Hausdorff dimension of subsets of the parameter space for families of rational maps

(A generalization of Shishikura's result)

TAN Lei

Department of Mathematics, University of Warwick, Coventry CV4 7AL, U.K.

Abstract

In this paper we extend Shishikura's result on the Hausdorff dimension of the boundary of the Mandelbrot set to higher dimensional parameter spaces, for example the space of degree d polynomials, and show that some parameter subsets, including the boundary of the connectedness locus, have Hausdorff dimension equal to the real dimension of the parameter space (which is 4 for cubic polynomials).

1 Introduction

A remarkable work of Shishikura shows that ∂M, the boundary of the Mandelbrot set, has Hausdorff dimension 2 (see [11],[12],[13]). In order of generalize this result to higher-dimensional parameter spaces, we need to find parameter subsets having similar properties as ∂M. The set ∂M can be described as: (a) the Julia-unstable locus, (b) a subset of the Julia-unstable locus, (c) the boundary of the critical non-escape locus, or (d) the boundary of the connectedness locus. For a general family of rational maps (for example the family of cubic polynomials) these sets may not coincide any more. We would like to study not only the largest set (a), but also some subset of (a) (for example (d) in the family of degree d polynomials). We will show that many subsets of (a), including (d), have Hausdorff dimension equal to the real dimension of the space.[1] Note that the result about (a) has no interests for the entire family of rational maps of degree d, since Mary Rees has shown that the set (a) has positive Lebesgue measure ([10]).

We will also show that each baby Mandelbrot set sitting inside of M has a Hausdorff dimension 2 boundary. For this we need to study not only families of rational maps, but also families of rational-like maps, which are often realized as renormalizations of rational maps.

[1] C. McMullen ([8]) has obtained similar results, but with a different approach.

The part of the work related to the dynamical planes is already done by Shishikura. We will just transfer it to the parameter space. The method is very similar to Shishikura's original proof for (complex) one-dimensional parameter spaces.

Theorem 1.1 below is the restatement of a theorem of Shishikura (cf. [11], Theorem 2, although the theorem there is stated for rational maps, the proof is actually valid for rational-like maps, see Remarks (iv) on page 28 of [11]). We say that a rational-like map g is an *S-map* (S for Shishikura), of type (k, p, q), if g has a parabolic k-periodic cycle ξ with multiplier $\exp(2\pi ip/q)$ ($p, q \in \mathbb{N}$, $(p, q) = 1$) and that the immediate parabolic basin attached to ξ (i.e. the union of periodic Fatou components U such that $g^k|_U$ converges to ξ locally uniformly) contains only one critical point of g^k. The other terminologies will be defined in the next section.

Theorem 1.1 *(Shishikura, dynamical plane) Suppose that an analytic family* $\mathbf{f}_\Lambda$ *of rational-like maps contains a non-persistent S-map* $g = f_{\lambda'}$*. Then for any* $\varepsilon > 0$*, there are many* λ_0 *near* λ' *such that* $hyp - dim(f_{\lambda_0}) > 2 - \varepsilon$*. More precisely, if* g *is of type* (k, p, q)*, for any* $\varepsilon > 0$ *and* $b > 0$*, there is a neighborhood* $\mathcal{N}$ *of* λ' *in* Λ*, a neighborhood* V *of* ξ *in* $\overline{\mathbb{C}}$*, positive integers* N_1 *and* N_2 *such that if* $\lambda_0 \in \mathcal{N}$ *and if* f_{λ_0} *has a* k*-periodic cycle in* V *with multiplier* $\exp(2\pi i\alpha)$*, where*

$$q\alpha = p \pm \cfrac{1}{a_1 \pm \cfrac{1}{a_2 + \beta}} \tag{1}$$

with integers $a_1 \geq N_1$*,* $a_2 \geq N_2$ *and* $\beta \in \mathbb{C}$*,* $0 \leq Re(\beta) < 1$*,* $|Im(\beta)| \leq b$*, then there is a homogeneous hyperbolic subset* $X \subset J_{\lambda_0}$ *such that* $\text{H-dim}(X) > 2 - \varepsilon$*.*

This paper consists of mainly the following two results:

Theorem 1.2 *Assume that* Λ *is an open subset of* $\mathbb{C}^k$*. Given an analytic family* $\mathbf{f}_\Lambda$ *of rational-like maps, denote by* A_0 *the Julia-unstable locus.*

1 (adaptation). If there is an S-map in A_0*, then there are many* λ_0 *as in Theorem 1.1 which are also in* A_0*.*

2 (transfer to the parameter space, the case $k = 1$ *is due to Shishikura). If* $A_0 \neq \emptyset$*, then*

a) A_0 *contains a homeomorphic image of* $\Delta_\# \times Y$*, with* $\Delta_\#$ *a non-empty polydisc of* $\mathbb{C}^{k-1}$ *and* Y *a non-empty subset of* $\overline{\mathbb{C}}$*.*

b) For any $\lambda_0 \in A_0$, and any neighborhood Δ of λ_0, we have

$$\text{H-dim}(A_0) \geq \text{H-dim}(A_0 \cap \Delta) \geq 2(k-1) + hyp - dim(f_{\lambda_0}) \ .$$

c) If there is an S-map $f_{\lambda'}$ with λ' in A_0, then for any neighborhood Δ' of λ', H-dim$(A_0 \cap \Delta') = 2k$*. If the parameter values corresponding to S-maps are dense in A_0,* H-dim$(A_0 \cap U) = 2k$ *for any open set U intersecting A_0.*

Later on we will define a special subset A_1 of the boundary of the non-escape locus and prove the above theorem for both A_0 and A_1. The precise statement will be given in Theorem 3.9. This generalized version will allow us to prove:

Theorem 1.3 *(application to polynomials) For C_d the connectedness locus of the family of degree d monic centered polynomials (which is also the non-escape locus), we have* H-dim$(\partial C_d) = 2(d-1)$*.*

We will also prove some density properties of the unstable locus, and give the Hausdorff dimension of some other subsets of Julia-unstable loci.

Section 2 consists of basic definitions. In Sections 3.1 and 3.2 we prove the density of non-primitive parabolic points in the unstable locus and a consequence of this allows us to use a description, due to C. McMullen, of the unstable locus by critical points. In Section 3.3 we define the set A_1, give the statement of Theorem 3.9 and prove the first part of Theorem 3.9. Section 4 contains a general transfer result, which then leads to the second part of Theorem 3.9. The section ends with a corollary showing that Misiurewicz points are dense. In Section 5 we prove Theorem 1.3.

Acknowledgement. Special thanks go to B. Branner, who, by "forcing" the author to write a report on Shishikura's work for the seventh EWM meeting, has initiated and encouraged this research. The author has also benefited a lot from discussions with A. Douady, A. Epstein, A. Manning, C. McMullen, H.H. Rugh and M. Shishikura.

2 Definitions

Definition 1. Let U' and U be two open sets of $\overline{\mathbb{C}}$ and $f : U' \to U$ be a holomorphic proper map. We say that (f, U', U) is *rational-like* if

Case 1. $U' = U = \overline{\mathbb{C}}$, and f is a rational map. Denote by J_f the Julia set of f.

Case 2. U and U' are open sets with non-empty but finitely many boundary components, $\overline{U'} \subset U$, and, by defining $K_f = \{z \in U' | f^n(z) \in U' \ \forall n \geq 0\}$

(*) $f : U' \to U,\ K_f \to K_f$ is hybrid equivalent to a rational map in the sense of Douady-Hubbard.

Define $J_f = \partial K_f$. Note that condition (*) guarantees that the set of preimages of any point in J_f as well as the set of repelling periodic points are dense in J_f.

Note that U (resp. U') has only finitely many components and each boundary component of U (resp. U') is infinite. We say that a point $z \in U'$ *does not escape* if $z \in K_f$. Note that condition (*) is satisfied if

1. Both U and U' are connected and $U - \overline{U'}$ consists of finitely many annuli and discs, a special case is that both U and U' are topological discs (polynomial-like mapping).

