—it was just too complicated to handle in their language.

In the Middle Ages the idea of coordinates emerged in a different way in the work of Oresme (around 1323–1382). Coordinates had been used in astronomy and geography since Hipparchus (around 150 BCE); in fact,

Oresme called his coordinates "longitude" and "latitude," but he seems to have been the first to use them to represent functions such as velocity as a function of time. Setting up the coordinate system *before* determining the curve was Oresme's step beyond the Greeks, but he too lacked the algebra to go further.

The step that finally made analytic geometry feasible was the solution of equations and the improvement of notation in the sixteenth century, which we discussed in the previous chapter. This step made it possible to consider equations, and hence curves, in some generality and to have confidence in one's ability to manipulate them. As we shall see in the next section, the two founders of analytic geometry, Fermat and Descartes, were both strongly influenced by these developments.

For more details on the development of analytic geometry, the reader is referred to an excellent book by Boyer (1956).

EXERCISE

7.1.1 Generalize the idea of Menaechmus to show that any cubic equation

$$ax^3 + bx^2 + cx + d = 0 \quad \text{with} \quad d \neq 0$$

may be solved by intersecting the hyperbola $xy = 1$ with a parabola.

7.2 Fermat and Descartes

There have been several occasions in the history of mathematics when an important discovery was made independently and almost simultaneously by two individuals: noneuclidean geometry by Bolyai and Lobachevsky, elliptic functions by Abel and Jacobi, the calculus by Newton and Leibniz, for example. To the extent that we can rationally explain these remarkable events, it must be on the basis of ideas already "in the air," of conditions becoming favorable for their crystallization. As I tried to show in the previous section, conditions were favorable for analytic geometry at the beginning of the seventeenth century. Thus it is not completely surprising that the subject was independently discovered by Fermat (1629) and Descartes (1637). (Descartes' work *La Géométrie* may in fact have been started in the 1620s. In any case it is independent of Fermat, whose work was not published until 1679.)

It is a surprise to learn, however, that both Fermat and Descartes began with an analytic solution of the same classical geometric problem, the four-line problem of Apollonius, and that the main discovery of each was that second-degree equations correspond to conic sections. Up to this point Fermat was more systematic than Descartes, but that was as far as he went. He was content to leave his work in a "simple and crude" state, confident that it would grow in stature when nourished by new inventions.

Descartes, on the other hand, treated many higher-degree curves and clearly understood the power of algebraic methods in geometry. He wanted to withhold this power from his contemporaries, however, particularly the rival mathematician Roberval, as he admitted in a letter to Mersenne [see Boyer (1956), p. 104]. *La Géométrie* was written to boast about his discoveries, not to explain them. There is little systematic development, and proofs are frequently omitted with a sarcastic remark such as, "I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself" (p. 10). Descartes' conceit is so great that it is a pleasure to see him come a cropper occasionally, as on p. 91: "The ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds." He was referring to the then-unsolved problem of determining the length of curves, but he spoke too soon, for in 1657 Neil and van Heuraet found the length of an arc of the semicubical parabola $y^2 = x^3$, and the calculus soon made such problems routine. [A full and interesting account of the story of arc length may be found in Hofmann (1974), Ch. 8.]

EXERCISES

As we now know, all conic sections may be given by the following standard form equations (from Section 2.4):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ (ellipse)},\qquad y=ax^2 \text{ (parabola)},\qquad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ (hyperbola)}.$$

The reduction of an arbitrary quadratic equation in x and y to one of these forms depends on suitable choice of origin and axes, as Fermat and Descartes discovered. The main steps are outlined in the following exercises.

7.2.1 Show that a *quadratic form* $ax^2+bxy+cy^2$ may be converted to a form $a'x'^2+b'y'^2$ by suitable choice of θ in the substitution

$$x = x'\cos\theta - y'\sin\theta,$$
$$y = x'\sin\theta + y'\cos\theta,$$

by checking that the coefficient of $x'y'$ is $(c-a)\sin 2\theta + b\cos 2\theta$.

7.2.2 Deduce from Exercise 7.2.1 that, by a suitable rotation of axes, any quadratic curve may be expressed in the form $a'x'^2+b'y'^2+c'x'+d'y'+e'=0$.

