RATIONAL MAPS WITH DISCONNECTED JULIA SET

by

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Abstract. — We show that if f is a hyperbolic rational map with disconnected Julia set $\mathcal{J}$, then with the possible exception of finitely many periodic components of $\mathcal{J}$ and their countable collection of preimages, every connected component of $\mathcal{J}$ is a point or a Jordan curve. As a corollary, every component of $\mathcal{J}$ is locally connected. We also discuss when a Jordan curve Julia component is a quasicircle and give an explicit example of a hyperbolic rational map with a Jordan curve Julia component which is not a quasicircle.

Résumé. — Soit f une fraction rationnelle hyperbolique. On suppose que son ensemble de Julia $\mathcal{J}$ n'est pas connexe. Nous allons montrer que, à l'exception d'un nombre fini de composantes périodiques de $\mathcal{J}$, et la collection dénombrable de leurs composantes préimages, toute composante de $\mathcal{J}$ est soit un point soit une courbe de Jordan. Par conséquent, toute composante de $\mathcal{J}$ est localement connexe. Nous discutons également quand une telle courbe de Jordan est aussi un quasi-cercle. Nous donnerons un exemple explicite d'une fraction rationnelle ayant une composante de Julia qui est une courbe de Jordan mais pas un quasi-cercle.

1. Introduction

For a rational map f of the Riemann sphere $\overline{\mathbf{C}}$ to itself with disconnected Julia set $\mathcal{J}$, we investigate the topological and geometric possibilities for a connected component of $\mathcal{J}$. If $\mathcal{J}$ is disconnected, then f maps components of $\mathcal{J}$ onto components of $\mathcal{J}$, and there are uncountably many such components (cf. [**Mi**] and [**Be**]). The *postcritical set* of f

$$\mathcal{P} := \overline{\bigcup_{\substack{n>0 \\ f'(c)=0}} f^{\circ n}(c)}$$

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plays a crucial role in our study. We say that f is *hyperbolic* if $\mathcal{P}\cap\mathcal{J}=\emptyset$, *geometrically finite* if $\mathcal{P}\cap\mathcal{J}$ is finite, and *nice* if $\mathcal{P}\cap\mathcal{J}$ is contained in finitely many connected components of $\mathcal{J}$.

Theorem 1.1. — *Let f be a polynomial with disconnected filled Julia set $\mathcal{K}$. Assume that only finitely many connected components of $\mathcal{K}$ intersect $\mathcal{P}$. Then, with the possible exception of finitely many periodic components and their countable collection of preimages, every connected component of $\mathcal{K}$ is a point.*

Using a variant of this theorem, we establish

Theorem 1.2. — *Let f be a hyperbolic rational map with disconnected Julia set $\mathcal{J}$. Then, with the possible exception of finitely many periodic components and their countable collection of preimages, every connected component of $\mathcal{J}$ is either a point or a Jordan curve.*

The same result for geometrically finite maps can be proved by similar methods as well. We will only sketch the necessary modifications at the end of the paper.

We establish also weaker results for nice maps. The precise statements are given in Propositions [Case 2], [Case 3], [Case 4] and Theorem 9.2.

It is known that $\mathcal{J}$ can be a Cantor set ([**Be**], §1.8), or homeomorphic to the product of a Cantor set with a quasicircle, where each component is a K-quasicircle for some fixed K independent of the component ([**Mc1**]). In §8 we give an explicit example of a hyperbolic rational map which has a Jordan curve Julia component which is not a quasicircle.

Results from plane topology imply that at most countably many Julia components contain an embedding of the letter "Y". Our theorems make precise which ones they are. It is also interesting to see that at most countably many Julia components can be a segment, which a priori is not a restriction from plane topology alone.

By a theorem of McMullen ([**Mc1**], Corollary 3.5), there are at most countably many periodic components of $\mathcal{J}$. Since periodic points of f are dense in $\mathcal{J}$, if $\mathcal{J}$ is disconnected there must be exactly countably many periodic Julia components. Since the degree of f is finite there must be exactly countably many preperiodic Julia components. Hence there are uncountably many wandering Julia components. Our theorems show that under the stated assumptions, no wandering Julia component can be a segment or contain an embedding of the letter "Y", since they must either be points or Jordan curves.

Combining the above theorem with a result of Tan-Yin ([**TY**]), which shows that every preperiodic Julia component for a hyperbolic rational map is locally connected, we get:

Corollary 1.3. — *For a hyperbolic rational map, each Julia component is locally connected.*

This corollary completes another entry in the growing dictionary between the theories of rational maps and Kleinian groups. The analogs of a hyperbolic rational map f and its Julia set $\mathcal{J}$ are a convex compact (or expanding) Kleinian group Γ and its limit set Λ. It is known that each component of Λ is locally connected; the proof depends on the fact that a "wandering" component of Λ, i.e. a component with trivial stabilizer, is necessarily a point. See [**AM**] and [**Mc3**] (Theorem 4.18).

The main ideas in our proof are a canonical decomposition $\overline{\mathbf{C}} = E \sqcup \mathcal{U}$, where E is a finite collection of Julia components such that $f(E) \subset E$, and the fact that a hyperbolic rational map f is uniformly expanding on a neighborhood of its Julia set with respect to the Poincaré metric on $\overline{\mathbf{C}} - \mathcal{P}$. Our goal is to show that any Julia component which does not land in E is either a point or a Jordan curve. To this end, we further decompose the sphere into several canonical pieces and measure the itinerary of the orbit of a Julia component J_0 under f with respect to these pieces. We use a combinatorial analysis combined with the lemma below to show that components with certain kinds of itineraries are points. A separate argument treats the case of Jordan curves.

We say that $K \subset \overline{\mathbf{C}}$ is a *full continuum* if K is compact, connected, and $\overline{\mathbf{C}} - K$ is connected. The following Lemma was essentially known to Fatou (see [**Br**], Thm. 6.2)

Lemma 1.4. — *(Fatou) Let f be a rational map, $Q = \bigsqcup_{i=1}^{k} Q_i$ be the union of finitely many disjoint full continua, such that $Q \cap \mathcal{P} = \emptyset$. Then any connected set $J \subset \mathcal{J}$ satisfying $f^n(J) \subset Q$ for infinitely many n is a point.*

Contents. In §2 we give some motivating examples, define the above mentioned decomposition and state four basic lemmas for nice maps, and give a more precise statement (Theorem 1.2') of Theorem 1.2. We then reduce the proof of Theorem 1.2' to three cases, Cases 2, 3 and 4. Related results for nice maps are stated as well. In §3 we analyze the topology and dynamics of the decomposition and prove the four basic lemmas. §4 contains analytic preliminaries for use in §§5 and 6. In §§5, 6, and 7 we prove the Propositions in Cases 2, 3, and 4, respectively; §7 contains also the proof of Theorem 1.1. §8 lists related results and discusses when a Jordan curve Julia component is a quasicircle. §9 contains sketches of proofs–a generalization our results to the geometrically finite case and some further results for nice maps. §10 is an appendix of technical topological results used in our proofs.

Acknowledgments. Recently G. Cui, Y. Jiang, and D. Sullivan [**CJS**] have also proven Theorem 1.2 for geometrically finite maps in a different context. Their methods are in some respects similar, but they do not make use of a canonical decomposition. The authors would like to thank Cui for providing a copy of their manuscript, A. Douady, D. Epstein, M. Lyubich, C. McMullen, B. Sevennec and M. Shishikura for many useful discussions, and MSRI for financial support.

2. The decomposition and the reduction to three cases

Let f be a rational map with Julia set $\mathcal{J} = \mathcal{J}(f)$ and postcritical set $\mathcal{P} = \mathcal{P}(f)$.

For J' a continuum (i.e. a compact, connected set) in $\overline{\mathbf{C}}$, and P a compact set disjoint from J', we say that J' *separates* P if either $J' \cap P \neq \emptyset$, or $J' \cap P = \emptyset$ and there are at least two components of $\overline{\mathbf{C}} - J'$ intersecting P. We say that J' *separates*

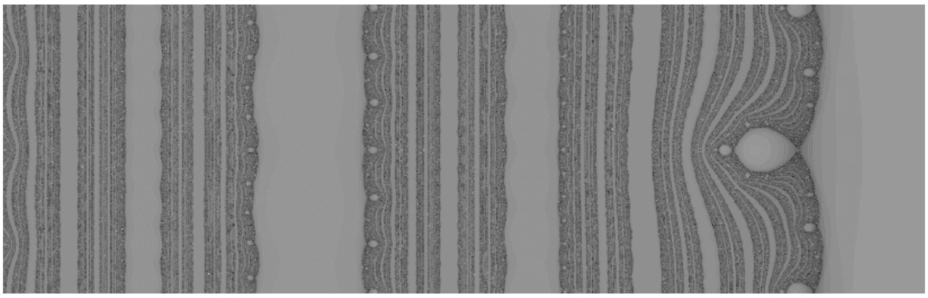

FIGURE 1. Julia set of $\frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + 10^{-11} z^{-3}$

P *into exactly* q *parts* if $J' \cap P = \emptyset$ and $\overline{\mathbf{C}} - J'$ has exactly q components intersecting P.

Let J' be a component of $\mathcal{J}$. We say that J' is *critically separating* if J' separates $\mathcal{P}$. We say that two distinct components J' and J'' of $\mathcal{J}$ are *parallel*, if they are both critically separating and the unique annulus component in $\overline{\mathbf{C}} - (J' \cup J'')$ (see Lemma 3.1) does not intersect $\mathcal{P}$.

Definition (decomposition $\overline{\mathbf{C}} = E \sqcup \mathcal{U}$, first step). Let E be the union of Julia components J' such that either

1. $J' \cap \mathcal{P} \neq \emptyset$, or
2. $J' \cap \mathcal{P} = \emptyset$ and J' separates $\mathcal{P}$ into three or more parts, or
3. J' separates $\mathcal{P}$ into exactly two parts and J' separates no two Julia components which are parallel to J', i.e. all Julia components J'' parallel to J' are contained in the same component of $\overline{\mathbf{C}} - J'$.

We think of J' as an *extremal* Julia component. Set $\mathcal{U} = \overline{\mathbf{C}} - E$. The set E may be empty, e.g. if $\mathcal{J}$ is a Cantor set.

Examples. Let $f_0(z) = z^2 + 10^{-9} z^{-3}$ (McMullen) and $f_1(z) = \frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + 10^{-11} z^{-3}$. The Julia set of f_1, in $\log(z)$-coordinates, is shown in Figure 1. The Julia set of f_0 is homeomorphic to product of a Cantor set with a quasicircle ([**Mc1**]).

For f_1, the point at infinity and -1 form the unique attracting cycle. There are five critical points in the annular Fatou component near the center of the picture, which maps to the Fatou component containing zero (at left) by degree five. $\mathcal{P}$ is contained in the union of the disc Fatou component containing zero (at left) and the immediate basins of infinity (at far right) and -1 (the prominent disc at right) . The set E consists of a homeomorphic copy J^+ of the Julia set of $z^2 - 1$, at right, and its preimage J^- (at left) which is a threefold cover of J^+. J^+ and J^- are parallel, J^- maps to J^+, and J^+ is fixed.

Lemma 2.1. — *If f is nice, the set E consists of at most finitely many Julia components, $f(E) \subset E$, $f^{-1}\mathcal{U} \subset \mathcal{U}$, and every component of E is either periodic or preperiodic.*

This lemma will be proved in the next section. One can also show that E is closed and forward-invariant for any rational map f; we omit the proof.

Definition (decomposition of $\mathcal{U}$, second step) Define
$\mathcal{A} = \bigcup$ { annular components of $\mathcal{U}$ disjoint from $\mathcal{P}$ };
$\mathcal{D} = \bigcup$ { disc components of $\mathcal{U}$ disjoint from $\mathcal{P}$ };
$\mathcal{L} = \bigcup$ {components of $\mathcal{U}$ not in $\mathcal{A}$ or $\mathcal{D}$ }
$= \bigcup$ { non-disc non-annular components of $\mathcal{U}$, or components of $\mathcal{U}$ intersecting $\mathcal{P}$ }.
$\mathcal{D}' = \bigcup$ { disc components of $\mathcal{U}$ disjoint from $\mathcal{P}$ but intersect $f^{-1}E$};
$\mathcal{D}'' = \bigcup$ { disc components of $\mathcal{U}$ disjoint from $\mathcal{P} \cup f^{-1}E$ }.

Note that $\mathcal{U} = \mathcal{A} \sqcup \mathcal{D} \sqcup \mathcal{L} = \mathcal{A} \sqcup \mathcal{D}' \sqcup \mathcal{D}'' \sqcup \mathcal{L}$.

Example. Let $f = f_1$. Then $\mathcal{U} = \overline{\mathbf{C}} - E$ is decomposed into:

- $\mathcal{A}$ = a single annulus, bounded by J^- and J^+ ($\overline{\mathcal{A}}$ is not a closed annulus).
- $\mathcal{L}$ = three disc components, each intersecting $\mathcal{P}$. Two are disc components of $\mathcal{U}$ containing the attractor at infinity and -1 with boundaries contained in J^+, and the other is the component of $\mathcal{U}$ containing zero with boundary contained in J^-.
- $\mathcal{D} = \mathcal{D}''$ = the countable set of remaining components of $\mathcal{U}$, all discs with boundaries in E.
- $\mathcal{D}' = \emptyset$.

The set $f^{-1}E$ consists of E plus two other components contained in $\mathcal{A}$ and parallel to components in E.

Definition (decomposition of $f^{-1}\mathcal{A}$, third and final step) We denote by

- $\mathcal{A}^S$, the union of components A' of $f^{-1}\mathcal{A}$ such that $A' \subset \mathcal{A}$ and $A' \hookrightarrow \mathcal{A}$ is not homotopic to a constant map, and
- $\mathcal{A}^O$, the union of components A' of $f^{-1}\mathcal{A}$ such that $A' \subset \mathcal{A}$ and $A' \hookrightarrow \mathcal{A}$ is homotopic to a constant map.

For f_1, the set $\mathcal{A}^S$ consists of two essential subannuli of $\mathcal{A}$, and the set $\mathcal{A}^O$ is empty.

Here is a more precise statement of Theorem 1.2.

Theorem 1.2' *Let f be a hyperbolic rational map and $\overline{\mathbf{C}} = E \sqcup \mathcal{U}$ be the decomposition above. Let J_0 be a Julia component. Set $J_n = f^n J_0$. Then exactly one of the following occurs:*

1. *$J_n \subset E$ for $n \geq n_0$, in which case J_0 is preperiodic, or*
2. *$J_n \subset \mathcal{A}^S$ for $n \geq n_0$, in which case J_0 is a Jordan curve, or*
3. *there is a sequence $n_k \to \infty$ such that $J_{n_k} \subset \mathcal{U} - \mathcal{A}^S$, in which case J_0 is a point.*

If $E = \emptyset$, the Julia set $\mathcal{J}$ is totally disconnected.

This decomposition is canonical and natural with respect to conjugation by Möbius transformations. While the first decomposition $\overline{\mathbf{C}} = \mathcal{U} \sqcup E$ is the same for any iterate of f, the further decomposition can change.