2. U is a topological disc different from $\mathbb{C}$.

Definition 2. We say that a compact set $X \subset \overline{\mathbb{C}}$ (or $X \subset \mathbb{C}^k$) is *homogeneous* if, X has no isolated points, and H-dim$(U \cap X)$ = H-dim(X) for any open set U intersecting X. Let f be a rational-like map. We say that $X \subset J_f$ is a *hyperbolic set*, if $f(X) \subset X$ and there are $C > 0, \mu > 1$ such that $\|(f^n)'(x)\| \geq C\mu^n$ for all $x \in X$ and $n \geq 0$ (where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric).

Definition 3. For a rational-like map f, we define

$$hyp - dim(f) = \sup\{\text{H-dim}(X) \mid X \text{ a homogeneous hyperbolic set of } f\}\ .$$

Definition 4. Let Λ be a complex manifold of dimension k. We say that $\mathbf{f}_\Lambda = \{f_\lambda : U'_\lambda \to U_\lambda | \lambda \in \Lambda\}$ is an *analytic family of rational-like maps*, if each f_λ is rational-like, and for $f(\lambda, z) = (\lambda, f_\lambda(z))$, either

Case 1: $U'_\lambda = U_\lambda = \overline{\mathbb{C}}$ and f is complex analytic on (λ, z); or

Case 2 (similar to Douady-Hubbard's analytic family of polynomial-like mappings, [4]): for $\mathcal{U} = \{(\lambda, z) | z \in U_\lambda\}$, $\mathcal{U}' = \{(\lambda, z) | z \in U'_\lambda\}$ and λ_0 a point in Λ,

(a) $U_{\lambda_0} \neq \overline{\mathbb{C}}$;

(b) $\mathcal{U}$ is homeomorphic to $\Lambda \times U_{\lambda_0}$ and $\mathcal{U}'$ is homeomorphic to $\Lambda \times U'_{\lambda_0}$;

(c) the projection from the closure of $\mathcal{U}'$ in $\mathcal{U}$ to Λ is proper;

(d) the mapping $f : \mathcal{U}' \to \mathcal{U}$ is complex analytic and proper.

Set $K_\lambda = K_{f_\lambda}$ and $J_\lambda = J_{f_\lambda}$.

Denote by $\mathcal{C}$, *the non-escape locus*, the set of $\lambda \in \Lambda$ such that $f_\lambda^n(c) \in U'_\lambda$ for all n and all critical points $c \in U'_\lambda$ of f_λ. In Case 1, $\mathcal{C} = \Lambda$. In Case 2, if U'_{λ_0} is a topological disc (i.e. $\mathbf{f}_\Lambda$ becomes a family of polynomial-like mappings), the set $\mathcal{C}$ coincides with the *connectedness locus*, i.e. the set of λ such that K_λ is connected.

Definition 5. For an analytic family $\mathbf{f}_\Lambda$ defined as above, we say that λ_0 is a *parabolic parameter or point* if f_{λ_0} has a non-persistent parabolic cycle in the sense of Mañé-Sad-Sullivan, that is, there is a p-periodic point $z_0 \in J_{\lambda_0}$ with $(f_{\lambda_0}^p)'(z_0) = e^{2\pi i t/s}$, $t, s \in \mathbb{N}$ minimal , such that either

I. $s \neq 1$ and $(f_\lambda^p)'(z(\lambda)) \not\equiv e^{2\pi i t/s}$, where $z(\lambda)$ is the unique p-periodic point near z_0 of f_λ for λ close to λ_0, or

II. $s = 1$ and the projection $M_p \to \Lambda$ onto the first coordinate is not injective in any neighborhood of (λ_0, z_0), where $M_p = \{(\lambda, z) | z \ p\text{-periodic points of } f_\lambda\}$.

We say that λ_0 is *non-primitive* if the first case happens for some z_0.

We say that an S-map $f_{\lambda'}$ is non-persistent if the parabolic cycle ξ is non-persistent.

Definition 6. Let X be an arbitrary subset of $\overline{\mathbb{C}}$ and Λ' be a complex manifold. Choose $\lambda_0 \in \Lambda'$ to be a base point. We say that $i : \Lambda' \times X \to \overline{\mathbb{C}}$ is a *holomorphic motion* if

(i) $i(\lambda_0, \cdot) = id$,

(ii) $i(\lambda, \cdot) : X \to \overline{\mathbb{C}}$ is injective,

(iii) $i(\cdot, z) : \Lambda' \to \overline{\mathbb{C}}$ is holomorphic for each $z \in X$. Set $X_\lambda = i(\lambda, X)$.

Definition 7. Let $\mathbf{f}_\Lambda$ be an analytic family of rational-like maps. Let $\lambda_0 \in \Lambda$. Let $X \subset \overline{\mathbb{C}}$ be an invariant set for f_{λ_0} (i.e. $X \subset U'_{\lambda_0}$ and $f_{\lambda_0}(X) \subset X$). We say that X admits a *dynamical holomorphic motion* $\lambda \mapsto X_\lambda$ on an open neighborhood Δ of λ_0 if there is a holomorphic motion $i : \Delta \times X \to \overline{\mathbb{C}}$ with base point λ_0 such that

(iv) $f_\lambda(i(\lambda, z)) = i(\lambda, f_{\lambda_0}(z))$ for any $(\lambda, z) \in \Delta \times X$.

We have (cf. Shishikura, [11], property (1.2)):

Lemma 2.1 *Any hyperbolic set X of f_{λ_0} admits a unique dynamical holomorphic motion in some neighborhood of λ_0.*

Definition 8. We say that λ_1 is a *J-stable* parameter if J_{λ_1} admits a dynamical holomorphic motion $i : \Delta \times J_{\lambda_1} \to \overline{\mathbb{C}}$ on a neighborhood Δ of λ_1, with $i(\lambda, J_{\lambda_1}) = J_\lambda$.

Denote by $\mathcal{E}^u$, *the unstable locus*, the set of J-unstable parameters in Λ.

3 Basic properties of the unstable locus

Assume in this whole section that $\mathbf{f}_\Lambda$ is an analytic family of rational-like maps, with Λ an open set of $\mathbb{C}^k$. Lemma 3.2 is developed in order to obtain Lemma 3.4, which will allow us to use a result of McMullen (Theorem 3.5 below) characterizing the unstable locus.

3.1 Non-primitive parabolic points are dense

Here is the first characterization of the unstable locus, due to Mañé, Sad and Sullivan ([9], Theorems A and B):

Theorem 3.1 *(MSS) The unstable locus $\mathcal{E}^u$ coincides with the closure of parabolic parameters. Moreover $\mathcal{E}^u$ is closed and nowhere dense in Λ.*

Lemma 3.2 *Non-primitive parabolic points are dense in $\mathcal{E}^u$ and $\mathcal{E}^u$ has no isolated points.*

Proof. Let $\lambda_0 \in \mathcal{E}^u$. Given any neighborhood Δ' of λ_0, we want to find non-primitive values $\lambda \neq \lambda_0$ in Δ' (whether λ_0 is itself a primitive parabolic parameter or not).