7.2.3 If $b'=0$, but $a'\neq 0$, show that the substitution $x'=x''+f$ gives either a standard-form parabola, or the "double line" $x''^2=0$.

(Why is this called a "double line," and is it a section of a cone?)

7.2.4 If both a' and b' are nonzero, show that a shift of origin gives the standard form for either an ellipse or a hyperbola, or else a pair of lines.

7.3 Algebraic Curves

> I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order is by recognizing the fact that all points of those curves which we may call "geometric," that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed by means of a single equation.
>
> [Descartes (1637), p. 48]

In this passage Descartes defines what we now call *algebraic curves*. The fact that he calls them "geometric" shows his attachment to the Greek idea that curves are the product of geometric constructions. He is using the notation of equations not to define curves directly but to restrict the notion of geometric construction more severely than the Greeks did, thereby restricting the concept of curve. As we saw in Section 2.5, the Greeks considered some constructions, such as rolling one circle on another, which are capable of producing transcendental curves. Descartes called such curves "mechanical" and found a way to exclude them by his restriction to curves "expressed by means of a single equation." It becomes clear in the lines following the preceding quotation that he means polynomial equations, since he gives a classification of equations by degree.

Descartes' rejection of transcendental curves was short-sighted, as the calculus soon provided techniques to handle them, but nevertheless it was fruitful to concentrate on algebraic curves. The notion of degree, in particular, was a useful measure of complexity. First-degree curves are the simplest possible, namely, straight lines; second-degree are the next simplest, conic sections. With third-degree curves one sees the new phenomena of inflections, double points, and cusps. Inflection and cusp are familiar from $y = x^3$ and $y^2 = x^3$, respectively; we also saw a cusp on the cissoid (Section 2.5). A classical example of a cubic with a double point is the *folium* (leaf) *of Descartes* (1638),

$$x^3 + y^3 = 3axy.$$

The "leaf" is the closed portion to the right of the double point; Descartes misunderstood the rest of the curve through neglect of negative coordinates. The true shape of the folium was first given by Huygens (1692). Figure 7.1 is Huygens' drawing, which also shows the asymptote to the curve.

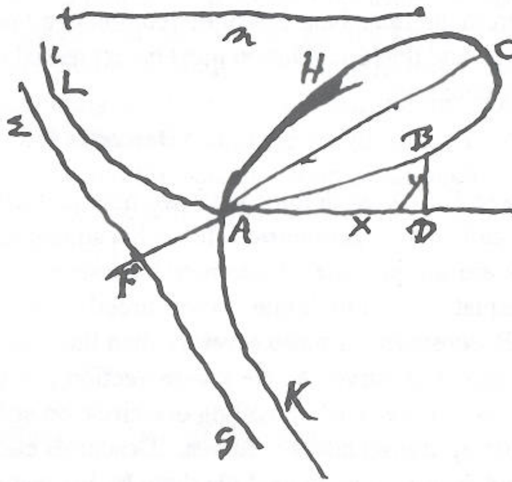

Figure 7.1: Huygens' drawing of the folium

An excellent account of the early history of curves can be found in Brieskorn and Knörrer (1981), Chapter 1. Many individual curves, with diagrams, equations, and historical notes, can be found in Gomes Teixeira (1995a,b,c). The development of Descartes' concept of curve has been studied by Bos (1981).

EXERCISES

The folium is a cubic curve to which Diophantus' chord method (Section 3.5) applies. One takes the line $y = tx$ through the "obvious" rational point $(0,0)$ on the curve, and finds its other point of intersection. This construction also enables us to express an arbitrary point (x,y) on the curve in terms of the parameter t.

7.3.1 Show that the folium of Descartes has parametric equations

$$x = \frac{3at}{1+t^3}, \qquad y = \frac{3at^2}{1+t^3}$$

and use these equations to show that it is tangential to the axes at 0.

7.3.2 Show that the equation $x^3 + y^3 = 3axy$ of the folium may be written in the form

$$x + y = \frac{3a}{\frac{x}{y} + \frac{y}{x} - 1}.$$

7.3.3 Show that x/y and y/x tend to -1 as $x \to \pm\infty$ on the folium, and hence deduce the equation of its asymptote from Exercise 7.3.2.

