# Analytic coordinates recording cubic dynamics. *

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**Abstract**

Let $\mathcal{H}$ be the central hyperbolic component of cubic polynomials (i.e. the one containing $z \mapsto z^3$). Works of Milnor show that $\mathcal{H}$ enjoys a kind of universality property. We construct an analytic coordinate on $\mathcal{H}$ recording the dynamical invariants. We use quadratic dynamics as parameter models. This coordinate has good extension properties to at least a large part of the boundary of $\mathcal{H}$. We illustrate this by proving some of these extension properties.

This paper concerns parameter spaces of rational maps viewed as dynamical systems through iteration. The parameter space of a family of rational maps has often a natural decomposition into the hyperbolic locus and the non-hyperbolic locus: a rational map is **hyperbolic** if and only if all its critical points are attracted to attracting periodic orbits. The hyperbolic locus is open (and conjecturally dense, in any reasonable family). A **hyperbolic component** is a connected component of the hyperbolic locus. Maps within the same hyperbolic component have essentially the same macroscopic dynamics.

However, this quantitative change of dynamics becomes qualitative and often drastic when one moves to the boundary of the hyperbolic component. Therefore a major research interest in this field is to understand the boundary structure of a hyperbolic component, including its topology, geometry and the bifurcations of the dynamics that occurs.

Given such a component $H$, in order to study its boundary, the first step is actually to study in detail the inner structure of $H$, to put our hand on an effective measuring of the (infinitessimal) changes of the dynamics within $H$.

One way to address this problem is to find some suitable model space $\mathcal{X}$ together with a map $\Phi : H \to \mathcal{X}$, satisfying simultaneously the following properties:

1. (coordinate) $\Phi$ is injective in $H$ and $\Phi(f)$ depends analytically on $f$;

2. (dynamics) $\Phi(f)$ records a complete set of dynamical invariants of $f$. In other words from these invariants (together with some combinatorial data) one can reconstruct $f$ up to conformal conjugacy.

Once we have such a coordinate, we may use $\Phi(H)$ to represent $H$ and use $\partial\Phi(H)$ to study $\partial H$. This will depend on boundary extension properties of $\Phi$. Thus the next step will be to study how well $\Phi$ or $\Phi^{-1}$ extends to the boundary and to which extent it still reflects the dynamical invariants.

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We will call $\Phi$ a *dynamical-analytic* coordinate on $H$.

Douady and Hubbard have already constructed such coordinates for the hyperbolic components of the family of quadratic polynomials. Furthermore their construction generalizes easily to other families with only one critical point or singular value.

Concerning hyperbolic components with two or more critical points, there is a remarkable pioneering work by Mary Rees on the family of quadratic rational maps with marked critical points ([Re], 1990). Rees provides topological-dynamical (non-analytic) coordinates for the hyperbolic components of the family, together with some radial extensions to the boundary.

The main purpose of this paper is to construct a dynamical-analytic coordinate on the principal hyperbolic component of the family of cubic polynomials, and then prove that this coordinate extends continuously to a large part of the boundary and records the bifurcations of the dynamics. Once this is done, we would obtain automatically a similar coordinate for many other hyperbolic components, via a previous work of Milnor (however the boundary extensions may vary from one another, and may require extra studies).

Our coordinate is similar in spirit to that of Mary Rees. However instead of using Blaschke products we will use quadratic polynomial dynamics as part of dynamical invariants (similar ideas can be found in [GK], [M3]). There are several advantages of doing so. On one hand, the quadratic polynomials are well-understood and are more convenient to deal with. On the other hand they lead to the analyticity of our coordinate, thus reflect better the analytic structure of the parameter space. This analyticity is also absolutely essential in our study of the boundary, as it will allow us to do holomorphic motions between parameter slices. In particular it will allow us to extend a powerful result of Faught-Roesch on a single slice to all the other slices. Moreover we use parts of dynamical planes of quadratics to model cubic parameter slices. In the case of quadratic polynomials with an indifferent fixed point there are still many unsolved questions. Any future progress on the study of such quadratic polynomials will contribute to our further understanding of cubic polynomials.

## Cubic parameter space and quadratic models

Let $\mathcal{M}^3$ denote the space of affine conjugacy classes $[P]$ of cubic polynomials, and let $\mathcal{H}$ denote the central hyperbolic component of $\mathcal{M}^3$, i.e. the component containing the class $[z \mapsto z^3]$. This $\mathcal{H}$ can also be characterized as the set of $[P]$ of cubic polynomials such that $P$ has an attracting fixed point in $\mathbb{C}$, whose immediate basin contains two critical points, or equivalently as the set of $[P]$ of hyperbolic cubic polynomials with a Jordan curve Julia set.

Denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$.

For $(\lambda, a) \in \mathbb{C}^2$, define $P_{\lambda,a}(z) = \lambda z + \sqrt{a} z^2 + z^3$. Then the map $(\lambda, a) \mapsto [P_{\lambda,a}]$ is well defined from $\mathbb{C}^2$ onto $\mathcal{M}^3$ (as the two choices of $\sqrt{a}$ will lead to conjugate polynomials), but it is not globally injective (as one may normalize different fixed points to be at 0). However, if we assume $\lambda \in \mathbb{D}$, then any $P$ in $[P_{\lambda,a}]$ will have an attracting fixed point with multiplier $\lambda$. It follows then easily that we may parametrize $\mathcal{H}$ as

$$\mathcal{H} = \{\mathbf{a} = (\lambda, a) \mid \lambda \in \mathbb{D},\ \text{the immediate basin of 0 for } P_{\mathbf{a}} \text{ contains two critical points}\}\ ,$$

and this makes $\mathcal{H}$ a connected and bounded open subset of $\mathbb{C}^2$. Define also for each $\lambda \in \mathbb{D}$ the slice $\mathcal{H}_\lambda := \{(\lambda, a) \in \mathcal{H}\}$.

Our first main result in this paper will be:

**Theorem A** (coordinatization). *There is a complex manifold $\mathcal{X}$ isomorphic to $\mathbb{D} \times \overline{\mathbb{C}}$ and a dynamical-analytic coordinate $\Phi$ of $\mathcal{H}$ mapping $\mathcal{H}$ onto an open subset $\mathcal{Y}$ of $\mathcal{X}$.*

A more precise statement will be give in Theorem A' below.

The definition of $\Phi$ is in fact quite easy to describe. It is based on the idea that a cubic polynomial $P_{\mathbf{a}} = P_{\lambda,a} : z \mapsto \lambda z + \sqrt{a} z^2 + z^3$ should behave like the quadratic polynomial $Q_\lambda : z \mapsto \lambda z + z^2$ together with an extra critical point. More precisely we will produce a semi-conjugacy $\eta_{\mathbf{a}}$ from $P_{\mathbf{a}}$ to $Q_\lambda$ whose domain of definition contains both critical points, and whose image of the 'first' attracted critical point is the critical point of $Q_\lambda$. Now the 'second' critical point $c^1_{\mathbf{a}}$ will have some position under $\eta_{\mathbf{a}}$. Our map $\Phi(\mathbf{a})$ will be roughly $(\lambda, \eta_{\mathbf{a}}(c^1_{\mathbf{a}})) \in \mathbb{D} \times \overline{\mathbb{C}}$.

But the notation of 'first' and 'second' attracted critical point is not always well defined. It is to rule out this ambiguity that we will have to perform a kind of cut and zip surgery on $\mathbb{D} \times \overline{\mathbb{C}}$, and to obtain a quotient space $\mathcal{X}$ for $\Phi(\mathcal{H})$ to live in.

There is a close relation between our coordinate $\Phi$ and Douady-Hubbard theory on the quadratic family $\{f_c : z \mapsto z^2 + c\}$:

The unique unbounded hyperbolic component $H_\infty$ of this family is characterized as the set of $c$ such that the critical value escapes to $\infty$ under the iterations of $f_c$. Douady and Hubbard defined a conformal isomorphism $\varphi : H_\infty \to \mathbb{C} \setminus \overline{\mathbb{D}}$, with $\varphi(c)$ recording the Böttcher position of the escaping critical value. It turns out that this $\varphi$ is a dynamical-analytic coordinate of $H_\infty$.

Now a parameter $c \notin H_\infty$ is in the hyperbolic locus if $f_c$ has an attracting periodic orbit $z_0, \ldots, z_{n-1}$. For each hyperbolic component $H \neq H_\infty$, Douady and Hubbard proved that the multiplier map $\lambda_H : H \to \mathbb{D}; \quad c \mapsto (f_c^n)'(z_0) = \prod_{i=0}^{n-1} f_c'(z_i)$ is a dynamical-analytic coordinate on $H$.

In a way the first coordinate in our map $\Phi : \mathcal{H} \to \mathcal{X}$ is the Douady-Hubbard multiplier map and the second is an adapted version of the Douady-Hubbard $\varphi$-map.

## The boundary extensions

The Douady-Hubbard multiplier map $\lambda_H$ has a homeomorphic extension to the closure, and, conjecturally, the inverse of the $\varphi$-map has a continuous extension to the closure. Many partial results have been obtained in this direction.

In our setting denote by $\overline{\mathcal{Y}}$ the closure of $\mathcal{Y}$ in $\mathcal{X}$. We will split the boundary $\partial\mathcal{H} \subset \mathbb{C}^2$ into two parts. The *tame* part denoted $\partial\mathcal{H}_{\mathbb{D}} := \{(\lambda, a) \in \partial\mathcal{H} | \lambda \in \mathbb{D}\}$, and the *wild* part $\partial\mathcal{H}_{\mathbb{S}^1} := \{(\lambda, a) \in \partial\mathcal{H} || \lambda| = 1\}$. Set $\overline{\mathcal{H}}_{\mathbb{D}} := \mathcal{H} \cup \partial\mathcal{H}_{\mathbb{D}}$.

Our second main result in this paper will be:

**Theorem B** (boundary extension) *The map $\Phi : \mathcal{H} \to \mathcal{Y}$ given by Theorem A extends as a homeomorphism $\Phi : \overline{\mathcal{H}}_{\mathbb{D}} \longrightarrow \overline{\mathcal{Y}}$.*

In forth coming papers we will study the wild part of the boundary of $\mathcal{H}$, using (often highly non-trivial) results on its quadratic counter parts. See Section A for a more detailed description.

## Outline of the paper

In Section 1 we will give precise definitions and a restatement of Theorem A. The theorem is proved in the following three sections. In Section 6 we prove Theorem B.

In the appendix we will translate Theorem A to the settings of monic centered polynomials (Theorem C). We then recall results of Milnor and use them to carry our coordinate to other hyperbolic components of the same type (Corollary 17).

At the end of the paper we will provide a table of notations.

# 1 Basic definitions

Let $f : W \longrightarrow \mathbb{C}$ be holomorphic and suppose $f(\alpha) = \alpha \in W$ is an attracting fixed point with multiplier $f'(\alpha) = \lambda \in \mathbb{D}$. Define the attracting basin $B(\alpha) = \{z \in W | f^n(z) \underset{n\to\infty}{\longrightarrow} \alpha\}$. We say that $B(\alpha)$ is a proper basin, if the restriction $f : B(\alpha) \longrightarrow B(\alpha)$ is a proper map. Similar definitions apply when $\alpha$ belongs to an attracting cycle, i.e. $f^k(\alpha) = \alpha \in W$ for some $k \geq 1$.

Attracting basins and attracting cycles come in two flavors: $\lambda \in \mathbb{D}^*$ called *attracting* and $\lambda = 0$ called *super attracting*. If $\alpha$ is super attracting then

$$f(z) = \alpha + a_k(z - \alpha)^k + \mathcal{O}(z - \alpha)^{k+1},$$

with $a_k \neq 0$, where $k > 0$ is the local degree at $\alpha$. Moreover there exists a Böttcher coordinate around $\alpha$, that is a univalent map $\phi : V \longrightarrow \mathbb{C}$, $V \subseteq B(\alpha)$, with $\phi(\alpha) = 0$ and $\phi \circ f = (\phi)^k$. The Böttcher coordinate is unique up to multiplication by a $(k-1)$-th root of unity. In the particular case of $k = 2$ (the case we are considering in this paper), $\phi$ is unique. If $\alpha$ is attracting there exists a Schröder/linearizing coordinate or in short a linearizer for $f$. It is a univalent map $\phi : V \longrightarrow \mathbb{C}$, $V \subseteq B(\alpha)$ with $\phi \circ f = \lambda\phi$. A linearizer extends to a holomorphic, locally finite branched covering $\phi : B'(\alpha) \longrightarrow \mathbb{C}$ (where $B'(\alpha)$ is the connected component of $B(\alpha)$ containing $\alpha$). A linearizer on $B'(\alpha)$ is unique up to multiplication by a non-zero complex number.

Please refer the upper figure of Figure 1, as well as Figure 2 for the following definitions.

**Definition of $U^0$ and $L^0$ in the case $\lambda \in \mathbb{D}^*$.** There is a unique domain $U^0 \subset B(\alpha)$ containing $\alpha$ and characterized by being the largest set which is mapped univalently by a linearizer $\phi$ onto a round disc centered at 0. Define $L^0$ as the connected component of $f^{-1}(\overline{f(U^0)})$ containing $\alpha$. It is a pinched closed disk. The sets $U^0$, $L^0$ are evidently independent of the choice of $\phi$. For a choice of $\phi$ denote by $\psi$ the local inverse of $\phi$, defined on $\phi(U^0)$ and mapping 0 to $\alpha$.

**Definition of $c^0$, $v^0$ and normalization of the linearizer**. In case $\lambda = 0$, $\alpha$ is also a critical point. Set $c^0 = \alpha$. In case $\lambda \in \mathbb{D}^*$, $\partial U^0$ contains at least one critical point. We choose one of them and name it by $c^0$, it is a *first attracted critical point*. Let $v^0 = f(c^0)$ be the corresponding critical value. It is contained in $U^0$. Unless explicitly stated otherwise we shall assume $\phi$ is normalized so that $\phi(c^0) = 1$ and hence $U^0 = \psi(\mathbb{D})$ and $v^0 = \psi(\lambda)$.

**Definition of the (filled) potential function $\kappa$ in case $\lambda \in \mathbb{D}^*$.** The Schröder **potential function** is the sub-harmonic function $\widehat{\kappa} : B'(\alpha) \longrightarrow [-\infty, \infty[$ defined by

$$\widehat{\kappa}(z) = \frac{\log|\phi(z)|}{\log\frac{1}{|\lambda|}} .$$

It satisfies $\widehat{\kappa}(f(z)) = \widehat{\kappa}(z) - 1$ and $\widehat{\kappa}(\alpha) = -\infty$ as well as $\widehat{\kappa}(z) = -\infty$ for any preimage $z$ of $\alpha$. For $t \in \mathbb{R}$ we let $U(t)$ denote the connected component of $\widehat{\kappa}^{-1}([-\infty, t[)$ containing $\alpha$ and similarly $L(t)$ denote the connected component of $\widehat{\kappa}^{-1}([-\infty, t])$ containing $\alpha$ or equivalently $L(t) = \cap_{s>t} U(s)$. Each $U(t)$ is a Jordan domain, contained in the compact subset $L(t) \subset B'(\alpha)$. We have $\overline{U(t)} = L(t)$ if and only if $\partial U(t)$ does not contain a (pre)critical point for $f$. Note that $U(0)$ (resp. $L(0)$) coincides with $U^0$ (resp. $L^0$) previously defined, due to our normalization of $\phi$. The **Milnor filled potential function** $\kappa : B'(\alpha) \longrightarrow [-\infty, \infty[$ is defined by

$$\kappa(z) = \inf\{s | z \in U(s)\}.$$

It has the virtue $\kappa^{-1}([-\infty, t[) = U(t) = f(U(t+1))$, $\kappa^{-1}([-\infty, t]) = L(t) = f(L(t+1))$, $\kappa^{-1}(t) = L(t) \setminus U(t)$. It is sub-harmonic with $\alpha$ as its sole pole.

Consider the family of quadratic polynomials $Q_\lambda(z) = \lambda z + z^2$ and the family of cubic polynomials $P_{\mathbf{a}}(z) = P_{\lambda,a}(z) = \lambda z + \sqrt{a} z^2 + z^3$. For $\lambda \in \mathbb{D}$ the point $\alpha = 0$ is a (super)attracting fixed point for both $Q_\lambda$ and $P_{\mathbf{a}}$. Denote by $B_\lambda$ (resp. $B_{\mathbf{a}}$) the attracting basin of 0 for $Q_\lambda$ (resp. for $P_{\mathbf{a}}$). Objects (such as $\phi$, $\psi$, $U^0$, $c^0$, $v^0$ etc) with a subscript $\lambda$ will be those related to $Q_\lambda$, and those with a subscript $\mathbf{a}$ will be related to $P_{\mathbf{a}}$. However we omit the superscript 0 on $c^0_\lambda$ and $v^0_\lambda$ since they are unique, in particular $c_\lambda = -\frac{\lambda}{2}$ and $\phi_\lambda(c_\lambda) = 1$. For the special parameter $\lambda = 0$ we define $\phi_0 = \psi_0 = \mathrm{id}$. Recall that

$$\mathcal{H} = \{\mathbf{a} \mid B'_{\mathbf{a}} = B_{\mathbf{a}} \text{ contains both finite critical points of } P_{\mathbf{a}}\} .$$

Set $\mathcal{H}^* = \mathcal{H} \setminus \{\mathbf{0}\}$ and $\mathcal{H}^*_0 = \mathcal{H}_0 \setminus \{\mathbf{0}\}$.