Let $\Delta \subset \Delta'$ be a small neighborhood of λ_0 such that all attracting cycles of f_{λ_0} remain attracting as λ varies in Δ. Due to McMullen ([7], page 54), the number $n(\lambda)$ of attracting cycles of f_λ is not locally constant near λ_0. But $n(\lambda) \geq n(\lambda_0)$ for $\lambda \in \Delta$. So some attracting periodic point becomes non-attracting as λ varies in Δ. Hence there is a $\lambda \in \Delta - \{\lambda_0\}$, with a periodic point α of f_λ of period, say p, such that $(f_\lambda^p)'(\alpha) = e^{2\pi i s/t} \neq 1$, with $s, t \in \mathbb{N}$. Therefore λ is non-primitive by Definition 5. ■

Corollary 3.3 *If $\mathcal{E}^u \neq \emptyset$, then $\mathcal{E}^u$ contains a real hypersurface.*

Proof. By the above proof there is a $\lambda_0 \in \mathcal{E}^u$ and a $z_0 \in J_{\lambda_0}$ such that $f_{\lambda_0}^p(z_0) = z_0$, $(f_{\lambda_0}^p)'(z_0) = e^{2\pi i t/s} \neq 1$ and $(f_\lambda^p)'(z(\lambda)) \not\equiv e^{2\pi i t/s}$, where $z(\lambda)$ denotes the local dynamical holomorphic motion of z_0. The set $\{\lambda | \; |(f_\lambda^p)'(z(\lambda))| = 1\}$ is a real hypersurface at non-singular values. ■

3.2 View from the critical points

Denote by Λ_0 the set of $\lambda \in \Lambda$ such that there are local holomorphic parametrizations of the critical points of f_λ. The set Λ_0 is an open set of $\mathbb{C}^k$, and Λ_0 is open and dense in Λ.

Lemma 3.4 *If $\mathcal{E}^u \neq \emptyset$, then $\mathcal{E}^u \cap \Lambda_0 \neq \emptyset$. The set $\mathcal{E}^u \cap \Lambda_0$ is open and dense in $\mathcal{E}^u$.*

Proof. By Corollary 3.3 the set $\mathcal{E}^u$ contains real hypersurfaces. On the other hand $\partial\Lambda_0$ is a subset of a dimension $< k$ analytic set. ■

Let $\lambda_0 \in \Delta' \subset \Lambda_0$, where Δ' is a polydisc. There are holomorphic maps $c_i : \Delta' \to \overline{\mathbb{C}}$ parameterizing the critical points of f_λ. We say that c_i is equivalent to c_j, if there are integers $n > 0, m > 0$ such that $f_\lambda^n(c_i(\lambda)) \equiv f_\lambda^m(c_j(\lambda))$. In each equivalence class, choose one representative. Let $a_1(\lambda), \cdots, a_m(\lambda)$ be a set of representatives. These maps are called *local critical maps*.

Each a_i generates a family of holomorphic maps

$$\mathcal{F}_i = \{\lambda \mapsto f_\lambda^n(a_i(\lambda)), \Delta' \to \overline{\mathbb{C}}, 0 \leq n \leq N_i(\lambda)\} ,$$

where $N_i(\lambda)$ is the minimal integer such that $f_\lambda^{N_i(\lambda)}(a_i(\lambda)) \notin U'_\lambda$. Set $N_i(\lambda) = \infty$ if $a_i(\lambda)$ does not escape. Each $\mathcal{F}_i$ is called a critical family at λ_0.

Definition 9. We say that $\mathcal{F}_i$ is a *(generalized) normal family* at λ_0 if either $N_i(\lambda_0) = n_0 < \infty$ (this only makes sense in Case 2 of Definition 4, and would imply $N_i(\lambda) \equiv n_0$ for λ in a neighborhood of λ_0); or $N_i(\lambda) \equiv \infty$ in a neighborhood $\Delta'' \subset \Delta'$ of λ_0, and (this is automatic in Case 2 of Definition 4 by Montel's theorem) $\mathcal{F}_i$ is normal in the usual sense.

We will need the following result of McMullen (cf. [7], page 54-55):

Theorem 3.5 *(McMullen) $\mathcal{E}^u \cap \Lambda_0 = \{\lambda_0 \in \Lambda_0 \mid$ at least one $\mathcal{F}_i$ is not normal at λ_0 $\}$.*

Remark. Although the original theorem was stated for rational maps, the proof applies easily to rational-like maps and generalized normal families defined as in this paper.

Definition 10. Denote by $\mathcal{E}$ the set of $\lambda_0 \in \mathcal{E}^u \cap \Lambda_0$ for which exactly one family, say $\mathcal{F}_1$, is not normal at λ_0.

Lemma 3.6 *The set $\mathcal{E}$ is open in $\mathcal{E}^u$.*

Proof. For any $\lambda_0 \in \mathcal{E}$, and any $\lambda_1 \notin \mathcal{E}$ close enough to λ_0, every critical point generates a normal family in a neighborhood of λ_1, therefore f_λ is J-stable at λ_1. ■

Lemma 3.7 *The non-escape locus $\mathcal{C}$ is closed in Λ. In Case 2 of Definition 4, we have $\mathcal{E} \cap \mathcal{C} \subset \mathcal{E}^u \cap \mathcal{C} = \partial\mathcal{C}$.*

Proof. The condition on $\lambda \in \Lambda$ such that some critical point of f_λ escapes is an open condition. Therefore $\mathcal{C}$, as the complement of the set of such parameters, is closed.

By Lemma 3.4, if $\mathcal{E}^u \cap int(\mathcal{C}) \neq \emptyset$, then $\mathcal{E}^u \cap int(\mathcal{C}) \cap \Lambda_0 \neq \emptyset$. However, for $\lambda_0 \in int(\mathcal{C}) \cap \Lambda_0$, and $a(\lambda)$ any local critical map near λ_0, we have $f_\lambda^n(a(\lambda)) \in U'_\lambda$ for all n and all λ in a neighborhood of λ_0 (Definition 4). Since $U'_\lambda \subset U_\lambda \neq \overline{\mathbb{C}}$, by Montel's theorem the family of holomorphic maps $\lambda \mapsto f_\lambda^n(a(\lambda))$ is normal at λ_0. Therefore $\lambda_0 \notin \mathcal{E}^u$ (Theorem 3.5). So $\mathcal{E}^u \cap \mathcal{C} \subset \partial\mathcal{C}$.

To show $\mathcal{E}^u \cap \mathcal{C} \supset \partial\mathcal{C}$, we choose a point $\lambda_0 \in \partial\mathcal{C}$ and a sequence $\lambda_n \notin \mathcal{C}$ with $\lim_{n\to\infty} \lambda_n = \lambda_0$. For each λ_n choose $c(\lambda_n)$ an escaping critical point for f_{λ_n}. Subtracting a subsequence if necessary, we may assume that $\lim_{n\to\infty} c(\lambda_n)$ exists, and denote it by $c(\lambda_0)$. By continuity $c(\lambda_0)$ is a critical point of f_{λ_0}. Since $\lambda_0 \in \mathcal{C}$, $c(\lambda_0) \in K_{\lambda_0}$.

Assume that λ_0 is also a J-stable parameter, i.e. J_{λ_0} admits a dynamical holomorphic motion. In this case none of the critical points $a(\lambda_0)$ of f_{λ_0} belonging to the Julia set J_{λ_0} can bifurcate, since the leaf $a(\lambda)$ of the holomorphic motion containing $a(\lambda_0)$ is a local critical map, moreover $a(\lambda) \in J_\lambda$ and the local degree of f_λ at $a(\lambda)$ remains constant.

If $c(\lambda_0) \in J_{\lambda_0}$ then for n large $c(\lambda_n)$ is contained in the leaf of the holomorphic motion containing $c(\lambda_0)$. Therefore $c(\lambda_n) \in J_{\lambda_n}$. This is a contradiction to the assumption that $c(\lambda_n)$ escapes.

If $c(\lambda_0) \in int(K_{\lambda_0})$, then for n large we must have $c(\lambda_0) \in int(K_{\lambda_n})$, since K_{λ_n} has the same topology as K_{λ_0} and is close to K_{λ_0}. This is also impossible since $c(\lambda_n)$ escapes.