The following lemmas will be proved in the next section:

Lemma 2.2. — *If f is nice, the sets $\mathcal{L}$, $\mathcal{A}$, $\mathcal{D}'$, $\mathcal{A}^S$ and $\mathcal{A}^O$ all have finitely many components.*

Lemma 2.3. — *Let f be a nice map. Then each component L of $\mathcal{L}$ contains a unique Fatou component W such that $\partial W \supset \partial L$ and $W \cap \mathcal{P} = L \cap \mathcal{P}$.*

Lemma 2.4. — *Let f be a nice map. Then every Julia component J_0 is in one of the following four cases: let $J_n = f^n(J_0)$,*
Case 1. *There is n_0 such that $J_n \subset E$ for $n \geq n_0$.*
Case 2. *There is n_0 such that $J_n \subset \mathcal{A}^S$ for $n \geq n_0$.*
Case 3. *$J_n \subset \mathcal{A}^O \cup \mathcal{D}'$ for infinitely many n.*
Case 4. *$J_n \subset \mathcal{L}$ for infinitely many n.*
While these cases cover all possibilities, the last two are not mutually disjoint.

This result tegether with Lemma 2.2 means that some finite part of the decomposition encodes a significant portion of the orbit of each Julia components. We are going to prove:
Proposition [Case 2]. *In Case 2, J_0 is a Jordan curve if f is hyperbolic, or $\overline{\mathbf{C}} - J_0$ has exactly two components if f is nice.*
Proposition [Case 3]. *In Case 3, J_0 is a point if f is hyperbolic, or $\overline{\mathbf{C}} - J_0$ is connected if f is nice.*
Proposition [Case 4]. *In Case 4, J_0 is a point if f is nice (in particular if f is hyperbolic).*

Our cases are also distinguished by our methods of proof. In Case 2, we extract a dynamical system consisting of a finite collection of annuli and covering maps and analyze this restricted system. Case 3 is similar to Case 2. Case 4 is more delicate. We actually prove a stronger result, Theorem 7.1, from which both Proposition [Case 4] and Theorem 1.1 follow as corollaries.

Theorem 1.2 and Theorem 1.2' are direct consequences of the above Propositions.

3. Topology and dynamics of the decomposition

Throughout this section, f denotes a nice rational map. We first prove Lemmas 2.1 and 2.2, and then analyze the topological and dynamical possibilities for the sets in our decomposition in order to prove Lemmas 2.3 and 2.4. We will frequently use the following result from plane topology (see [**Ne**] for a proof):

Lemma 3.1. — *For a nonempty set $\mathcal{J}$ of disjoint continua J' in S^2, every component of $S^2 - J'$ is a disc (simply connected). Given J' and J'' two disjoint continua, the set $S^2 - (J' \cup J'')$ has a unique annulus component $A(J', J'')$, the component J'' is contained in a component U' of $S^2 - J'$, and*

$$U' = A(J', J'') \sqcup J'' \sqcup \bigcup \{V \mid V \text{ is a component of } S^2 - J'' \text{ and } V \cap J' = \emptyset\} .$$

If J^0 is a continuum disjoint from $J' \cup J''$ but separating J' and J'', then
$$A(J', J'') = A(J', J^0) \sqcup A(J^0, J'') \sqcup J^0 \sqcup \bigcup \{V \mid V \text{ is a component of } S^2 - J^0 \text{ and } V \cap (J' \cup J'') = \emptyset\} .$$

Proof of Lemma 2.1. — Since f is nice, there exists a compact set $B \subset \overline{\mathbf{C}}$ such that
1. B has finitely many connected components,
2. $B \supset \mathcal{P}$, and each connected component of B intersects $\mathcal{P}$,
3. B contains every Julia component intersecting $\mathcal{P}$ and no other Julia components.

B may be taken to be the union of Julia components intersecting $\mathcal{P}$ together with the suitable preimages of the following: closed, forward-invariant neighborhoods of attracting and superattracting basins; closed, forward-invariant attracting parabolic petals, invariant closed sub-discs of Siegel discs containing points of $\mathcal{P}$, and invariant sub-rings of Herman rings containing points of $\mathcal{P}$. Then a Julia component J is critically separating if and only if it is either contained in B, or is disjoint from $\mathcal{P}$ and separates components of B.

An easy induction argument shows that the number of Julia components J' which separate B into three or more pieces is finite and bounded by $k-2$ if B has k components. Such Julia components are in E by definition.

Now we deal with the Julia components in E that separate B into exactly two parts. By a method similar to the above, one can prove that if each continuum of the set is disjoint from B and separates B into exactly two parts, and no two continua are parallel (relative to B), then this set of continua is finite. Now assume $J^0 \subset E$ and J^0 separates $\mathcal{P}$ into two parts. We will see that among the Julia components parallel to J^0 at most one of them is contained in E. Let J^1, J^2 be two distinct parallels of J^0. Since J^0 does not separates its parallels, J^1 and J^2 are contained in the same component of $\overline{\mathbf{C}} - J^0$. There are two possible cases:

a. The component J^1, say, separates J^0 from J^2. Then, according to Lemma 3.1, J^1 is contained in $A(J^0, J^2)$ and $J^1 \hookrightarrow A(J^0, J^2)$ is homotopically non trivial. Since $A(J^0, J^2) \cap \mathcal{P} = \emptyset$, the component J^1 is also parallel to J^2. So J^1 separates $\mathcal{P}$ into exactly two parts, and separates parallels of J^1. Thus J^1 is not contained in E.

b. We have $J^1 \subset A(J^0, J^2)$ but J^1 does not separate $J^0 \cup J^2$ (therefore $J^2 \subset A(J^0, J^1)$ but J^2 does not separate $J^0 \cup J^1$). This is impossible for the following reason: The annulus $A(J^0, J^2)$ contains J^1 and all but one (disc)-components of $\overline{\mathbf{C}} - J^1$. Since J^1 is critically separating, at least one of these components contains points of $\mathcal{P}$. So $A(J^0, J^2) \cap \mathcal{P} \neq \emptyset$. This is a contradiction to the assumption that J^0 and J^2 are parallel.

So E contains at most finitely many components that separate B into two or three parts, hence E has at most finitely many components.

The following lemma implies that $f(E) \subset E$; since $\mathcal{U} = \overline{\mathbf{C}} - E$, $f^{-1}\mathcal{U} \subset \mathcal{U}$. Therefore each component of E is either periodic or preperiodic. If E is not empty, it consists of finitely many periodic cycles of Julia components, and some (finitely many) of their preimages. □

Proof of Lemma 2.2. — If $E = \emptyset$, we have $\mathcal{L} = \overline{\mathbf{C}}$ and $\mathcal{A} = \mathcal{D}' = \emptyset$. There is nothing to prove.

Assume now $E \neq \emptyset$. Since $f^{-1}E$ has finitely many components, so is $\mathcal{D}'$. It remains to prove that $\mathcal{A} \cup \mathcal{L}$ has finitely many components. Using the notation in the proof of Lemma 2.1, the set $\mathcal{P}$ is contained in B, which consists of finitely many connected components and contains finitely many Julia components. Let U be a disc-component of $\mathcal{L}$ intersecting $\mathcal{P}$. Then either U contains a component of B, or U contains a preimage of a closed parabolic attracting petal containing points of $\mathcal{P}$ used in the construction of B. Since the number of components of B and the number of such petals is finite, the number of disc-components of $\mathcal{L}$ intersecting $\mathcal{P}$ is finite.

The other components of $\mathcal{A}\cup\mathcal{L}$ are precisely the non simply connected components of $\overline{\mathbf{C}}-E$. Since E consists of finitely many components, only finitely many components of $\overline{\mathbf{C}}-E$ can be non simply connected. □

Lemma 3.2. — *If a Julia component J' is critically separating then $f(J')$ is also critically separating. If $f(J'')$ of a Julia component J'' separates $\mathcal{P}$ into two parts and separates the parallels of $f(J'')$, then either J'' does not separate $\mathcal{P}$ or J'' separates the parallels of J'' and separates $\mathcal{P}$ into two parts.*

Proof. — We prove that if $f(J')$ is not critically separating then neither is J'. If $J'\cap\mathcal{P}\neq\emptyset$ then $f(J')\cap\mathcal{P}\neq\emptyset$. Hence we may assume that J' and $f(J')$ are disjoint from $\mathcal{P}$. Assume that $\mathcal{P}$ is contained in one (disc)-component of $\overline{\mathbf{C}}-f(J')$. There is a Jordan curve γ separating $f(J')$ and $\mathcal{P}$. Let C be the disc-component of $\overline{\mathbf{C}}-\gamma$ containing $f(J')$. Since $C\cap\mathcal{P}=\emptyset$, every component of $f^{-1}(C)$ is again a disc, and again disjoint from $\mathcal{P}$ since $f^{-1}\mathcal{P}\supset\mathcal{P}$. One of them, say bounded by γ', contains J'. Thus J' does not separate $\mathcal{P}$.

Now we prove the second statement of the lemma. By assumption $\overline{\mathbf{C}}-f(J'')$ has exactly two components U^1 and U^2 intersecting $\mathcal{P}$, and there are Julia components $J^1\subset U^1$ and $J^2\subset U^2$ parallel to $f(J'')$. Then $A(J^1,J^2)\cap\mathcal{P}=\emptyset$, since, according to Lemma 3.1, $A(J^1,J^2)$ is the union of $A(J^1,f(J''))$, $A(f(J''),J^2)$, $f(J'')$ and the components of $\overline{\mathbf{C}}-f(J'')$ distinct from U^1 and U^2, and none of these sets intersects $\mathcal{P}$.

Now each component of $f^{-1}(A(J^1,J^2))$ is again an annulus, and again disjoint from $\mathcal{P}$. One of them, say A'', contains J'', and the components of $\partial A''$ are contained in two distinct Julia components J_1' and J_2' which are separated by J''.

Since all but two components of $\overline{\mathbf{C}}-J''$ are contained in A'', and $A''\cap\mathcal{P}=\emptyset$, the component J'' separates $\mathcal{P}$ into at most two parts. If it does separate $\mathcal{P}$, so do J_1' and J_2'. Hence J_1' and J_2' are parallel to J''. □

Lemma 3.3. — *Suppose $f:S^2\to S^2$ is a branched covering, $U,V\subset S^2$ are finitely connected open subsets with $f(U)=V$ and $f|U:U\to V$ proper. Then $f(\partial U)=\partial V$ and f maps connected components of ∂U onto connected components of ∂V.*

Remark. A subtlety is that the map $f:\partial U\to\partial V$ need not be open in the subspace topology, in other words, a component of ∂U may be a proper subset of a component of $f^{-1}(\partial V)$.

Proof. — That $f(\partial U)\subset\partial V$ follows by properness. Since $f(U)=V, f(\partial U)\subset\partial V$, and $f(\overline{U})$ is a closed subset containing V, we have $f(\partial U)=\partial V$. Finally, let K' be a boundary component of U and let K be the component of ∂V containing $f(K')$. Since f is a branched covering and V has finitely many boundary components, there are open annuli $A'\subset U, A\subset V$ such that K',K are boundary components of A',A respectively and $f:A'\to A$ is a covering map. Then $f|A':A'\to A$ satisfies the hypotheses of the first conclusion, so $f(\partial A')=\partial A$. It follows easily that $f(K')=K$. □

Lemma 3.4. — *Given any integer n and any component V of $f^{-n}\mathcal{U}$, any Julia component is either contained in V or disjoint from V. The boundary ∂V has finitely many components, and each component is contained in a different Julia component which is disjoint from V.*

Proof. — $f^n(V)$ is a component of $\mathcal{U}$ and the mapping $f^n : V \to f^n(V)$ is proper and finite-to-one. Hence V has finitely many boundary components since $f^n V \in \mathcal{U}$ has finitely many boundary components. Since f^n preserves the Julia components, a Julia component intersecting both V and ∂V would be mapped to a Julia component intersecting both $f^n(V)$ and $\partial f^n(V)$, which is impossible. □

Lemma 3.5. — *Any Julia component J^0 not in E separating $E \cup \mathcal{P}$ is disjoint from $\mathcal{P}$ and critically separating.*

Proof. — Since J^0 is not in E, if it separates E it must separate components of E. Assume that J^1 and J^2 are two Julia components in E, separated by J^0. Then there is a component U^i of $\overline{\mathbf{C}} - J^0$ containing J^i, $i = 1, 2$, and $U^1 \cap U^2 = \emptyset$. By Lemma 3.1 the set U^1 contains all but one component of $\overline{\mathbf{C}} - J^1$. Since J^1 is critically separating, $U^1 \cap \mathcal{P} \neq \emptyset$. Similarly $U^2 \cap \mathcal{P} \neq \emptyset$. Thus J^0 separates $\mathcal{P}$.

Assume now J^1 is a Julia component in E and $x \in \mathcal{P}$ such that J^1 and x are separated by J^0. One can show similarly that J^0 is also critically separating. □

Lemma 3.6. — *Critically separating Julia components are contained in $E \sqcup \mathcal{A}E \sqcup \mathcal{A}^S$.*

Proof. — Here we denote by UV the set of $z \in U$ for which $f(z) \in V$.

We show at first that those Julia components are contained in $E \sqcup \mathcal{A}$. If $J' \cap \mathcal{P} \neq \emptyset$ then $J' \in E$ by definition. Thus we may assume that $J' \cap \mathcal{P} = \emptyset$ and that J' is critically separating and is not in E. Then there are exactly two components U^1 and U^2 of $\overline{\mathbf{C}} - J'$ meeting $\mathcal{P}$, and each U^i contains parallels of J'. Set $P^i = U^i \cap \mathcal{P}$. We will construct a component of $\mathcal{A}$ containing J'.

For $i = 1$ and 2, let $\mathcal{W}_i =$

$$\{\, U \mid U \text{ is a component of the complement of some Julia component and } U \cap \mathcal{P} = P^i \,\}.$$

Set $W^i = \bigcup_{U \in \mathcal{W}_i} U$.

Here we apply the topological result Lemma A.1 to conclude that W^i is an open disc which is also an element of $\mathcal{W}^i$ and is in fact the unique maximal element. Because each U^i contains parallels of J', we have $\partial W^1 \cap \partial W^2 = \emptyset$ and $W^1 \cup W^2 = \overline{\mathbf{C}}$. Thus $W^1 \cap W^2$ is an annulus. To show that this annulus is a component of $\mathcal{A}$, we just need to show $\partial(W^1 \cap W^2) \subset E$ and $W^1 \cap W^2 \cap E = \emptyset$.

Denote by J^i the Julia component containing ∂W^i, $i = 1, 2$.

Assume by contradiction that J^1, say, is not contained in E. Since W^1, as a component of $\overline{\mathbf{C}} - J^1$, meets only part of $\mathcal{P}$, the Julia component J^1 is critically separating. By definition of E, the only possibility for J^1 not being in E is that there is another component W of $\overline{\mathbf{C}} - J^1$ such that $P^2 = \mathcal{P} - P^1$ is contained in W, and there is a Julia component $J^0 \subset W$ parallel to J^1. Thus $\overline{\mathbf{C}} - J^0$ has a component U containing $J^1 \cup W^1$. Furthermore $U \cap \mathcal{P} = W^1 \cap \mathcal{P} = P^1$ (Lemma 3.1). In other words, U is also an element of $\mathcal{W}^1$. This contradicts the fact that W^1 is the maximal element of $\mathcal{W}^1$.