**Definition of $co^0_{\mathbf{a}}$ and $c^1_{\mathbf{a}}$.** For $\mathbf{a} \in \mathcal{H}^*$ the critical point labelled $c^0_{\mathbf{a}}$ is the choice (generically the unique choice) of a first attracted critical point. The other critical point is labelled $c^1_{\mathbf{a}}$. We denote by $co^0_{\mathbf{a}}$, *the co-critical point* of $c^0_{\mathbf{a}}$, that is the unique point with $P_{\mathbf{a}}(co^0_{\mathbf{a}}) = P_{\mathbf{a}}(c^0_{\mathbf{a}})$ for which $co^0_{\mathbf{a}} \neq c^0_{\mathbf{a}}$ if and only if $c^0_{\mathbf{a}} \neq c^1_{\mathbf{a}}$ and $co^0_{\mathbf{a}} = c^0_{\mathbf{a}}$ if and only if $c^0_{\mathbf{a}} = c^1_{\mathbf{a}}$ is a double critical point.

**Definition of $t^1_{\mathbf{a}}$, $U_{\mathbf{a}}$ and $L_{\mathbf{a}}$.** Assume at first $\lambda \neq 0$. For $\mathbf{a} \in \mathcal{H} \setminus \mathcal{H}_0$ define $t^1_{\mathbf{a}} = \kappa_{\mathbf{a}}(c^1_{\mathbf{a}})$, $U_{\mathbf{a}} = U_{\mathbf{a}}(t^1_{\mathbf{a}})$ and $L_{\mathbf{a}} = L_{\mathbf{a}}(t^1_{\mathbf{a}})$. Note that $c^1_{\mathbf{a}} \in L^0_{\mathbf{a}}$ iff $t^1_{\mathbf{a}} = 0$. Assume now $\lambda = 0$. For $\mathbf{a} \in \mathcal{H}^*_0$ and $\phi_{\mathbf{a}}$ the unique Böttcher coordinate, define $t^1_{\mathbf{a}} = \log|\phi_{\mathbf{a}}(c^1_{\mathbf{a}})|$, $U_{\mathbf{a}} \subset B_{\mathbf{a}}$ to be the largest set mapped univalently by $\phi_{\mathbf{a}}$ to a round disc, and $L_{\mathbf{a}} = \overline{U_{\mathbf{a}}}$. Necessarily $\phi_{\mathbf{a}}(U_{\mathbf{a}}) = D(0, e^{t^1_{\mathbf{a}}})$. For $\mathbf{a} = \mathbf{0}$, set $t^1_{\mathbf{a}} = -\infty$, $U_{\mathbf{a}} = \emptyset$ and $L_{\mathbf{a}} = \{0\}$. Define $\psi_0 = id$ and $L_0(t) = \overline{\mathbb{D}(0, e^t)}$.

For $\mathbf{a} \in \mathcal{H}^*_0$ define $\Omega_{\mathbf{a}} = \overline{U_{\mathbf{a}}}$, then $\eta_{\mathbf{a}} = \psi_0 \circ \phi_{\mathbf{a}} = \phi_{\mathbf{a}} : \Omega_{\mathbf{a}} \longrightarrow L_0(t^1_{\mathbf{a}})$ is a homeomorphic conjugacy, which extends to a holomorphic semi-conjugacy on a neighborhood of $\Omega_{\mathbf{a}}$. The following Proposition extends this idea to the case $\lambda \in \mathbb{D}^*$.

**Proposition 1** *For $\mathbf{a} \in \mathcal{H} \setminus \mathcal{H}_0$, the local conjugacy $\eta_\mathbf{a} := \psi_\lambda \circ \phi_\mathbf{a}$ from $P_\mathbf{a}$ to $Q_\lambda$ extends to a continuous surjective semi-conjugacy $\eta_\mathbf{a} : \Omega_\mathbf{a} \longrightarrow L_\lambda(t^1_\mathbf{a})$, where $\Omega_\mathbf{a}$ is a closed, connected and simply connected set with $U_\mathbf{a} \cup \{c^1_\mathbf{a}\} \subset \Omega_\mathbf{a} \subset L_\mathbf{a}$. Moreover $\eta_\mathbf{a}$ extends holomorphically to a neighborhood of $\Omega_\mathbf{a} \setminus \{co^0_\mathbf{a}\}$ and $\kappa_\lambda(\eta_\mathbf{a}(c^1_\mathbf{a})) = t^1_\mathbf{a} = \kappa_\mathbf{a}(c^1_\mathbf{a})$.*

*The map $\mathbf{a} \mapsto \eta_\mathbf{a}(c^1_\mathbf{a})$ takes values in $B_\lambda \setminus U^0_\lambda$. It is single-valued if $c^1_\mathbf{a} \notin \partial U^0_\mathbf{a}$, and double-valued if $c^1_\mathbf{a} \in \partial U^0_\mathbf{a}$ and $\phi_\mathbf{a}(c^1_\mathbf{a}) \neq \pm 1$ (due to the two ways of labeling $c^0_\mathbf{a}$ and $c^1_\mathbf{a}$). In the later case its two values $z_0, z_1$ belong to $\partial U^0_\lambda$ and satisfy $\phi_\lambda(z_0) \cdot \phi_\lambda(z_1) = 1$.*

This proposition will be proved in the next section, where more details on $\Omega_\mathbf{a}$ is also provided.

We want to use $\mathbf{a} \mapsto \eta_\mathbf{a}(c^1_\mathbf{a})$ to parametrize $\mathcal{H}$. In order to turn this multi-valued function into a single valued one, we define an equivalence relation $\sim_\lambda$ on $\overline{\mathbb{C}} \setminus U^0_\lambda$ (see Figure 1) by:

$$\begin{array}{ll} \lambda \in \mathbb{D}^* , & z_1 \sim_\lambda z_2 \Leftrightarrow [z_1 = z_2] \text{ or } [(z_1, z_2 \in \partial U^0_\lambda) \text{ and } (\phi_\lambda(z_1) = \overline{\phi_\lambda(z_2)})] ; \\ \lambda = 0 , & \sim_\lambda \text{ is trivial.} \end{array}$$

Note that $\sim_\lambda$ can be given an intrinsic meaning, depending only on $\partial U^0_\lambda$ (but not on the normalization of $\phi_\lambda$).

(Incidentally, $\sim_\lambda$ can be also thought of as an equivalence relation on $\overline{\mathbb{C}}$, where $z_1 \sim_\lambda z_2 \Leftrightarrow [z_1 = z_2]$ or $[(z_1, z_2 \in \overline{U^0_\lambda})$ and $Re(\phi_\lambda(z_1)) = Re(\phi_\lambda(z_2))]$, thus eliminating the need for the excision of $U^0_\lambda$.)

Define $\mathcal{U}^0 = \{(\lambda, z) | \lambda \in \mathbb{D}^* \text{ and } z \in U^0_\lambda\}$ and an equivalence relation on $\mathbb{D} \times \overline{\mathbb{C}} \setminus \mathcal{U}^0$ as follows

$$(\lambda_1, z_1) \sim (\lambda_2, z_2) \Leftrightarrow (\lambda_1 = \lambda_2) \text{ and } (z_1 \sim_{\lambda_1} z_2) .$$

Let $\Pi : \mathbb{D} \times \overline{\mathbb{C}} \setminus \mathcal{U}^0 \longrightarrow (\mathbb{D} \times \overline{\mathbb{C}} \setminus \mathcal{U}^0)/\sim$ denote the natural projection, and let $\pi_\lambda$ denote its restriction to the $\lambda$-slice.

**Definition 2**

$$\begin{array}{rcl} \mathcal{X} & = & \Pi(\mathbb{D} \times \overline{\mathbb{C}} \setminus \mathcal{U}^0) \quad \textit{the cut-glued space}, \\ \mathcal{Y} & = & \Pi(\mathcal{B} \setminus \mathcal{U}^0) \quad \textit{the cut-glued basin}, \\ \Gamma & = & \Pi(\partial \mathcal{U}^0) \quad \textit{the scar.} \end{array}$$

*Denote by $\mathcal{X}_\lambda := \Pi(\{\lambda\} \times \overline{\mathbb{C}} \setminus U_\lambda)$, $\mathcal{Y}_\lambda := \Pi(\{\lambda\} \times B_\lambda \setminus U_\lambda)$ and $\Gamma_\lambda := \Pi(\{\lambda\} \times \partial U_\lambda)$, $\lambda \neq 0$ and $\Gamma_0 := \Pi(0,0)$ the corresponding slices related to $\{\lambda\} \times \overline{\mathbb{C}}$. Set $\mathcal{X}^* := \mathcal{X} \setminus \Pi(\{\mathbf{0}\})$, $\mathcal{Y}^* = \mathcal{Y} \setminus \Pi(\{\mathbf{0}\})$.*

These spaces comes naturally with the (Hausdorff) quotient topology. The space $\mathcal{X}$ (and $\mathcal{Y} \subset \mathcal{X}$) has a canonical complex structure off $\Gamma$ for which $\Pi$ is holomorphic.

We may now restate Theorem A:

**Theorem A'** (modelization).
a) *The space $\mathcal{X}$ equipped with the quotient topology is Hausdorff, $\Pi(\mathbf{0}) \in \mathcal{Y}$ and $\mathcal{Y}$ is open in $\mathcal{X}$.*
b1) *$\mathcal{X}$ is a 2-dimensional complex manifold.*
b2) *$\mathcal{X}$ is isomorphic to $\mathbb{D} \times \overline{\mathbb{C}}$, with the isomorphism commuting with the projection to the $\lambda$ coordinate.*

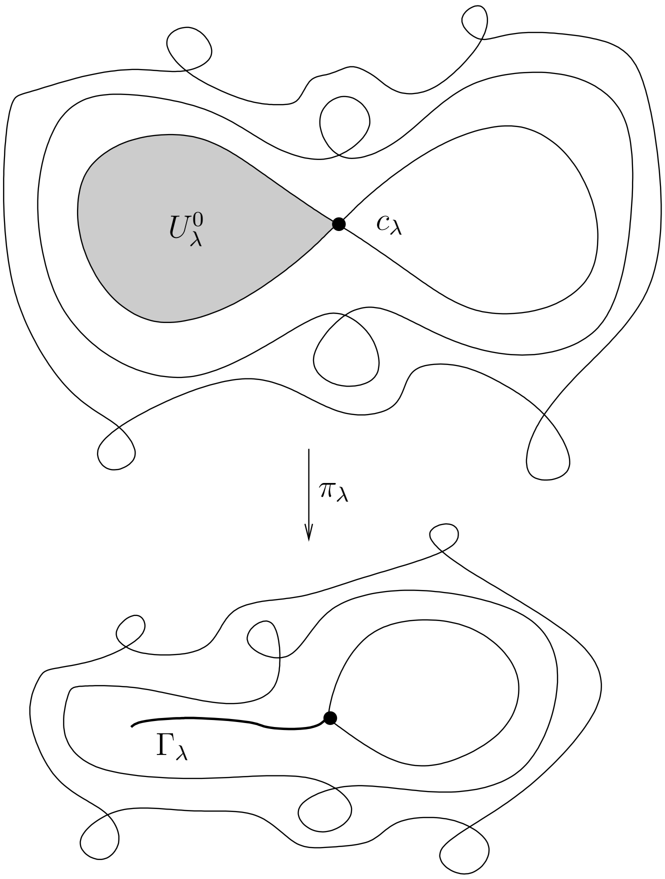


Figure 1: The sets $U^0_\lambda$, $L^0_\lambda$ (the closed set bounded by the figure eight) and the cut-glue operation.

c) *The map* $\Phi : \mathcal{H} \longrightarrow \mathcal{Y} : \begin{cases} \mathbf{0} & \mapsto \ \Pi(\mathbf{0}) \\ \mathbf{a} & \mapsto \ \Pi(\lambda, \eta_{\mathbf{a}}(c^1_{\mathbf{a}})) \ \textit{for } \mathbf{a} \neq \mathbf{0} \end{cases}$ *is a well defined bi-holomorphic homeomorphism.*

Part a) is trivial. Parts b1) and c) are proved in §2-5, in the following way: We first prove that there is a natural complex structure on $\mathcal{X}^*$ (and therefore on the open set $\mathcal{Y}^*$). We then show that $\Phi$ is a homeomorphism, mapping $\mathbf{0}$ to $\Pi(\mathbf{0})$ and is analytic from $\mathcal{H}^*$ onto $\mathcal{Y}^*$. This completes the atlas on $\mathcal{X}$ and gives both b1) and c). We then prove Theorem B using a result of Faught-Roesch together with Słodkovski extensions of holomorphic motions. It is the content of §6. Finally we establish part b2) of Theorem A' as a consequence of Theorem B.

# 2 Definition of $\Omega_{\mathbf{a}}$ and extension of $\eta_{\mathbf{a}}$

The main purpose of this section is to prove Proposition 1. We shall however also introduce some auxiliary notions and some preliminary results which will serve also in subsequent sections.

The following properties of the sets $L_\lambda(s)$ are easily verified. For later reference we state them as a Lemma, whose proof we leave for the reader.

**Lemma 3** *(structure of $L_\lambda(s)$) Assume $\lambda \in \mathbb{D}^*$.*

- *For $s \in \mathbb{R}$, the set $\overline{U_\lambda(s)}$ is a closed Jordan domain.*
- *For $s \notin \mathbb{N}$, $L_\lambda(s) = \overline{U_\lambda(s)}$.*
- *For $s \in \mathbb{N}$ (starting from $s = 0$), $\overline{U_\lambda(n)} \subseteq L_\lambda(n)$ and $L_\lambda(n) \setminus \overline{U_\lambda(n)}$ consists of $2^n$ components each one attached to $\overline{U_\lambda(n)}$ by a point in $Q_\lambda^{-n}(c_\lambda)$.*
- *In particular, $L_\lambda(0)$ is bounded by a figure eight whose branching point is the critical point $c_\lambda$.*
- *The restrictions $Q_\lambda : \overline{U_\lambda(s)} \to \overline{U_\lambda(s-1)}$ are homeomorphisms for $s \leq 0$ and coverings of degree two branched over $v_\lambda$ for $s > 0$; the restrictions $Q_\lambda : L_\lambda(s) \to L_\lambda(s-1)$ are homeomorphisms for $s < 0$ and coverings of degree two branched over $v_\lambda$ for $s \geq 0$.*

The following Lemmas show that the structure of $L_{\mathbf{a}}(s)$ is biholomorphically the same as the structure of $L_\lambda(s)$, when $s < t^1_{\mathbf{a}} := \kappa_{\mathbf{a}}(c^1_{\mathbf{a}})$.

**Lemma 4** *(local bi-holomorphic conjugacy) For $\mathbf{a} = (\lambda, a) \in \mathcal{H} \setminus \mathcal{H}_0$ the conformal isomorphism*

$$\eta_{\mathbf{a}} := \psi_\lambda \circ \phi_{\mathbf{a}} : U^0_{\mathbf{a}} \longrightarrow U^0_\lambda$$

*satisfies $\eta_{\mathbf{a}}(v^0_{\mathbf{a}}) = v_\lambda$ and $\eta_{\mathbf{a}} \circ P_{\mathbf{a}} = Q_\lambda \circ \eta_{\mathbf{a}}$ on $U^0_{\mathbf{a}}$. It continuously extends to a semi-conjugacy $\eta_{\mathbf{a}} : L^0_{\mathbf{a}} \longrightarrow L^0_\lambda$, is analytic on a neighborhood of $L^0_{\mathbf{a}} \setminus \{co^0_{\mathbf{a}}\}$, and is unique if $c^0_{\mathbf{a}} \neq co^0_{\mathbf{a}}$. We have $c^0_{\mathbf{a}} = co^0_{\mathbf{a}}$ iff $c^1_{\mathbf{a}} = c^0_{\mathbf{a}}$, and $co^0_{\mathbf{a}} \in L^0_{\mathbf{a}}$ iff $c^1_{\mathbf{a}} \in L^0_{\mathbf{a}}$. It is a conjugacy if $t^1_{\mathbf{a}} > 0$.*

The proof is easy and is left to the reader. See Figure 2 for a schematic illustration.

We want to extend $\eta_{\mathbf{a}}$ further if $c^1_{\mathbf{a}} \notin L^0_{\mathbf{a}}$ (i.e $t^1_{\mathbf{a}} > 0$). For the rest of this section we shall omit $\mathbf{a}$ as a subscript.

**Lemma 5** *For $\mathbf{a} \in \mathcal{H} \setminus \mathcal{H}_0$ the local conjugacy $\eta$ has a unique bi-holomorphic extension $\eta : U = U(t^1) \longrightarrow U_\lambda(t^1)$ where $t^1 = \max\{0, \kappa(v^1) + 1\}$.*

**Proof :** By construction the local conjugacy $\eta : U(0) \longrightarrow U_\lambda(0)$ is bi-holomorphic and preserves critical values: $\eta(P(c^0)) = Q_\lambda(c_\lambda)$. Write $\hat{t} = \kappa(v^1)$. If $\hat{t} \leq -1$ then $c^1 \in L(0) \setminus U(0)$, hence $t^1 = 0$ and the conclusion of the lemma holds.