Therefore $\mathcal{E}^u \cap \mathcal{C} \supset \partial\mathcal{C}$. ■

Lemma 3.8 *Let $\lambda_0 \in \Lambda_0$. Let $X \subset J_{\lambda_0}$ be a hyperbolic set. If there is a local critical map $a(\lambda)$ and an integer $N > 0$ such that $f_{\lambda_0}^N(a(\lambda_0)) = z_0 \in X$, but $f_\lambda^N(a(\lambda)) \not\equiv z(\lambda)$ (where $z(\lambda)$ denotes the leaf passing through z_0 of the holomorphic motion of X). Then $a(\lambda)$ genereates a non normal family at λ_0.*

Proof. Taking an one-dimensional slice if necessary, we may assume $\dim_{\mathbb{C}}(\Lambda_0) = 1$. Furthermore we assume that $\lambda_0 = 0$ and the orbit of z_0 is disjoint from ∞.

Given two arbitrary holomorphic maps $b(\lambda)$ and $c(\lambda)$ with $b(0) = c(0) = z_0$ and $b(\lambda) \not\equiv c(\lambda)$, we define $b_n(\lambda) = f_\lambda^n(b(\lambda))$ and $c_n(\lambda) = f_\lambda^n(c(\lambda))$. Assume $b(\lambda) - c(\lambda) = t\lambda^p + O(\lambda^{p+1})$. An easy calculation and induction show that

$$b_n(\lambda) - c_n(\lambda) = (f_0^n)'(z_0)t\lambda^p + O(\lambda^{p+1}) \ .$$

In our case $|(f_0^n)'(z_0)| \to \infty$ and the p-th derivative of $z_n(\lambda)$ at 0 remains bounded. Therefore the p-th derivative of $\lambda \mapsto f_\lambda^{n+N}(a(\lambda))$ at 0 diverges. Hence the family $\{\lambda \mapsto f_\lambda^{n+N}(a(\lambda)),\ n \in \mathbb{N}\}$ is not normal at 0. ■

3.3 The set A_1 and the statement of Theorem 3.9

Definition 11. Assume that Λ is an open subset of $\mathbb{C}^k$. Given an analytic family $\mathbf{f}_\Lambda$ of rational-like maps, set $A_0 = \mathcal{E}^u$ and $A_1 = \mathcal{E} \cap \mathcal{C}$.

The following theorem is a generalized version of Theorem 1.2.

Theorem 3.9 *Assume we are in the setting of Definition 11. For $i = 0, 1$,*

1 (adaptation). If there is an S-map in A_i, then there are many λ_0 as in Theorem 1.1 which are also in A_i.

2 (transfer to the parameter space, the case $k = 1$ is due to Shishikura). If $A_i \neq \emptyset$, then

a) A_i contains a homeomorphic image of '$\Delta \times Y$, with 'Δ a non-empty polydisc of $\mathbb{C}^{k-1}$ and Y a non-empty subset of $\overline{\mathbb{C}}$.

b) For any $\lambda_0 \in A_i$, and any neighborhood Δ of λ_0, we have

$$\text{H-dim}(A_i) \geq \text{H-dim}(A_i \cap \Delta) \geq 2(k-1) + hyp - dim(f_{\lambda_0}) \ .$$

c) If there is an S-map $f_{\lambda'}$ with λ' in A_i, then for any neighborhood Δ' of λ', H-dim$(A_i \cap \Delta') = 2k$. *If the parameter values corresponding to S-maps are dense in A_i,* H-dim$(A_i \cap U) = 2k$ *for any open set U intersecting A_i.*

Remark. It is possible that the parameter values corresponding to S-maps are dense in $\partial\mathcal{C}$. A consequence of this would be that $\partial\mathcal{C}$ is homogeneous, i.e. H-dim$(U \cap \partial\mathcal{C}) = 2k$ for any open set U intersecting $\partial\mathcal{C}$. This is true for the boundary of the Mandelbrot set, due to Shishikura.

Proof of Theorem 3.9.1. Fix $i \in \{0, 1\}$.

Let $g = f_{\lambda'}$ be an S-map in A_i. Note that in the formula (1) , if λ_0 is chosen such that $\beta = 0$, then $\alpha = p'/q'$ is a real rational number. So the corresponding cycle $\xi(\lambda_0)$ of f_{λ_0} is again parabolic and non-persistent. Therefore $\lambda_0 \in \mathcal{E}^u = A_0$.

The case $i = 0$ is therefore straight forward.

Assume now $i = 1$. That is $g = f_{\lambda'}$ and $\lambda' \in A_1 \subset \Lambda_0$. Denote by $a_1(\lambda), \cdots, a_m(\lambda)$ the local critical maps. Assume that $a_1(\lambda)$ is the one that generates a non-normal family at λ', and $a_j(\lambda)$ for $j > 1$ generates a normal family in a neighborhood Δ of λ'. By Definition 4, for $j = 1, \cdots, k$, $a_j(\lambda')$ does not escape. So by Definition 9, for $j > 1$, $a_j(\lambda)$ does not escape for all $\lambda \in \Delta$.

Choose $\lambda_0 \in \Delta$ such that $\beta = 0$. Then f_{λ_0} has again a non-persistent parabolic cycle $\xi(\lambda_0)$. So $\lambda_0 \in \mathcal{E}^u \cap \Lambda_0$ (by Theorem 3.1). Hence by Theorem 3.5 at least one local critical map does not generate a normal family at λ_0. It must be $a_1(\lambda)$. By Definition 9 again $a_1(\lambda_0)$ does not escape. Hence $\lambda_0 \in \mathcal{E} \cap \mathcal{C} = A_1$. ∎

4 Transfer to the parameter space

Theorem 3.9.2 will be a consequence of the following more general result. Denote by $\Delta(\lambda, r)$ the polydisc in $\mathbb{C}^k$ of center λ and radius r. That is, in case $\lambda = 0$,

$$\Delta(0, r) = \{(\lambda_1, \cdots, \lambda_k) | \ \max |\lambda_i| < r\} \ .$$

Set also

$$\Delta_\#(0, s) = \{(\eta_1, \cdots, \eta_{k-1}) \ | \ \max |\eta_i| < s\} \ , \quad \text{and } D(0, t) = \{z \in \mathbb{C} | \ |z| < t\} \ .$$

4.1 A general transfer result

Proposition 4.1 *Let $X \subset \overline{\mathbb{C}}$ be a compact set. Assume that we have a holomorphic motion $i : \Delta \times X \to \overline{\mathbb{C}}$ (where Δ denotes the unit polydisc in $\mathbb{C}^k$) and an analytic mapping $v : \Delta \to \overline{\mathbb{C}}$, with $v(0) = z_0 \in X$, $v(\lambda) \not\equiv i(\lambda, z_0)$. Set $M_v = \{\lambda \in \Delta \mid v(\lambda) \in X_\lambda\}$. Assume either $dv|_{\lambda=0} \neq di(\lambda, z_0)|_{\lambda=0}$ (transversality), or z_0 is not isolated in X. Then*

a) M_v contains a homeomorphic image of $\Delta_\# \times (X \cap D(z', r))$ for some $z' \in X$, $r > 0$ and $\Delta_\#$ a non-empty polydisc of dimension $k-1$ (with $z' = z_0$ in the transversal case), and

b) $\text{H-dim}(M_v) \geq 2(k-1) + \lim_{r\to 0} \text{H-dim}(X \cap D(z_0, r))$ in the transversal case, or, if X is homogeneous, $\text{H-dim}(M_v) \geq 2(k-1) + \text{H-dim}(X)$.

Proof. Set $v_1(\lambda) = v(\lambda) - i(\lambda, z_0)$. Assume at first that the differential $dv_1 \neq 0$.