Thus $\partial(W^1 \cap W^2) \subset E$. Now any critically separating Julia component in $W^1 \cap W^2$ would also separate J^1 and J^2, therefore separate $\mathcal{P}$ into two parts and be parallel to both J^1 and J^2. So $W^1 \cap W^2$ contains no component of E. As a consequence, $W^1 \cap W^2$ is an annulus component of $\overline{\mathbf{C}} - E$ disjoint from $\mathcal{P}$. By definition, $W^1 \cap W^2$ is a component of $\mathcal{A}$.

So $J' \subset \mathcal{A}$. Thus every critically separating Julia component is contained in $E \sqcup \mathcal{A}$.

Now $\mathcal{A}$ is decomposed into $\mathcal{A}E \sqcup \mathcal{A}^S \sqcup \mathcal{A}^O \sqcup \mathcal{A}\mathcal{L} \sqcup \mathcal{A}\mathcal{D}$. For any Julia component J' in $\mathcal{A}\mathcal{L} \sqcup \mathcal{A}\mathcal{D}$, we have $f(J') \subset \mathcal{L} \sqcup \mathcal{D}$. Hence $f(J')$ is not critically separating. By Lemma 3.2, the component J' is not critically separating either. Since the inclusion map of each component of $\mathcal{A}^O$ into $\mathcal{A}$ is homotopic to a constant map, and $\mathcal{A}$ is disjoint from $\mathcal{P}$, no continuum in $\mathcal{A}^O$ can be critically separating.

Thus all critically separating Julia components are contained in $E \sqcup \mathcal{A}E \sqcup \mathcal{A}^S$. □

Lemma 3.7. — *For U a component of $\mathcal{L} \sqcup \mathcal{D}$, there is a unique component U^R of $f^{-1}\mathcal{U}$ which we call a* **reduced component** *with the following properties:*

1. *$U^R \subset U$, $\partial U \subset \partial U^R$, and each component of ∂U is a component of ∂U^R;*
2. *If $U \cap f^{-1}E = \emptyset$ then $U^R = U$. Otherwise U^R is the complement in U of the union of finitely many disjoint full continua, each of which is contained in U;*
3. *$(U - U^R) \cap \mathcal{P} = \emptyset$;*
4. *If U is a component of $\mathcal{L}$ then $f(U^R)$ is also a component of $\mathcal{L}$. In particular, $f(\partial\mathcal{L}) \subset \partial\mathcal{L}$.*
5. *$f(U^R)$ is a component V of $\mathcal{U}$, $f(\partial U^R) = \partial V$, and f maps connected components of ∂U^R onto connected components of ∂V.*
6. *There are finitely many components U in $\mathcal{D}$ such that $U \cap f^{-1}(\mathcal{P} \cup E) \neq \emptyset$. For any such U, $f(U^R)$ is a component of $\mathcal{L} \cup \mathcal{A}$.*

Proof. — **1 and 2.** Note that $f^{-1}\mathcal{U} = \overline{\mathbf{C}} - f^{-1}E$. If $U \cap f^{-1}E = \emptyset$, the set U is also a component of $\overline{\mathbf{C}} - f^{-1}E$ (since $f^{-1}E \supset E$), and we set $U^R = U$. Otherwise, let $C_1, \ldots, C_k$ denote the components of $f^{-1}E$ which are contained in U. Lemmas 3.5 and 3.6 imply that no C_i separates components of ∂U. Hence for each i, there is a unique component V_i of $\overline{\mathbf{C}} - C_i$ containing ∂U. Let $K_i = \overline{\mathbf{C}} - V_i$ and $K = \cup_i K_i$. Lemma 3.1 implies that either $K_i \cap K_j = \emptyset$ or $K_i \subset K_j$ or $K_j \subset K_i$. Each K_i is full since V_i is connected. Then $U^R := U - K$ has the first two properties in the lemma.
3. Now we show that no component of ∂U^R separates $(\mathcal{P} \cap U) \cup \partial U$. This is trivial for components of ∂U^R which are also components of ∂U. For the other components of ∂U^R, if this does not hold, there would be a Julia component J' in U separating $\mathcal{P} \cup E$, and thus J' would be critically separating (Lemma 3.5). This is impossible by Lemma 3.6. Therefore $U^R \cap \mathcal{P} = U \cap \mathcal{P}$ and $(U - U^R) \cap \mathcal{P} = \emptyset$.
4. Assume now that U is a component of $\mathcal{L}$. $f(U^R)$ is a component of $\mathcal{U}$. By definition of $\mathcal{L}$, either $U \cap \mathcal{P} \neq \emptyset$ or U is neither a disc nor an annulus. In the first case $f(U^R) \cap \mathcal{P} \supset f(U^R \cap \mathcal{P}) = f(U \cap \mathcal{P}) \neq \emptyset$, so $f(U^R)$ is again a component of $\mathcal{L}$. In the second case, either $U^R \cap f^{-1}\mathcal{P} \neq \emptyset$ (in which case $f(U^R) \cap \mathcal{P} \neq \emptyset$ and hence $f(U^R)$ is in $\mathcal{L}$), or $f : U^R \to f(U^R)$ is an unbranched covering. Since U is not a disc

or annulus neither is U^R. Any covering over a disc (resp. an annulus) is again a disc (resp. an annulus), so $f(U^R)$ is neither a disc nor an annulus and hence $f(U^R)$ is a component of $\mathcal{L}$.
5. This follows immediately from the properness of $f : U^R \to V$ and Lemma 3.3.
6. Suppose $U \cap f^{-1}(\mathcal{P} \cup E) \neq \emptyset$. If $U \cap f^{-1}E \neq \emptyset$, then U is a component of $\mathcal{D}'$. Otherwise U contains a Fatou component intersecting $\mathcal{P}$. By the No Wandering Domains theorem, the number of such Fatou components is finite, hence the number of components U of the latter type is finite. Combining with the fact that $\mathcal{D}'$ has only finitely many components (Lemma 2.2), we get the finiteness. Now if $f(U^R)$ was an element of $\mathcal{D}$, then $f(U^R)$ would be an open disc disjoint from $\mathcal{P}$ and E, hence U would be an open disc disjoint from $f^{-1}(\mathcal{P} \cup E)$. Hence $f(U^R)$ is a component of $\mathcal{A} \cup \mathcal{L}$. □

Lemma 3.8. — *Let A be a component of $\mathcal{A}$ and δ^+ be a component of ∂A. Then there is a unique component A^+ of $f^{-1}\mathcal{U}$ with the following properties:*

1. *$A^+ \subset A$ and $\delta^+ \subset \partial A^+$;*
2. *Either A^+ is a component of $\mathcal{A}^S$ or A^+ is a component of $\mathcal{AL}$, i.e. $f(A^+)$ is a component of $\mathcal{A}$ or $\mathcal{L}$;*
3. *$f(\partial A^+) = \partial f(A^+)$ and f maps connected components of ∂A^+ onto connected components of $\partial f(A^+)$. Thus f maps boundary components of components of $\mathcal{A}^S$ onto boundary components of components of $\mathcal{A}$ or $\mathcal{L}$.*

The proof is similar to the one above.

Lemma 3.9. — *Assume $E \neq \emptyset$. For U a component of $\mathcal{U}$, the set $f(U)$ is again a component of $\mathcal{U}$ if and only if $U \cap f^{-1}E = \emptyset$. Either there is a minimal integer $k \geq 0$ such that $f^kU \cap f^{-1}E \neq \emptyset$, or some iterate V of U is a periodic Fatou component. In the latter case, V is finitely connected, is itself a component of $\mathcal{U}$, and either*

1. *$V \cap \mathcal{P} \neq \emptyset$, and V is a component of $\mathcal{L}$ which is either a simply-connected attracting or parabolic basin, or a Siegel disc or Herman ring intersecting $\mathcal{P}$, or*
2. *$V \cap \mathcal{P} = \emptyset$, and V is either a Siegel disc and a component of $\mathcal{D}$, or a Herman ring and a component of $\mathcal{A}$.*

Proof. — If $U \cap f^{-1}E \neq \emptyset$ then $f(U) \cap E \neq \emptyset$, and so $f(U)$ can not be a component of $\mathcal{U} = \overline{\mathbf{C}} - E$. Otherwise $U \cap f^{-1}E = \emptyset$ and so U is a component of $f^{-1}\mathcal{U}$ which maps properly under f onto a component of $\mathcal{U}$.

Assume now that for every $n \geq 0$, $f^n(U) \cap f^{-1}E = \emptyset$. Then f^nU is a component of $\mathcal{U}$ for every $n \geq 0$. By Montel's theorem the family $\{f^n|_U\}_n$ is then normal (since E is uncountable if it is nonempty), so U coincides with a Fatou component (since $\partial U \subset \mathcal{J}$). By the No Wandering Domains Theorem, some iterate V of U is a periodic Fatou component, and so V is a component of $\mathcal{L}$. Since components of $\mathcal{U}$ are finitely connected, V is finitely connected. The Lemma then follows from the classification of periodic Fatou components and the fact that attracting or parabolic basins are either simply connected or infinitely connected ([**Be**], §7.5). □

The following lemma is a more precise version of Lemma 2.3:

Lemma 3.10. — *Assume $E \neq \emptyset$. Let $L_0, L_1, \cdots, L_m$ be the (finitely many) components of $\mathcal{L}$. Then each L_i contains a unique Fatou component W_i such that $\partial W_i \supset \partial L_i$. Moreover the components of ∂L_i are precisely the components of ∂W_i separating $\mathcal{P}$.*

We say $f_*(L_i) = L_j$ if $f(L_i^R) = L_j$. In this case every L_i is preperiodic under f_* and $f(W_i) = W_j$. Furthermore, if $\{L_0, \cdots, L_{p-1}\}$ is a periodic cycle of f_*, then either

1. *$L_i^R = L_i$ for all $0 \leq i \leq p-1$, in which case $W_i = L_i$ and either*
   (a) *$W_i \cap \mathcal{P} \neq \emptyset$ for all i, in which case W_i is a simply connected attracting or parabolic basin, a Siegel disc or a Herman ring intersecting $\mathcal{P}$, or*
   (b) *$W_i \cap \mathcal{P} = \emptyset$ for all i, in which case W_i is a Siegel disc or Herman ring disjoint from $\mathcal{P}$; or*
2. *$L_i^R \neq L_i$ for some i, in which case W_i is an infinitely connected attracting or parabolic basin for each $0 \leq i \leq p-1$.*

Proof. — By Lemma 2.2 the set $\mathcal{L}$ has only finitely many components. By Lemma 3.6 no Julia component separates ∂L_i (resp. ∂L_i^R) or is critically separating. Corollary A.5 implies that there is a unique Fatou component W_i (resp. W_i^R) such that $W_i \subset L_i$ and $\partial W_i \supset \partial L_i$ (resp. $W_i^R \subset L_i^R$ and $\partial W_i^R \supset \partial L_i^R$).

By Lemma 3.7, we have $\partial L_i^R \supset \partial L_i$, thus by uniqueness with respect to the property of containing ∂L_i, we have $W_i^R = W_i$.

Note that every connected component of ∂W_i is either contained in ∂L_i or is contained in L_i. Since no Julia component in L_i is critically separating, and every component of ∂L_i is critically separating, the components of ∂W_i separating $\mathcal{P}$ are precisely those in ∂L_i.

If $f_*(L_i) = L_j$, by uniqueness of W_j, we have $f(W_i) = f(W_i^R) = W_j$.

By Lemma 3.7, for any i, $f(L_i^R)$ is again a component of $\mathcal{L}$, thus coincides with some L_j. So $f_*(L_i)$ is well defined for each i. Since there are only finitely many components in $\mathcal{L}$, each of them is eventually periodic under f_*.

Let $\{L_0, \cdots, L_{p-1}\}$ be a periodic cycle of f_*. Then $\{W_0, \cdots, W_{p-1}\}$ forms a periodic cycle of Fatou components.

If $L_i^R = L_i, 0 \leq i \leq p-1$, then the conclusion (1) follows by Lemma 3.9. Otherwise, L_i^R has at least two boundary components, hence W_i has at least two boundary components. W_i cannot be a Siegel disc or Herman ring. For in these cases, $\partial W_i \cap \mathcal{P} \neq \emptyset$ and so $L_i = W_i$ is itself a component of $\mathcal{U}$, contradicting $L_i^R \neq L_i$ for some i. Hence W_i is either an attracting or parabolic basin with at least two boundary components, hence is infinitely connected. □

For our example $f = f_1$ above, $\mathcal{L}$ has a periodic cycle of period 2 formed by the Fatou components containing infinity and -1.

Proof of Lemma 2.4. — Since $f(E) \subset E$, if $J_{n_0} \subset E$ for some n_0, then $J_n \subset E$ for all $n \geq n_0$. This is our case 1.

Assume now $J_n \subset \mathcal{U}$ for all n. Assume furthermore that J_0 is not in Case 2, that is, $J_n \cap \mathcal{A}^S = \emptyset$ for infinitely many n. We are going to show $J_n \subset \mathcal{A}^O \cup \mathcal{D}' \cup \mathcal{L}$ for infinitely many n (so J_0 is in Case 3 or 4 or both).

We show at first that $J_n \subset \mathcal{A}^O \cup \mathcal{D} \cup \mathcal{L} = \mathcal{A}^O \cup \mathcal{D}' \cup \mathcal{D}'' \cup \mathcal{L}$ for infinitely many n. Denote by AB the set of $z \in A$ for which $f(z) \in B$. We have $J_n \cap \mathcal{A}E = \emptyset$ for all n. If $J_n \subset \mathcal{A}\mathcal{D} \cup \mathcal{A}\mathcal{L}$ for some n, then $J_{n+1} \subset \mathcal{D} \cup \mathcal{L}$. Since $\mathcal{U} = \mathcal{A}E \cup \mathcal{A}^S \cup \mathcal{A}^O \cup \mathcal{A}\mathcal{D} \cup \mathcal{A}\mathcal{L} \cup \mathcal{D} \cup \mathcal{L}$, we are done.

We now show that if $J_n \subset \mathcal{D}''$ for infinitely many n, then $J_n \subset \mathcal{L} \cup \mathcal{D}'$ for infinitely many n. Assume $J_{n_1} \subset D$ for D a component of $\mathcal{D}''$. By definition, $D \cap f^{-1}E = \emptyset$. By Lemma 3.7, either $f(D)$ is a component of $\mathcal{D}' \cup \mathcal{D}''$ (this corresponds to the case $D \cap f^{-1}\mathcal{P} = \emptyset$), or $f(D)$ is a component of $\mathcal{L}$. As a consequence of Sullivan's non-wandering domain theorem, there is an integer $0 < k < \infty$ such that D, $f(D)$, $\cdots$, $f^{k-1}(D)$ are components of $\mathcal{D}''$ and $f^k(D)$ is a component of $\mathcal{D}' \cup \mathcal{L}$. Therefore $J_{n_1+k} \subset \mathcal{D}' \cup \mathcal{L}$. □

4. Analytic preliminaries

We now restrict to the case when f is hyperbolic. The results generalize to geometrically finite maps; the Poincaré metric ρ is replaced by a more complicated metric for which the map is still expanding (cf. [**TY**] and §9).