A whole family of "multileaved" curves was studied by Grandi (1723):

7.3.4 The *roses of Grandi* are given by the polar equations

$$r = a\cos n\theta$$

for integer values of n. [Figure 7.2 shows some of these curves, as given by Grandi (1723)]. Show that the roses of Grandi are algebraic.

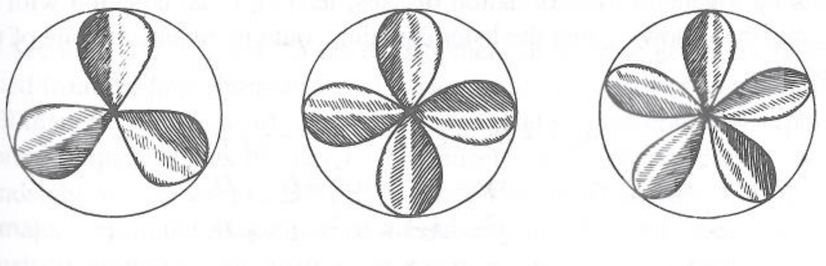

Figure 7.2: Roses of Grandi

7.3.5 Show that the "rose" for $n = 1$ is a circle and that the "rose" for $n = 2$ has cartesian equation

$$(x^2 + y^2)^3 = a^2(x^2 - y^2)^2.$$

7.4 Newton's Classification of Cubics

Since first- and second-degree curves are straight lines and conics, they were well understood before the advent of analytic geometry. Up to the end of the eighteenth century most mathematicians considered them not amenable to further clarification, and hence an unsuitable subject for the new methods. A famous example is the Greek-style treatment of planetary orbits in Newton's *Principia* [Newton (1687)]. The classical attitude to low-degree curves was summed up by d'Alembert in his article on geometry in the *Encylopédie* (1751):

> Algebraic calculation is not to be applied to the propositions of elementary geometry because it is not necessary to use this calculus to facilitate demonstrations, and it appears that there are no demonstrations which can really be facilitated by this calculus except for the solution of problems of second degree by the line and circle.

Thus the first new problem opened up by analytic geometry, and also the first considered properly to belong to the subject, was the investigation of cubic curves. These curves were classified, more or less completely, by Newton (1695) [see Ball (1890) for a commentary].

Newton (1667) began this work with the general cubic in x and y,

$$ay^3+bxy^2+cx^2y+dx^3+ey^2+fxy+gx^2+hy+kx+l=0,$$

making a general transformation of axes, leading to an equation with 84 terms, then showing that the latter equation could be reduced to one of the forms

$$\begin{aligned}Axy^2+By&=Cx^3+Dx^2+Ex+F,\\ xy&=Ax^3+Bx^2+Cx+D,\\ y^2&=Ax^3+Bx^2+Cx+D,\\ y&=Ax^3+Bx^2+Cx+D.\end{aligned}$$

Newton then divided the curves into species according to the roots of the right-hand side, obtaining 72 species (and overlooking 6). His paper does not contain detailed proofs; these were supplied by Stirling (1717), along with four of the species Newton had missed. Newton's classification was

criticized by some later mathematicians, such as Euler, for lacking a general principle. A unifying principle was certainly desirable, to reduce the complexity of the classification. And such a principle was already implicit in one of Newton's passing remarks, Section 29, "On the Genesis of Curves by Shadows." This principle, which will be explained in the next chapter, reduces cubics to the five types seen in Figure 7.3 [taken from an English translation of Newton's paper published in 1710; see Whiteside (1964)].

The reader may wonder where the most familiar cubic, $y=x^3$, appears among these five. The answer is that it is equivalent to the one with a cusp, in Newton's Figure 75. This is explained in the next chapter.

Exercises

The cubic curves that Newton called "cuspidate" and "nodated" are algebraically simpler than the others. In particular, they can be parameterized by rational functions.

7.4.1 Find a parameterization $x=p(t), y=q(t)$ of the semicubical parabola $y^2=x^3$ by polynomials p and q, (i) by inspection, (ii) by finding the second intersection point of the line $y=tx$ through the cusp $(0,0)$.