Suppose $\hat{t} > -1$ and let $n$ be the unique integer with $n - 1 \leq \hat{t} < n$. Then the restriction $P : U(n) \longrightarrow U(n-1)$ has degree 2 and hence we can extend $\eta$ recursively to unique bi-holomorphic conjugacies $\eta : U(k) \longrightarrow U_\lambda(k)$ by lifting $\eta \circ P$ on $U(k)$ to $Q_\lambda$ on $U_\lambda(k)$ for $k = 0 \cdots n$. The degree of $P : U(\hat{t} + 1) \longrightarrow U(\hat{t})$ is 2 and the degree of $P : U(s + 1) \longrightarrow U(s)$ is 3 for any $s > \hat{t}$, because $v^1 \in U(s) \setminus U(\hat{t})$ for any such $s$. It follows from the latter that $t^1 \leq 1 + \hat{t}$. And it follows from the former that we can extend $\eta$ to a uniquely determined bi-holomorphic conjugacy $\eta : U(\hat{t} + 1) \longrightarrow U_\lambda(\hat{t} + 1)$ by lifting as above. Hence $0 < \hat{t} + 1 \leq t^1$. **q.e.d.**

The bi-holomorphic conjugacy above extends uniquely as a homeomorphic conjugacy of the closures $\eta : \overline{U(t^1)} \longrightarrow \overline{U_\lambda(t^1)}$, because both the open sets $U(t^1)$ and $U_\lambda(t^1)$ are Jordan domains. Moreover this extension is (except when $co^0 \in \partial U(t^1)$, or equivalently $c^1 \in \partial U(t^1)$) locally the restriction of a holomorphic function.

$$\begin{array}{ccccccc} (c^0, 0),\ \overline{U(t)} & \xrightarrow{\eta} & \overline{U_\lambda(t)},\ (c_\lambda, 0) & & L(t) & \xrightarrow{\eta} & L_\lambda(t) \\ P\downarrow & & \downarrow Q_\lambda & \text{and} & P\downarrow & & \downarrow Q_\lambda \\ (v^0, 0),\ \overline{U(t-1)} & \xrightarrow{\eta} & \overline{U_\lambda(t-1)},\ (v_\lambda, 0) & & L(t-1) & \xrightarrow{\eta} & L_\lambda(t-1) \\ \text{for} & t \leq t^1 & & & \text{for} & t < t^1 & \end{array}$$

## Proof of Proposition 1

We shall construct $\Omega$ and $\eta$.

**Case 0.** $\lambda = 0$. We take $\Omega = \overline{U_{\mathbf{a}}}$ then $\eta = \phi : \Omega \longrightarrow \overline{\mathbb{D}(0, \mathrm{e}^{t^1})}$ is a homeomorphism, which is the restriction of an analytic map.

Assume from now on $\lambda \neq 0$.

Recall that we are looking for a closed, connected, simply connected set $\Omega$ such that $c^1 \in \Omega$, $\overline{U(t^1)} \subseteq \Omega \subseteq L(t^1)$, and such that the following diagram commutes:

$$\begin{array}{ccc} \Omega & \xrightarrow{\eta} & L_\lambda(t^1) \\ P\downarrow & & \downarrow Q_\lambda \\ L(t^1 - 1) & \xrightarrow{\eta} & L_\lambda(t^1 - 1) \end{array} .$$

Recall also that $U = U(t^1)$, $U(0) = U^0$ and $L(0) = L^0$.

**Case 1.** $t^1 = 0$. We take $\Omega = L(0) = L^0$. Then $\eta$ has a continuous extension $\eta : \Omega \longrightarrow L_\lambda(t^1)$, which is analytic except at the boundary point $co^0$ of $\Omega$. Its value at $c^1$ is uniquely determined.

**Case 2.** $t^1 > 0$ *and* $c^1 \in \overline{U(t^1)} = \overline{U}$. Then $c^1$ is a separating point for $L(t^1)$ and the critical value $v^1$ is a point in $\partial L(t^1 - 1)$. Let $co\Omega$ denote the connected component of $P^{-1}(L(t^1 - 1) \setminus \{v^1\})$ containing the co-critical point $co^0$. Define $\Omega = L(t^1) \setminus co\Omega$, then $\eta \circ P : \Omega \longrightarrow L_\lambda(t^1 - 1) \subset U(t^1)$ has a unique lift $\eta : \Omega \longrightarrow L_\lambda(t^1)$ to $Q_\lambda$, extending $\eta$ to $U(t^1)$.

**Case 3.** $t^1 > 0$ *and* $c^1 \notin \overline{U(t^1)}$. In this case the truncation of $L(t^1)$ is slightly more delicate: We have $t^1 = n > 0$ an integer, $c^1 \in L(n) \setminus \overline{U(n)}$ and $v^1 \in L(n-1) \setminus \overline{U(n-1)}$ by Lemma 5. The first attracted critical point $c^0$ is a separation point for $L(0)$, i.e. $L(0) \setminus \{c^0\}$ has two components. It follows by induction that for $m \leq n$ there are $2^m$ points of $P^{-m}(c^0)$ on the boundary of $U(m)$ and for $m < n$ they are precisely the separation points of $L(m)$. Let $w^0$ be the separation point of $L(n-1)$ which separates 0 from the critical value $v^1$ and let $L'(n-1)$ denote the connected component of $L(n-1) \setminus w^0$ containing 0. Moreover let $co\Omega$ denote the connected component of $P^{-1}(L'(n-1))$ containing $co^0$. We set $\Omega = L(n) \setminus co\Omega$. Then $\eta \circ P : \Omega \longrightarrow L_\lambda(n-1) \subset U(n)$ has a unique lift $\eta : \Omega \longrightarrow L_\lambda(n)$

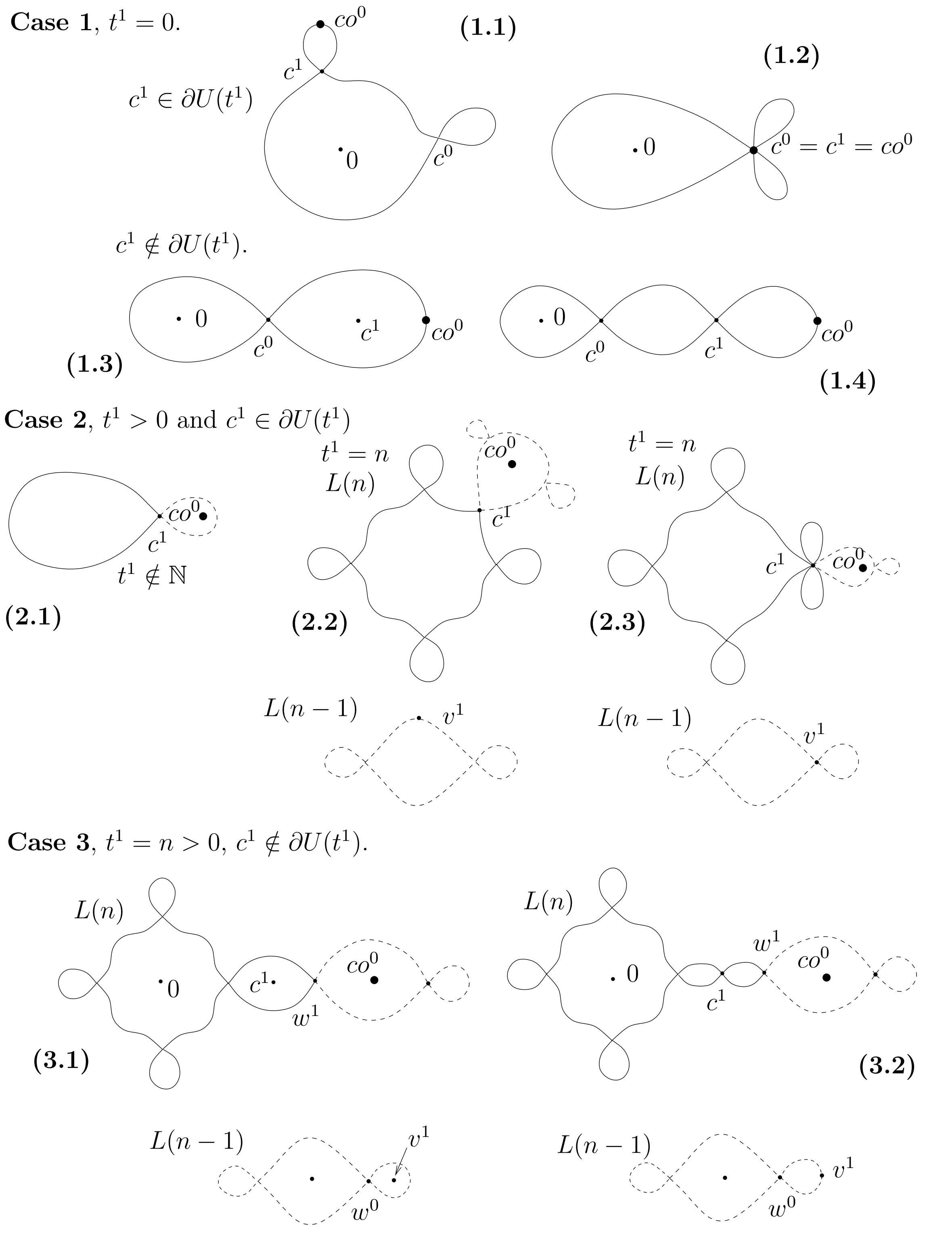


Figure 2: The set $\Omega$ is the closed region bounded by the solid curves. We have $co\Omega = L(n) \setminus \Omega$ and $co^0 \notin \Omega$ in cases 2 and 3, and $co\Omega = \emptyset$ and $co^0 \in \partial\Omega$ in case 1.

to $Q_\lambda$, extending $\eta$ on $U(n)$. This extension is locally the restriction of a holomorphic function.

This ends the construction of $\Omega$ and $\eta$.

Clearly $c^1 \notin \partial U^0$ if and only if $c^0$ is the unique first attracted critical point and $\mathbf{a} \mapsto \eta_{\mathbf{a}}(c^1_{\mathbf{a}})$ is single valued. Thus suppose $\mathbf{a}$ is such that both critical points belongs to $\partial U^0$, and let $z_0$ and $z_1$ denote the two values of $\eta(c^1) = \psi_\lambda \circ \phi(c^1) \in \partial U^0_\lambda$. That is fix a labelling $c^0, c^1$ of the critical points and let $\phi_0, \phi_1$ denote the two normalizations of $\phi$ with $\phi_i(c^i) = 1$ for $i = 0, 1$, then $z_i = \eta_i(c^{1-i}) = \psi_\lambda \circ \phi_i(c^{1-i})$ and $\phi_\lambda(z_i) = \phi_i(c^{1-i})$, $i = 0, 1$. Moreover $\phi_1 = \rho \cdot \phi_0$ for some $\rho$ of modulus 1. In particular $\rho = \rho \cdot 1 = \phi_\lambda(z_1)$ and $\rho \cdot \phi_\lambda(z_0) = 1$. So $\phi_\lambda(z_1) \cdot \phi_\lambda(z_0) = 1$.

Finally $\kappa_\lambda(\eta_{\mathbf{a}}(c^1_{\mathbf{a}})) = t^1_{\mathbf{a}} = \kappa_{\mathbf{a}}(c^1_{\mathbf{a}})$ follows from the constructions above. This ends the proof of Proposition 1.

**Corollary 6** *The map $\Phi : \mathcal{H} \longrightarrow \mathcal{Y}$, $\mathbf{0} \mapsto \Pi(\mathbf{0})$ and $\mathbf{a} \mapsto \Pi(\lambda, \eta_{\mathbf{a}}(c^1_{\mathbf{a}}))$ is well defined. Moreover if $\mathbf{a} \neq \mathbf{0}$ then $\Phi(\mathbf{a}) \neq \Pi(\mathbf{0})$.*

**Proof :** The only ambiguity in the definition of $\Phi$ occurs when $\partial U^0_{\mathbf{a}}$ contains two distinct critical points. In this case different choices of the labeling $c^0_{\mathbf{a}}, c^1_{\mathbf{a}}$ lead to two different maps $\eta_{\mathbf{a}}$ and to two values $z_0, z_1$ for $\eta_{\mathbf{a}}(c^1_{\mathbf{a}})$, with $\phi_\lambda(z_1) \cdot \phi_\lambda(z_0) = 1$. So $\Pi(\lambda, z_0) = \Pi(\lambda, z_1)$, and is independent of the choices involved. Hence $\Phi$ is well defined.

Clearly $\Phi(\mathbf{a}) \neq \Pi(\mathbf{0})$ if $\mathbf{a} \neq \mathbf{0}$, and $\Phi(\mathcal{H}) \subset \mathcal{Y}$. **q.e.d.**

# 3 Analyticity of the map $(\mathbf{a}, z) \mapsto \eta_{\mathbf{a}}(z)$.

The purpose of this section is to prove Proposition 10 below. For this we will rely on a result of Petersen (Proposition 7 below).

For $\mathbf{a} = (0, a) \in \mathcal{H}^*_0$ define the potential function $\widehat{g}_{\mathbf{a}} : B_{\mathbf{a}} \to [-\infty, 0\,[$ by $\widehat{g}_{\mathbf{a}}(z) = \log|\phi_{\mathbf{a}}(z)|$ near 0, and extend it to the entire basin by the recursive relation

$$\widehat{g}_{\mathbf{a}}(z) = \widehat{g}_{\mathbf{a}}(P_{\mathbf{a}}(z))/2. \tag{3.1}$$

For $t \in \mathbb{R}_-$ let $U_{\mathbf{a}}(t)$ denote the connected component of $\widehat{g}_{\mathbf{a}}^{-1}([-\infty, t[)$ containing 0. Define also the filled potential

$$g_{\mathbf{a}}(z) = \inf\{s | z \in U_{\mathbf{a}}(s)\}.$$

Then $t^1_{\mathbf{a}} = g_{\mathbf{a}}(c^1_{\mathbf{a}})$, $U_{\mathbf{a}} = U_{\mathbf{a}}(t^1_{\mathbf{a}})$ and $\Omega_{\mathbf{a}} = \overline{U_{\mathbf{a}}(t^1_{\mathbf{a}})}$.

Define

$$\mathcal{F} := \Phi^{-1}(\Gamma) = \{\mathbf{0}\} \cup \{\mathbf{a} \in \mathcal{H} \setminus \mathcal{H}_0 \mid \text{both critical points are on } \partial U^0_{\mathbf{a}}\}$$

and $\mathcal{F}_\lambda = \mathcal{F} \cap \mathcal{H}_\lambda$. Set

$$\mathcal{U} = \{(\lambda, a, z) | \mathbf{a} = (\lambda, a) \in \mathcal{H} \setminus \mathcal{F} \text{ and } z \in U_{\mathbf{a}}\}\ .$$

**Proposition 7** *The set $\mathcal{U}$ is open in $\mathbb{C}^3$ and the map $(\mathbf{a}, z) \mapsto \eta_{\mathbf{a}}(z)$ is complex analytic on $\mathcal{U}$.*

**Proof :** See Petersen [Pe], where the Proposition is proved for the family $\widehat{P}_{\lambda,b}(z) = \lambda z + bz^2 + z^3$, which double covers the family $P_{\mathbf{a}}$, and there is no obstruction to passing the result to the quotient, because we are avoiding the parameters in $\mathcal{F}$. **q.e.d.**

**Definition.** In case $\lambda \neq 0$ and $t^1_{\mathbf{a}} > 0$ set $pC_\lambda = \overline{\{Q^n_\lambda(c_\lambda) | n \geq 1\}}$, $pC_{\mathbf{a}} = \overline{\{P^n_{\mathbf{a}}(c^0_{\mathbf{a}}) | n \geq 1\}}$ and $copC_{\mathbf{a}} = P^{-1}_{\mathbf{a}}(pC_{\mathbf{a}}) \setminus L^0_{\mathbf{a}}$. The last two sets depend analytically on $\mathbf{a}$.

**Lemma 8** *Fix $\mathbf{a} \in \mathcal{H}$ with $\lambda \neq 0$ and $t^1_{\mathbf{a}} > 0$. Let $X \subset B_{\mathbf{a}}(0) \setminus copC_{\mathbf{a}}$ be any simply connected forward invariant ($P_{\mathbf{a}}(X) \subset X$) domain, with $U_{\mathbf{a}}(T) \subset X$ for some $0 < T \leq t^1_{\mathbf{a}}$.*

*Then the restriction of $\eta_{\mathbf{a}}$ to $U^0_{\mathbf{a}}$ has an analytic extension $\widehat{\eta}$ to $X$ satisfying $\widehat{\eta} \circ P_{\mathbf{a}} = Q_\lambda \circ \widehat{\eta}$. This extension is uniquely determined by $X$, but in general depends on $X$. However if $\widehat{\eta}(X) \subset U_\lambda(t^1_{\mathbf{a}})$, then $X \subset U_{\mathbf{a}}(t^1_{\mathbf{a}})$ and $\widehat{\eta} = \eta_{\mathbf{a}}$ on $X$. And inversely if $U_{\mathbf{a}}(t) \subset X$ for some $0 < t \leq t^1_{\mathbf{a}}$, then $\eta_{\mathbf{a}} = \widehat{\eta}$ on $U_{\mathbf{a}}(t)$. Furthermore if $\Omega_{\mathbf{a}} \subset X$ then $\eta_{\mathbf{a}} = \widehat{\eta}$ on $\Omega_{\mathbf{a}}$.*

Let us remark that $\eta$ has no continuous extension to a neighborhood of $co^0_{\mathbf{a}}$ as a semi-conjugacy: Assume at first $co^0_{\mathbf{a}} \neq c^0_{\mathbf{a}}$. Then $P_{\mathbf{a}}$ is locally univalent around $co^0_{\mathbf{a}}$. But $\eta_{\mathbf{a}}(P_{\mathbf{a}}(co^0_{\mathbf{a}})) = v_\lambda$ and $\eta_{\mathbf{a}}$ is locally univalent at $P(co^0_{\mathbf{a}})$, and $v_\lambda$ has a unique preimage by $Q_\lambda$ which is $c_\lambda$, around which $Q_\lambda$ is non-univalent. The other case $co^0_{\mathbf{a}} = c^0_{\mathbf{a}}$ is similar ($P_{\mathbf{a}}$ is locally of degree 3 this time).