Recall that f is hyperbolic if and only if $\mathcal{J} \cap \mathcal{P} = \emptyset$. If $|\mathcal{P}| = 2$ then f is conjugate to z^n and $\mathcal{J}$ is connected. Moreover, $\overline{\mathbf{C}} - \mathcal{P}$ is connected. Let $\rho|dz|$ denote the Poincaré Riemannian metric on $\overline{\mathbf{C}} - \mathcal{P}$, $d_\rho(x,y)$ the corresponding distance, and $l_\rho(\gamma)$ the length of a curve with respect to ρ. Then $f : \overline{\mathbf{C}} - f^{-1}(\mathcal{P}) \to \overline{\mathbf{C}} - \mathcal{P}$ is expanding with respect to ρ. If B is the subset given in the proof of Lemma 2.1, then since f is hyperbolic we have $\mathcal{P} \subset B \subset \mathcal{L} \cap (\overline{\mathbf{C}} - \mathcal{J})$, and $f : \overline{\mathbf{C}} - f^{-1}(\mathrm{int}(B)) \to \overline{\mathbf{C}} - \mathrm{int}(B)$ expands ρ uniformly by some definite factor $\lambda > 1$. The inverse of f is then uniformly contracting, in the following sense: if $\gamma : [0,1] \to \overline{\mathbf{C}} - \mathrm{int}(B)$, then $l_\rho(\widetilde{\gamma}) < (1/\lambda^n) l_\rho(\gamma)$ for any lift $\widetilde{\gamma}$ of γ under f^n. This observation will be the main tool in our proofs of Propositions [Case 2] and [Case 3].

If U is a path-connected subset of $\overline{\mathbf{C}} - \mathcal{P}$, we define the path metric $\mathrm{dpath}_U(x,y)$ on U with respect to ρ by

$$\mathrm{dpath}_U(x,y) = \inf_\gamma \{ l_\rho(\gamma) | \rho : [0,1] \to U, \gamma(0) = x, \gamma(1) = y \}.$$

Lemma 4.1. — *Let J' be a periodic or preperiodic Julia component and U a component of $\overline{\mathbf{C}} - J'$. Then there is a C^1 Jordan curve $\gamma : S^1 \to U - B$, a continuous surjective map $h : S^1 \to \partial U$ and a constant L depending on U, such that for each $t \in S^1$, one can find a path $\eta_t : [0,1] \to \overline{U}$ with ρ-length at most L, such that $\eta_t(]0,1]) \subset U$, $\eta_t(0) = h(t)$ and $\eta_t(1) = \gamma(t)$.*

Recall that we have a partition $E \sqcup \mathcal{U}$ of $\overline{\mathbf{C}}$, and $\mathcal{A} \sqcup \mathcal{D}$ is the union of disc and annulus components of $\mathcal{U}$ disjoint from $\mathcal{P}$. Moreover $\partial(\mathcal{A} \sqcup \mathcal{D}) \subset E$, and E consists of finitely many preperiodic Julia components.

Corollary 4.2. — *Each component U of $\mathcal{A}, \mathcal{D}, \mathcal{A}^O$, and $\mathcal{A}^S$ has finite path-diameter relative to ρ, and each has locally connected boundary.*

Proof. — We first assume that J' is fixed and that the ideal boundary (cf. [**Mc1**]) of U is also fixed by f. In other words, there is a component U' of $f^{-1}U$ such that $U' \subset U$, but $\partial U \subset \partial U'$ and $f(\partial U) = \partial U$. We have $f(U') = U$. Choose $\gamma = \gamma_0 : S^1 \to U$ such that the annulus A between ∂U and $\gamma(S^1)$ contains no points of $f^{-1}B$. Denote by A' the component of $f^{-1}A$ such that $A' \subset U'$ and $\partial U \subset \partial A'$. We may adjust γ so that $A' \subset A$, see [**Mc1**].

Choose $x' \in A'$ such that $f(x') = \gamma(0)$. The degree $d = \deg(f : A' \to A)$ is a positive integer. We define $\gamma_1(t)$ so that $\gamma_1(0) = x'$ and $f(\gamma_1(t)) = \gamma(d \cdot t)$. Let $H_0 : [0,1] \times S^1 \to A$ be a C^1 map such that $H_0(0, \cdot) = \gamma_1$ and $H_0(1, \cdot) = \gamma$.

Since $f : A' \to A$ is a covering, one can lift H_0 to get $H_1 : [0,1] \times S^1 \to A'$ such that $H_1(1, \cdot) = H_0(0, \cdot)$. Define $\gamma_2(\cdot) = H_1(0, \cdot)$. One can then define H_{n-1} and γ_n by induction.

To control the convergence, we proceed as follows. For each $t_0 \in S^1$, the ρ-length of the curve $\{H_0(s, t_0), s \in [0,1]\}$ is finite, depending continuously on $t_0 \in S^1$. So it has a finite maximum, say C'. For $t \in S^1$, $d_\rho(\gamma_n(t), \gamma_{n-1}(t))$ is smaller than or equal to the length of the curve $\{H_n(s,t), s \in [0,1]\}$ which is smaller than or equal to C'/λ^{n-1}. So $\{\gamma_n\}$ forms a Cauchy sequence.

Therefore $\gamma_n(t)$ converges uniformly to a limit map, h, and the path distance between $\gamma(t)$ and $\gamma_n(t)$ is uniformly bounded by $C'\lambda/(\lambda - 1)$.

To define η_t for each $t \in S^1$, one note that the set $h(t) \cup (\bigcup_{n \geq 0} \bigcup_{s \in [0,1]} H_n(s,t))$ is an embedded closed arc with finite length. Reparametrizing it we get $\eta(t)$.

For periodic J' or periodic ideal boundary, we consider an iterate of f. For preperiodic cases, we pull back the curves given by the result for the periodic cases. □

5. Proof of Proposition [Case 2]

This is the case where a Julia component J_0 satisfies $J_n = f^n(J_0) \subset \mathcal{A}^S$ for $n \geq n_0$. We may assume $n_0 = 0$.

Part I, f is hyperbolic. We will show that J_0 is a Jordan curve.

Recall that $\mathcal{A}^S$ consists of components of $f^{-1}\mathcal{A}$ parallel to some components of $\mathcal{A}$.

Denote by $A_1, A_2, \cdots, A_p$ the components of $\mathcal{A}^S$, and by A_{ij} the set of points z such that $z \in A_i$, $f(z) \in A_j$.

If $A_{ij} \neq \emptyset$, it is an essential subannulus of A_i, and $f : A_{ij} \to A_j$ is a covering. For $f : A_i \to f(A_i)$ is a covering, $f(A_i)$ is a component of $\mathcal{A}$ and A_j is a subannulus of $f(A_i)$ parallel to $f(A_i)$.

For each A_j, choose $\gamma_j : S^1 \to A_j$ an injective homotopically non-trivial C^1 curve. For each i such that $A_{ij} \neq \emptyset$, choose $x_{ij} \in A_{ij}$ such that $f(x_{ij}) = \gamma_j(0)$. Then the homotopy classes of γ_i and γ_j determine uniquely generators for $\pi_1(A_{ij})$ and $\pi_1(A_j)$. The degree $d_{ij} = \deg(f : A_{ij} \to A_j)$ is then a positive or negative integer. Define a lift $\gamma_{ij}(t)$ of γ_j so that $\gamma_{ij}(0) = x_{ij}$ and $f(\gamma_{ij}(t)) = \gamma_j(d_{ij} \cdot t)$. Let $H_{ij} : [0,1] \times S^1 \to A_i$ be a C^1 map such that $H_{ij}(0, \cdot) = \gamma_i$ and $H_{ij}(1, \cdot) = \gamma_{ij}$.

Let $\alpha = (a_0 a_1 \cdots)$ be any infinite sequence such that for all $n \geq 0$ we have $1 \leq a_n \leq p$ and $A_{a_n a_{n+1}} \neq \emptyset$ and call such a sequence *admissible*.

For $n \geq 0$, denote by $T_n = T_n(\alpha)$ the set of points z such that $f^k(z) \in A_{a_k}$ for $0 \leq k \leq n$. Then T_{n+1} is an essential subannulus of T_n for $n \geq 0$ and $f^n : T_n \to A_{a_n}$ is a covering. The curve γ_{a_0} determines a generator for $\pi_1(T_n)$ and we let $d_n = \deg(f^n : T_n \to A_{a_n})$; it can be a positive or negative integer.

Set $J' = J'(\alpha) = \bigcap_n \overline{T}_n$. Since $\overline{T}_n$ forms a nested sequence of compact connected sets which are critically separating, J' is also compact connected and critically separating. We will show that either $J' \subset \partial T_N$ for some N, or J' is a Jordan curve, and a Julia component. The proof is split into several lemmas.

Lemma 0. For each $n \geq 0$, there is a (parametrized) C^1 curve $\zeta_n(t)$, and for $n \geq 1$, a homotopy $G_n : [0,1] \times S^1 \to T_{n-1}$ such that $\zeta_n(S^1) \subset T_n \subset A_{a_0}$, $G_n(0,t) = \zeta_{n-1}(t)$, $G_n(1,t) = \zeta_n(t)$. Moreover $f^n(\zeta_n(t)) = \gamma_{a_n}(d_n \cdot t)$.

Proof. Set $\zeta_0 = \gamma_{a_0}$. Assume we have constructed ζ_{n-1} and G_{n-1}. Since the map $f^{n-1} : T_{n-1} \to A_{a_{n-1}}$ is a covering, mapping T_n onto A_{a_{n-1},a_n}, one can lift the homotopy H_{a_{n-1},a_n} to a map $G_n : [0,1] \times S^1 \to T_{n-1}$ with $G_n(0,t) = \zeta_{n-1}(t)$. Set $\zeta_n(t) = G_n(1,t)$. These are the maps required by the lemma. Finally, by our choice of generators of fundamental groups, we have $f^n(\zeta_n(t)) = \gamma_{a_n}(d_n \cdot t)$.

Next, by our choice of B (in particular $B \cap \mathcal{J} = \emptyset$) we have $\bigcup \overline{A}_i \subset \overline{\mathbf{C}} - B$ and $\overline{A}_{ij} \subset \overline{\mathbf{C}} - f^{-1}(B)$ for all possible pairs (i,j). By Corollary 4.2, there is a positive number M such that for $i = 1, \cdots, p$, the path diameter of $A_i \in \mathcal{A}^S$ with respect to ρ is at most M. Moreover, on $\overline{\mathcal{A}^S} \subset \overline{\mathbf{C}} - f^{-1}(\mathrm{int}(B))$, f expands ρ by a definite factor $\lambda > 1$.

Lemma 1. The curves $\zeta_n(t)$ defined in Lemma 0 converge uniformly to a continuous map $\zeta : S^1 \to \overline{A}_{a_0}$.

Proof. For each possible pair (i,j), the ρ-length of the curve $\{H_{ij}(s,t_0), s \in [0,1]\}$ is finite, depending continuously on $t_0 \in S^1$. So it has a finite maximum. Let C' be the maximum of the ρ-length of the curves among all possible couples (i,j) and all $t_0 \in S^1$; it is again finite. For $t \in S^1$, $d_\rho(\zeta_n(t), \zeta_{n-1}(t))$ is less than or equal to the length of the curve $\{G_n(s,t), s \in [0,1]\}$ which is smaller than or equal to C'/λ^{n-1}. So $\{\zeta_n\}$ forms a Cauchy sequence.

Choose a base point $x^\pm$ in each component of ∂A_{a_0}. Denote by δ_n^+ (resp. δ_n^-) the component of ∂T_n which either contains x^+ (resp. x^-) or which separates x^+ (resp. x^-) and T_n. Denote by D_H the Hausdorff distance on compact subsets of $\overline{\mathbf{C}} - \mathrm{int}(B)$ with respect to the metric d_ρ. By definition

$$D_H(F,G) = \max\ (\ \max_{x \in F} \min_{y \in G} d_\rho(x,y)\ ,\ \max_{y \in G} \min_{x \in F} d_\rho(x,y)\ )\ .$$

Lemma 2. For every $\varepsilon > 0$, there exists an N independent of α such that for every $n \geq N$, $D_H(J', \delta_n^+) < \varepsilon$, $D_H(J', \delta_n^-) < \varepsilon$ and $D_H(J', \overline{T}_n) < \varepsilon$.

Lemma 3. $D_H(J', \zeta_n(S^1)) \to 0$.

Proof of Lemmas 2 and 3. Fix $y \in \delta_n^+$. We first show

$$\min_{x \in J'} d_\rho(x,y) < M/\lambda^n.$$

Let $y_n = f^n(y), n \geq 0$, and for each n choose $x_n \in f^n(J')$. Then for each n, there is a path $\eta_n : [0,1] \to \overline{A}_{a_n}$ such that $\eta_n(0) = y_n, \eta_n(1) = x_n, \eta_n(]0,1[) \subset A_{a_n}$, and $l_\rho(\eta_n) \leq M$. For any n, $f^{-n}(f^n(J')) \cap \overline{T}_n = J'$, since $f^n(\bigcap_k \overline{T}_k) = \bigcap_k f^n \overline{T}_k$. Hence there is a lift $\widetilde{\eta}_n : [0,1] \to \overline{T}_n$ of η_n under f^n joining y to some point $x'_n \in J'$. Hence by expansion

$$\min_{x\in J'} d_\rho(x,y) \le d_\rho(x'_n,y) \le l_\rho(\widetilde{\eta}_n) \le M/\lambda^n.$$

Hence

$$\max_{y\in\delta_n^+}\min_{x\in J'} d_\rho(x,y) \le M/\lambda^n.$$

A similar argument bounds $\max_{x\in J'}\min_{y\in\delta_n^+} d_\rho(x,y)$ by the same quantity. The remainder of the two lemmas are proved similarly.

Lemmas 1 and 3 imply that $\zeta(S^1) = J'$. As a consequence, J' is locally connected.

Lemma 4. Either J' coincides with one boundary component of T_N for some N, or $J' \subset T_n$ for all n. In the second case, J' is a Jordan curve, and a Julia component.

Proof. We first show $J' \subset \mathcal{J}$. First, $\partial J' \subset \mathcal{J}$ since $\partial T_n \subset \mathcal{J}$ for all n and $\mathcal{J}$ is closed. Second, $J' = \partial J'$. For otherwise there is a nonempty component W of $\mathrm{int}(J')$. Then $W \subset T_n$ for all n, and as a consequence $f^n(W) \subset \mathcal{A}$ for all n. Thus $\{f^n\}$ is normal on W. So W coincides with a Fatou component. But for a hyperbolic map every Fatou component is eventually attracting, and meets eventually $\mathcal{P}$, contradicting $\mathcal{A}\cap\mathcal{P} = \emptyset$.

There are thus two possibilities: either $J' \subset \partial T_N$ for some N, or $J' \subset T_n$ for all n. In the first case J' coincides with one boundary component of T_k for all $k \ge N$ (Lemma 2). In the second case, J' must be a Julia component. For otherwise, J' is a nonempty proper closed subset of some Julia component J''; if $x \in J'' - J'$ then $D_H(x, J') > 0$, and hence Lemma 2 implies that for some n, ∂T_n either separates x and J' or ∂T_n contains x. But this implies that J'' intersects ∂T_n, hence J' is contained in a boundary component of T_n, violating our assumption.

Moreover, Lemma 2 implies that $\overline{\mathbf{C}} - J'$ has exactly two components U_1, U_2, and $\partial U_1 = \partial U_2 = J'$. The lemma below (pointed out to us by M. Lyubich) allows us to conclude that J' is a Jordan curve.