7.4.2 Find rational functions $x=r(t), y=s(t)$ that parameterize $y^2=x^2(x+1)$, by finding the second intersection of the line $y=tx$ through the double point of the curve.

7.5 Construction of Equations and Bézout's Theorem

In Sections 7.1, 7.2, and 7.3 the development of analytic geometry is outlined from the first observations of equations as properties of curves to the full realization that equations *defined* curves and that the concept of (polynomial) equation was the key to the concept of (algebraic) curve. With hindsight, we can say that Descartes' *La Géométrie* [Descartes (1637)] was a major step in the maturation of the subject, but the book does not conclusively establish what analytic geometry is. In fact, it is largely devoted to two transitional topics in the development of the subject: the sixteenth-century theory of equations and the now almost forgotten discipline called "construction of equations."

The paradigm construction of an equation was Menaechmus' construction of $\sqrt[3]{2}$ by intersecting a parabola and hyperbola. From a geometric

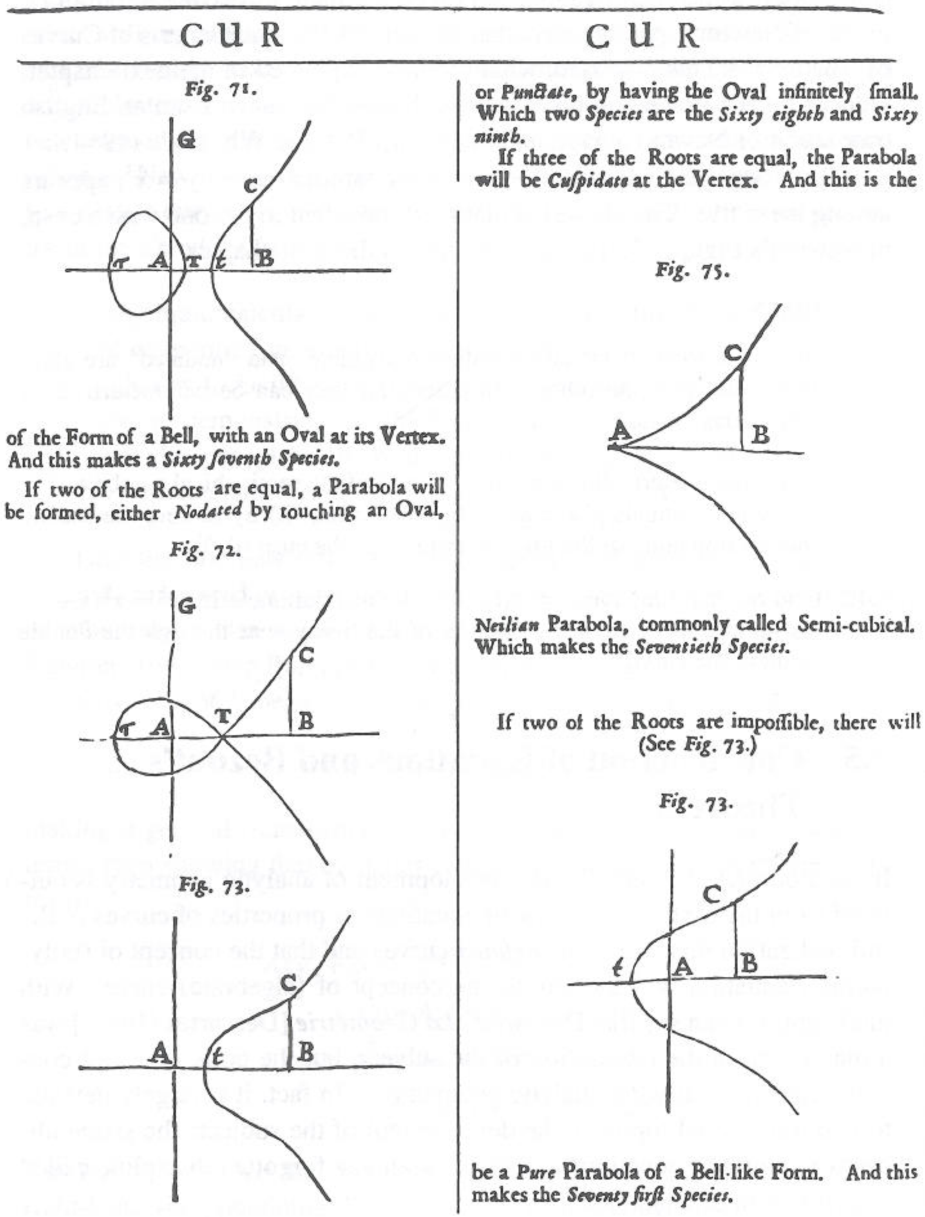
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Fig. 71.