**Proof :** Uniqueness comes from the uniqueness of analytic continuations on simply connected domains. In the following we omit the subscript $\mathbf{a}$ for simplicity.

For $t \in \mathbb{R}$ define $X_t$ as the connected component of $U(t) \cap X$ containing 0. Then each $X_t$ is a simply connected forward invariant domain (in fact $P(X_t) \subseteq X_{t-1} \subseteq X_t$) and $X_t \cap copC = \emptyset$. We define $\widehat{\eta} = \eta$ on $X_0 = U(0)$ and will define $\widehat{\eta}$ recursively on $X_n$, $n \geq 0$ as the unique lift of $\widehat{\eta} \circ P$ on $X_n$ to $Q_\lambda$ fixing 0.

In order to get the existence of the lifting, we make use of the following classical result (see for example Massey [Ma]):

**Lemma 9** *Let $Y, Z, W$ be path connected and locally path connected Hausdorff spaces with base points $y \in Y$, $z \in Z$ and $w \in W$. Suppose $p : W \to Y$ is an unbranched covering and $f : Z \to Y$ is a continuous map such that $f(z) = y = p(w)$.*

$$\begin{array}{ccc} Z, z & \xrightarrow{\tilde{f}} & W, w \\ & f \searrow & \downarrow p \\ & & Y, y \end{array}$$

*Then there exists a continuous lift $\tilde{f}$ of $f$ to $p$ with $\tilde{f}(z) = w$ if and only if*

$$f_*(\pi_1(Z, z)) \subset p_*(\pi_1(W, w)) , \tag{3.2}$$

*where $\pi_1$ denotes the fundamental group. This lift is unique.*

By assumption $U(T) \subset X$ for some $0 < T \leq t^1$ and by Lemma 5 the conjugacy $\eta$ has a bi-holomorphic extension $\widehat{\eta} : U(T) \longrightarrow U_\lambda(T)$. We shall prove by induction on $n \geq T$ that $\widehat{\eta}$ has a unique analytic extension to $X_n$, with $\widehat{\eta}^{-1}(pC_\lambda) = pC \subset U^0$.

Suppose that $\widehat{\eta}$ is analytically extended to $X_{n-1}$ with $\widehat{\eta}^{-1}(\widehat{\eta}(pC)) = pC \subset U^0$ and suppose $n \geq T$. We want to prove that there exists a lift $\widetilde{\eta}$ of the analytic, hence continuous map $\widehat{\eta} \circ P$ on $X_n$ to $Q_\lambda$ extending $\widehat{\eta}$. Moreover we want to show that $\widetilde{\eta}$ is injective above $pC_\lambda$, i.e. $\widetilde{\eta}^{-1}(pC_\lambda) = pC \subset U^0$.

$$\begin{array}{ccc} X_n \setminus \{c^0\} & \xrightarrow{\widetilde{\eta}} & B_\lambda \setminus \{c_\lambda\} \\ P \downarrow & & \downarrow Q_\lambda \\ X_{n-1} \setminus \{v^0\} & \xrightarrow{\widehat{\eta}} & B_\lambda \setminus \{v_\lambda\} \end{array} .$$

We set all base points to be 0. Any closed loop $\gamma$ in $X_n \setminus \{c^0\}$ is homotopic to a closed loop $\gamma'$ in $U(T) \setminus \{c^0\}$, because $X_n$ is simply connected and contains $U(T)$, for which $c^0$ is an interior point. Thus $(\widehat{\eta} \circ P)_*(\pi_1(X_n \setminus \{c^0\}, 0)) = (\widehat{\eta} \circ P)_*(\pi_1(U(T) \setminus \{c^0\}, 0)) \subset (Q_\lambda)_*(\pi_1(B_\lambda \setminus \{c_\lambda\}, 0))$ is satisfied, by the pure existence of $\widehat{\eta}$. Hence the lift $\widetilde{\eta}$ fixing 0 exists and is unique. In particular $\widetilde{\eta} = \widehat{\eta}$ on the connected subset $X_{n-1} \smallsetminus \{c^0\} \subset X_n \smallsetminus \{c^0\}$, hence extends to a continuous map on $X_n$. To see that $\widetilde{\eta}^{-1}$ is injective above $pC_\lambda$, we note that since $\widehat{\eta}$ is injective above $pC_\lambda$, we have $\widetilde{\eta}(z) \in pC_\lambda$ if and only if $z \in P^{-1}(pC)$. However since $\widetilde{\eta} = \widehat{\eta}$ is injective on $L^0 \subset U(T)$, we have either $z \in pC$ or $z \in copC$. By assumption however $copC \cap X_n = \emptyset$. Thus $\widetilde{\eta}$ is injective above $pC_\lambda$. In particular $\widetilde{\eta}(X_n \setminus \{v^0\}) \subset B_\lambda \setminus \{v_\lambda\}$ and we can continue the induction. **q.e.d.**

**Proposition 10** *The set $\Omega_{\mathbf{a}}$ varies upper-semi-continuously for $\mathbf{a} \in \mathcal{H}^*$ for the Hausdorff topology. The map $\mathbf{a} \mapsto \eta_{\mathbf{a}}(c^1_{\mathbf{a}})$ is analytic on $\mathcal{H} \smallsetminus \mathcal{F}$.*

**Proof :** Semi-continuity of $\Omega_{\mathbf{a}}$. This is carried out by inspecting each case in Figure 2 (and using Proposition 7 for the case $\lambda = 0$).

Assume $\lambda \neq 0$. Suppose at first that $t^1_{\mathbf{a}} > 0$. Choose an open connected and simply connected neighborhood $X$ of $\Omega_{\mathbf{a}}$ with $P_{\mathbf{a}}(\overline{X}) \subset U_{\mathbf{a}}(t^1_{\mathbf{a}} - \frac{1}{2})$ and with $copC_{\mathbf{a}} \cap \overline{X} = \emptyset$. There exists a neighborhood $\Lambda$ of $\mathbf{a}$ such that for all $\mathbf{a}' \in \Lambda$ we have $P_{\mathbf{a}'}(X) \subset U_{\mathbf{a}'}(t^1_{\mathbf{a}'}) = U_{\mathbf{a}'}$, $\Omega_{\mathbf{a}'} \subset X$ and $copC_{\mathbf{a}'} \cap \overline{X} = \emptyset$, because $\Omega_{\mathbf{a}'}$ varies upper semi-continuously and $copC_{\mathbf{a}'}$ varies continuously with $\mathbf{a}'$ for the Hausdorff metric on compact subsets. Lemma 8 implies that for every $\mathbf{a}' \in \Lambda$ the map $\eta_{\mathbf{a}'}$ has a unique analytic extension $\widehat{\eta}_{\mathbf{a}'}$ to $X$, which agrees with $\eta_{\mathbf{a}'}$ on $\Omega_{\mathbf{a}'}$. We need to prove that these lifts define a continuous and hence analytic map.

Denote by $p_1$ the projection $\mathbf{a}' = (\lambda', a') \mapsto \lambda'$.

$$\begin{array}{ccc} \bigcup_{\mathbf{a}' \in \Lambda} (\mathbf{a}', X \smallsetminus \{c^0_{\mathbf{a}'}\}) & \xrightarrow{(p_1, \widetilde{\eta}_{\mathbf{a}'})} & \bigcup_{\mathbf{a}' \in \mathbb{D}^*} (\lambda', B_{\lambda'} \smallsetminus \{c_{\lambda'}\}) \\ (id, P_{\mathbf{a}'}) \downarrow & & \downarrow (id, Q_{\lambda'}) \\ \bigcup_{\mathbf{a}' \in \Lambda} (\mathbf{a}', U_{\mathbf{a}'} \smallsetminus \{v^0_{\mathbf{a}'}\}) & \xrightarrow{(p_1, \eta_{\mathbf{a}'})} & \bigcup_{\mathbf{a}' \in \mathbb{D}^*} (\lambda', B_{\lambda'} \smallsetminus \{v_{\lambda'}\}) \end{array} .$$

Here we set the base points to be $(\mathbf{a}, 0)$ for the left hand sets, and $(\lambda, 0)$ for the right hand sets. By Proposition 7, the map $(p_1, \eta_{\mathbf{a}'})$ is analytic, therefore continuous. The right hand map is a covering. Now any loop in $\bigcup_{\mathbf{a}' \in \Lambda} (\mathbf{a}', X \smallsetminus \{c^0_{\mathbf{a}'}\})$ based at $(\mathbf{a}, 0)$ is homotopic to a loop on the slice $\mathbf{a}' = \mathbf{a}$. So the condition (3.2) is satisfied due to the existence of $\widehat{\eta}_{\mathbf{a}}$. Thus we can apply Lemma 9 to obtain a continuous lift in the above diagram, obviously in the form $(p_1, \widetilde{\eta}_{\mathbf{a}'})$. Using the uniqueness of lifts on each slice, we get $\widetilde{\eta}_{\mathbf{a}'} = \widehat{\eta}_{\mathbf{a}'}$.

This implies that $(\mathbf{a}', z) \to \widehat{\eta}_{\mathbf{a}'}(z)$ on $\Lambda \times X$ is continuous, and analytic on each variable, therefore analytic on the joint variable $(\mathbf{a}', z)$. In particular $\mathbf{a}' \mapsto \widehat{\eta}_{\mathbf{a}'}(c^1_{\mathbf{a}'}) = \eta_{\mathbf{a}'}(c^1_{\mathbf{a}'})$ is analytic.

Suppose next that $t^1_{\mathbf{a}} = 0$ and $c^1_{\mathbf{a}} \in L^0_{\mathbf{a}} \smallsetminus \overline{U^0_{\mathbf{a}}}$. Then for $\epsilon > 0$ sufficiently small the set $\Omega_{\mathbf{a}} \smallsetminus \mathbb{D}(co^0_{\mathbf{a}'}, \epsilon)$ is a connected closed set containing $\overline{U^0_{\mathbf{a}}}$ and $c^1_{\mathbf{a}}$. Using a similar argument as above with a sufficiently small neighborhood $X$ of this set we may conclude similarly.

Case $\lambda = 0$. Choose a connected and simply connected neighborhood $X$ of $\Omega_{\mathbf{a}}$ with $co^0_{\mathbf{a}} \notin \overline{X}$. Then the same argument as above works. **q.e.d.**

# 4 The natural complex structure of $\mathcal{X} \smallsetminus \{\mathbf{0}\}$

Let $\widetilde{\Gamma} = \{(\lambda, z), z \in \overline{U^0_\lambda}\} \cup \{\mathbf{0}\}$. Then the restriction $\Pi : \mathbb{D} \times (\overline{\mathbb{C}} \smallsetminus \widetilde{\Gamma}) \longrightarrow \mathcal{X} \smallsetminus \Gamma$ is a homeomorphism of open sets. Thus it defines a complex analytic structure $\widehat{\mathcal{A}}$ on $\mathcal{X} \smallsetminus \Gamma$. We will show that this complex structure is in fact the restriction of a complex analytic structure on $\mathcal{X}^* = \mathcal{X} \smallsetminus \Pi(\mathbf{0})$:

**Proposition 11** *The space $\mathcal{X}^*$ has a complex analytic structure $\mathcal{A}^*$ containing $\widehat{\mathcal{A}}$. Moreover this structure gives each fiber $\mathcal{X}_\lambda := \{\Pi(\lambda, z) | z \in \overline{\mathbb{C}}\}$ a unique complex structure and the corresponding Riemann surface is isomorphic to $\overline{\mathbb{C}}$.*

**Proof :** We prove first that each fiber $\mathcal{X}_\lambda$, $\lambda \in \mathbb{D}$ has a unique complex structure containing the chart $\pi_\lambda^{-1} : \mathcal{X}_\lambda \smallsetminus \Gamma_\lambda \longrightarrow \overline{\mathbb{C}}$, which is complex analytic for the structure $\widehat{\mathcal{A}}$.

For $\lambda = 0$, the map $\pi_0^{-1} : \mathcal{X}_0 \longrightarrow \overline{\mathbb{C}}$ is a homeomorphism, and thus a global chart.

Fix now $\lambda \in \mathbb{D}^*$. We define a chart $k_\lambda$ on $W_\lambda := \pi_\lambda(U_\lambda(1))$, a neighborhood of $\Gamma_\lambda$ as follows:

$$\begin{array}{ccccccc}
U_\lambda(1) \smallsetminus U^0_\lambda & & \overset{\theta_\lambda}{\hookrightarrow} & & \mathbb{C} \smallsetminus U & \overset{f}{\longrightarrow} & \mathbb{C} \smallsetminus \mathbb{D} \\
 & \phi_\lambda \searrow & & & \swarrow_g & & \\
\pi_\lambda \downarrow & & \mathbb{C} \supset D(\frac{1}{|\lambda|}) \supset \mathbb{D} & & & & \downarrow h \\
W_\lambda & & & \overset{k_\lambda}{\longrightarrow} & & & \mathbb{C}
\end{array}$$

We will use a **model double cover** $g : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ which is the quadratic polynomial $g(z) = -z^2 + 1$. Note that $g$ has critical point at 0, with critical value at 1, and $g(-1) = 0$.

The restriction $\phi_\lambda : U_\lambda(1) \longrightarrow \mathbb{D}(1/|\lambda|)$ is proper of degree 2 and with critical point $c_\lambda$ and critical value 1. There is thus a univalent lift $\theta_\lambda : U_\lambda(1) \longrightarrow \mathbb{C}$ of $\phi_\lambda$ to $g$ with $\theta_\lambda(c_\lambda) = 0$ and $\theta_\lambda(0) = -1$.

Denote by $U$ the connected component of $g^{-1}(\mathbb{D})$ containing $-1$. Then $\theta_\lambda(U^0_\lambda) = U$. Let $f : \overline{\mathbb{C}} \smallsetminus U \longrightarrow \overline{\mathbb{C}} \smallsetminus \mathbb{D}$ with $f(\infty) = \infty$ be the homeomorphically extended Riemann map normalized by $f(0) = 1$. Then $f(z_1) = \overline{f(z_2)} \in \mathbb{S}^1$ for $z_1, z_2 \in \partial U$ with $g(z_1) = \overline{g(z_2)}$, because both $g$ and $f$ have only real coefficients. Denote by $h$ the branched double covering $h(z) = z + 1/z : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$. It satisfies $h(z) = h(1/z)$ for all $z$, and in particular

$h(z) = h(\overline{z})$ whenever $z \in \mathbb{S}^1$. The map

$$k_\lambda := h \circ f \circ \theta_\lambda \circ \pi_\lambda^{-1} : W_\lambda \longrightarrow \mathbb{C} \tag{4.1}$$

is a well defined homeomorphism onto its image. Moreover the composition $k_\lambda \circ \pi_\lambda = h \circ f \circ \theta_\lambda : U_\lambda(1) \smallsetminus \overline{U_\lambda^0} \longrightarrow \overline{\mathbb{C}} \smallsetminus [-2,2]$ is holomorphic. Hence the two charts $k_\lambda$ and $\pi_\lambda^{-1}$ together define a complex analytic atlas on $\mathcal{X}_\lambda$.

Finally let $\chi : \omega \longrightarrow \mathbb{C}$ be a chart for a complex structure on $\mathcal{X}_\lambda$ and suppose $\chi \circ \pi_\lambda$ is holomorphic on $\pi_\lambda^{-1}(\omega) \smallsetminus \partial U_\lambda^0$. Then $\chi \circ k_\lambda^{-1}$ is continuous where defined and holomorphic on $k_\lambda(\omega) \smallsetminus [-2,2]$. But then it is holomophic on all of $k_\lambda(\omega)$ by Morera's theorem. Thus $\chi$ is in the atlas defined by $k_\lambda$. This proves the uniqueness of the complex structure on $\mathcal{X}_\lambda$ containing the chart $\pi_\lambda^{-1}$ on $\mathcal{X}_\lambda \smallsetminus \Gamma_\lambda$.

It remains to be proved that the complex structures on the fibers line up to form a complex structure on $\mathcal{X}^*$. The set $W := \cup_{\lambda \in \mathbb{D}^*} \{\lambda\} \times W_\lambda$ is an open neighborhood of $\Gamma^* = \Gamma \smallsetminus \Pi(\mathbf{0})$. Moreover $K : W \longrightarrow \mathbb{C}^2$ given by $K(\lambda, z) = (\lambda, k_\lambda(z))$ is a homeomorphism onto its image, because the map $(\lambda, z) \mapsto k_\lambda(z)$ is continuous on $W$ and a homeomorphism of $W_\lambda$ onto its image for each fixed $\lambda \in \mathbb{D}^*$.