Consider now our Julia component J_0 such that $J_n \subset \mathcal{A}^S$ for all n. It determines an admissible sequence $\alpha = (a_0 a_1 \cdots)$ by setting $a_n = m$ if $J_n \subset A_m$. Then $J_0 = J'(\alpha)$, and it is a Jordan curve.

(**Remark.** The proof actually shows much more; see §8.)

Part II, f is nice. Let J_0 be a Julia component such that $J_n \subset \mathcal{A}^S$ for all n. Define T_n to be the component of $f^{-n}\mathcal{U}$ containing J_0 (it is in fact the same T_n as in Part I). Then each T_n is an open annulus, contained essentially in T_{n-1}. With the help of Sullivan's non-wandering domain theorem, one can show easily that $J_0 = \bigcap_n T_n$. On the other hand, since J_0 is disjoint from $\partial T_n \subset f^{-n}(E)$ for all n, there is a sequence $n_k \to \infty$ such that $\overline{T_{n_k}} \subset T_{n_{k-1}}$. Therefore $\overline{\mathbf{C}} - J_0$ has exactly two components.

Lemma 5.1. — *Assume that K is a closed subset of $\overline{\mathbf{C}}$ satisfying either conditions a) and b) or condition c):*

a) K is the common boundary of two disjoint open connected sets U_1 and U_2.

b) K is locally connected.

c) $\overline{\mathbf{C}} - K$ has exactly two components V_1 and V_2 and each point of K is accessible from both V_1 and V_2.

Then K is a Jordan curve. There are counter examples if one of the above conditions is not satisfied.

Proof. — Condition a) shows that U_1 and U_2 are simply connected and K is compact connected. By b) and Carathéodory's theorem, a Riemann map $\phi : \Delta \to U_1$ extends continuously to the boundary, and the extension is locally non constant. But the extension is also injective, for otherwise the image by ϕ of a pair of distinct radial segments of Δ would form a Jordan curve ν and each component of $\overline{\mathbf{C}} - \nu$ would intersect U_2. The contradicts the fact that $\nu \cap U_2 = \emptyset$ and U_2 is connected.

For a proof in condition c) case, see Newman ([**Ne**]), Theorem 16.1. □

6. Proof of Proposition [Case 3]

This is the case where a Julia component J_0 satisfies $J_n = f^n(J_0) \subset \mathcal{A}^O \cup \mathcal{D}'$ for infinitely many n.

Part I. f is hyperbolic. We will need the following variant of Lemma 1.4; the difference is that we do not assume $\overline{Q}_i$ is full.

Lemma 6.1. — *Let f be a hyperbolic rational map and $Q = \sqcup_{i=1}^k Q_i$ be a finite family of disjoint path-connected subsets of $\overline{\mathbf{C}}$ such that*

1. *$\overline{Q}_i \cap \mathcal{P} = \emptyset$ for each i,*
2. *the inclusion maps $\iota_i : Q_i \to \overline{\mathbf{C}} - \mathcal{P}$ are homotopic to constant maps, and*
3. *the path-diameter of each Q_i with respect to $dpath_{Q_i}$ is finite.*

Then any connected set J satisfying $f^n(J) \subset Q$ for infinitely many n is a point.

Proof. — We may assume $\overline{Q}_i \cap B = \emptyset$, where B is a closed neighborhood of $\mathcal{P}$ constructed previously. Then since f is hyperbolic, f expands ρ uniformly by some factor $\lambda > 1$ on $\overline{\mathbf{C}} - f^{-1}(\mathrm{int}(B))$. Let $J_n = f^n(J)$ and let Q^n denote the component of Q containing J_n. Choose $x_0, y_0 \in J_0$ and let $x_n = f^n(x_0), y_n = f^n(y_0)$. Let M be an upper bound on the path-diameters of the Q_i with respect to ρ. Then for each n, there is a path $\eta_n : [0,1] \to Q^n, \eta_n(0) = x_n, \eta_n(1) = y_n$ for which $l_\rho(\eta_n) \leq M$. Since $Q^n \hookrightarrow \overline{\mathbf{C}} - \mathcal{P}$ is homotopic to a constant map, for each n there is a lift $\widetilde{\eta}_n$ of η_n under f^n joining x_0 to y_0. By expansion,

$$d_\rho(x_0, y_0) \leq l_\rho(\widetilde{\eta}_n) \leq l_\rho(\eta_n)/\lambda^n \leq M/\lambda^n \to 0.$$

Hence $x_0 = y_0$ and J is a point. □

Now assume that J_0 is a Julia component for f, and $J_n \subset \mathcal{A}^O \cup \mathcal{D}'$ for infinitely many n. By Lemma 2.2 the set $\mathcal{A}^O \cup \mathcal{D}'$ has finitely many components, each of finite path diameter by Corollary 4.2. The above Lemma then applies and hence J_0 is a point.

Part II. f is nice. For each component U of $\mathcal{A}^O \subset \mathcal{A}$, take a simple closed curve γ which is a generator of the fundamental group of U. Then $\overline{\mathbf{C}} - \gamma$ has exactly one disc component V contained in $\mathcal{A}$. The set $\widehat{U} = U \cup V$ is an open disc contained in $\mathcal{A}$ (so is disjoint from $\mathcal{P}$. Therefore the enlarged set $\widehat{\mathcal{A}^O} \cup \mathcal{D}'$ consists of finitely many open discs disjoint from $\mathcal{P}$. The rest of the proof is very similar to Part II in the proof of Proposition [Case 2]. We omit the details here.

7. Proof of Proposition [Case 4] and Theorem 1.1

In this section we derive Theorem 1.1 and Proposition [Case 4] from

Theorem 7.1. — *Let f be a rational map (not necessarily hyperbolic or nice). Let $\mathcal{W}$ be the union of finitely many Fatou components $W_0, \cdots, W_m$ such that*

1. $f(\mathcal{W}) \subset \mathcal{W}$;
2. *each W_i eventually lands on an attracting or parabolic periodic basin under iteration of f, and $W_i \cap \mathcal{P} \neq \emptyset$ for all i;*
3. *for each W_i, only finitely many components $K_{1,i}, \cdots, K_{m_i,i}$ of $\overline{\mathbf{C}} - W_i$ intersect $\mathcal{P}$.*

Then there is a finite union Q of disjoint full continua in $\overline{\mathbf{C}} - \mathcal{P}$ such that any Julia component J_0 satisfying $f^n(J_0) \subset \bigcup_i \left(\overline{\mathbf{C}} - \bigcup_{j=1}^{m_i} K_{j,i}\right)$ for infinitely many n passes infinitely often through Q, and is a point.

Proof of Theorem 1.1. — Let W_0 be the basin of infinity of the polynomial f. It is a fixed attracting component, and infinitely connected. Since only finitely many components of $\mathcal{K} = \overline{\mathbf{C}} - W_0$ intersect $\mathcal{P}$, we may apply the above theorem to prove that every Julia component of f passing infinitely many times through U_0 is a point, where

$$U_0 = W_0 \bigcup \{K \mid K \text{ is a } \mathcal{K}\text{-component disjoint from } \mathcal{P}\} .$$

On the other hand, every Julia component is the boundary of a $\mathcal{K}$-component. So every $\mathcal{K}$-component passing infinitely many times through U_0 is a point. But in this particular case, we know also that the orbit of a $\mathcal{K}$-component either stays entirely in U_0 or lands eventually on a $\mathcal{K}$-component intersecting $\mathcal{P}$, which is preperiodic, since there are only finitely many of such $\mathcal{K}$-components and the union of them is forward invariant. □

Proof of Proposition [Case 4]. — Let f be a nice map and J_0 be a Julia component. Assume that $J_n = f^n(J_0) \subset \mathcal{L}$ for infinitely many n. For each component L_i of $\mathcal{L}$, Lemma 3.10 provides a unique Fatou component W_i such that $W_i \subset L_i$ and $\partial L_i \subset \partial W_i$. The union of these W_i's satisfies the conditions of Theorem 7.1. Moreover, in the notation of the statement and the proof of the theorem, we have $L_i = U_i = \overline{\mathbf{C}} - \bigcup_{j=1}^{m_i} K_{j,i}$. Therefore we can apply Theorem 7.1 to conclude that J_0 is a point. □

We now start the proof of Theorem 7.1. — We will use the following notation. Let S be a closed subset of S^2 and W be an open connected subset of S^2. Define

$$U(W, S) = W \cup \bigcup \{K \mid K \text{ a component of } S^2 - W \text{ such that } K \cap S = \emptyset\} .$$

Then $U(W, S)$ is an open set.

For $i = 0, \cdots, m$, set $U_i = U(W_i, \mathcal{P}) = \overline{\mathbf{C}} - \bigcup_{j=1}^{m_i} K_{j,i}$. Our aim is to find a compact set Q satisfying the properties require in the theorem. We then apply Lemma 1.4 to conclude.

The proposition below is proved in the Appendix.

Proposition 7.2. — *Let W be a Fatou component of a rational map f. Suppose that $\mathcal{P} \cap (\overline{\mathbf{C}} - W)$ is contained in the union $K_1 \sqcup \cdots \sqcup K_r$ of finitely many connected components of $S^2 - W$. Let W' be a connected component of $f^{-1}(W)$ and let $U = U(W, \mathcal{P})$ and $U' = U(W', f^{-1}(\mathcal{P}))$. Then*

1. *U and U' are connected open subsets with finitely many boundary components;*
2. *$f : U' \to U$ is proper, surjective and a branched covering;*
3. *$f : \partial U' \to \partial U$ is surjective, and given any connected component K_i' of $S^2 - U'$, f maps $\partial K_i'$ surjectively onto ∂K_j for some unique component K_j of $S^2 - U$.*

Since each component W_i contains points of $\mathcal{P}$, $U_i \cap U_j = \emptyset, i \neq j$. For each i, set $U_i' = U(W_i, f^{-1}\mathcal{P})$. Then $U_i' \subset U_i$ since $f^{-1}(\mathcal{P}) \supset \mathcal{P}$. By Proposition 7.2, U_i' has finitely many boundary components, hence $Q_i := U_i - U_i'$ consists of finitely many full continua disjoint from $f^{-1}(\mathcal{P})$, hence disjoint from $\mathcal{P}$. This will be one piece of our set Q.

Proposition 7.2 also implies that $f : U_i' \to U_j$ is proper if $f(W_i) = W_j$. We analyze the dynamical system $f : \sqcup_i U_i' \to \sqcup_i U_i$. Define $f_*(U_i) = U_j$ if $f(U_i') = U_j$ (equivalently, if $f(W_i) = W_j$).

Assume that a Julia component J_0 satisfies that $f^n(J_0) \subset \bigcup_i U_i$ for infinitely many n. Then either $f^n(J_0) \subset \bigcup_i Q_i$ for infinitely many n or $f^n(J_0) \subset \bigcup_i U_i'$ for all $n \geq n_1$. In the former case $f^n(J_0) \subset Q$ for infinitely many n too since Q is going to be defined as a set containing $\bigcup_i Q_i$. In the latter case the orbit of J_0 lands eventually into a periodic cycle of f_*. We now analyze this second case.

Let $U_0, \cdots, U_{p-1}$ be a periodic cycle of f_*. Set $g = f^p$. Then $\mathcal{J}(g) = \mathcal{J}(f)$ and $U_i = U(W_i, \mathcal{P}(f)) = U(W_i, \mathcal{P}(g))$.

For any Julia component J_0 such that $J_0 \subset U_0'$ and $f^n(J_0) \subset \bigcup_i U_i'$ for all $n \geq 0$, we have $f(J_0) \subset U_1'$, $\cdots$, $f^{p-1}(J_0) \subset U_{p-1}'$ and $f^p(J_0) \subset U_0'$ and so on. Therefore $g^n(J_0) = f^{np}(J_0) \subset U_0' \subset U_0$ for all n.

Set $W = W_0$ and $U = U_0 = U(W, \mathcal{P})$. Then W is a fixed attracting or parabolic Fatou component of g.

We are going to find a compact set Q_0' which is the union of finitely many disjoint full continua such that, if the g-orbit of a Julia component J_0 passes infinitely many times through U, then it passes infinitely many times through Q_0' as well.

Denote by X the empty set in case W is an attracting basin, or the set of one single element which is the fixed parabolic point of W, in case W is a parabolic basin. Set $X_n = \bigcup_{0 \leq j \leq n} g^{-j} X$.

One can find a disc $V \subset W$ with Jordan curve boundary such that $\overline{V} \subset W \cup X$ and $g(\overline{V}) \subset V \cup X$. We may choose V such that $(\partial V - X) \cap \mathcal{P} = \emptyset$. For any n, let V_n be the unique component of $g^{-n}(V)$ containing $V_0 = V$. Then $\overline{V}_n \subset V_{n+1} \cup X_n$ for any n. There is an integer N guaranteed by Lemma 7.3, such that every component of $\overline{\mathbf{C}} - V_N$ contains at most one of the finitely many components of $\overline{\mathbf{C}} - W$ which intersect $\mathcal{P}$, and $\mathcal{P} \cap W \subset V_N$.

The set $\overline{\mathbf{C}} - V_N$ is a disjoint union of finitely many full continua with $\mathcal{J} \subset int(\overline{\mathbf{C}} - V_N) \cup X_N$. Denote the components of $\overline{\mathbf{C}} - V_N$ by $\overline{D_1}, \cdots, \overline{D_l}, \cdots, \overline{D_n}$ such that, for $j > l$, $\overline{D_j} \cap \mathcal{P} = \emptyset$; and for $1 \leq j \leq l$, $\overline{D_j}$ contains a unique component K_j of $\overline{\mathbf{C}} - W$ such that $K_j \cap \mathcal{P} \neq \emptyset$.

For $1 \leq j \leq l$, set $A_j = int(\overline{D_j}) - K_j$. Then A_j is an open annulus with possibly finitely many pinched points at points in X_N. Moreover A_j contains no critical value (by the choice of N).

Now we look at the level $N+1$. For $U = U(W, \mathcal{P})$ and $U' = U(W, g^{-1}\mathcal{P})$, the map $g : U' \to U$ is a branched covering (Proposition 7.2). Since $V_N \cup (\bigcup_{j=1}^{l} A_j \cup \bigcup_{j>l} \overline{D_j}) = U$, we conclude that $U' - g^{-1}(\bigcup_{j=1}^{l} \overline{A_j} \cup \bigcup_{j>l} \overline{D_j})$ is connected, and coincides with $g^{-1}V_N \cap U'$. So $g^{-1}V_N$ has a unique component in U', which is V_{N+1}.

Denote the components of $\overline{\mathbf{C}} - V_{N+1}$ by $\overline{D'}_1, \cdots, \overline{D'}_l, \cdots, \overline{D'}_k$ such that $K_j \subset \overline{D'}_j \subset \overline{D}_j$ for $1 \leq j \leq l$, and $\overline{D'}_j \cap \mathcal{P} = \emptyset$ for $j > l$.

We set $g_*(K_i) = K_j$ if $g(\partial K_i) = \partial K_j$, which is well-defined by Proposition 7.2.

Set $A'_j = int(\overline{D'_j}) - K_j$, $j = 1, \cdots, l$. If $g_*(K_i) = K_j$, then $g(A'_i) = A_j$ and $g : A'_i \to A_j$ is a covering map.