Step 0. To fix our ideas we first treat a caricature case. Assume that $X_\lambda \equiv X_0 = X$ for $\lambda \in \Delta$, $z_0 = 0$ and $\partial v / \partial \lambda_k\ (0) \neq 0$, where λ_k is the k-th coordinate of λ. Then

$$\begin{aligned} M_v &= v^{-1}(X_0) = \{\lambda \in \Delta \mid v(\lambda) \in X\} = \bigcup_{w \in X} \{\lambda \in \Delta \mid v(\lambda) - w = 0\} \\ &\supset \bigcup_{w \in X \cap D(0,t_0)} \{(\lambda_\#, \lambda_k) \in \Delta(0, t_0) \mid \lambda_k = h(\lambda_\#, w)\} = H(\Delta_\#(0, t_0) \times (X \cap D(0, t_0))\ , \end{aligned}$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$, $\lambda_\# = (\lambda_1, \cdots, \lambda_{k-1})$, $h(\lambda_\#, w)$ is the solution of the Weierstrass preparation theorem: $v(\lambda) - w = (\lambda_k - h(\lambda_\#, w))g(\lambda, w)$, with $g(\lambda, w) \neq 0$ in a polydisc around 0 of $\mathbb{C}^{k+1}$, and $H(\lambda_\#, w) = (\lambda_\#, h(\lambda_\#, w))$.

Since $v(\lambda_\#, h(\lambda_\#, w)) = w$, we have $\partial h / \partial w\ (0) \neq 0$. So H is a local diffeomorphism near 0. For t small enough, we have

$$\text{H-dim}\,(H(\Delta_\#(0, t) \times (X \cap D(0, t))) = \text{H-dim}(\Delta_\#(0, t) \times (X \cap D(0, t)))\ .$$

Thus $\text{H-dim}(M_v) \geq \text{H-dim}(\Delta_\#(0, t) \times (X \cap D(0, t)) \geq 2(k-1) + \text{H-dim}(X \cap D(0, t))$, where the last inequality is due to a dimension formula for products (cf. Falconer, [5], Corollary 7.4, page 95).

Step 1, proof of a). We claim that there is a holomorphic change of coordinates H (this H is different from the H in Step 0), and constants $r_0 > 0, s > 0$, such that for $r \in]0, r_0[$ and $Y^r = X \cap D(z_0, r)$, there is an $R_r > 1$ (with $R_r \to \infty$ as $r \to 0$) and a holomorphic motion: $j^r : (\{|\mu| < R_r\} \times \Delta_\#(0, s)) \times Y^r \to \overline{\mathbb{C}}$ such that $H(\eta_\#, j^r(1, \eta_\#, y)) \in M_v$ for any $\eta_\# \in \Delta_\#(0, s)$ and $y \in Y^r$. As a consequence the map $(\eta_\#, y) \to H(\eta_\#, j^r(1, \eta_\#, y))$ is a homeomorphism between the product space $\Delta_\#(0, s) \times Y^r$ and a subset of M_v.

We may assume $z_0 = 0$. Recall that $v_1(\lambda) = v(\lambda) - i(\lambda, 0)$ and $dv_1(0) \neq 0$.

Let $H : \eta \mapsto \lambda$, $0 \mapsto 0$ be a local bi-holomorphic map such that

(v) $\widehat{v} := v_1 \circ H(\eta) = \eta_k$.

Fix $s' > 0$ small enough such that H is well defined in $\Delta(0, s')$ and $H(\Delta(0, s')) \subset \Delta$.

Define $\widehat{i} : \Delta(0, s') \times X \to \overline{\mathbb{C}}, (\eta, z) \mapsto i(H(\eta), z) - i(H(\eta), 0)$. Then $\widehat{i}$ is again a holomorphic motion of X. Moreover $\widehat{i}(\eta, 0) \equiv 0$. Define

$$M_{\widehat{v}} = \{\eta \in \Delta(0, s') \mid \widehat{v}(\eta) = \eta_k \in \widehat{i}(\eta, X)\} .$$

Since the family of holomorphic maps $\{\widehat{i}(\cdot, z) : \Delta(0, s') \to \overline{\mathbb{C}}\}_{z \in X}$ is normal, and $\widehat{i}(\lambda, z) \neq 0$ for any $z \neq 0$ and any η (by (ii) of Definition 6), any limit function of the family corresponding to a sequence $z_n \to 0$ must be the constant function 0. Set $\|\eta\| = \max\{|\eta_1|, \cdots, |\eta_k|\}$ and

(vi) Fix $0 < s < s'$. Set $b_r := \sup\{|\widehat{i}(\eta, y)| \mid y \in X \cap D(0, r),\ \|\eta\| \leq s\}$. We have $b_r \to 0$ as $r \to 0$.

Fix r_0 such that $s > b_r$ for $0 < r < r_0$.

(vii) Fix $r \in]0, r_0[$, set $R_r = s/b_r$. We have $R_r > 1$, and $R_r \to \infty$ as $r \to 0$.

For every $\mu \in \Delta(0, R_r) \subset \mathbb{C}$, $\eta_\# \in \Delta_\#(0, s)$ and $y \in X \cap D(0, r)$, we claim that there is a unique solution $\eta_k(\mu, \eta_\#, y)$ in the disc $\Delta(0, \min\{s, \frac{s}{|\mu|}\})$ of the equation

$$\eta_k = \widehat{i}((\eta_\#, \mu\eta_k), y) \ . \tag{2}$$

To show this, set $F(\eta_k) = \eta_k$, and $G(\eta_k) = \eta_k - \widehat{i}((\eta_\#, \mu\eta_k), y)$. The map F has a unique zero in the disc $|\eta_k| < \min\{s, \frac{s}{|\mu|}\}$, which is 0. On the other hand,

$$\begin{aligned} |F - G|_{|\eta_k| = \min\{s, s/|\mu|\}} &= \left| \widehat{i}((\eta_\#, \mu\eta_k), y) \right|_{|\eta_k| = \min\{s, s/|\mu|\}} \\ &\leq b_r = s/R_r \\ &< \min\left\{s, \tfrac{s}{|\mu|}\right\} = |F|_{|\eta_k| = \min\{s, s/|\mu|\}} , \end{aligned}$$

where the first inequality is due to (vi) and the second inequality is due to (vii). So by Rouché's theorem G has a unique zero in the disc $|\eta_k| < \min\{s, \frac{s}{|\mu|}\}$.

Set $Y^r = X \cap D(0,r)$. Define $j^r : \{|\mu| < R_r\} \times \Delta_\#(0,s) \times Y^r \to \overline{\mathbb{C}}$ by $j^r(\mu, \eta_\#, y) = \eta_k(\mu, \eta_\#, y)$.

• Check that j^r is a holomorphic motion:

$j^r(0,0,y)$ is the solution of $\eta_k = \widehat{i}(0,y)$. But $\widehat{i}(0,y) = y$ (by (i) of Definition 6), so $j^r(0,0,y) = y$.

Assume $j^r(\mu, \eta_\#, y) = j^r(\mu, \eta_\#, y') = \eta_k$. By (2) , $\eta_k = \widehat{i}((\eta_\#, \mu\eta_k), y) = \widehat{i}((\eta_\#, \mu\eta_k), y')$. Therefore $y = y'$ (by (ii) of Definition 6). So $j^r(\mu, \eta_\#, \cdot)$ is injective.

Now we show that $j^r(\cdot,\cdot,y) : \Delta(0,R_r) \times \Delta_\#(0,s) \to \overline{\mathbb{C}}$ is holomorphic. The value $j^r(\mu, \eta_\#, y)$ is the solution of $\eta_k - \widehat{i}((\eta_\#, \mu\eta_k), y) = 0$. The left hand side is a holomorphic function of $\mu, \eta_\#, \eta_k$ (by (2)). So the solution $\eta_k(\mu, \eta_\#, y)$ is holomorphic in $\mu, \eta_\#$ (by Weierstrass' preparation theorem).

• Set $Z^r = \Delta_\#(0,s) \times Y^r$. Define $\widehat{j}^r : Z^r \to \mathbb{C}^k$ by $(\eta_\#, y) \mapsto (\eta_\#, j^r(1, \eta_\#, y))$. Due to the fact that j^r is a holomorphic motion we know that $\widehat{j}^r$ is a homeomorphism from Z^r onto its image. We check now $\widehat{j}^r(Z^r) \subset M_{\widehat{v}}$. Since $\eta_k = j^r(1, \eta_\#, y)$ is the solution of $\eta_k - \widehat{i}((\eta_\#, \eta_k), y) = 0$ where $y \in X \cap D(0,r)$, we have $\eta_k \in \widehat{i}((\eta_\#, \eta_k), X)$. So $(\eta_\#, \eta_k) \in M_{\widehat{v}}$.