Finally the composition

$$K \circ \Pi : \bigcup_{\lambda \in \mathbb{D}^*} \{\lambda\} \times \left(U_\lambda(1) \smallsetminus \overline{U_\lambda(0)}\right) \longrightarrow \mathbb{C}^2$$

is holomorphic. Thus $K$ and $\Pi^{-1}$ together define a complex analytic atlas on $\mathcal{X}^*$. **q.e.d.**

# 5 Proof of Theorem A'

## Injectivity of $\Phi$ (rigidity), a lifting process

$$\begin{array}{ccccccccccccc}
\cdots & \overline{\mathbb{C}} & \xrightarrow{P_1} & \overline{\mathbb{C}} & \xrightarrow{P_1} & \overline{\mathbb{C}} & \supset & \Omega_1 & \eta_1 & \xrightarrow{P_1} & \eta_1 & P_1(\Omega_1) \\
 & & & & & & & & \searrow & & \swarrow & \\
\cdots & h_2 \downarrow & & h_1 \downarrow & & h_0 \downarrow & & h_0 \downarrow & \eta_2 & L_\lambda(t) & \eta_2 & \downarrow h_0 \\
 & & & & & & & & \nearrow & & \nwarrow & \\
\cdots & \overline{\mathbb{C}} & \xrightarrow{P_2} & \overline{\mathbb{C}} & \xrightarrow{P_2} & \overline{\mathbb{C}} & \supset & \Omega_2 & & \xrightarrow{P_2} & & P_2(\Omega_2)
\end{array}$$

Given any two (in the following fixed) points $\mathbf{a_1} = (\lambda_1, a_1)$ and $\mathbf{a_2} = (\lambda_2, a_2)$ with $\Phi(\mathbf{a_1}) = \Phi(\mathbf{a_2})$, we shall prove that $\mathbf{a_1} = \mathbf{a_2}$. To reduce notation we use the indices $1, 2$ for all the involved quantities. In particular $P_1 = P_{\mathbf{a_1}}$ and $P_2 = P_{\mathbf{a_2}}$. It follows immediately that $\lambda_1 = \lambda_2 = \lambda$ and by Proposition 1 that $t := t^1_{\mathbf{a_1}} = \kappa_\lambda(\eta_1(c_1^1)) = \kappa_\lambda(z) = \kappa_\lambda(\eta_2(c_2^1)) = t^1_{\mathbf{a_2}}$. Hence also $\eta_1(\Omega_1) = \eta_2(\Omega_2) = L_\lambda(t)$. Possibly exchanging the labels of $c_2^0$ and $c_2^1$, we can assume that $\eta_1(c_1^1) = z = \eta_2(c_2^1)$, and $\Phi(\mathbf{a_1}) = \Phi(\mathbf{a_2}) = \Pi(\lambda, z)$.

The injectivity of $\Phi$ on $\mathcal{H}_0$ is proved by Milnor. The proof in the case $\lambda \neq 0$ is similar, the core is the so called pull-back argument. We shall however go through the details. We shall recursively define quasi conformal maps $h_n$, which are conformal on larger and larger domains. Extracting a convergent subsequence will provide a conformal conjugacy between $P_1$ and $P_2$.

**Definition of** $h|_{\Omega_1}$. We claim that there exists a homeomorphic conjugacy $h : \Omega_1 \longrightarrow \Omega_2$ of $P_1$ to $P_2$, such that $h$ is the restriction of a bi-holomorphic conjugacy between neighborhoods of the two sets.

Assume at first $t = t^1_{\mathbf{a_1}} = t^1_{\mathbf{a_2}}$ is not an integer. Then both $\Omega_1$ and $\Omega_2$ are closed Jordan domains. Set $h = \eta_2^{-1} \circ \eta_1$, it is a well defined homeomorphism and maps $c^1_1 \in \partial U_1(t)$ to $c^1_2 \in \partial U_2(t)$.

Assume next $t = n \geq 0$. Note that $L_\lambda(n) \setminus \overline{U_\lambda(n)}$ consists of $2^n$ components, each one is attached to $U_\lambda(n)$ by a point in ${Q_\lambda}^{-n}(c_\lambda)$. Each component, except the one containing $z$, is homeomorphically covered under each of the maps $\eta_1$ and $\eta_2$ by similar components of $\Omega_i \setminus \overline{U_i(n)}$, $i = 1, 2$. The component containing $z$ is however double covered under both maps with $z$ as critical value and $c_i$ as resp. critical points. Thus the existence of $h$ follows also in this case. This proves the claim.

**Extension of** $h$. We shall extend $h$ to a homeomorphism $h : L_1(t) \longrightarrow L_2(t)$ conjugating $P_1$ to $P_2$. For this we remark first that $L_i(t) = \Omega_i \cup co\Omega_i$, where the sets $co\Omega_i$, $i = 1, 2$ were defined in the proof of Proposition 1. And secondly that $\eta_i \circ P_i$ maps $co\Omega_i$ homeomorphically onto a common image. The existence of the homeomorphic extension easily follows.

Denote by $A_i$ the annuli $U_i(t+1) \setminus L_i(t)$ and by $B_i$ the annuli $U_i(t) \setminus L_i(t-1)$, $i = 1, 2$. Then $P_i : A_i \longrightarrow B_i$ are covering maps of degree 3. Moreover by the above the restriction $h|_{\overline{B_1}} : \overline{B}_1 \longrightarrow \overline{B}_2$ is a homeomorphism and a conformal isomorphism of the annuli. Also the restriction $h|_{\partial L_1} : \partial L_1 \longrightarrow \partial L_2$ is a homeomorphism. It easily follows that $h \circ P_1$ on $\overline{A_1}$ has a unique homeomorphic (bi-holomorphic) lift to $P_2$ extending $h$ on $L_1(t)$. One way to see this is to uniformize the annuli as round annuli with concentric boundaries and transport the maps to the uniformized annuli, trivially solve the problem and transport the solution back. Since all the involved maps are continuous on the closures and all boundaries are locally connected, continuity and hence analyticity of $h$ on $\partial L_1$ is assured.

**Definition of** $h_0$. Fix a smooth equipotential $\kappa_1^{-1}(T)$ with $t < T < t + 1$ and define $h_0 := h$ on the domain $U_1(T)$ inside this equipotential. Then $\kappa_2^{-1}(T)$ is also smooth bounding a domain $U_2(T)$ and $h_0 := h : U_1(T) \longrightarrow U_2(T)$ is a bi-holomorphic conjugacy.

For $i = 1, 2$ fix neighborhoods $V_i = \phi_{i,\infty}^{-1}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}(R)})$ of infinity, where $\phi_{i,\infty}$ are Böttcher coordinates at infinity and $R > 1$. Define $h_0 := \phi_{2,\infty}^{-1} \circ \phi_{1,\infty} : V_1 \longrightarrow V_2$. Extend $h_0$ to a global quasi conformal homeomorphism of $\overline{\mathbb{C}}$, by a, say $C^1$ homeomorphism between the two remaining annuli.

Our map $h_0$ is now globally defined and is conformal on the forward invariant domains $V_1$ and $U_1(T)$.

**The lifting** $h_1$. We claim that there is a homeomorphism $h_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that
1) $P_2 \circ h_1 = h_0 \circ P_1$, i.e.

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{h_1} & \overline{\mathbb{C}} \\ P_1 \downarrow & & \downarrow P_2 \\ \overline{\mathbb{C}} & \xrightarrow[h_0]{} & \overline{\mathbb{C}} \end{array}$$

2) $h_1$ restricted to $P_1^{-1}(V_1 \cup U_1(T))$ is holomorphic,

3) $h_1|_{U_1(T)} = h_0|_{U_1(T)}$,
4) $h_1$ is isotopic to $h_0$ relative to the postcritical set of $P_1$.

(This is to say that $P_1$ and $P_2$ are *c-equivalent*, a notion that generalizes Thurston's combinatorial equivalence to postcritically infinite hyperbolic maps. See [CJS] and [CT] for a detailed study of the notion.)

*Proof.* We want to apply Lemma 9 to $h_0 \circ P_1$. Notice that for $i = 1, 2$, the restrictions $P_i : \mathbb{C} \setminus P_i^{-1}(\{v_i^0, v_i^1\}) \longrightarrow \mathbb{C} \setminus \{v_i^0, v_i^1\}$ are covering maps. Set $Z := U_1(T) \setminus P_1^{-1}(\{v_1^0, v_1^1\})$. Now any loop based at 0 in $\mathbb{C} \setminus P_1^{-1}(\{v_1^0, v_1^1\})$ is homotopic to a loop based at 0 in $Z$. This means that $\pi_1(\mathbb{C} \setminus P_1^{-1}(\{v_1^0, v_1^1\}), 0)$ is canonically identified with $\pi_1(Z, 0)$.

The map $h_0$ restricted to $Z$ satisfies $P_2 \circ h_0 = h_0 \circ P_1$ and $h_0(0) = 0$. So

$$(h_0 \circ P_1)_* \pi_1(\mathbb{C} \setminus P_1^{-1}(\{v_1^0, v_1^1\}), 0) = (h_0 \circ P_1)_* \pi_1(Z, 0) =$$
$$= (P_2 \circ h_0)_* \pi_1(Z, 0) = P_{2*} \pi_1(\mathbb{C} \setminus P_2^{-1}(\{v_2^0, v_2^1\}), 0) \ .$$

So we can apply Lemma 9 to $h_0 \circ P_1$ to guarantee the existence of a continuous $h_1$ satisfying 1) together with the base point condition $h_1(0) = 0$.

By uniqueness of the lifting, we have also 3). The point 2) follows.

Now we may exchange the role of $P_1$ and $P_2$ to show that $h_1$ has a continuous inverse map. In other words $h_1$ is in fact a homeomorphism.

The postcritical set of $P_1$ is contained in $\underline{U_1(T)} \cup \{\infty\}$. We will use the fact that a homeomorphism on the closed unit disc $\overline{\mathbb{D}}$ which is the identity on $\underline{\{0\} \cup \partial\mathbb{D}}$, is isotopic to the identity relative to $\{0\} \cup \partial\mathbb{D}$. This implies that $h_0$ and $h_1$ are isotopic rel $\overline{U}_1(T) \cup \{\infty\}$, in particular rel the postcritical set of $P_1$. This proves 4) and ends the proof of the Claim.

Notice that we can not guarantee $h_1|_{V_1} = h_0|_{V_1}$. And this fact is usually not true.

**The subsequent liftings** $h_n$. From now on it follows from a general rigidity principle, see for example [Mc]. We may either apply a general theorem or redo the argument in our setting. We choose the latter option.

Define recursively $h_n : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ for each $n$ by $P_2 \circ h_n = h_{n-1} \circ P_1$ and $h_n(0) = 0$.

Each $h_n$ is quasi conformal with the same maximal dilatation $K$ as $h_0$ and satisfies $h_n = h_0$ on $U_1(T)$ and that $h_n$ is conformal on $P_1^{-n}(U_1(T) \cup V_1)$. The family $(h_n)$ is thus normal. Let $(h_{n_k})$ be a subsequence converging uniformly to some $K$-quasi conformal map $H : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$. Notice that $\overline{\mathbb{C}} \setminus \cup_n P_1^{-n}(U_1(T) \cup V_1)$ coincides with the Julia set $J_1$ of $P_1$. Therefore $H$ is conformal except possibly on $J_1$. However $J_1$ has Lebesgue measure zero, because $P_1$ is hyperbolic (in fact $J_1$ is a quasi-circle). Thus $H$ is globally conformal. But since $H = h_0$ on $U_1(T)$ we have $H \circ P_1 = P_2 \circ H$ on $U_1(T)$. By the isolated zero theorem of holomorphic maps we have actually $H \circ P_1 = P_2 \circ H$ on $\overline{\mathbb{C}}$.

This shows that $P_1$ and $P_2$ are conformally conjugate.

Finally we have parametrized our family so that each conformal conjugacy class is represented only once. So $\mathbf{a_1} = \mathbf{a_2}$.

## Analyticity of $\Phi$

**Analyticity of $\Phi$ on $\mathcal{H} \setminus \{(\lambda, 3\lambda), (\lambda, 0) | \lambda \in \mathbb{D}\}$.**

From Proposition 10 we obtain immediately that $\Phi$ is complex analytic on $\mathcal{H} \setminus \mathcal{F}$, as it is a composition of complex analytic maps.

As $\mathcal{H} \setminus \{(\lambda, 3\lambda), (\lambda, 0) | \lambda \in \mathbb{D}\} = (\mathcal{H} \setminus \mathcal{F}) \sqcup (\mathcal{F} \setminus \{(\lambda, 0), (\lambda, 3\lambda) | \lambda \in \mathbb{D}\})$, it remains to show the analyticity of $\Phi$ at $\mathbf{a}_0 \in \mathcal{F} \setminus \{(\lambda, 0), (\lambda, 3\lambda), \lambda \in \mathbb{D}\}$. (The parameters $(\lambda, 3\lambda)$ and $(\lambda, 0)$ in $\mathcal{F}$ corresponds to maps with a double critical point, respective maps with a symmetry (commute with $z \mapsto -z$).)

Denote by $\hat{\varphi}_{\mathbf{a}}$ the linearizer of $P_{\mathbf{a}}$ at 0 normalized so that $\hat{\varphi}'_{\mathbf{a}}(0) = 1$. Note that this normalization does not depend on which critical point is the 'closest', and is analytic on the parameter $\mathbf{a}$.

Recall that $P_{\mathbf{a}}(z) = z^3 + \sqrt{a} z^2 + \lambda z$. The two critical points are: $c_{\mathbf{a}}^{\pm} = (-\sqrt{a} \pm \sqrt{a - 3\lambda})/3$. Assume that $\mathbf{a}_0 \in \mathcal{H}$ and $\mathbf{a}_0 \neq (\lambda, 0), (\lambda, 3\lambda)$. Then the maps

$$\mathbf{a} \mapsto c_{\mathbf{a}}^{-}, \quad \mathbf{a} \mapsto c_{\mathbf{a}}^{+}, \quad \Sigma^{+} : \mathbf{a} \mapsto \frac{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{-})}{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{+})}, \quad \Sigma^{-} : \mathbf{a} \mapsto \frac{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{+})}{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{-})} = \frac{1}{\Sigma^{+}(\mathbf{a})}$$

are locally holomorphic in a neighborhood of $\mathbf{a}_0$, as long as the denominators are non-zero at $\mathbf{a}_0$. This is the case when $\mathbf{a}_0 \in \mathcal{F}$, $\mathbf{a}_0 \neq (\lambda, 3\lambda)$, in which case $\Sigma^k(\mathbf{a}_0) \in S^1 \setminus \{1\}$, $k = +, -$.

Assume now $\mathbf{a}_0 \in \mathcal{F} \setminus \{(\lambda, 0), (\lambda, 3\lambda), \lambda \in \mathbb{D}\}$. Then $\hat{\varphi}_{\mathbf{a}_0}(c_{\mathbf{a}_0}^{+}) = re^{i\theta^{+}}$, $\hat{\varphi}_{\mathbf{a}_0}(c_{\mathbf{a}_0}^{-}) = re^{i\theta^{-}}$, $\theta^{+} \neq \theta^{-}$, and for $k = +, -$, the component of 0 in $\hat{\varphi}_{\mathbf{a}_0}^{-1}(\{se^{i\theta^k}, 0 \leq s < r\})$ reaches $c_{\mathbf{a}_0}^{k}$ at one end.

**Claim 1.** There is a small polydisc neighborhood $\Delta$ of $\mathbf{a}_0$ so that $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k}) \in \{r\sqrt{|\lambda|} < |z| < r/\sqrt{|\lambda|}\}$ for $k = +, -$, the two points $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$, $k = +, -$ are on distinct radial lines, and for the half closed radial segment $K_{\mathbf{a}}^{k}$ from 0 to $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$ (excluding $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$ but including 0), the component of 0 in $\hat{\varphi}_{\mathbf{a}}^{-1}(K_{\mathbf{a}}^{k})$ reaches $c_{\mathbf{a}}^{k}$ at one end.

Proof. Only the last part is non-trivial. Fix $k = +$ or $-$. Note at first $c_{\mathbf{a}}^{k}$ are simple critical points of $\hat{\varphi}_{\mathbf{a}}$. Take $U^k$ a small disc neighborhood of $c_{\mathbf{a}_0}^{k}$ such that $\hat{\varphi}_{\mathbf{a}_0}$ has no other critical points in $\overline{U^k}$. This remains true for $\hat{\varphi}_{\mathbf{a}}$ with $\mathbf{a} \in \Delta$ and $\Delta$ small. By Rouché's theorem $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$ has two preimages (counting with multiplicity) in $U^k$. But $c_{\mathbf{a}}^{k}$ is one of them, and it has multiplicity two. So it is in fact the only preimage. Similarly one can show that the component of 0 in $\hat{\varphi}_{\mathbf{a}}^{-1}(K_{\mathbf{a}}^{k})$ ends in $U^k$ and ends at a preimage of $\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$. By uniqueness, the end is $c_{\mathbf{a}}^{k}$.