of the Form of a Bell, with an Oval at its Vertex. And this makes a *Sixty seventh Species.*

If two of the Roots are equal, a Parabola will be formed, either *Nodated* by touching an Oval,

Fig. 72.

Fig. 73.

or *Punctate*, by having the Oval infinitely small. Which two *Species* are the *Sixty eighth* and *Sixty ninth.*

If three of the Roots are equal, the Parabola will be *Cuspidate* at the Vertex. And this is the

Fig. 75.

Neilian Parabola, commonly called Semi-cubical. Which makes the *Seventieth Species.*

If two of the Roots are impossible, there will (See *Fig.* 73.)

Fig. 73.

be a *Pure* Parabola of a Bell-like Form. And this makes the *Seventy first Species.*

Figure 7.3: Newton's classification of cubic curves

point of view, one is using familiar curves (parabola and hyperbola) to construct a less familiar length ($\sqrt[3]{2}$). This becomes sharper when expressed algebraically: curves of degree 2 are being used to solve an equation of degree 3, $x^3 = 2$. In the 1620s Descartes discovered something more general: a method of solving any third- or fourth-degree equation by intersecting curves of degree 2, a parabola and a circle. His friend Beeckman (1628) reported in a note that "M. Descartes made so much of this invention that he confessed never to have found anything superior himself and even that nobody else had ever found anything better" [translation by Bos (1981), p. 330]. Descartes was not as superior as he thought, since Fermat independently made the same discovery in an unpublished work [Fermat (1629)], strengthening the already extraordinary coincidence between his work and that of Descartes. However, Fermat apparently did not pursue the idea further, and Descartes did.

In *La Géométrie* Descartes found a particular cubic curve, the so-called cartesian parabola, whose intersections with a suitable circle yield the solution of any given fifth- or sixth-degree equation. Descartes concludes the book with this result, blithely telling the reader that

> it is only necessary to follow the same method to construct all problems, more and more complex, ad infinitum; for in the case of a mathematical progression, whenever the first two or three terms are given, it is easy to find the rest.
>
> [Descartes (1637), p. 240]

In reality it was not easy, and efforts to find a satisfactory general construction for *n*th-degree equations petered out around 1750. The story of the rise and fall of this field of mathematics has been told by Bos (1981, 1984).

In their search for a general construction, mathematicians had casually assumed that a curve of degree *m* meets a curve of degree *n* in *mn* points. The first statement of this principle, which became known as Bézout's theorem, seems to have been made by Newton on May 30, 1665:

> For y^e number of points in w^ch two lines may intersect can never bee greater y^n y^e rectangle of y^e numbers of their dimensions. And they always intersect in soe many points, excepting those w^ch are imaginarie onely.
>
> [Newton (1665b), p. 498]

Bézout's theorem leads one to expect that solutions of an equation $r(x) = 0$ of degree $k = m \cdot n$ might be obtainable from the intersections of a suitable degree m curve with a suitable degree n curve. In algebraic terms, one seeks equations

$$p(x,y) = 0, \quad (1)$$
$$q(x,y) = 0 \quad (2)$$

of degrees m, n respectively, from which elimination of y yields the given equation

$$r(x) = 0 \quad (3)$$

as "resultant." This is how mathematicians in the West first encountered the problem of elimination, which the Chinese had solved some centuries earlier (Section 6.2).