Clearly $M_v = H(M_{\widehat{v}})$. So M_v contains $H \circ \widehat{j}(Z^r)$.

Step 2, proof of b) . Set $Y_1 = j^r(1,0,Y^r)$ and for $0 < t \le s$, $Z^r_t = \Delta_\#(0,t) \times Y^r$.

Then due to the fact that H is bi-Lipschitz, we have

$$\text{H-dim}(M_v) = \text{H-dim}(M_{\widehat{v}}) \ge \text{H-dim}(\widehat{j}^r(Z^r)) \ge \text{H-dim}(\widehat{j}^r(Z^r_t)) \text{ for } 0 < t < s \ .$$

Now we claim that

$$\text{H-dim}(\widehat{j}^r(Z^r_t)) \ge C(t) \cdot \text{H-dim}(\Delta_\#(0,t) \times Y_1) = C(t) \cdot (2(k-1) + \text{H-dim}(Y_1)) \tag{3}$$

$$\text{and} \quad \text{H-dim}(Y_1) \ge C'(r) \cdot \text{H-dim}(Y^r) \ , \tag{4}$$

$$\text{with} \quad C(t) = \left(\frac{s-t}{s+t}\right)^{k-1} \quad \text{and} \quad C'(r) = \frac{R_r - 1}{R_r + 1} \ .$$

Letting $t \to 0$ and $r \to 0$ (so that $R_r \to \infty$ by (vii)), we get

$$\text{H-dim}(M_v) \ge 2(k-1) + \lim_{r \to 0} \text{H-dim}(X \cap D(0,r)) \ . \tag{5}$$

To prove inequalities in (3) and (4) , we apply Słodkowski's theorem (see for example [2] and [3]). Note at first that $\xi : \Delta_\#(0,s) \times Y_1 \to \overline{\mathbb{C}}$, $(\eta_\#, j^r(1,0,y)) \mapsto j^r(1, \eta_\#, y)$ is also a holomorphic motion. So by Słodkowski's theorem the map $Y_1 \to \overline{\mathbb{C}}$, $j^r(1,0,y)) \mapsto j^r(1,\eta_\#,y)$ extends to a $K(\eta_\#)$-quasi-conformal mapping with

$$K(\eta_\#) = \left(\frac{s+\|\eta_\#\|}{s-\|\eta_\#\|}\right)^{k-1} .$$

In particular it is a $1/C(t)$-quasi-conformal mapping for each $\eta_\# \in \Delta_\#(0,t)$. It is then $C(t)$-bi-Hölder continuous for each $\eta_\# \in \Delta_\#(0,t)$ by Mori's inequality for quasi-conformal mappings.

One can then check easily that the bijection

$$\widehat{\xi} : \Delta_\#(0,t) \times Y_1 \to \widehat{j}^r(Z_t^r) \ , \ (\eta_\#, j^r(1,0,y)) \mapsto (\eta_\#, j^r(1,\eta_\#,y))$$

is also $C(t)$-bi-Hölder continuous. Therefore by classical results on Hausdorff dimensions

$$\text{H-dim}(\widehat{j}^r(Z_t^r)) \geq C(t) \cdot \text{H-dim}(\Delta_\#(0,t) \times Y_1) \ .$$

We get the first inequality in (3) . The second inequality in (3) is due to the dimension formula for products (see Step 0).

To prove (4) , we just argue as above and remark that the mapping $Y^r \to Y_1$, $y \mapsto j^r(1,0,y)$ extends to a $1/C'(r)$-quasi-conformal mapping, therefore a $C'(r)$-bi-Hölder continuous mapping.

Assume now dv_1 is degenerate at 0. Assume at first that there is a sequence $x_n \in X$, $x_n \neq z_0$ with $\lim_{n\to\infty} x_n = z_0$. We are going to find a sequence $t_n \in M_v$ with $t_n \to 0$ such that $v(\lambda)$ and $i(\lambda, x_n)$ meet at t_n transversally.

We may assume that $z_0 = 0$ and $i(\lambda, 0) \equiv 0$. So $v(\lambda) = v_1(\lambda)$. Since $v(\lambda) \not\equiv 0$, there is a one-dimensional slice T with coordinate t such that for $t \in T$ we have $v(t) = t^d h(t)$, and $h(0) \neq 0$. Let $\Omega \subset T$ be a neighborhood of 0 compactly contained in the domain of definition of the holomorphic motion. Denote by $x_n(t)$ the leaf of the holomorphic motion passing x_n. We have $x_n(t) \neq 0$ for all $t \in \Omega$ and $n > 0$, and $\lim_{n\to\infty} \sup_{t\in\Omega} |x_n(t)| = 0$ (see (vi)). By Rouché's theorem, for n large enough, there is $t \in \Omega$ such that $v(t) = x_n(t)$. The following two lemmas will show that for any n large enough and t_n a point such that $v(t_n) = x_n(t_n)$, we have $v'(t_n) \neq x_n'(t_n)$. Clearly $t_n \in M_v$ and $t_n \to 0$ as $n \to \infty$.

We get then the desired transversality and can apply the result above. This gives a) for $z' = x_n$ and

$$\text{H-dim}(M_v) \geq 2(k-1) + \lim_{r\to 0} \text{H-dim}(X_{t_n} \cap D(x_n(t_n), r)) \ . \tag{6}$$

Moreover by the above Słodkowski-Hölder argument, there is $r > r' > 0$ such that

$$\text{H-dim}(X_{t_n} \cap D(x_n(t_n), r)) \geq \frac{1-|t_n|}{1+|t_n|} \text{ H-dim}(X \cap D(x_n, r')) \ . \tag{7}$$

Finally we make use of the extra assumption that X is homogeneous, and get

$$\text{H-dim}(M_v) \geq 2(k-1) + \frac{1-|t_n|}{1+|t_n|} \text{ H-dim}(X) \ .$$

Letting $n \to \infty$ we get b). ■

Lemma 4.2 *Let* $A > 0$ *and* $g : \{|z| < R\} \to \{Rew < A\}$ *be holomorphic, with* $g(0) = 0$*. Then* $|g'(0)| \leq \frac{2}{R}A$*.*

Proof. The estimate follows from Schwarz Lemma after pre-composing and post-composing appropriate Möbius transformations. ■

Lemma 4.3 *Let* $\Omega \subset \mathbb{C}$ *be a simply connected neighborhood of* 0 *and* $v : \Omega \to \mathbb{C}$ *be holomorphic in the form* $v(t) = t^d h(t)$*, with* $|h(t)| \geq C$ *on* Ω*. Let* $x : \Omega \to \mathbb{C}$ *satisfy: for some* $\varepsilon > 0$*,* $|x(t)| \leq \varepsilon$ *and* $x(t) \neq 0$ *for all* $t \in \Omega$*. Then, if* ε *is small enough, and if* t_0 *is a point such that* $x(t_0) = v(t_0)$*, then* $x'(t_0) \neq v'(t_0)$*.*

Proof. There is a holomorphic map $g(t)$ such that $x(t) = e^{g(t)}$. Moreover $Re(g(t)) = \log|x(t)|$ and $Re(g(t_0)) = \log|x(t_0)| = \log|v(t_0)|$. Since $|x(t)| \leq \varepsilon$ we have $Re(g(t)) \leq \log\varepsilon$. Therefore there is a disque D_R centered at t_0 with R uniformly bounded from below (as $\varepsilon, t_0 \to 0$) such that $g(D_R) \subset \{Re(w - g(t_0)) \leq \log(\varepsilon/|v(t_0)|)\ \}$.