Note that $A_j - A'_j = int(\overline{D_j}) - int(\overline{D'_j})$. Moreover $\mathcal{J} \cap (int(\overline{D_j}) - \overline{D'}_j)$ is contained in $\bigcup_{s>l} \overline{D'}_s$.

We claim then every Julia point x in $\bigcup_{j=0}^{l} A'_j$ must have some iterate in $\bigcup_{j=0}^{l} (A_j - A'_j)$. If not, there is $n_0 > 0$ such that for all $n > n_0$, $g^n(x) \in \bigcup \{A'_j \mid K_j$ is periodic for g_* $\}$. On the other hand, for $K_0, \cdots, K_{q-1}$ a periodic cycle of g_*, and for any $y \in A'_0$, there is a minimal integer $s > 0$ such that $g^s(y) \notin \bigcup_{j=0}^{q-1} A'_j$, for Lemma 7.3 below implies that each K_j is the nested intersection of sets of the form $\cap_n B_n$, where B_n is a component of $\overline{\mathbf{C}} - V_n$, $n \geq 0$.

Therefore if a Julia component J_0 satisfies that $g^n(J_0) \subset \bigcup_{j=0}^{l} A'_j \cup \bigcup_{s>l} \overline{D'_s}$ for infinitely many n, then $g^n(J_0) \subset \bigcup_{s>l} \overline{D'_s}$ for infinitely many n.

On the other hand, for our set $U = U(W, \mathcal{P})$, we have $\mathcal{J} \cap U \subset \bigcup_{j=0}^{l} A'_j \cup \bigcup_{s>l} \overline{D'_s} \cup X_{N+1}$. Thus saying $g^n(J_0) \subset U$ for infinitely many n is the same as saying $g^n(J_0) \subset \bigcup_{j=0}^{l} A'_j \cup \bigcup_{s>l} \overline{D'_s}$ for infinitely many n.

Return to our original map f now. The components of $Q_0 = U_0 - U'_0$ are disjoint full continua contained in $\bigcup_{j=0}^{l} A'_j \cup \bigcup_{s>l} \overline{D'_s}$. Therefore $Q'_0 = Q_0 \cup \bigcup_{s>l} \overline{D'_s}$ is again a union of finitely many disjoint full continua.

For each periodic cycle of f_*, choose one representative U_i in the cycle, and define Q'_i in the same way, for the other i in the cycle, define $Q'_i = Q_i$. We write $\{0, 1, \ldots, m\} = I' \sqcup I$, where $j \in I'$ if U_j is f_*-periodic and $j \in I$ otherwise.

Set $Q = \bigcup_{i \in I'} Q'_i \cup \bigcup_{i \in I} Q_i$. This is again a union of finitely many disjoint full continua, disjoint from $\mathcal{P}$. Now let J_0 be a Julia component passing infinitely many times through $\bigcup_i U_i$. Then either it passes infinitely many times through $\bigcup_i Q_i \subset Q$, or there is some $i \in I'$, such that J_0 passes infinitely many times through $Q'_i \subset Q$. In both cases J_0 must pass infinitely many times through Q.

Finally we apply Lemma 1.4 to conclude that such Julia components are points.

We mention here two particular cases.

1. The component W is simply connected. In this case $U = W$ and no Julia component passes through U.

2. The set $U(W, \mathcal{P})$ coincides with $\overline{\mathbf{C}}$, that is $\mathcal{P} \subset W$. In this case $l = 0$ and each Julia component is a point. That is $\mathcal{J}$ is totally disconnected. $\square$

A variant of the following lemma can be found in [**St**], page 63 and 117.

Lemma 7.3. — *Let W be an fixed attracting or parabolic basin of a rational map f. Let V be a either a disc neighborhood of the attracting fixed point with Jordan curve boundary or a Fatou petal in W of the parabolic fixed point. Then for V_n the component of $f^{-n}(V)$ containing V, we have $W = \bigcup_n V_n$, and each component of $\overline{\mathbf{C}} - V_n$ is closed disc with possibly finitely many pinching points. Furthermore, given any two components K_1 and K_2 of $\overline{\mathbf{C}} - W$, there is an integer N such that V_N separates K_1 and K_2.*

8. Further results

8.1. Diameter of Julia components. — Let f be a hyperbolic rational map.

Corollary 8.1. — *A Julia component J_0 of $\mathcal{J}$ is not a point if and only if there is an integer N such that $J_n = f^n(J_0) \subset \mathcal{A}^S \sqcup f^{-1}E$ for any $n \geq N$.*

If J_0 is a point, we may regard N as $+\infty$.

One can show that the set $f^{-1}(\mathcal{A}^S \sqcup f^{-1}E) - (\mathcal{A}^S \sqcup f^{-1}E)$ is contained in finitely many discs. Thus the same technique as in the above sections can prove also the following results: to each integer N, there is a positive number $\varepsilon(N)$, with $\varepsilon(N) \to 0$ as $N \to \infty$, such that for J_0 a Julia component, and N the minimal integer such that $f^n(J_0) \subset \mathcal{A}^S \sqcup f^{-1}E$ for $n \geq N$, the spherical diameter of J_0 is at most $\varepsilon(N)$.

8.2. Symbolic dynamics of Julia components in $\mathcal{A}^S$. — Let f be a hyperbolic rational map. The arguments given in §5 actually show much more. The space Σ of admissible sequences $\{a_0 a_1 \ldots\}$ equipped with the product topology and the one-sided shift map σ is a subshift of finite type. Let $\mathrm{Cl}(\overline{\mathbf{C}})$ denote the space of all closed subsets of $\overline{\mathbf{C}}$ in the Hausdorff topology. The map f induces a map from this space to itself which we again denote by f. Given $\alpha \in \Sigma$, we have defined in §5 a nested sequence of annuli $T_n(\alpha)$ and a continuum $J'(\alpha) = \cap_n \overline{T_n(\alpha)}$. The set $J'(\alpha)$ may or may not be a Julia component.

Theorem 8.2. — *The map*

$$\Phi : \alpha \mapsto J'(\alpha); \Sigma \to Cl(\overline{\mathbf{C}})$$

defines a uniformly continuous injective map conjugating σ to $f|\Phi(\Sigma)$. Moreover,

1. *there is a (at most) countable subset Σ_e and a finite subset $\Sigma_{e,0} \subset \Sigma$ such that*
   - (a) *$\alpha \in \Sigma_{e,0}$ if and only if $J'(\alpha)$ is a boundary component of A_{a_0} for some $A_{a_0} \in \mathcal{A}^S$, i.e. is a boundary component of $T_0(\alpha)$.*
   - (b) *$\alpha \in \Sigma_e$ if and only if $J'(\alpha)$ is a boundary component of $T_n(\alpha)$ for some n.*
   - (c) *$\sigma(\Sigma_{e,0}) \subset \Sigma_{e,0}$.*
   - (d) *$\Sigma_e = \cup_{n \geq 0} \sigma^{-n}(\Sigma_{e,0})$.*
2. *$\alpha \notin \Sigma_e$ if and only if $J'(\alpha)$ is a Jordan curve Julia component satisfying $f^n(J'(\alpha)) \subset \mathcal{A}^S$ for all $n \geq 0$.*

3. *Let* $J_{e,0} = \Phi(\Sigma_{e,0})$. *Then as a subset of* $\overline{\mathbf{C}}$,

$$\bigcup_{\alpha\in\Sigma} \Phi(\alpha) = \{z | f^n(z) \in \mathcal{A}^S \sqcup (\cup_{\delta\in J_{e,0}}\delta) \forall n \geq 0\}.$$

The elements of the set of continua $J_e := \Phi(\Sigma_e)$ may be thought of as "exposed" boundary components, in the sense that they are boundary components of T_n.

Corollary 8.3. — *The following are equivalent:*

1. Σ *is uncountable.*
2. *There is a wandering component of the Julia set which is a Jordan curve.*
3. *There are uncountably many wandering components of the Julia set which are Jordan curves.*
4. *There are infinitely many periodic Jordan curve Julia components.*
5. *There exists a component* C' *of* $\mathcal{A}$, *disjoint essential subannuli* $A, B \subset C'$, *and integers* $m, n > 0$ *such that* $f^m : A \to C', f^n : B \to C'$ *are covering maps.*

Proof. — For any subshift of finite type (Σ, σ), the following are equivalent: 1. the space of admissible sequences Σ is uncountable; 2. there is a wandering sequence; 3. there are uncountably many wandering sequences; 4. there are infinitely many periodic sequences; 5. there are two finite-length sequences $\alpha = (a_0, \ldots, a_m), \beta = (b_0, \ldots, b_n)$ with $c := a_0 = a_m = b_0 = b_n$ and $a_i \neq b_i$ for some $0 \leq i \leq \min m, n$.

By the above Theorem and the preceding facts, the first three conditions are thus equivalent, and (1) implies (4). The set E contains finitely many Julia components, and each periodic Jordan curve Julia component which is not in E must be contained in $\mathcal{A}^S$. Each such component is $J'(\alpha)$ for some periodic α, by the above Theorem, hence Σ has infinitely many periodic sequences and so (4) implies (1). We now show the equivalence of (4) and (5). First, note that (5) is equivalent, by pulling back, to the same condition with C', a component of $\mathcal{A}$, replaced by C, a component of $\mathcal{A}^S$. The equivalence of (4) and (5) then follows immediately from the above Theorem and the preceding paragraph. □

Proof of Theorem 8.2. — The continuity of Φ follows immediately from Lemma 2 of §5. That Φ is a semiconjugacy follows from the fact that $f : T_n(\alpha) \to T_{n-1}(\sigma(\alpha))$ is a covering map, hence $f : \overline{T_n(\alpha)} \to \overline{T_{n-1}(\sigma(\alpha))}$ is surjective, and so $f(\Phi(\alpha)) = f(J'(\alpha)) = f(\cap_n \overline{T_n(\alpha)}) = \cap_n \overline{T_{n-1}(\sigma(\alpha))} = J'(\sigma(\alpha)) = \Phi(\sigma(\alpha))$.

To see that Φ is injective, first note that given a boundary component δ of $A \in \mathcal{A}$, no other component $B \in \mathcal{A}$ has δ as a boundary component. For δ is locally connected (e.g. by Corollary 4.2) and hence δ is homeomorphic to S^1 if it is the common boundary of two disjoint open annuli, by Lemma 5.1. But if this occurs then δ is a component of the Julia set separating its parallels, violating the construction of $\mathcal{A}$. More generally, since f sends boundary components of $T_n(\alpha)$ to boundary components of $T_{n-1}(\sigma(\alpha))$, if δ is the common boundary of $T_n(\alpha)$ and $T_n(\beta)$, then $T_n(\alpha) = T_n(\beta)$, i.e. if $\alpha = \{a_0a_1 \ldots\}, \beta = \{b_0b_1 \ldots\}$ then $a_i = b_i, 0 \leq i \leq n$. If $J'(\alpha) = J'(\beta)$, then either $J'(\alpha) = J'(\beta)$ is a boundary component of $T_n(\alpha)$, some n, or $J'(\alpha) \subset T_n(\alpha)$ for all n. In the former case we have $T_n(\alpha) = T_n(\beta)$ for all n by the above observation while in the latter we must have $T_n(\alpha) = T_n(\beta)$ for all n since if $\alpha \neq \beta$ then for some n, $T_n(\alpha) \cap T_n(\beta) = \emptyset$. Hence $\alpha = \beta$.

(1). Define $\Sigma_{e,0}$ as in 1(a). We first prove that $J_{e,0} = \Phi(\Sigma_{e,0})$ is forward-invariant under f. Let $J_0 \in J_{e,0}$ and let $J_1 = f(J_0)$. Then J_0 is a boundary component of $T_0(\alpha)$ for some α. The proof of Lemma 4 of §5 shows that then J_0 is a boundary component of $T_k(\alpha)$ for all $k \geq 0$, hence J_0 is a boundary component of $T_1(\alpha)$. Since $f : T_1(\alpha) \to T_0(\sigma(\alpha))$ is a covering, $f(J_0) = J_1$ is a boundary component of $T_0(\sigma(\alpha))$ and so $J_1 \in J_{e,0}$. Now 1(a) holds by definition, 1(c) follows from the above result and the fact that Φ is a semiconjugacy. Define Σ_e by 1(d). 1(b) then follows immediately from the fact that Φ is a semiconjugacy and the fact that $f^n : T_n(\alpha) \to T_0(\sigma^n(\alpha))$.

(2). Note that $\alpha \notin \Sigma_e$ if and only if $J'(\alpha) \subset T_n(\alpha)$ for all n, by 1b). The result then follows by Lemma 4 of §5.

(3). First, let $J_0 = J'(\alpha)$. If $\alpha \notin \Sigma_e$, then $J_0 \subset T_n(\alpha)$ for all $n \geq 0$, hence $f^n(J_0) \subset T_0(\sigma^n(\alpha)) \subset \mathcal{A}^S$ for all $n \geq 0$. Otherwise, there is a minimal $N \geq 0$ such that for $0 \leq i < N$, $J_0 \subset T_i(\alpha)$ and for $i = N$, J_0 is a boundary component of $T_i(\alpha)$. Hence for $0 \leq i < N$, $f^i(J_0) \subset T_0(\sigma^i(\alpha)) \subset \mathcal{A}^S$, and for $i = N$, $f^i(J_0)$ is a boundary component of $T_0(\sigma^i(\alpha)) \in J_{e,0}$. Since $J_{e,0}$ is forward-invariant under f the inclusion in this direction is proved.

To prove the other direction, given z an element of the right-hand side we must produce α so that $z \in J'(\alpha)$. Let $z_n = f^n(z)$. If $z_n \in A_{b_n} \in \mathcal{A}^S$ for all $0 \leq n < N$ we set $a_n = b_n$. Hence we may assume $z_n \in \partial\mathcal{A}^S$ for all $n \geq 0$. Choose a component A_{a_0} so that $z_0 \in J_0$, a boundary component of A_{a_0} (there may be more than one such component, since the annuli in $\mathcal{A}^S$ may have closures which intersect). Then $J_0 \in J_{e,0}$. Since $J_{e,0}$ is forward-invariant, $J_n = f^n(J_0)$ is a boundary component of a unique component A_{a_n} of $\mathcal{A}^S$. We now claim that $\alpha = (a_0 a_1 \ldots)$ chosen in this fashion is admissible, i.e. that $A_{a_i a_{i+1}} \neq \emptyset$ for all i. For on the one hand, $f(A_{a_i}) = A \in \mathcal{A}$, hence $J_{i+1} = f(J_i)$ is a boundary component of A. On the other hand, J_{i+1} is a boundary component of $A_{a_{i+1}} \subset B \in \mathcal{A}$. If $A \neq B$ then we must have that J_{i+1} is the common boundary of A and B, which we have previously noted is impossible. Hence $A = B$ and so $A_{a_i a_{i+1}} \neq \emptyset$. Hence $J_0 = J'(\alpha)$ contains z. □

8.3. Moduli restrictions and description of the shift. — Let f be a nice rational map. Let $\{C_1, \ldots, C_k\}$ be the set of annuli in $\mathcal{A}$. Denote by m_j the modulus of C_j. Note that $0 < m_j < \infty$. For each j choose a Jordan curve $\gamma_j \subset C_j$ which is a homotopically non trivial in C_j. Set $\Gamma = \{\gamma_1, \cdots, \gamma_k\}$. Let $\Lambda(i,j)$ denote the set of components $C_{i,j,\lambda}$ of $f^{-1}(C_j)$ homotopic to C_i and let $d_{i,j,\lambda}$ be the positive degree of $f : C_{i,j,\lambda} \to C_j$.