However, apart from the fact that construction of equations was inverse to elimination, and much harder, Western mathematicians needed two additional facts about elimination itself: first, that elimination between equations of degrees m and n gave a resultant of degree mn; second, that an equation of degree mn has mn roots. The second statement, as mentioned in Section 6.7, becomes a fact only when complex numbers are admitted. The first becomes a fact only when "points at infinity" are admitted. If, for example, (1) and (2) are equations of parallel lines, then (3) is of "degree 0" and has *no* solutions. However, one can consider parallel lines to meet "at infinity," and the geometric framework for this idea, projective geometry, developed at about the same time as analytic geometry. Unfortunately, it was not realized until the nineteenth century that projective geometry and analytic geometry needed each other. Until then, projective geometry developed without coordinates, and all attempts to prove Bézout's theorem [notably by Maclaurin (1720), Euler (1748b), Cramer (1750), and Bézout (1779)] foundered for want of a proper method for counting points at infinity. As a result, Bézout's theorem, which turned out to be the main achievement of the theory of construction of equations, was not properly proved until long after the theory itself had been abandoned.

The origins of projective geometry, and the fruits of its merger with analytic geometry, are discussed in Chapter 8.

EXERCISES

We know from Section 6.7 that an arbitrary quartic equation is equivalent to one of the form

$$x^4 + px^2 + qx + r = 0.$$

7.5.1 Show that any such equation may be solved by finding the intersection of the parabola $y = x^2$ with another quadratic curve (hence with a conic section).

7.5.2 Find two parabolas whose intersections give the solutions of $x^4 = x + 1$, and hence show that this quartic equation has two real roots.

7.5 The Arithmetization of Geometry

We have stressed that early analytic geometers—Descartes in particular—did not accept that geometry could be *based* on numbers or algebra. Perhaps the first to take the idea of arithmetizing geometry seriously was Wallis (1616–1703). Wallis (1657), Chs. XXIII and XXV, gave the first arithmetic treatment of Euclid's Books II and V, and he had earlier given the first purely algebraic treatment of conic sections [Wallis (1655b)]. He initially derived equations from the classical definitions by sections of the cone but then proceeded conversely to derive their properties from the equations, "without the embranglings of the cone," as he put it.

Wallis was ahead of this time. Thomas Hobbes, introduced at the beginning of Chapter 2, described Wallis' treatise on conics as a "scab of symbols" and denounced "the whole herd of them who apply their algebra of geometry" [Hobbes (1656), p. 316, and Hobbes (1672), p. 447]. The example and authority of Newton probably reinforced the opinion that algebra was inappropriate in the geometry of lines or conic sections; we saw in Section 7.4 how this remained the accepted view until at least 1750.

Algebra did not catch on in elementary geometry until it was taken up by Lagrange (1773b) and supported by influential textbooks of Monge and Lacroix around 1800. But by the time elementary geometry had been brought into the theory of equations, higher geometry had broken out, depending more and more on calculus and the emerging theories of complex functions, abstract algebra, and topology, which bloomed in the nineteenth century. Higher geometry broke away to form differential geometry and algebraic geometry, leaving the elementary residue we call "analytic geometry" today.

Despite its lowly status, analytic geometry was given an important foundational role by Hilbert (1899). Hilbert took Wallis' arithmetization to its logical conclusion by assuming only the real numbers and sets as given and constructing *Euclidean geometry* from them.

Thus from the set $\mathbb{R}$ of reals, one constructs the *Euclidean plane* as the set of ordered pairs (x, y) ("points") where $x, y \in \mathbb{R}$. A *straight line* is a set of points (x,y) in the plane such that $ax+by+c=0$ for some constants a,b,c. Lines are *parallel* if their x and y coefficients are proportional. The *distance* between points (x_1,y_1) and (x_2,y_2) is defined to be $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. As explained in Section 1.6, this definition is motivated by Pythagoras' theorem, which is the keystone in the bridge from arithmetic to geometry.

With these definitions, all axioms and propositions of Euclid's geometry become provable propositions about equations. For example, the axiom that nonparallel lines have a point in common corresponds to the theorem that linear equations

$$a_1x+b_1y+c_1=0,$$
$$a_2x+b_2y+c_2=0$$

have a solution when $a_1b_2-b_1a_2\neq 0$.

Hilbert did not believe, any more than Newton did, that numbers were the true subject matter of geometry. He strongly supported geometric intuition as a method of discovery, as the book Hilbert and Cohn-Vossen (1932) makes clear. The purpose of his arithmetization was to give a secure logical foundation to geometry after the nineteenth-century developments that discredited geometry and installed arithmetic as the ultimate authority in mathematics. This foundation is no longer quite as secure as it seemed in 1900, as we shall see in Chapter 23; nevertheless, it is still the most secure foundation we know.