Therefore by Lemma 4.2, $|g'(t_0)| \leq \frac{2}{R}\log\varepsilon/|v(t_0)|$. Hence there is a constant C_1 independent of ε such that

$$\frac{2d}{R}\log\frac{1}{|t_0|} + C_1 \geq \frac{2}{R}\log\frac{\varepsilon}{|t_0|^d|h(t_0)|} = \frac{2}{R}\log\frac{\varepsilon}{|v(t_0)|} \geq |g'(t_0)| = \left|\frac{x'(t_0)}{x(t_0)}\right| \ .$$

On the other hand, for $t \in \Omega$, we have

$$\left|\frac{v'(t)}{v(t)}\right| = \left|\frac{d}{t} + \frac{h'(t)}{h(t)}\right| \geq \frac{d}{|t|} - \left|\frac{h'(t)}{h(t)}\right| \geq \frac{d}{|t|} - C_2$$

with C_2 a constant independent of ε. Since $1/|t|$ grows faster than $\frac{2d}{R} \cdot \log(1/|t|) + C_1$ when $|t| \to 0$, for sufficiently small ε, so that $|t_0| \approx \varepsilon^{1/d}$ is also small, we have

$$\left|\frac{v'(t_0)}{v(t_0)}\right| \ > \ \left|\frac{x'(t_0)}{x(t_0)}\right| \ .$$

Since $v(t_0) = x(t_0)$, so $|v'(t_0)| > |x'(t_0)|$. ■

4.2 Proof of Theorem 3.9.2.

Let $\mathbf{f}_\Lambda$ be an analytic family of rational-like maps with Λ an open set of $\mathbb{C}^k$. Recall that $A_0 = \mathcal{E}^u$ and $A_1 = \mathcal{E} \cap \mathcal{C}$.

Fix $i \in \{0,1\}$ and $r_0' > 0$. Let $\lambda_0 \in A_i$. Choose $\varepsilon > 0$ such that $hyp-dim(f_{\lambda_0}) > 2-\varepsilon$. We will show that H-dim$(A_i \cap \Delta(\lambda_0, r_0')) > 2k - \varepsilon$.

By Definition 3, there is a homogeneous hyperbolic set $X \subset J_{\lambda_0}$ such that H-dim$(X) > 2 - \varepsilon$.

Take $r_0 < r_0'$ such that the dynamical holomorphic motion of X given in Lemma 2.1 is defined in $\Delta(\lambda_0, r_0)$. Denote by $i : \Delta(\lambda_0, r_0) \times X \to \overline{\mathbb{C}}$ this motion. For λ close to λ_0 the mapping $i(\lambda, \cdot)$ does not change too much the Hausdorff dimension (see the proof of the above proposition). Thus there is a small neighborhood $\Delta'' \subset \Delta(\lambda_0, r_0)$ of λ_0 such that

$$\text{H-dim}\,(X_\lambda \cap D(i(\lambda,x),r)) > 2-\varepsilon \text{ for all } \lambda \in \Delta'',\ x \in X \text{ and } r > 0\ . \tag{8}$$

By Lemma 3.4, $\mathcal{E}^u \cap \Lambda_0$ is dense in $\mathcal{E}^u$. In case $i = 0$, since $A_0 = \mathcal{E}^u$, we can choose $\lambda_0 \in A_0 \cap \Lambda_0$. In case $i = 1$, since $A_1 = \mathcal{E} \cap \mathcal{C} \subset \mathcal{E} \subset \mathcal{E}^u \cap \Lambda_0$ and by assumption $\lambda_0 \in A_1 \neq \emptyset$, we may choose $\lambda_0 \in A_1 \cap \Lambda_0$.

Let Δ' be a polydisc neighborhood of λ_0, with $\Delta' \subset \Delta'' \cap \Lambda_0$. Fix $z_0 \in X$.

We claim that there is $\lambda' \in \Delta'$ and an integer $N > 0$ such that at least one local critical map, say $a(\lambda)$, satisfies that $f^N_{\lambda'}(a(\lambda')) = i(\lambda', z_0)$ and $f^N_\lambda(a(\lambda)) \not\equiv i(\lambda, z_0)$. Suppose no such λ' exists, i.e. either $f^N_\lambda(c) \neq i(\lambda, z_0)$ for all $\lambda \in \Delta''$, all integers N and all critical points c of f_λ, or the equality occurs for some λ', c' and N, and there is a unique parametrization $a(\lambda)$ of critical points of f_λ such that $a(\lambda') = c'$, moreover $a(\lambda) \equiv i(\lambda, z_0)$. One can then pull-back the holomorphic leaf $i(\lambda, z_0)$ to create a holomorphic motion of the set $X' = \bigcup_n f^{-n}_{\lambda_0}(z_0)$ ([9], Lemma III.2), and then, since X' is dense in J_{λ_0} (see remark in Definition 1), extend it to a holomorphic motion of J_{λ_0} (by λ-lemma of [9]). This contradicts the assumption that λ_0 is an J-unstable value. (Note that in Case 1 of Definition 4 for any $a_i(\lambda)$ not normal at λ_0 there are N, λ' with the same property.)

Choose one such $a(\lambda)$. So $a(\lambda) \not\equiv i(\lambda, z_0)$. Set $v(\lambda) = f^N_\lambda(a(\lambda))$ for $\lambda \in \Delta'$. We have

$$v(\lambda') = i(\lambda', z_0),\ \text{and}\ v(\lambda) \not\equiv i(\lambda, z_0)\ . \tag{9}$$

Now we check that $M_v := \{\lambda \in \Delta' \mid v(\lambda) \in X_\lambda\} \subset A_i$. For any $\lambda'' \in M_v$, we have $v(\lambda'') = i(\lambda'', z')$ for some $z' \in X$ and $v(\lambda) \not\equiv i(\lambda, z')$. By Lemma 3.8, the critical map $a(\lambda)$ generates a non-normal family at λ''. Therefore $\lambda'' \in \mathcal{E}^u$ by Theorem 3.5. So $M_v \subset \mathcal{E}^u = A_0$.

The case $i = 0$ is again straight forward.

Assume now $i = 1$. Denote by $a_1(\lambda), \cdots, a_m(\lambda)$ the local critical maps such that $a(\lambda) = a_1(\lambda)$. Since $\lambda_0 \in A_1 \subset \mathcal{E}$, we may decrease r_0 if necessary such that for all $j > 1$ the map $a_j(\lambda)$ generates a normal family in $\Delta(\lambda_0, r_0)$, i.e. $\mathcal{E}^u \cap \Delta(\lambda_0, r_0) \subset \mathcal{E}$. Since $M_v \subset \Delta(\lambda_0, r_0) \cap \mathcal{E}^u$, we have $M_v \subset \mathcal{E}$. For $j > 1$, since $a_j(\lambda_0)$ does not escape, so by Definition 9 the point $a_j(\lambda)$ does not escape for all $\lambda \in \Delta'$. On the other hand, for $\lambda \in M_v \subset \Delta'$, the critical point $a_1(\lambda)$ does not escape either, since $a_1(\lambda) \in J_\lambda = \partial K_\lambda$. Therefore $M_v \subset \mathcal{C}$. As a consequence $M_v \subset \mathcal{E} \cap \mathcal{C} = A_1$.

In both cases $i = 0, 1$, we have $\text{H-dim}(A_i \cap \Delta(\lambda_0, r_0')) \geq \text{H-dim}(A_i \cap \Delta') \geq \text{H-dim}(M_v)$.