**Claim 2.** Let $h(z) = z + 1/z$. For any $\mathbf{a} = (\lambda, a) \in \Delta$, we have

$$\Phi_\lambda(\mathbf{a}) = (h \circ \phi_\lambda \circ \pi_\lambda^{-1})^{-1}(\Sigma^{+}(\mathbf{a}) + \Sigma^{-}(\mathbf{a})) =$$
$$= (h \circ \phi_\lambda \circ \pi_\lambda^{-1})^{-1} \circ h(\Sigma^{+}(\mathbf{a})) = (h \circ \phi_\lambda \circ \pi_\lambda^{-1})^{-1} \circ h(\Sigma^{-}(\mathbf{a}))\ .$$

Proof. For every $\lambda \in \mathbb{D}^*$, one can find $R_\lambda = 1/|\lambda|$ such that $\psi_\lambda$, the inverse of the linearizer $\phi_\lambda$ for $Q_\lambda$ (normalized so that $\phi_\lambda(c_\lambda) = 1$), has a univalent extension to $D(R_\lambda) \setminus [1, R_\lambda]$. Denote its image by $U'_\lambda$. Set $W'_\lambda = \pi_\lambda(U'_\lambda)$. It is a neighborhood of $\Gamma_\lambda \setminus \pi_\lambda(0)$. Then for the complex structure of $\mathcal{X}_\lambda$ the map $h \circ \phi_\lambda \circ \pi_\lambda^{-1}$ maps $W'_\lambda$ univalently onto $h(A')$, where $A' = \{z | 1 \leq |z| < R_\lambda, \quad z \notin \mathbb{R}_{+}\}$. Therefore $\pi_\lambda \circ \psi_\lambda \circ h^{-1} = (h \circ \phi_\lambda \circ \pi_\lambda^{-1})^{-1}$ maps $h(A')$ holomorphically onto $W'_\lambda$, where $h^{-1} : h(A') \to A'$ is the multi-valued function.

Fix $\mathbf{a} = (\lambda, a) \in \Delta$. For $k = +, -$, denote by $\phi_{\mathbf{a}}^{k}$ the linearizer that maps $c_{\mathbf{a}}^{k}$ to 1. We have $\phi_{\mathbf{a}}^{k}(z) = \hat{\varphi}_{\mathbf{a}}(z)/\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{k})$.

Now at least one of the critical points $c_{\mathbf{a}}^{\pm}$ is the closest. Assume, say, it is $c_{\mathbf{a}}^{+}$. Denote by $U'_{\mathbf{a}}$ the connected component of 0 in $(\phi_{\mathbf{a}}^{+})^{-1}(D(R_\lambda) \setminus [1, R_\lambda[)$. Then $\eta_{\mathbf{a}} = \psi_\lambda \circ \phi_{\mathbf{a}}^{+}$ on $U'_{\mathbf{a}}$. By Claim 1, $c_{\mathbf{a}}^{-} \in U'_{\mathbf{a}} \setminus U_{\mathbf{a}}$. Therefore $h \circ \phi_{\mathbf{a}}^{+}(c_{\mathbf{a}}^{-}) \in h(A')$ and

$$\Phi_\lambda(\mathbf{a}) = \pi_\lambda \eta_{\mathbf{a}}(c_{\mathbf{a}}^{-}) = (\pi_\lambda \psi_\lambda h^{-1}) h(\phi_{\mathbf{a}}^{+}(c_{\mathbf{a}}^{-})) = (h \circ \phi_\lambda \circ \pi_\lambda^{-1})^{-1} h(\phi_{\mathbf{a}}^{+}(c_{\mathbf{a}}^{-})) \in W'_\lambda \ .$$

But $h(\phi_{\mathbf{a}}^{+}(c_{\mathbf{a}}^{-})) = h(\frac{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{-})}{\hat{\varphi}_{\mathbf{a}}(c_{\mathbf{a}}^{+})}) = \Sigma^{+}(\mathbf{a}) + \Sigma^{-}(\mathbf{a})$. We are done with Claim 2.

From Claim 2 it is clear that $\mathbf{a} \mapsto \Phi_\lambda(\mathbf{a})$ is analytic in $\Delta$, as it is the composition of analytic functions.

This proves the analyticity of $\Phi(\mathbf{a}) = (\lambda, \Phi_\lambda(\mathbf{a}))$ in $\mathcal{H} \setminus \{(\lambda, 0), (\lambda, 3\lambda) | \lambda \in \mathbb{D}\}$.

**Analyticity of $\Phi$ on $\mathcal{H}^* := \mathcal{H} \setminus \{\mathbf{0}\}$.**

Assume now $\lambda \in \mathbb{D}^*$.

At first $\Phi_\lambda$ is holomorphic on $\mathcal{H}_\lambda \setminus \{(\lambda, 0), (\lambda, 3\lambda)\}$. The two singularities are clearly removable (by boundedness). Therefore $\Phi_\lambda : \mathcal{H}_\lambda \to \mathcal{Y}_\lambda$ is holomorphic. Next $\Phi(\mathbf{a})$ is also holomorphic in $\lambda$ on $\{(\lambda, 3\lambda_0), \lambda \in D$ a small disc about $\lambda_0\}$, as it is so on the punctured disc $D \setminus \{\lambda_0\}$ and the puncture is again removable. To get the analyticity at $(\lambda, 0)$ we just need to make the change of coordinates $(\lambda, a) \to (\lambda + a, a)$ and proceed as above. Therefore $\Phi|_{\mathcal{H}^*}$ is holomorphic in each variable, hence holomorphic as a function of the two variables $(\lambda, a)$.

**Surjectivity of $\Phi$.**

It suffices to prove that for every $\lambda \in \mathbb{D}$, the map $\Phi_\lambda : \mathcal{H}_\lambda \longrightarrow \mathcal{Y}_\lambda$ is surjective. For $\lambda = 0$ this was proved by Milnor.

Assume $\lambda \in \mathbb{D}^*$. By the above $\Phi_\lambda : \mathcal{H}_\lambda \to \mathcal{Y}_\lambda$ is holomorphic. We just need to show that $\Phi_\lambda$ is proper (as proper holomorphic maps over a connected domain are always surjective). It is easily seen that $\mathcal{H}_\lambda$ is bounded in $\mathbb{C}$ and has therefore compact closure. Let $\mathbf{a_n} = (\lambda, a_n) \in \mathcal{H}_\lambda$ be a sequence diverging to the boundary of $\mathcal{H}_\lambda$. We need to prove that $\Phi_\lambda(\mathbf{a_n}) \in \mathcal{Y}_\lambda$ diverges to the boundary of $\mathcal{Y}_\lambda$ or equivalently that $\kappa_\lambda(\Phi_\lambda(\mathbf{a_n})) \to +\infty$. Assume not and note that $\kappa_\lambda(\Phi_\lambda(\mathbf{a_n})) = \kappa_{\mathbf{a_n}}(c^1_{\mathbf{a_n}})$ by the very definitions. Hence there exists a subsequence $\mathbf{a_{n_k}}$ with $\kappa_{\mathbf{a_{n_k}}}(c^1_{\mathbf{a_{n_k}}})$ uniformly bounded, say by some $K > 0$ and by relative compactnes with $\mathbf{a_{n_k}} \to \mathbf{a}'$. We contend that $\kappa_{\mathbf{a}'}(c^1_{\mathbf{a}'}) \le K < \infty$ and thus $\mathbf{a}' \in \mathcal{H}_\lambda$, contradicting that $\mathbf{a_n}$ diverges to the boundary of $\mathcal{H}_\lambda$. If $\kappa_{\mathbf{a}'}(c^1_{\mathbf{a}'}) = 0$ we are through so assume not. Then $c^0_{\mathbf{a_{n_k}}}$ converges to $c^0_{\mathbf{a}'}$ and $c^1_{\mathbf{a_{n_k}}}$ converges to $c^1_{\mathbf{a}'}$. Hence the linearizers $\phi_{\mathbf{a_{n_k}}}$ converge uniformly on compact subsets of $B_{\mathbf{a}'}$ to $\phi_{\mathbf{a}'}$. But then also $\kappa_{\mathbf{a_{n_k}}}$ converge uniformly to $\kappa_{\mathbf{a}'}$ on compact subsets of $B_{\mathbf{a}'}$. Thus $\kappa_{\mathbf{a_{n_k}}}(c^1_{\mathbf{a_{n_k}}}) \to \kappa_{\mathbf{a}'}(c^1_{\mathbf{a}'}) \le K$.

**Bi-holomorphicity of $\Phi$ on $\mathcal{H}^*$.**

This follows from a general result about analytic and bijective maps (cf. [Ch], page 53). However the fiber structures give a direct proof: On $\mathcal{H} \setminus \mathcal{F}$, the map $\Pi^{-1} \circ \Phi$ has the form $(\lambda, a) \mapsto (\lambda, \eta_{\mathbf{a}}(c^1_{\mathbf{a}}))$. So

$$Jac(\Pi^{-1} \circ \Phi) = \begin{pmatrix} 1 & 0 \\ * & \frac{d}{da}\eta_{\mathbf{a}}(c^1_{\mathbf{a}}) \end{pmatrix} .$$

As $\Phi_\lambda$ is univalent, $\frac{d}{da}\eta_{\mathbf{a}}(c^1_{\mathbf{a}}) \neq 0$. So the determinant of the Jacobian is non zero, i.e. $\Phi$ is locally bi-holomorphic.

On $\mathcal{F}$, the argument is similar, by locally uniformizing $\mathcal{Y}$ to a poly-disc (preserving the fibers) via a bi-holomorphic map $F$ and by studying the Jacobian of $F \circ \Phi$.

**A consequence**. The set $\mathcal{F}_\lambda$ coincides with $\Phi^{-1}(\Gamma_\lambda)$, it is a real-analytic arc with end-points $(\lambda, 3\lambda)$ and $(\lambda, 0)$.

**Continuity of $\Phi$ and $\Phi^{-1}$ at $\mathbf{0}$.**

First the continuity of $\Phi$. We proceed by contradiction. Assume that there is a sequence $\mathbf{a_n} = (\lambda_n, a_n) \to \mathbf{0}$ such that $\Phi(\mathbf{a_n}) \to (0, b)$, $b \neq 0$. Then $\lambda_n \neq 0$ since $\Phi_0$ is continuous with $\Phi_0 = 0$, and $(0, b) \in \overline{\mathcal{Y}_0}$.

**Claim.** We may assume that $(0, b) \in \mathcal{Y}_0$.

Proof. We may check by hand that $\Phi(\lambda_n, 3\lambda_n) \to \mathbf{0}$. Choose $K_n \subset \mathcal{H}_{\lambda_n}$ a continuum containing $\mathbf{a}_n$ and $(\lambda_n, 3\lambda_n)$, such that $K_n \to \{\mathbf{0}\}$ in the Hausdorff topology. Then a subsequence of $\Phi(K_n)$ converges to a continuum $K$ in $\overline{\mathcal{Y}_0}$. The set $K$ contains $\mathbf{0}$ and $(0, b)$. So $K$ contains a point in $\mathcal{Y}_0 \setminus \{\mathbf{0}\}$. End of the proof.

On the other hand, $\Phi : \mathcal{H}^* \to \mathcal{Y}^*$ is bijective. So there is $(0, a)$ with $a \neq 0$ such that $\Phi(0, a) = (0, b)$. But $\Phi$ is also open near $(0, a)$ thus maps a small neighborhood $\Delta$ of $(0, a)$ onto a neighborhood of $(0, b)$. We may choose $\Delta$ to avoid $\mathbf{a_n}$ for large $n$. These facts together contradict the fact that $\Phi$ is injective.

Therefore $\Phi$ is continuous at $\mathbf{0}$, with $\Phi(\mathbf{0}) = \mathbf{0}$.

A similar argument shows that $\Phi^{-1}$ is continuous at $\mathbf{0}$.

Conclusion: $\Phi : \mathcal{H} \to \mathcal{Y}$ is a homeomorphism, bi-holomorphic on $\mathcal{H}^*$.

**An atlas for $\mathcal{X}$.** On $\mathcal{X} \setminus \{\mathbf{0}\}$ we use the natural complex structure given in the previous section. On $\mathcal{Y}$ we use $\Phi^{-1} : \mathcal{Y} \to \mathcal{H} \subset \mathbb{C}^2$. Due to the analyticity of $\Phi$ on $\mathcal{H} \setminus \{\mathbf{0}\}$ we get a collection of charts on $\mathcal{X}$, making it a 2-dimensional complex variety.

This ends the proof of parts a),b1) and c) of Theorem A'.

The remaining part b2) follows from b1) by a general result about analytic varieties fibered over $\mathbb{D}$ with fibers isomorphic to $\overline{\mathbb{C}}$. At the end of the next section we will also provide a direct proof of b2).

# 6 The extension of $\Phi$ to $\overline{\mathcal{Y}}$. Proof of Theorem B

The objective here is to prove Theorem B, which claims that the biholomorphism $\Phi : \mathcal{H} \to \mathcal{Y}$ extends to a homeomorphism $\Phi : \overline{\mathcal{H}_{\mathbb{D}}} \to \overline{\mathcal{Y}}$, where subscript $\mathbb{D}$ indicates the intersection with the cylinder above $\mathbb{D}$.

Note that $\partial\mathcal{Y}_\lambda$ is a quasi-circle and $\partial\mathcal{H}_\lambda$ is not (it contains many cusps pointing outwards). We may thus conclude

**Corollary 12** *$\Phi_\lambda$ is not a quasi-conformal map for any $\lambda \in \mathbb{D}$.*

In order to prove the theorem we will need information from outside of $\mathcal{H}$ and $\mathcal{Y}$. The essential tool we shall use is holomorphic motions, a notion developed by Sullivan and Thurston. A *(classical) holomorphic motion* of a set $\mathbb{X} \subset \overline{\mathbb{C}}$ over a complex analytic manifold $\Lambda$ is a mapping $H : \Lambda \times \mathbb{X} \to \Lambda \times \overline{\mathbb{C}}$, $(\lambda, z) \mapsto (\lambda, h(\lambda, z))$, such that

1. For all $z \in \mathbb{X}$ the map $\lambda \mapsto h(\lambda, z) =: h_z(\lambda)$ is analytic on $\Lambda$.

2. For all $\lambda \in \Lambda$ the map $z \mapsto h(\lambda, z) =: h^\lambda(z)$ is injective on $X$.

3. The time 0 map $h^0$ is the identity map.

Being a holomorphic motion is a strong property and holomorphic motions have some elementary properties which can be easily proved using normal family arguments, for example that $H$ has a unique extension as a holomorphic motion over $\Lambda$ of the closure $\overline{\mathbb{X}}$ of $\mathbb{X}$, and that $H$ is continuous with respect to $(\lambda, z)$. The next important property is that for each time $\lambda$ the map $h^\lambda$ is (the restriction of a) quasi conformal (homeomorphism). Moreover if $\Lambda = \mathbb{D}$ then $h^\lambda$ is $\log(1+|\lambda|)/(1-|\lambda|)$ quasi conformal. A far deeper property is however the celebrated Słodkovski theorem which states that a holomorphic motion of some set $\mathbb{X}$ over $\mathbb{D}$ extends to a holomorphic motion of all of $\overline{\mathbb{C}}$ over the same disc $\mathbb{D}$, the extension is by no means unique in the interior of the complement of $\mathbb{X}$ however. For details, see for example [BR], [D1].

We shall use a slightly generalized definition of holomorphic motions. Let $\mathcal{X}$ be a complex analytic manifold and $p : \mathcal{X} \to \Lambda$ be a holomorphic projection. Denote by $\mathcal{X}_\lambda$ the fiber $p^{-1}(\lambda)$. Let $E \subset \mathcal{X}_0$. A holomorphic motion of $E$ over $\Lambda$ into $\mathcal{X}$ is $H : \Lambda \times E \to \mathcal{X}$, $(\lambda, z) \mapsto H(\lambda, z)$ such that

1. For any fixed $\lambda$, $H(\lambda, \cdot)$ is injective on $E$ and maps $E$ into $\mathcal{X}_\lambda$.
2. For any fixed $z \in E$, $H(\cdot, z)$ is analytic.
3. $H(0, \cdot)$ is the identity.

Fix $P_{\lambda,b}(z) = z^3 + bz^2 + \lambda z$. Set $a = b^2$. Denote by $\mathcal{C}_0$ the connectedness locus in $\{(0, a), a \in \overline{\mathbb{C}}\}$, by $\mathcal{C}$ the connected locus in $\{(\lambda, a), (\lambda, a) \in \mathbb{D} \times \overline{\mathbb{C}}\}$, and by $\mathcal{C}^c$ the complement $(\mathbb{D}\times\overline{\mathbb{C}})\setminus\mathcal{C}$. We shall use the following ad hoc notation $\sqrt{\mathcal{A}} = \{(\lambda, b) \mid (\lambda, b^2) \in \mathcal{A}\}$ whenever $\mathcal{A} \subset \mathbb{D} \times \overline{\mathbb{C}}$.