Define two linear maps

$$f_\Gamma, f_{\Gamma,\#} : \mathbf{R}^\Gamma \to \mathbf{R}^\Gamma$$

by

$$f_\Gamma(\gamma_j) = \sum_{\gamma_i \in \Gamma} \Big(\sum_{\lambda \in \Lambda(i,j)} \frac{1}{d_{i,j,\lambda}} \Big) \gamma_i$$

and

$$f_{\Gamma,\#}(\gamma_j) = \sum_{\gamma_i \in \Gamma} \Big(\sum_{\lambda \in \Lambda(i,j)} 1 \Big) \gamma_i.$$

If $\Lambda(i,j) = \emptyset$ we take the coefficient of γ_i to be zero. The map f_Γ is the Thurston linear transformation defined by the multicurve Γ. The Grötzsch inequality implies that if $\vec{m} = (m_j) \in \mathbf{R}^\Gamma$ is the vector of moduli m_j, then

$$(f_\Gamma(\vec{m}))_j < m_j, 1 \le j \le k.$$

As a consequence, the leading eigenvalue of f_Γ is smaller than 1 (consider the product $v f_\Gamma m$ with v a leading eigenvector of $(f_\Gamma)^t$).

The map $f_{\Gamma,\#}$ is almost the transition matrix for the subshift (Σ, σ). The components of $\mathcal{A}^S$ are in one-to-one correspondence with the collection of elements $\{\lambda | \lambda \in \Lambda(i,j)\}$. If $A_\lambda, A_\mu \in \mathcal{A}^S$ where $\lambda \in \Lambda(i,j) \neq \emptyset$ and $\mu \in \Lambda(k,l) \neq \emptyset$ then $A_{\lambda\mu} = \{z | z \in A_\lambda, f(z) \in A_\mu\} \neq \emptyset$ if and only if $j = k$. An admissible sequence $\alpha \in \Sigma$ is thus given by an infinite sequence λ_n where successive terms satisfy the above condition. Hence by removing any basis elements γ_j for which the jth row of $f_{\Gamma,\#}$ consists only of zeros we obtain a new matrix. Iterating this removal process we obtain a matrix which gives exactly the subshift (Σ, σ).

8.4. Quasicircles and non-quasicircles. — A Jordan curve J in the sphere is a *K-quasicircle* if it is the image of a round circle under a K-quasiconformal map, and it is said to have *K'-bounded turning* if the ratio

$$\operatorname{diam}(L)/d(x,y) \le K'$$

for all $x, y \in J$, where L is the component of $J - \{x, y\}$ with smallest diameter. Here distance is measured with respect to the spherical metric d. J is said to be a *quasicircle* if it is a K-quasicircle for some K. It is known that J is a quasicircle if and only if it has bounded turning ([**LV**] II, §8).

If a Jordan curve component J of $\mathcal{J}$ is preperiodic and f is hyperbolic, it is a quasicircle by the surgery argument of McMullen [**Mc1**]. However, a wandering Jordan curve component J need not be a quasicircle. We first show

Theorem 8.4. — *Let f be a hyperbolic rational map. If the orbit of α under σ does not accumulate on $\Sigma_{e,0}$ then $\Phi(\alpha)$ is a quasicircle.*

(Here we use the same notation as in §8.2). The proof also shows the following corollary. However, we have no example where the hypotheses are satisfied.

Corollary 8.5. — *If $\Sigma_{e,0}$ is empty (i.e. if every boundary component of a component of $\mathcal{A}^S$ maps to a boundary component of $\mathcal{L}$ which is not also a boundary component of any component of $\mathcal{A}^S$), then there exists a K such that for every $\alpha \in \Sigma$, $\Phi(\alpha)$ is a Julia component which is a $K-$quasicircle.*

Proof. — Let $J_0 = \Phi(\alpha)$. If α does not accumulate on $\Sigma_{e,0}$ then there is a positive integer N_0 such that

$$\mathcal{T}_0' = \{T_{N_0}(\sigma^i(\alpha)), i \ge 0\}$$

is a disjoint union of open annuli which is compactly contained in $\mathcal{A}^S$, and which contains the entire forward orbit of J_0. Since the orbit of J_0 does not accumulate on a boundary component of $\mathcal{A}^S$, it does not accumulate on boundary components of $\mathcal{T}_0'$. Hence there is an integer $N_1 > N_0$ such that

$$\mathcal{T}_1' = \{T_{N_1}(\sigma^i(\alpha)), i \geq 0\}$$

is a disjoint union of open annuli which is compactly contained in $\mathcal{T}_0'$ and which contains the entire forward orbit of J_0. Then $f^{N_1-N_0}(\mathcal{T}_1') = \mathcal{T}_0'$.

Enlarge $\mathcal{T}_0'$ slightly to a disjoint union $\mathcal{T}_0$ of open annuli whose boundaries are real-analytic Jordan curves and which is compactly contained in $\mathcal{A}^S$. Let $\mathcal{T}_1$ be the union of preimages of components of $\mathcal{T}_0$ under $f^{N_1-N_0}$ containing components of $\mathcal{T}_1'$. Then $\mathcal{T}_1$ is compactly contained in $\mathcal{T}_0$, and

$$g = f^{N_1-N_0} : \mathcal{T}_1 \to \mathcal{T}_0$$

is an expanding conformal dynamical system.

On the other hand, we may build another model for this dynamical system of the form

$$h : \mathcal{R}_0 \to \mathcal{R}_1$$

where $\mathcal{R}_i$ consist of round annuli and the map h on each component is $z \mapsto z^d$, some d. We may also find smooth maps $\phi_i : \mathcal{R}_i \to \mathcal{T}_i$ such that $\phi_0 \circ h = g \circ \phi_1$ on $\mathcal{R}_1$ and $\phi_0 \simeq \phi_1$ *rel* $\partial\mathcal{R}_0$. That is, the pair (ϕ_0, ϕ_1) gives a *combinatorial equivalence* between the two dynamical systems. By Theorem A.1 of [**Mc2**], ϕ_i are isotopic *rel* $\partial\mathcal{R}_0$ to a quasiconformal conjugacy ψ. Since the Julia set of h is the product of a Cantor set with a round circle, J_0 is a component of the Julia set of g and hence J_0 is a quasicircle.

If $\Sigma_{e,0}$ is empty, then $f^{-1}(\mathcal{A}^S) \cap \mathcal{A}^S$ is compactly contained in $\mathcal{A}^S$. We may then apply the argument above with $\mathcal{T}_0' = \mathcal{A}^S$ and $\mathcal{T}_1' = f^{-1}(\mathcal{A}^S) \cap \mathcal{A}^S$ to prove the Corollary.

□

Example. We illustrate this by our example $f = f_1$. In this case $\mathcal{A}^S$ has two components A_0 and A_1. We choose A_0 to be the outermost one, i.e. the one with a fixed boundary (the component J^+). There are no components of $\mathcal{J}$ which are points, and indeed $\mathcal{J} = \{z | f^n(z) \in \overline{\mathcal{A}^S} \forall n\}$. Moreover, the boundary components of components of $\mathcal{A}^S$ are entire Julia components, and $\overline{A}_0 \cap \overline{A}_1 = \emptyset$. It follows that the connected components of $\mathcal{J}$ are precisely the sets of the form $J'(\alpha), \alpha \in \Sigma$. Hence by Theorem 8.2, the dynamics on the space of connected components of $\mathcal{J}$ in the Hausdorff topology is conjugate to the one-sided shift on two symbols 0 and 1. Note that then $\Sigma_{e,0} = \{(000\ldots)\}$ and hence that $\Sigma_e = \{(a_0a_1\ldots) | a_n = 0 \forall n \geq n_0\}$.

The dynamical system (Σ, σ) is conjugate via ϕ to (C, g), where C is the invariant Cantor set for the interval map $g : [0, 1/3] \sqcup [2/3, 1] \to [0, 1]$ given by $g(c) = 3c$ for $0 \leq c \leq 1/3$ and $g(c) = 3(1-c)$ for $2/3 \leq c \leq 1$. The conjugacy ϕ is defined by $\phi(c) = (a_0a_1a_2\ldots)$ where $a_n = 0$ if $g^n(c) \in [0, 1/3]$ and $a_n = 1$ otherwise. Note that the point $0 \in C$ corresponds to $(0000\ldots)$ and the point $1 \in C$ to $(1000\ldots)$. Order the components of $\mathcal{J}$ so that $J > J'$ if J' separates J and the point at infinity. Then the composition $\Phi \circ \phi$ is order-preserving with respect to the usual ordering on the interval and $>$.

Say $c \in C$ is *exposed* if c is a boundary point of a component of the complement of C and is *buried* otherwise. Similarly, we say a Julia component is *exposed* if it meets the boundary of a Fatou component of f and is *buried* otherwise. Then c is exposed if and only if $\phi(c) \in \Sigma_e$ if and only if $\Phi \circ \phi(c)$ is exposed if and only if $\Phi \circ \phi(c)$ is a covering of J^+ if and only if $\Phi \circ \phi(c)$ is not a Jordan curve.

Finally, we note that Σ_e is dense in Σ.

We now show

Theorem 8.6. — *$\Phi \circ \phi(c)$ is a quasicircle if and only if c does not accumulate at 0 under g, i.e. if and only if $\alpha = \phi(c)$ does not contain arbitrarily long strings of consecutive zeros.*

Thus quasi-circle components form a dense subset in the space of Julia components. On the other hand, the non quasi-circle components form a residual set in Baire's category.

Proof. — Let $J_0 = \Phi(c_0)$ and $J_n = f^n(J_0) = \Phi(g^n(c_0)), n \geq 0$. By Theorem 8.4 it suffices to show that if zero is a limit point of the orbit of c_0 then J_0 is not a quasicircle. On a compact neighborhood of $\overline{\mathcal{A}^S}$ avoiding the point at infinity the spherical and planar metrics are equivalent, hence J_0 is a quasicircle if and only if it has bounded turning with respect to the planar metric. For simplicity we conjugate the map by $1/z$ so that J^+ is near zero and the J_n separate J^+ from infinity.

By Theorem 8.2, if $g^n(c_0)$ has zero as limit point, then there is a subsequence $J_{n_k} \to J^+$ in the Hausdorff topology. In J^+ we may find a cut point p and small open discs W' and W with $\overline{W} \subset W'$ and such that $W' \cap \mathcal{P} = \emptyset$, $p \in W$, and W contains a connected component L of $J^+ - \{p\}$. Since p is a cut point p is accessible from the component of $\overline{\mathbf{C}} - J^+$ containing infinity (recall we conjugated our map by $1/z$) via two distinct accesses η_x, η_y. Since $J_{n_k} \to J^+$ in the Hausdorff topology and the J_{n_k} are critically separating, for k sufficiently large, there are points $x_k \in \eta_x \cap J_{n_k} \cap W$ and $y_k \in \eta_y \cap J_{n_k} \cap W$ such that $x_k, y_k \to p$ and for which the component L_k of $J_{n_k} - \{x_k, y_k\}$ of smallest diameter is also contained in W and has diameter bounded below by $D = \operatorname{diam}(L)$.

Since $W' \cap \mathcal{P} = \emptyset$, there is a univalent branch h_k of $(f^{n_k})^{-1}$ on W' sending L_k to a subarc of J_0. Let $x_{0k} = h_k(x_k), y_{0k} = h_k(y_k)$, and $L_{0k} = h_k(L_k)$. Then the Koebe distortion theorem implies that since W is compactly contained in W', there is a constant $C \geq 1$ independent of k such that h_k distorts ratios of planar distances between points of W by at most a factor of C. Hence for all k

$$C^{-1}\frac{\operatorname{diam}(L_k)}{|x_k - y_k|} \leq \frac{\operatorname{diam}(L_{0k})}{|x_{0k} - y_{0k}|} \leq C\frac{\operatorname{diam}(L_k)}{|x_k - y_k|}.$$

But $\operatorname{diam}(L_k)$ is bounded from below by D while $|x_k - y_k| \to 0$. Hence J_0 has unbounded turning and so is not a quasicircle. □

8.5. Constructing other examples. — The map f_1 was initially found by using quasiconformal surgery to glue $z^2 - 1$ with $1/z^3$ in the appropriate fashion. This construction and its generalizations are the subject of work in progress. For example, one may apply the same construction by replacing $z^2 - 1$ with any polynomial p whose filled Julia set has interior, and glue it to $1/z^3$.
Question If p is a quadratic polynomial with non-locally connected Julia set and a Siegel disc, can one obtain, by this gluing, a map with wandering Julia components which are critically separating but not Jordan curves?

8.6. How to find the set E. — There are finitely many Fatou components $W_0, \cdots, W_m$ either intersecting $\mathcal{P}$ or separating $\mathcal{P}$ into at least three parts (Lemma 2.2). For each W_i take the finitely many boundary components which are critically separating. Saturated them into Julia components. Call the union of them E'. It is finite and forward invariant. Fill in the disc components of $\overline{\mathbf{C}} - E'$ disjoint from $\mathcal{P}$. We get a fattened E'' of E'. We distinguish three types of components of $\overline{\mathbf{C}} - E''$: type I, components containing exactly one W_i (these are also the components intersecting $\bigcup W_i$); type II, annulus components disjoint from $\bigcup W_i$; and type III, non-annulus components disjoint from $\bigcup W_i$. In each of type III component, there are finitely many Julia components separating $\mathcal{P}$ into at least three parts. Adding them to E', we get E. Note that the set $\mathcal{L}$ is precisely the union of components of type I (see also Lemma 3.10).

Let W be a fixed attracting basin for a rational map f. How do we know that only finitely many components of $\overline{\mathbf{C}} - W$ intersect $\mathcal{P}$? And if so how can we find V_N in Lemma 7.3 and in the proof of Theorem 7.1? Here is a constructive answer. Take V_0 a open disc with Jordan curve boundary in W such that $f(\overline{V_0}) \subset V_0$. Let V_n be the component of $f^{-n}(V_0)$ containing V_0. Then only finitely many components of $\overline{\mathbf{C}} - W$ intersect $\mathcal{P}$ if and only if there is a minimal integer N such that each boundary curve of V_{N+1} is either not critically separating, or is parallel to a boundary curve of V_N. And for this V_N, each component of $\overline{\mathbf{C}} - V_N$ contains at most one component of $\overline{\mathbf{C}} - W$ intersecting $\mathcal{P}$.

9. Generalizations

9.1. Geometrically finite maps. — Our techniques allow a generalization of Theorem 1.2 to the case of geometrically finite maps f. In [**TY**] (§1, Step 4 and Prop. 1.3)) (cf. also [**DH2**]) a Riemannian metric is constructed for which f is uniformly expanding on a neighborhood of its Julia set. The arguments given above for Propositions [Case 2] and [Case 3] then apply in this more general setting. Proposition [Case 4] is proved for nice maps, therefore applies automatically to geometrically finite maps.