Applying Proposition 4.1 to M_v (with 0 corresponding to λ'), we obtain Theorem 3.9.2.a). Now apply Formulae (5) , (6) and (7) to M_v, with $X = X_{\lambda'}$, we get

$$\text{H-dim}(M_v) \geq 2(k-1) + \lim_{r \to 0} \text{H-dim}(X_{\lambda'} \cap D(i(\lambda', z_0), r))$$

in the transversal case, or otherwise, for some sequences $x_n \in X_{\lambda'}$, $x_n \to i(\lambda', z_0)$ and $|t_n| \to 0$ as $n \to \infty$,

$$\text{H-dim}(M_v) \geq 2(k-1) + \frac{1 - |t_n|}{1 + |t_n|} \lim_{r' \to 0} \text{H-dim}(X_{\lambda'} \cap D(x_n, r')) \quad \text{for any large } n \ .$$

Applying (8) to $X_{\lambda'}$ in both cases (recall that $\lambda' \in \Delta' \subset \Delta''$), we get

$$\text{H-dim}(A_i \cap \Delta(\lambda_0, r_0')) \geq \text{H-dim}(M_v) > 2(k-1) + \frac{1 - |t_n|}{1 + |t_n|} \cdot (2 - \varepsilon) \ .$$

Since $A_i \cap \Delta(\lambda_0, r_0')$ is independent of n and ε, and $2 - \varepsilon$ is arbitrarily close to $hyp - dim(f_{\lambda_0})$, we obtain Theorem 3.9.2.b).

Part c) of the theorem is a consequence of Theorem 1.1, Theorem 3.9.1 and Theorem 3.9.2.b). ∎

Definition. We say that λ is a Misiurewicz point if one critical point of f_λ lands eventually into a repelling periodic cycle, and this critical relation is not locally constant.

Lemma 4.4 *Assume $\mathcal{E}^u \neq \emptyset$. The set of Misiurewicz points contains a complex sub-manifold of dimension $k-1$ and forms a dense subset of $\mathcal{E}^u$.*

Proof. By Lemma 3.4, $\mathcal{E}^u \cap \Lambda_0 \neq \emptyset$. Let $\lambda_0 \in \mathcal{E}^u \cap \Lambda_0$. Let $z_0 \in J_{\lambda_0}$ be periodic repelling and not in the orbit of a critical point. Let X be the orbit of z_0. Then X is hyperbolic, forward invariant and admits a holomorphic motion X_λ. Let Δ be a neighborhood of λ_0 such that X_λ remains repelling in Δ and disjoint from critical values of f_λ.

As in the proof of Theorem 3.9.2, there is a function $v(\lambda) = f_\lambda^N(a(\lambda))$ such that M_v contains a sub-manifold of dimension $k-1$, and $M_v \subset \mathcal{E}^u$. Each point of M_v is a Misiurewicz point. As a consequence, Misiurewicz points are dense in $\mathcal{E}^u \cap \Lambda_0$, thus are dense also in Λ. ■

5 Application to polynomials

This final section proves a generalized version of Theorem 1.3. According to Branner-Hubbard ([1], Chapter I, Section 2), we parametrize the family of degree d monic centered polynomials by $(a,b) \in H \times \mathbb{C}$, where $H \subset \mathbb{C}^{d-1}$ is the hyperplane $\{a = (a_1, \cdots, a_{d-1}) \in \mathbb{C}^{d-1} \mid \sum_i a_i = 0\}$, $\{a_1, \cdots, a_{d-1}\}$ is the set of critical points of $f_{(a,b)}$ and $b = f_{(a,b)}(0)$. Let $\Lambda = \mathbb{C}^{d-1} = \{(a_1, \cdots, a_{d-2}, b), a_i \in \mathbb{C}, b \in \mathbb{C}\}$. For $\lambda \in \Lambda$, set $f_\lambda = f_{(a_1,\cdots,a_{d-1},b)}$ with $a_{d-1} = -(a_1 + \cdots + a_{d-2})$. In this case $\Lambda = \Lambda_0$. Theorem 1.3 is a particular case (the case $k=0$) of the following result.

Proposition 5.1 *For $k = 0, \cdots, d-2$, denote by $\mathcal{S}^k \subset \Lambda$ the set of degree d monic centered polynomials such that exactly k critical points (counted with multiplicity) escape to ∞. Then there are S-maps $f_{\lambda'}$ with $\lambda' \in \mathcal{E} \cap \mathcal{S}^k$ and for any $r' > 0$,* H-dim$(\partial\mathcal{S}^k \cap \Delta(\lambda', r')) = 2(d-1)$.

Proof. We start with the case $k=0$. Then $\mathcal{S}^0$ coincides with the connectedness locus, and $\partial\mathcal{S}^0$ is compact and connected (this is a consequence of $\mathcal{S}^{(0)}$ being cellular, a result of Branner-Hubbard and Lavaurs, see [1] and [6]).

There are S-maps in $\mathcal{E} \cap \partial\mathcal{S}^0$, for example there is a polynomial $f_{\lambda'}$ conformally conjugate to $g_d : z \mapsto z^{d-1}(z+c)$, with

$$c = -\frac{d-1}{d-2}(d-2)^{\frac{1}{d-1}} e^{\pi i \frac{1}{d-1}} \ , \quad \xi = -\frac{d-2}{d-1} c \ .$$

One can check easily that $g_d(\xi) = \xi$ and $g_d'(\xi) = 1$. Since the fixed point 0 is also a critical point of g_d with multiplicity $d-2$, there is a unique critical point that is attracted by ξ. So g_d is an S-map. Hence by Theorem 3.9, H-dim$(\partial\mathcal{S}^0 \cap \Delta(\lambda', r')) = 2(d-1)$ for any $r' > 0$.

As for $\mathcal{S}^k$, $k > 0$, one can construct S-maps in $\mathcal{S}^k$ by applying some quasi-conformal surgery to $g = g_{d-k}$ as follows. Take an open annulus A with smooth Jordan curves as boundary in the basin of infinity of g such that g maps one boundary curve onto the other one as a covering. Denote by D_0 the bounded closed disc of the complement of A. Put k disjoints closed discs $D_1, \cdots, D_k$ in A. Set $h|_{D_0} = g$, and define for $i > 0$, $h|_{D_i} : D_i \to \overline{A} \cup D_0$ to be any conformal mapping. Define $h_{\mathbb{C}-A\cup D_0} = \phi^{-1} \circ z^d \circ \phi$, where

$$\phi : \mathbb{C} - A \cup D_0 \to \{|z| \geq R > 1\}$$

is a conformal map. Finally define $h : \overline{A - \bigcup_i D_i} \to \phi^{-1}(\{R^d \geq |z| \geq R\})$ to be a degree d branched covering matching the boundary values of h, differentiable, and holomorphic in a neighborhood of the critical points. Then h preserves a bounded almost complex structure and is hybrid equivalent to a polynomial $f_{\lambda'} \in \mathcal{E} \cap \mathcal{S}^k$ which is also an S-map (cf. [4] for properties of hybrid equivalences).

Fix $r' > 0$. Let $\Delta \subset \Delta(\lambda', r') \subset \Lambda$ be a small neighborhood of λ' such that $\Delta \cap \mathcal{E}^u \subset \mathcal{E}$. Then $\Delta \cap \mathcal{E}^u = \Delta \cap \partial\mathcal{S}^k$. Therefore by Theorem 3.9 we have H-dim$(\Delta \cap \partial\mathcal{S}^k) = 2(d-1)$.

■

Corollary 5.2 *Given any analytic family of polynomial-like mappings* $\mathbf{f}_\Lambda$ *such that each* f_λ *has only one critical point, and that the boundary of the connectedness locus* $\mathcal{C}$ *is not empty, we have* H-dim$((\partial\mathcal{C}) \cap U) = \dim_{\mathbb{R}}(\Lambda)$ *for any open set* U *intersecting* $\partial\mathcal{C}$*. In particular for the connectedness locus* M_d *of the family* $z \mapsto z^d + c$*, and any open set* U *intersecting* ∂M_d*, we have* H-dim$((\partial M_d) \cap U) = 2$*.*

Proof. In both cases every parabolic map is an S-map since there is only one critical point, moreover such maps are dense in the unstable locus (which coincides with the boundary of the connectedness locus). Applying Theorem 3.9.2.c) to these families, we get the desired result. ■

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