Set $\mathbf{b} = (\lambda, b)$. For any $P_\mathbf{b}$, define $\varphi_\mathbf{b}(z)$ to be its normalized Böttcher coordinates at $\infty$, more precisely

$$\varphi_\mathbf{b}(z) := \lim_{n\to\infty} (P_\mathbf{b}^n(z))^{\frac{1}{3^n}} = \lim_{n\to\infty} z \left(\frac{P_\mathbf{b}(z)}{z^3}\right)^{\frac{1}{3}} \left(\frac{P_\mathbf{b}^2(z)}{P_\mathbf{b}(z)^3}\right)^{\frac{1}{3^2}} \cdots \left(\frac{P_\mathbf{b}^{n+1}(z)}{P_\mathbf{b}^n(z)^3}\right)^{\frac{1}{3^{n+1}}}$$

$$= z \prod_{k=0}^{\infty} \left(\frac{P_\mathbf{b}^{k+1}(z)}{P_\mathbf{b}^k(z)^3}\right)^{\frac{1}{3^{k+1}}} = z \prod_{k=0}^{\infty} \left(1 + \frac{b}{P_\mathbf{b}^k(z)} + \frac{\lambda}{P_\mathbf{b}^k(z)^2}\right)^{\frac{1}{3^{k+1}}}.$$

For $Q_\lambda(z) = z^2 + \lambda z$, define similarly

$$\varphi_\lambda(z) := \lim_{n\to\infty} (Q_\lambda^n(z))^{\frac{1}{2^n}} = \lim_{n\to\infty} z \left(\frac{Q_\lambda(z)}{z^2}\right)^{\frac{1}{2}} \left(\frac{Q_\lambda^2(z)}{Q_\lambda(z)^2}\right)^{\frac{1}{2^2}} \cdots \left(\frac{Q_\lambda^{n+1}(z)}{Q_\lambda^n(z)^2}\right)^{\frac{1}{2^{n+1}}}$$

$$= z \prod_{k=0}^{\infty} \left(\frac{Q_\lambda^{k+1}(z)}{Q_\lambda^k(z)^2}\right)^{\frac{1}{2^{k+1}}} = z \prod_{k=0}^{\infty} \left(1 + \frac{\lambda}{Q_\lambda^k(z)}\right)^{\frac{1}{2^{k+1}}}.$$

Denote by $\mathcal{V} = \{(\lambda, z), |\lambda| < 1, z \in B_\lambda^c\}$. Then

**Lemma 13** *The map $(\lambda, z) \mapsto (\lambda, \varphi_\lambda(z))$ is a homeomorphism from $\mathcal{V}$ onto $\mathbb{D} \times \mathbb{D}^c$ and is bi-holomorphic in the interior. Its inverse map is denoted by $\Psi$. We have $B_0^c = \overline{\mathbb{C}} \setminus \mathbb{D}$ and $\Psi(0, z) \equiv (0, z)$.*

Now $\Psi : \mathbb{D} \times B_0^c \to \mathcal{V}$ is a bi-holomorphic map, in particular a holomorphic motion of $B_0^c = \mathbb{D}^c$.

For any $(\lambda, b)$ with $(\lambda, b^2) \in \mathcal{C}^c$, there is a unique critical point $c$ of $P_{\lambda,b}$ in the basin of $\infty$. Denote by $c'(\lambda, b)$ the co-critical point. Then $\varphi_{\lambda,b}$ is well defined at $c'(\lambda, b)$.

**Proposition 14** *The map* $\sqrt{\mathcal{L}} : (\lambda, b) \mapsto (\lambda, \varphi_{\lambda,b}(c'(\lambda, b)))$ *is bi-holomorphic from* $\sqrt{\mathcal{C}^c}$ *onto* $\mathbb{D} \times \overline{\mathbb{D}}^c$*, odd with respect to* $b$*, and induces a bi-holomorphic map* $\mathcal{L} : \mathcal{C}^c \to \mathbb{D} \times \overline{\mathbb{D}}^c$*. In particular* $\sqrt{\mathcal{L}_0} : (0, b) \mapsto (0, \varphi_{0,b}(c'(0, b)))$ *is bi-holomorphic from* $\sqrt{\mathcal{C}_0^c}$ *onto* $\overline{\mathbb{D}}^c$*, odd with respect to* $b$*, and induces a bi-holomorphic map :* $\mathcal{C}_0^c \to \overline{\mathbb{D}}^c$*.*

For a proof see [B-H]. Define a bi-holomorphic map $\mathcal{M} = \mathcal{L}^{-1} \circ (Id \times \mathcal{L}_0) : \mathbb{D} \times \mathcal{C}_0^c \to \mathcal{C}^c$. It is a holomorphic motion.

**Proof of Theorem B:** We write also $\mathcal{M} : \mathbb{D} \times \overline{\mathbb{C}} \longrightarrow \mathbb{D} \times \overline{\mathbb{C}}$ for the extended holomorphic motion, whose existence is assured by the Słodkovski theorem ([BR], [D1]). This extension is uniquely determined on the boundary of $\mathcal{C}_0^c$, which contains the boundary of $\mathcal{H}_0$. Hence it easily follows that for each fixed $\lambda \in \mathbb{D}$ we have $\mathcal{M}(\lambda, \partial\mathcal{H}_0) = \partial\mathcal{H}_\lambda$ and $\mathcal{M}(\lambda, \mathcal{H}_0) = \mathcal{H}_\lambda$. Thus $\mathcal{M}$ restricts to $\mathbb{D} \times \overline{\mathcal{H}_0}$ is a homeomorphism from $\mathbb{D} \times \overline{\mathcal{H}_0}$ onto $\overline{\mathcal{H}_\mathbb{D}}$.

We define a new map $\mathcal{N} : \mathbb{D} \times \mathcal{X}_0 \to \mathcal{X}$ by

$$\mathcal{N}(\lambda, z) = \begin{cases} \Pi \circ \Psi(\lambda, z) & \text{if } z \in \mathcal{Y}_0^c = B_0^c = \overline{\mathbb{C}} \setminus \mathbb{D} \\ \Phi \circ \mathcal{M} \circ (Id \times \Phi_0^{-1})(\lambda, z) & \text{if } z \in \mathcal{Y}_0 = \mathbb{D} \quad . \end{cases}$$

It is easy to check that $\mathcal{N}$ is a generalized holomorphic motion.
**Claim**. The map $\mathcal{N}$ is a homeomorphism.

Assuming for the moment this claim, we have $\Phi = \mathcal{N} \circ (Id \times \Phi_0) \circ \mathcal{M}^{-1} : \mathcal{H} \to \mathcal{Y}$. However, a deep result of Faught-Roesch ([Fa, Ro]) assures that $\Phi_0$ extends to a homeomorphism (also denoted by $\Phi_0$) : $\overline{\mathcal{H}_0} \to \overline{\mathcal{Y}_0}$. Let us call it the **FR-extension**. Therefore the composition above defines a homeomorphic extension $\Phi : \overline{\mathcal{H}_\mathbb{D}} \to \overline{\mathcal{Y}}$. The big diagram below illustrates the various maps and their relationships, where the diagram $*$ is the final step.
Proof of the Claim: Clearly $\mathcal{N}$ is injective and a local homeomorphism at an $(\lambda, x)$ with $|x| \neq 1$. Fix $(\lambda_0, x_0)$ with $x_0 \in S^1$ and $|\lambda_0| = r_0 < 1$. Fix $r_0 < r < 1$. Now the set of $z \in \mathcal{Y}_0$ whose leaf intersects $\bigcup_{|\lambda| \leq r} \{\lambda\} \times \Gamma_\lambda = \Gamma_{|\lambda| \leq r}$ is compact in $\mathcal{Y}_0 = \mathbb{D}$, as it is the case in $\mathcal{H}_0$. Hence there is a neighborhood $V \subset \mathbb{C}$ of $x_0$ such that $\mathcal{N}(D_r \times V) \cap \Gamma = \emptyset$. Therefore $\Pi^{-1} \circ \mathcal{N} : D_r \times V \to D_r \times \overline{\mathbb{C}}$ is a holomorphic motion in the classical sense. It is therefore a homeomorphism from $D_r \times V$ onto its image ([BR], [D1]). So $\mathcal{N} = \Pi \circ (\Pi^{-1} \circ \mathcal{N})$ is a homeomorphism from $D_r \times V$ onto its image, in particular a local homeomorphism at $(\lambda_0, x_0)$. **q.e.d.**

**Corollary 15** *(=Theorem A'.b2)) The variety* $\mathcal{X}$ *is isomorphic to* $\mathbb{D} \times \overline{\mathbb{C}}$*.*

**Proof :** The map $\mathcal{N} : \mathbb{D} \times \overline{\mathbb{C}} = \mathbb{D} \times \mathcal{X}_0 \to \mathcal{X}$ is a holomorphic motion, and analytic on $\mathbb{D} \times \{|z| \geq R\}$ for some large $R$. Now for each $\lambda \in \mathbb{D}$ the maps $\mathcal{N}_\lambda$ pulls back the complex structure of $\mathcal{X}_\lambda$ to a complex structure $\sigma_\lambda$ on $\overline{\mathbb{C}}$, which has compact support. It follows that $\lambda \mapsto \sigma_\lambda$ is holomorphic. By the Measurable Riemann Mapping Theorem, there is a holomorphic motion

$$\mathbb{D}\times\overline{\mathbb{C}} \xleftarrow{\mathcal{O}} \mathbb{D}\times\overline{\mathbb{C}}$$

normalized so that $\mathcal{O}_\lambda$ is tangent to the identity at $\infty$ and fixes 0, such that $\mathcal{O}_\lambda$ integrates $\sigma_\lambda$. Now

$$\mathcal{N}\circ\mathcal{O}^{-1} : \mathbb{D}\times\overline{\mathbb{C}} \xleftarrow{\mathcal{O}} \mathbb{D}\times\overline{\mathbb{C}} \xrightarrow{\mathcal{N}} \mathcal{X}$$

is bi-holomorphic, and fiber-preserving. **q.e.d.**

$$\begin{array}{ccccccccccc}
 & & & & \mathbb{D}\times\overline{\mathbb{D}}^c & \xleftarrow{(\lambda,b^2)} & \mathbb{D}\times\overline{\mathbb{D}}^c & & & & \\
 & & & \nearrow_{Id\times\mathcal{L}_0} & \uparrow_{\mathcal{L}} & & \uparrow_{\sqrt{\mathcal{L}}} & & \text{Böttcher position of the} & & \\
 & & \mathbb{D}\times\mathcal{C}_0^c & \xrightarrow{\mathcal{M}} & \mathcal{C}^c & \xleftarrow{(\lambda,b^2)} & \sqrt{\mathcal{C}^c} & & \text{escaping co-critical point} & & \\
 & & \cap & & \cap & & & & & & \\
 & & \mathbb{D}\times\overline{\mathbb{C}} & \xrightarrow{\mathcal{M}} & \mathbb{D}\times\overline{\mathbb{C}} & & \boxed{\text{Słodkovski extension}} & & & & \\
 & & \cup & & \cup & & & & & & \\
\mathbb{D}\times\mathcal{H}_0 & \subset & \mathbb{D}\times\overline{\mathcal{H}_0} & \xrightarrow{\mathcal{M}} & \overline{\mathcal{H}}_{\mathbb{D}} & \supset & \mathcal{H} & \supset & \{\lambda\}\times\mathcal{H}_\lambda & \ni & \mathbf{a} \\
{}_{Id\times\Phi_0}\downarrow & & {}^{\text{FR-extension}}_{Id\times\Phi_0}\downarrow & * & \downarrow_{\Phi} & & \downarrow_{\Phi} & & \downarrow_{\Phi_\lambda} & & \eta_{\mathbf{a}}(c^1_{\mathbf{a}})\downarrow \\
\mathbb{D}\times\mathcal{Y}_0 & \subset & \mathbb{D}\times\overline{\mathcal{Y}_0} & \xrightarrow{\mathcal{N}} & \overline{\mathcal{Y}} & \supset & \mathcal{Y} & \supset & \mathcal{Y}_\lambda & \xleftarrow{\pi_\lambda} & B_\lambda \\
 & & \cup & & \cup & & \cap & & & & \\
 & & \mathbb{D}\times\partial\mathcal{Y}_0 & \xrightarrow{\mathcal{N}} & \partial\mathcal{Y} & & \mathcal{X} & & & & \\
 & & \cap & & \cap & & & & & & \\
 & & \mathbb{D}\times\mathcal{Y}_0^c & \xrightarrow{\mathcal{N}} & \mathcal{Y}^c & & & & \boxed{\mathcal{Y}_0=B_0=\mathbb{D}} & & \\
 & & Id\uparrow & & \uparrow_{\Pi} & & & & & & \\
 & & \mathbb{D}\times B_0^c & \xrightarrow{\Psi} & \mathcal{V} & & & & & & \\
 & & (\lambda,z) & \mapsto & (\lambda,h(\lambda,z)) & & \boxed{\text{Böttcher coordinates at }\infty} & & & &
\end{array}$$

# A Boundary extensions as $|\lambda| \to 1$

Fix $\lambda_0 \in \mathbb{S}^1$. In order to extend our coordinate $\Phi$ to the $\lambda_0$-slice the first task is to define a suitable $\mathcal{Y}_{\lambda_0}$ from the dynamical plane of $Q_{\lambda_0}$, and then study the limit of $\Phi$ as $\lambda \in \mathbb{D}$ and $\lambda \to \lambda_0$ radially.

In the parabolic case $\lambda_0 = e^{i2\pi p/q}$, $(p,q) = 1$ the sets $U^0_\lambda$, their boundaries and the equivalence relations $\sim_\lambda$ have describeable limits. This will lead to the definition of $\mathcal{Y}_{\lambda_0}$

together with a map $\Phi_{\lambda_0}$ from the corresponding part of $\partial\mathcal{H}$ to $\mathcal{Y}_{\lambda_0}$. We will prove that $\Phi_{\lambda_0}$ is a radially continuous extension of $\Phi$. This will be the content of [P-R-T].

The case $\lambda_0 = e^{i2\pi\theta}$ with $\theta$ irrational is a lot more subtle. Let us denote by $\omega(c_0)$ the $\omega$-limit of the critical orbit.

Yoccoz [Yo] has proved that, depending on whether $\theta$ satisfies the Brjuno condition or not, either $Q_{\lambda_0}$ has a Siegel disk $\Delta_{\lambda_0}$ and the pointed domains $(U^0_\lambda, 0)$ converge in the sense of Caratheodory to $(\Delta_{\lambda_0}, 0)$; or $Q_{\lambda_0}$ has a Cremer point and $(U^0_\lambda, 0)$ diverge in the sense of Caratheodory to the singleton $\{0\}$.

The easiest case to study is the case that $\Delta_{\lambda_0}$ exists, and has a Jordan curve boundary passing through the critical point $c_{\lambda_0}$. In this case $\omega(c_{\lambda_0}) = \partial\Delta_{\lambda_0}$ and the linearizer $\phi_{\lambda_0} : \Delta_{\lambda_0} \longrightarrow \mathbb{D}$ normalized by $\phi_{\lambda_0}(c_{\lambda_0}) = 1$ induces an equivalence relation $\sim_{\lambda_0}$ on $\overline{\mathbb{C}}\backslash\Delta_{\lambda_0}$ similar to the one in the case $\lambda \in \mathbb{D}$. Therefore $(K_{\lambda_0}\backslash\Delta_{\lambda_0})/\sim_{\lambda_0}$ would be a good candidate for our $\mathcal{Y}_{\lambda_0}$. For $\theta$ of bounded type S. Zakeri [Za] has indeed proved that the parameter slice $\{(\lambda_0, a) | a \in \mathbb{C}\}$ contains an embedded copy of $\mathcal{Y}_{\lambda_0}$. But it remains to prove that this copy sits on the boundary of $\mathcal{H}$ and that $\Phi$ extends radially continuously on it.

In the remaining cases, conjecturally, the compact sets $\partial U^0_\lambda$ converges in the Hausdorff topology to $\omega(c_{\lambda_0})$ and $\sim_\lambda$ converges to some equivalence relation $\sim_{\lambda_0}$. Then we can define $Y_{\lambda_0}$ accordingly. The difficulty here is to prove that the Hausdorff limit of $\partial U^0_\lambda$ does exist and does not exceed $\omega(c_{\lambda_0})$. We don't have yet definite results in these cases. However the new results by Inou-Shishikura on parabolic renormalization will surely contribute to future progress on this problem.

# B Application to other hyperbolic components

In order to apply Milnor's result ([M2]) we will need to work on a different parametrization of cubic polynomials.

For $d \geq 3$ let $\mathcal{P}^d = \{P | P(z) = z^d + a_{d-2}z^{d-2} + \ldots + a_0\} \approx \mathbb{C}^{d-1}$ denote the space of monic centered polynomials of degree $d$.

We want to carry our result on $\mathcal{H}$ to the central hyperbolic component $\mathcal{H}^3$ of $\mathcal{P}^3$. We show here that $\mathcal{H}^3$ is naturally a double cover of $\mathcal{H}$. We then define an appropriate square root, denoted by $\mathcal{Y}^3$, of $\mathcal{Y}$, and provide a dynamical-analytic coordinate from $\mathcal{H}^3$ onto $\mathcal{Y}^3$.

To be more precise, define the Moduli space $\mathcal{M}^d$ as the space of affine conjugacy classes $[P]$ of degree $d$ polynomials $P$ and let $proj^d : \mathcal{P}^d \longrightarrow \mathcal{M}^d$ denote the natural projection. If $\rho^{d-1} = 1$ and $P \in \mathcal{P}^d$, then $\rho P(z/\rho) \in \mathcal{P}^d$ and thus the conjugation by any of the rotations $z \mapsto \rho z$, $\rho^{d-1} = 1$ leaves $\mathcal{P}^d$ invariant. On the other hand any other affine map conjugates any element of $\mathcal{P}^d$ out of $\mathcal{P}^d$. It follows that $proj^d : \mathcal{P}^d \longrightarrow \mathcal{M}^d$ is a degree $(d-1)$ branched covering. Let $\mathcal{H}^d_{\mathcal{M}}$ denote the central hyperbolic component of $\mathcal{M}^d$, then also the restriction $proj^d : \mathcal{H}^d \longrightarrow \mathcal{H}^d_{\mathcal{M}}$ has degree $d-1$. For any $d > 1$ the complex line $\{P \in \mathcal{P}^d | a_1 \in \mathbb{C},\ a_0 = a_2 = a_3 = \ldots = a_{d-2} = 0\}$ is contained in the branching locus, and for $d = 3, 4$ it equals the branching locus for $proj^d$. For any $d > 2$ the polynomials on this line are characterized by having 0 as a fixed point with multiplier $a_1$ and a $d-1$ fold rotational symmetry. Since this line equals the branching locus when $d = 3$ (and 4) the projection onto moduli space is a simple squaring respectively cubing around this line.