We may also recover a variant of Theorem 8.2 for geometrically finite maps as well, though we may lose the injectivity of Φ–it may be possible to construct a map f with a periodic Jordan curve Julia component J_0 which intersects $\mathcal{P}$ and which is the common boundary of two components of $\mathcal{A}^S$. Then J_0 may be $J'(\alpha)$ for two distinct admissible sequences α. At worst, however, Φ is two-to-one. We define $\Sigma_{e,0}$ and Σ_e the same as for hyperbolic maps.

Theorem 8.4 holds for geometrically finite maps as well, and we may recover a result of Cui et. al. ([**CJS**], Prop. 6.2):

Proposition 9.1. — *If f is geometrically finite, then there are at most finitely many periodic Jordan curve Julia components J_0 which are not quasicircles.*

Proof. — Let $J_n = f^n(J_0)$. Then either $J_n \subset \mathcal{A}^{\mathrm{S}}$ for all n, or J_n is a boundary component of a component of $\mathcal{A}^S$ for all n, since J_0 is periodic. The latter set of such J_0 is finite, while in the former case $J_0 = J'(\alpha)$ for some unique $\alpha \in \Sigma$. The sequence α cannot accumulate on $\Sigma_{e,0}$ under σ since otherwise J_0 accumulates on a boundary component of $\mathcal{A}^S$, by Theorem 8.2. Hence J_0 is a quasicircle by Theorem 8.4. □

9.2. A further result for nice maps. —

Theorem 9.2. — *If f is nice and every component of $\mathcal{A}^S$ is also a component of $\mathcal{A}$, then every Julia component J_0 not eventually landing on a component of E is a point.*

Proof. — The hypotheses imply that every component of $\mathcal{A}^S$ is a Herman ring or a preimage of a Herman ring, therefore contains no Julia components. So Case 2 of Lemma 2.4 does not occur.

By Proposition [Case 4], if J_0 is a Julia component such that $J_n = f^n(J_0) \subset \mathcal{L}$ for infinitely many n, then J_0 is a point.

Assume now J_0 is a Julia component that is not in Cases 1, 2 and 4 of Lemma 2.4. Replacing J_0 by a forward iterate of it if necessary, we may assume that $J_n \cap (E \cup \mathcal{A}^S \cup \mathcal{L}) = \emptyset$ for all $n \geq 0$. Furthermore, by Lemma 2.4, $J_n \subset \mathcal{A}^O \cup \mathcal{D}'$ for infinitely many n. Therefore

$$J_n \subset \hat{Q} = (\mathcal{A}^O - \mathcal{A}^O\mathcal{A}^S \cup \mathcal{A}^O\mathcal{A}\mathcal{L}) \cup (\mathcal{D}' - \mathcal{D}'\mathcal{A}^S \cup \mathcal{D}'\mathcal{L} \cup \mathcal{D}'\mathcal{A}\mathcal{L})$$

for infinitely many n, where the notation ABC means $\{z | z \in A, f(z) \in B, f^2(z) \in C\}$.

We claim that $\hat{Q}$ is contained in the disjoint union of finitely many full continua disjoint from $\mathcal{P}$. We can then apply Lemma 1.4 to conclude that J_0 is a point.

Let A' be a component of $\mathcal{A}^O$. Either it coincides with a component of $\mathcal{A}^O\mathcal{A}^S$, or $A = f(A')$ is a component of $\mathcal{A} - \mathcal{A}^S$. Assume we are in the latter case. By Lemma 3.8, for $\delta^{\pm}$ the boundary components of A, there are components $\overline{V^+, V^-}$ of $\mathcal{A}\mathcal{L}$ (which may coincide) such that $\delta^+ \subset \partial V^+$ and $\delta^- \subset \partial V^-$. Thus $\overline{A - (V^+ \cup V^-)}$ is disjoint from ∂A. Hence $\overline{A' - A'\mathcal{A}\mathcal{L}}$ is contained in A'. Since the inclusion map from A' into $\overline{\mathbf{C}} - \mathcal{P}$ is homotopic to a constant map, $\overline{A' - A'\mathcal{A}\mathcal{L}}$ is contained in a full continuum disjoint from $\mathcal{P}$.

The case of a component of $\mathcal{D}'$ is similar, using Lemma 3.7. □

A
Technical results about plane topology

In this appendix we collect technical results used in the course of our proofs.

The following lemma was used in the proof of Lemma 3.6.

Lemma A.1. — *Let P^1 and P^2 be two disjoint non empty closed sets of S^2. Set $\mathcal{W} = \{U \mid U$ is a complement component of some component of J, $P^1 \subset U$ and $P^2 \cap U = \emptyset\}$. Then $\mathcal{W}$ is either empty or totally ordered with respect to inclusion, and $W = \bigcup_{U \in \mathcal{W}} U$ as the unique maximal element, and $\partial\mathcal{W}$ is connected.*

Proof. — That $\mathcal{W}$ is totally ordered with respect to inclusion if it is nonempty follows from Lemma 3.1. Since each $U \in \mathcal{W}$ is a disc and the U's are nested, we have that W is a disc. Hence ∂W is connected.

We now show that W is also an element of $\mathcal{W}$. Let $x \in \partial W$. For any integer n, there is a point x' in W such that the distance between x' and x is less than $1/n$, and there is $U \in \mathcal{W}$ such that $x' \in U$. The segment $[x, x']$ intersects ∂U, which is a subset of J. We conclude then either $x \in \partial U$ or there are points of J arbitrarily close to x. In both cases $x \in J$.

Since ∂W is a connected subset of J, it is contained in some component S of J.

Now W must be a component of $\overline{\mathbf{C}} - S$. This is because $S \cap U = \emptyset$ for any $U \in \mathcal{W}$, therefore $S \cap W = \emptyset$.

Moreover $W \supset P^1$, but $W \cap P^2 = \emptyset$. So W is indeed the maximal element of $\mathcal{W}$. □

The remaining results are essentially ingredients in the proof of Theorem 7.1. The logical dependencies are:

Lemma A.2 → Corollary A.3 → Corollary A.4 → Corollary A.5 → Lemma 3.10 → Theorem 7.1.

We will need the following fact from general topology.

Let X be a compact Hausdorff space. Then every component Y of X coincides with the intersection of open and closed subsets of X containing Y. Moreover, for any open neighborhood U of Y, there is a closed and open subset V of X such that $Y \subset V \subset U$. The component Y is always closed. In case Y is not open in X, there is a sequence of closed and open set V_n such that $Y \subset\neq V_n \subset\neq V_{n-1}$, and $\bigcap_n V_n = Y$.

If X is the Julia set of some map, and is disconnected, no component of X is open in X (see [**Be**]).

In the next three results, let $\mathcal{J} \sqcup \mathcal{F}$ be a decomposition of S^2 with $\mathcal{J}$ closed and disconnected, let d denote spherical distance and d_H the Hausdorff distance on closed sets with respect to d.

The following Lemma is known as Zoretti's Theorem ([**Wh**], p. 109).

Lemma A.2. — *Given J' and J'' two distinct components of $\mathcal{J}$, and $\epsilon > 0$, there is a Jordan curve γ in $\mathcal{F}$ separating J' and J'' such that* $\sup_{x\in\gamma} d(x, J') < \epsilon$.

The next two results are easy consequences of Lemma A.2.

Corollary A.3. —

1. *Let U be a connected component of $S^2 - J'$ for some connected component of $\mathcal{J}$. Then there is a sequence of closed discs D_n bounded by Jordan curves γ_n such that $\gamma_n \subset U \cap \mathcal{F}$, $D_n \subset int(D_{n-1})$, $\partial U \subset int(D_n)$, and $D_H(\partial U, \gamma_n) \to 0$ (where D_H denotes the Hausdorff distance of compact sets).*
2. *Given W a component of $\mathcal{F}$ and K a component of $S^2 - W$, there is a sequence of closed disks $\overline{D}_n$ such that $\overline{D}_n \subset int(D_{n-1})$, $\partial D_n \subset W$ and $\bigcap_n D_n = K$.*

Proof. — Let $\{K_m\}_{m=0}^{\infty}$ be a sequence of closed discs such that $K_{m-1} \subset \text{int}(K_m)$, $\cup_m K_m = U$, and $d_H(\partial U, \partial K_m) \to 0$. Let $\mathcal{J}_0 = \mathcal{J} \cup K_0, F_0 = S^2 - \mathcal{J}_0$. Then $\mathcal{J}_0$ is closed and disconnected and Lemma A.2 implies that there is a Jordan curve $\gamma_0 \subset F_0 \subset F$ separating ∂U from ∂K_0. Let D_0 be the disc bounded by γ_0 and containing ∂U. Inductively define γ_n as follows. There is an $m(n)$ such that $\gamma_{n-1} \subset \text{int}(K_{m(n)})$. Let $\mathcal{J}_n = \mathcal{J} \cup K_{m(n)}, F_n = S^2 - \mathcal{J}_n$. Lemma A.2 implies that there is $\gamma_n \subset F_n \subset F$ separating ∂U from $K_{m(n)}$ and bounding a closed disc D_n containing ∂U. This shows the first part; the second is similar. □

Corollary A.4. — *The following conditions are equivalent:*

1. *there is a component W of $\mathcal{F}$ contained in U such that $\partial W \cap \partial U \neq \emptyset$;*
2. *there is a component W of $\mathcal{F}$ contained in U such that ∂U is a component of ∂W;*
3. *there is an n, such that no component of $\mathcal{J}$ separates ∂U and γ_n, where γ_n is a sequence of Jordan curves as in Corollary A.3, Part 1.*

Proof. — That $2 \Longrightarrow 1$ is obvious; $2 \Longrightarrow 3$ follows directly from Part 2 of Corollary A.3 and the hypothesis that ∂U is a component of ∂W. To see $3 \Longrightarrow 2$, note that $\mathcal{F} \cap (U - D_n)$ must be connected, where the D_n are as in Part 1 above. Let $X = \partial W$. Then ∂U is contained in a nested intersection of sets which are open and closed in X, hence ∂U is a connected component of ∂W. Finally, that $1 \Longrightarrow 2$ follows easily from the observation that if C_0 is the connected component of ∂W which intersects ∂U, then C_0 is contained in some component J_0 of $\mathcal{J}$, hence C_0 cannot separate points of U. □

Corollary A.5. — *Let E be a nonempty set of finitely many disjoint components of $\mathcal{J}$ and let U be a connected component of $S^2 - E$.*

1. *If ∂U is connected, let W_0 be a component of $\mathcal{F}$ contained in U, and assume no component of $\mathcal{J}$ separates W_0 from ∂U.*
   Then ∂U is a connected component of ∂W_0, and is the unique component of $\mathcal{F}$ contained in U whose boundary intersects ∂U.
2. *If ∂U is not connected, assume no component of $\mathcal{J}$ separates components of E.*
   Then there exists a component W_0 of $\mathcal{F}$ contained in U such that every connected component of ∂U is a connected component of ∂W_0, and W_0 is the unique component of $\mathcal{F}$ contained in U whose boundary intersects each component of ∂U.

Proof. — Assume first that ∂U is connected. By Corollary A.3, Part 1, there is a sequence $\gamma_n \to \partial U$ of Jordan curves in $\mathcal{F} \cap U$; we may take these curves to separate some point w_0 of W_0 from ∂U. By hypothesis and Corollary A.4, Part 3, there exists a component W of $\mathcal{F}$ contained in U such that ∂U is a connected component of ∂W. If $W \neq W_0$, then since $W_0, W \subset U$ we must have that W and W_0 are separated by a component J' of $\mathcal{J}$ which is contained in U. Then $W \cup \partial U$ is a connected set which is separated from W_0 by J', hence ∂U is separated from W_0 by J', a contradiction. The uniqueness assertion follows from the equivalence of the first two parts in the previous lemma, and the proof of the second case is similar. □

B
Proof of Proposition 7.2

A subtlety to prove Proposition 7.2 is that the map $f : \partial W' \to \partial W$ need not be open in the subspace topology. We first establish

Claim Let f be a rational map and let W', W be two Fatou components of f such that $f(W') = W$. Let K (resp. K') be a component of $\overline{\mathbf{C}} - W$ (resp. $\overline{\mathbf{C}} - W'$) such that $f(\partial K') \cap \partial K \neq \emptyset$. Then

(1) $f(\partial K') = \partial K$.
(2) $f(K') \supset K$.
(3) $K \cap P = \emptyset$ if and only if $K' \cap f^{-1}P = \emptyset$. In this case $f(K') = K$.
(4) For any set $S \supset P$, $K' \cap f^{-1}S \neq \emptyset$ if and only if $K \cap S \neq \emptyset$.

Proof of Proposition 7.2. — **1.** By (1) above, if K' is a connected component of $\overline{\mathbf{C}} - W'$, then $f(\partial K') = \partial K$ where K is a connected component of $\overline{\mathbf{C}} - W$. By (1) and (4), there are finitely many such K' for which $K' \cap f^{-1}(\mathcal{P}) \neq \emptyset$. **2.** It suffices to show $f(U') \subset U$ and $f(\partial U') \subset \partial U$. Since U' is the union of W' with components of $\overline{\mathbf{C}} - W'$ disjoint from $f^{-1}\mathcal{P}$, this follows from (1) and (3). **3.** Again this follows directly from the finiteness of the number of boundary components and (1). □

Proof of Claim. — (1) Denote by J the Julia component containing ∂K.

Denote by V the component of $\overline{\mathbf{C}} - J$ containing W and by V' the component of $\overline{\mathbf{C}} - f^{-1}J$ containing W' . Then $f : V' \to V$ is proper and f maps each boundary component of V' onto the boundary of V, which is $\partial K'$ (see Lemma 3.3).

Clearly $f(\partial K') \subset \partial K$. So $\partial K' \cap V' = \emptyset$. Since $W' \subset V'$ and $\partial K' \subset \partial W'$, we have $\partial K' \subset \partial V'$. So $\partial K'$ is contained in a component S' of $\partial V'$. But S' is in the Julia set, and the Julia component of $\partial K'$ is contained in K', so $S' \subset K'$. Since $\partial K'$ separates W' from $\mathrm{int}(K')$, no point of $\mathrm{int}(K')$ can be in V'. Therefore $S' \subset \partial K'$. Thus $S' = \partial K'$ and $f(\partial K') = \partial K$.

(2) A rational map f maps connected components of preimages of a compact connected set surjectively onto its image (see [**Be**], Ch. 5). Let L' be the connected component of the preimage of K intersecting $\partial K'$. Then $f(L') = K$ and $\partial K' \subset L' \subset K'$. Hence $f(K') \supset K$.

(3) Assume at first that $K \cap P = \emptyset$. Since K is full the component L' in point (2) is also full. A simple topological argument then shows $K' = L'$. So $f(K') = K$ and $K' \cap f^{-1}P = \emptyset$.

Assume now that $K' \cap f^{-1}P = \emptyset$. If $K \cap P \neq \emptyset$, since $f(K') \supset K$ we would have $K' \cap f^{-1}P \neq \emptyset$ which is impossible. So $K \cap P = \emptyset$, $K' = L'$ and $f(K') = K$.

(4) Assume at first that $K \cap P = \emptyset$. Since $f(K') = K$, we get 4). Assume now $K \cap P \neq \emptyset$. Then $K \cap S \neq \emptyset$ and $K' \cap f^{-1}S \neq \emptyset$ (for otherwise $K' \cap f^{-1}P = \emptyset$ and $K \cap P = \emptyset$). □

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