7.6 Biographical Notes: Descartes

René Descartes (Figure 7.4) was born in La Haye (now called La Haye-Descartes) in the French province of Touraine in 1596 and died in Stockholm in 1650. His father, Joachim, was a councilor in the high court of Rennes in Brittany; his mother, Jeanne, was the daughter of a lieutenant general from Poitiers and the owner of property that was eventually to assure Descartes of financial independence. His mother died in 1597, and Descartes was raised by his maternal grandmother and a nurse. He does not seem to have been close to his father, brother, or sister, seldom mentioning them to others and writing to them only on matters of business.

Joachim Descartes was away from home for half the year because of his court duties, but he saw enough of René to observe his exceptional curiosity, calling him his "little philosopher." In 1606 he enrolled him in the

Jesuit College of La Flèche, which had recently been founded by Henry IV in Anjou. The young Descartes was given special privileges at school, in recognition of his intellectual promise and delicate health. He was one of the few boys to have his own room, was permitted books forbidden to other students, and was allowed to stay in bed until late in the morning. Spending several morning hours in bed thinking and writing became his lifelong habit, and when he finally had to break it in the Swedish winter, the consequences were fatal.

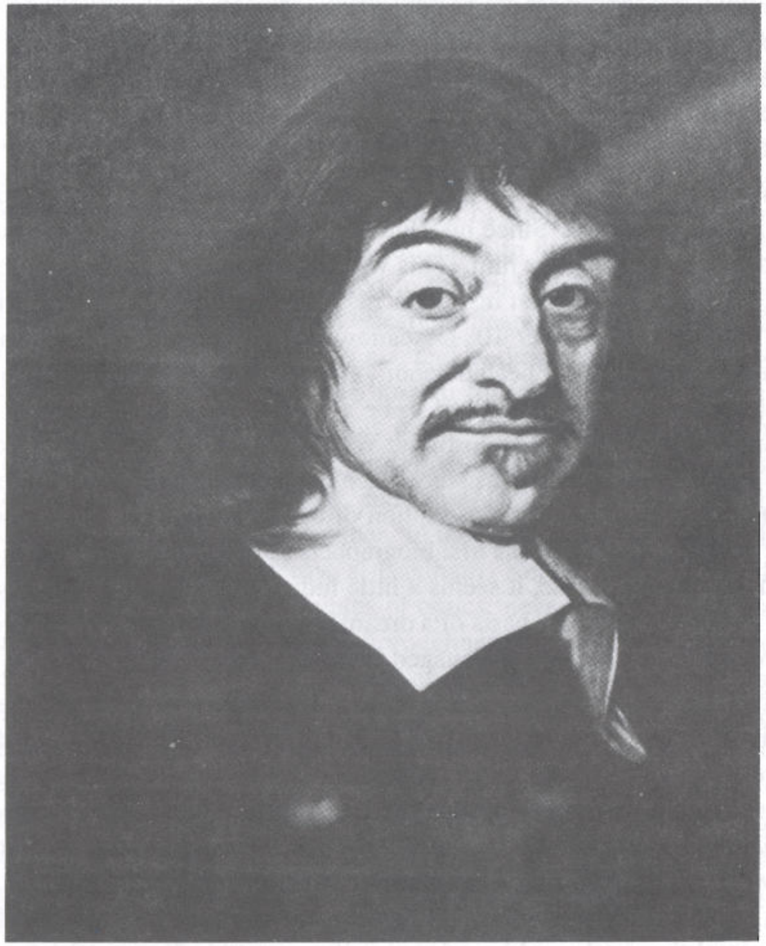

Figure 7.4: Descartes (Louvre Museum)

The most dramatic event of his schooldays was the assassination of Henry IV in 1610. Since Henry IV was not only the founder of the school but also the most popular king in French history, his death was a profound shock. La Flèche became the venue for an elaborate funeral ceremony, the climax of which was the burial of the king's heart. Descartes was one of 24 students chosen to participate in the ceremony.

He left La Flèche in 1614 and, after legal studies at Poitiers, which seem to have left no impression on him, went to Holland as an unpaid volunteer in the army of Prince Maurice of Nassau in 1618. This was not an unusual decision for a young Frenchman of means at the time, since the Dutch were fighting France's enemy, Spain, and Descartes seems to have joined the army to see the world, not because of any taste for barracks life or combat. As it happened, there was then a lull in the war, and Descartes had two years of virtual leisure to reflect on science and philosophy.

When in Breda, on November 10, 1618, he saw a mathematical problem posted on a wall. Since his Dutch was not yet fluent, he asked a bystander to translate it for him. This was how Descartes met Isaac Beeckman, who became his first instructor in mathematics and a lifelong friend. The following November 10, Descartes was in Bavaria. He spent a day of intense thought in a heated room ("stove" he called it) and that night had a dream he later considered to be a revelation of the path he should follow in developing his philosophy. Whether the dream also revealed the path to analytic geometry, as some have conjectured, will probably never be known. Descartes' own description of the dream has been lost, and we have only a summary by his first biographer, Baillet (1691), p. 85, which is not helpful. In any case, it seems a little ludicrous to award Descartes priority over Fermat on the basis of a dream. Could a counterclaim of priority be lodged if the dream of a teenaged Fermat came to light?

In 1628 Descartes moved to Holland, where he spent most of the rest of his life. He lived a simple but leisurely life and finally settled down to working out the ideas conceived nine years earlier. The relative isolation suited him, as he was hostile to other scientific giants of his time such as Galileo, Fermat, and Pascal and preferred to communicate with scholars who could understand him without challenging his superiority. One such was Marin Mersenne, who had been a senior student at La Flèche in Descartes' time and was his main scientific contact in France. Others were Princess Elizabeth of Bohemia and Queen Christina of Sweden, with both of whom Descartes had extensive correspondences.

A positive side to Descartes' intolerance of intellectual rivals was an apparently genuine interest in the affairs of his neighbors in Holland. He encouraged local youths who showed talent in mathematics, and he was known in the region as someone to turn to in times of trouble [see Vrooman (1970), pp. 194–196]. The one serious love of his life was a servant girl named Helen, who bore him a daughter, Francine, in 1635. Admittedly, his interest in this case did not extent to marrying Helen, but the death of Francine from scarlet fever in 1640 was the greatest sorrow of his life.

In 1649 Descartes agreed to journey to Stockholm to become tutor to Queen Christina. This was the culmination of his correspondence with her and of negotiations through Descartes' friend Chanut, the French ambassador. The queen, who was noted for her physical as well as mental vigor, slept no more than five hours a night and rose at 4 a.m. Descartes had to arrive at 5 a.m. to give her lessons in philosophy. The program commenced on January 14, 1650, during the coldest winter for over 60 years. One can imagine the shock to Descartes' system of such early rising followed by a journey from the ambassador's residence to the palace. However, it was actually Chanut who succumbed to the cold first. On January 18 he came down with pneumonia, and Descartes apparently caught it from him. Chanut recovered but Descartes did not, and he died on February 11, 1650.

Descartes is, of course, as well known for his philosophy as his analytic geometry. The *Geometry* was originally an appendix to his main philosophical work, the *Discourse on Method*. The other appendices were the *Dioptrics*, a treatise on optics, and the *Meteorics*, the first attempt to give a scientific theory of the weather. In the *Dioptrics*, Descartes did not inform his readers that Ptolemy, al-Haytham, Kepler, and Snell had already discovered the main principles of optics; nevertheless, he presented the subject with greater clarity and thoroughness than before, undoubtedly advancing both the theory and practice of optical instrumentation. As for the *Meteorics*, we now know how premature it was to attempt a theory of the weather in 1637, so it is understandable that this treatise has more misses than hits. His big hit was a correct explanation of rainbows (except for the colors, whose explanation was completed by Newton), which Descartes was able to give on the basis of his optics. More typical, unfortunately, was his explanation of thunder: it was caused by clouds bumping together, and not related to lightning. An excellent survey of Descartes' scientific work and philosophy, with a particularly detailed analysis of the *Geometry*, is given by Scott (1952).