For $\lambda \in \mathbb{D}^* := \mathbb{D} \setminus \{0\}$, recall that $\psi_\lambda : \overline{\mathbb{D}} \to \mathbb{C}$ denote the inverse of the linearizer $\phi_\lambda$ for $Q_\lambda$, and that $\psi_\lambda(1)$ is the critical point of $Q_\lambda$. The map $\beta : \mathbb{D} \longrightarrow \mathbb{C}$ given by $\beta(0) = 0$ and $\beta(\lambda) = \psi_\lambda(-1)$ for $\lambda \in \mathbb{D}^*$ is holomorphic by the removable singularities theorem. Define a holomorphic branched double covering $\chi : \mathbb{D} \times \overline{\mathbb{C}} \longrightarrow \mathbb{D} \times \overline{\mathbb{C}}$ by $\chi(\lambda, z) = (\lambda, \chi_\lambda(z)) = (\lambda, z^2 + \beta(\lambda))$ and define $\mathcal{S}^3 = \chi^{-1}(\mathbb{D} \times \overline{\mathbb{C}} \setminus \mathcal{U})$. On the space $\mathcal{S}^3$ we define an equivalence relation $\sim^3$ by declaring $(\lambda_1, z_1) \sim^3 (\lambda_2, z_2)$ iff $\lambda_1 = \lambda_2 = \lambda$ and either $z_1 = z_2$ or both $z_1$ and $z_2$ belongs to the same connected component of ${\chi_\lambda}^{-1}(\psi_\lambda(\mathbb{S}^1 \setminus \{-1\}))$ and $\chi_\lambda(z_1) \sim_\lambda \chi_\lambda(z_2)$. Let $\Pi^3 : \mathcal{S}^3 \longrightarrow \mathcal{S}^3 / \sim^3 := \mathcal{X}^3$ denote the natural projection and equip $\mathcal{X}^3$ with the quotient topology. This topology is easily seen to be Hausdorff. Define also $\mathcal{Y}^3 = \Pi^3(\chi^{-1}(\mathcal{B} \setminus \mathcal{U})) \subset \mathcal{X}^3$ and an involution $\kappa : \mathcal{X}^3 \longrightarrow \mathcal{X}^3$ by $\kappa(\Pi^3(\lambda, z)) = \Pi^3(\lambda, -z)$. The fixed point set for this involution is the set $\mathcal{C}^3 = \Pi^3(\mathbb{D}, 0)$. Define $\Sigma : \mathcal{Y}^3 \longrightarrow \mathcal{Y}$ by $\Sigma(\Pi^3(\lambda, z)) = \Pi(\chi(\lambda, z))$ so that $\Sigma$ is a degree 2 branched covering with covering transformation $\kappa$ and branching locus equal to $\mathcal{C}^3$ and branch value set equal to the complex disk $\mathcal{V} = \Pi(\mathbb{D}, 0) = \Pi(\{(\lambda, \beta(\lambda)) | \lambda \in \mathbb{D}\})$. Then $\mathcal{Y}^3$ and hence $\mathcal{X}^3$ has a (unique) complex structure for which the map $\Sigma$ is holomorphic. This is the unique structure for which the projection $\Pi^3 : \mathbb{D} \times \overline{\mathbb{C}} \setminus \overline{\mathcal{U}} \longrightarrow \mathcal{X}^3$ is holomorphic.

**Theorem C.** *There exists a bi-holomorphic map $\Phi^3 : \mathcal{H}^3 \longrightarrow \mathcal{Y}^3$ such that $\Sigma \circ \Phi^3 = \Phi \circ proj^3$.*

**Proof :** The restriction $proj^3 : \mathcal{H}^3 \longrightarrow \mathcal{H}$ is also a holomorphic degree 2 branched covering with the set of branch values equal to $\Phi^{-1}(\mathcal{V})$. It follows that there exists a bi-holomorphic map $\Phi^3 : \mathcal{H}^3 \longrightarrow \mathcal{Y}^3$ with $\Sigma \circ \Phi^3 = \Phi \circ proj^3$. This map satisfies $\Phi^3(-a_0, a_1) = \kappa(\Phi^3(a_0, a_1))$ and is unique up to post composition by $\kappa$ or equivalently by precomposition by the map $(a_0, a_1) \mapsto (-a_0, a_1) : \mathcal{H}^3 \to \mathcal{H}^3$. **q.e.d.**

Note that it follows from Theorem A' that there exists a bi-holomorphic map from $\mathcal{X}^3$ onto $\mathbb{D} \times \overline{\mathbb{C}}$, commuting with projection to the first coordinates.

## Universality of $\mathcal{H}$

Our coordinate on $\mathcal{H}$ and $\mathcal{H}^3$ can be carried to a similar coordinate to many other hyperbolic components, through a result of Milnor ([M2]). He has classified the hyperbolic components of $\mathcal{P}^d$ into a finite number of types. He then constructs one abstract complex analytic model for each type and proves that any hyperbolic component is canonically bi-holomorphic to its abstract model.

In order to make precise statements, let us recall Milnor's classification [M2, M2, pages 3–8] of hyperbolic components in terms of mapping schemes. By definition a (finite) (critical) *mapping schema* is a weighted finite directed graph $\mathbf{S} = (S, w)$ such that each vertex $v \in S$ is the origin of precisely one directed edge $e_v$, which is allowed to be a closed loop, and $w(v)$ is a non negative integer called the critical weight of $v$. The vertex $v$ is critical iff $w(v) > 0$. The positive integer $d(e_v) = 1 + w(v)$ is called the degree of the edge $e_v$. The mapping schema $\mathbf{S}$ is connected iff the underlying un-oriented graph is path connected. Clearly each connected component of $\mathbf{S}$ contains a unique cycle $v_0, \ldots, v_{p-1}$, i.e. the terminal point of $e_{v_i}$ is the vertex $v_{(i+1) \bmod p}$. It is further assumed that $\mathbf{S}$ is *critical* i.e. each cycle contains at least one critical vertex, and any entry vertex, i.e.

one which is not the terminal point of any edge, is critical. Two mapping schemes are isomorphic if there is a weight preserving bijection between the two underlying oriented graphs.

A mapping schema $\mathbf{S}$ is *reduced* iff every vertex is critical. There is a natural projection map from the space of mapping schemes to the space of reduced mapping schemes given by contracting each chain of degree 1 edges as well as their endpoints onto the terminal vertex of the chain.

(Reduced) Mapping Schemes encode the dynamical types of hyperbolic components in a canonical way: Firstly, associated with any hyperbolic polynomial $P$ is a mapping schema $\mathbf{S} = (S, w)$, where the set of vertices of $S$ equals the set of Fatou components $U$ for $P$, with $U$ containing at least one critical or postcritical point. For $U \in S$ a vertex the edge $e_U$ terminates on the vertex $P(U)$ and the critical weight equals the number of $P$ critical points in $U$ counted with multiplicity, so that the degree of the edge $e_U$ equals the degree of the restriction $P_{|U}$, by the Riemann-Hurwitz theorem. Secondly, if two polynomials $P_1$ and $P_2$ belong to the same hyperbolic component, then they have isomorphic mapping schemes, because they are quasi-conformally conjugate in a neighborhood of their Julia sets, or $J$-equivalent, in the language of Mañé-Sad-Sullivan ([M-S-S]). Thus any hyperbolic component for which all the Fatou components are simply connected has a unique (isomorphic class of) associated mapping schema.

For cubic polynomials with connected Julia set, there are four types of hyperbolic components according to their reduced mapping schema (the names are not completely identical to those of Milnor, there are also two types with disconnected Julia set that we do not mention, see [B-H, B-H2] for related results):

Type I, (truly) cubics, corresponds to polynomials with an attracting periodic orbit for which one component of the immediate basin contains both critical points and hence this component is mapped properly by degree 3 onto its image.

Type II, bi-transitive, corresponds to polynomials with an attracting periodic orbit for which both critical points are in the immediate basin of a single attracting cycle, but in two different connected components. The two components are thus mapped quadratically onto their images and the first return map on a component of the immediate basin has degree 4.

Type III, capture, corresponds to polynomials with an attracting periodic orbit which attracts both critical points, but only one of them is in the immediate basin, the other is contained in a strictly preperiodic preimage. Again the map is quadratic on the critical components, but the first return map on a connected component of the immediate basin is quadratic.

Type IV, bi-quadratic, corresponds to maps with two distinct attracting cycles, each one with a critical point in the immediate basin.

The corresponding reduced mapping schemes are illustrated in Figure 3.

A type IV (bi-quadratic) hyperbolic component $H$ is in some sense not cubic since the dynamics of the two critical points are completely decoupled. Such a component is simply the product of two one-dimensional hyperbolic components and the direct product

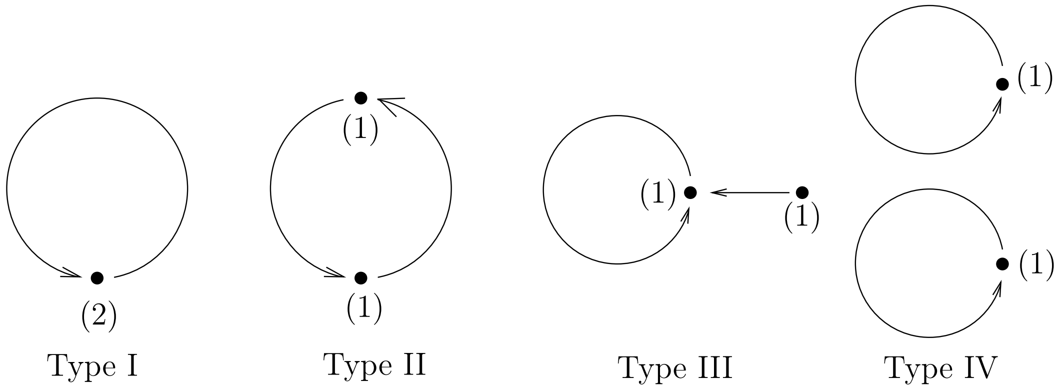


Figure 3: The reduced mapping schemes corresponding to the four types of cubic hyperbolic components. The vertex weight is indicated by the adjacent number in parenthesis. This figure is equivalent to [M2, Figure 2, p. 4] but with different notations.

of the multiplier maps of the two attracting periodic orbits defines a bi-holomorphic map onto the bi-disk $\mathbb{D} \times \mathbb{D}$. Bi-quadratics are also bi-renormalizable with two quadratic like renormalizations, see also the paper [E-Y] for extended results in this direction.

Let us return to polynomials of general degree. Milnor [M2, Definition 2.4, p. 6] defines a *universal polynomial model space* $\widehat{\mathcal{P}}^{\mathbf{S}}$ associated with every mapping schema $\mathbf{S}$ as follows: The elements of $\widehat{\mathcal{P}}^{\mathbf{S}}$ are the self maps $f$ of $|S| \times \mathbb{C}$ (where $|S|$ denotes the set of vertices of $S$) such that $f$ maps $v \times \mathbb{C}$ onto $v' \times \mathbb{C}$ as a monic centered polynomial of degree $d(e_v)$, where $v'$ is the terminal vertex of $e_v$. The hyperbolic locus $\widehat{\mathcal{H}}^{\mathbf{S}} \subset \widehat{\mathcal{P}}^{\mathbf{S}}$ is the subset of maps for which every critical point converges to an attracting periodic cycle. The central hyperbolic component $\widehat{\mathcal{H}}_0^{\mathbf{S}}$ of $\widehat{\mathcal{H}}^{\mathbf{S}}$ is the connected component of $\widehat{\mathcal{H}}^{\mathbf{S}}$ containing the map $f$ which has the form $z^{d(e_v)}$ on each complex plane $v \times \mathbb{C}$. Note that for $\mathbf{S}$ the weighted graph with only one vertex, with weight $d-1$, and hence with exactly one edge of degree $d$, we have $\widehat{\mathcal{P}}^{\mathbf{S}} = \mathcal{P}^d$, and $\widehat{\mathcal{H}}^{\mathbf{S}} = \mathcal{H}^d$, where $\mathcal{H}^d$ denotes **the central hyperbolic component** of $\mathcal{P}^d$, namely the one containing $z^d$. Note also that the dynamics of any $\widehat{\mathcal{P}}^{\mathbf{S}}$ factors into sub factors corresponding to each connected component $\mathbf{S}'$ of $\mathbf{S}$ and that each such factor is universal in the sense that it depends on $\mathbf{S}'$ only and in particular not on the other factors corresponding to the other components of $\mathbf{S}$. Evidently the attracting periodic cycles are in $1:1$ correspondence with the connected components of $\mathbf{S}$, e.g. such as type IV above.

Here is a schematic picture:

$$\begin{array}{c}
\{\text{hyperbolic polynomials}\} \subset \mathcal{P}^d \\
\downarrow \\
\{\text{H, hyperbolic components}\} \\
\downarrow \\
\{\text{Julia equivalent classes}\} \\
\downarrow \\
\{\text{mapping schemes}\} \\
\downarrow \\
\{\, \mathbf{S}, \text{reduced schemes}\} \longleftrightarrow \{\widehat{\mathcal{P}}^{\mathbf{S}}\} \longleftrightarrow \{\widehat{\mathcal{H}}_0^{\mathbf{S}}\} \\
\updownarrow \\
\left\{ \bigsqcup \mathbf{S_i} \;\middle|\; \mathbf{S_i} \text{ connected reduced schema} \right\} \longleftrightarrow \left\{ \prod \widehat{\mathcal{H}}_0^{\mathbf{S_i}} \right\}
\end{array}$$

Though not explicitly stated, Milnor establishes the following factorization theorem for hyperbolic components:

**Theorem 16 (Milnor)** *Let $H$ be a hyperbolic component in $\mathcal{P}^d$ for some $d \geq 2$ with reduced mapping schema $\mathbf{S}$ and let $\mathbf{S_1}, \ldots, \mathbf{S_n}$ be the connected components of $\mathbf{S}$. Then $H$ is bi-holomorphic to the direct product $\widehat{\mathcal{H}}_0^{\mathbf{S_1}} \times \ldots \times \widehat{\mathcal{H}}_0^{\mathbf{S_n}}$. In particular, if one $\mathbf{S_i}$ is the Type I reduced schema above (i.e. with only one vertex of weight $2$), then the central hyperbolic component $\mathcal{H}^3$ of $\mathcal{P}^3$ appears as a factor in $H$. Consequently every Type I hyperbolic component in $\mathcal{P}^3$ is canonically bi-holomorphic to $\mathcal{H}^3$.*

(Note that this statement is similar to the discussion of the Teichmüller space of a rational map by McMullen and Sullivan in [McM-S].

Combining this with our result, we obtain

**Corollary 17** *The dynamical-analytic coordinate for $\mathcal{H}^3$ constructed in Theorem C yields automatically a similar coordinate for any truly cubic (Type I) factor of any hyperbolic component in $\mathcal{P}^d$, $d \geq 3$; in particular for any Type I hyperbolic component of $\mathcal{P}^3$.*

Note that the tools and ideas we have developed in this paper can easily be adapted to the study of Type III components.

Milnor's result on isomorphism of hyperbolic components of the same type is in fact *only valid* for the spaces $\mathcal{P}^d$ of monic centered polynomials and the universal polynomial model spaces $\widehat{\mathcal{P}}^{\mathbf{S}}$. The reason is that for an arbitrary parameter space a hyperbolic component may be a branched covering of a corresponding component in $\mathcal{P}^d$. In this case it is not in general bi-holomorphic to the covered component. Or if some component in $\mathcal{P}^d$ has a symmetry the branching may go the other way. In fact some of the hyperbolic components in $\mathcal{P}^d$ do have symmetries. The most prominent example is the central hyperbolic component $\mathcal{H}^d \subset \mathcal{P}^d$, consisting of maps for which some attracting fixed point attracts all critical points in the immediate basin. In a terminology similar to the above these polynomials are truly degree $d$.

# C Table of notations

| $\lambda \in \mathbb{D}$ | $P_{\mathbf{a}}(z) = P_{\lambda,a}(z) =$ $\lambda z + \sqrt{a} z^2 + z^3$ | $Q_\lambda(z) = \lambda z + z^2$ |
|---|---|---|
| attracting fixed point | $0$ | $0$ |
| attracting basin | $B_{\mathbf{a}}$ | $B_\lambda$ |
| **case** $\lambda \neq 0$ | | |
| max. domain mapped univalently onto a round disc by a linearizer | $U^0_{\mathbf{a}}$ | $U^0_\lambda$ |
| critical points<br>critical values | $c^0_{\mathbf{a}} \in \partial U^0_{\mathbf{a}}$, $c^1_{\mathbf{a}}$<br>$v^0_{\mathbf{a}} \in U^0_{\mathbf{a}}$, $v^1_{\mathbf{a}}$ | $c^0_\lambda \in \partial U^0_\lambda$<br>$v^0_\lambda \in U^0_\lambda$ |
| normalized linearizer | $\phi_{\mathbf{a}}$, $\phi_{\mathbf{a}}(c^0_{\mathbf{a}}) = 1$<br>$\hat{\varphi}_{\mathbf{a}}$, $\hat{\varphi}'_{\mathbf{a}}(0) = 1$ | $\phi_\lambda$, $\phi_\lambda(c^0_\lambda) = 1$ |
| filled potential function<br>open potent. domain<br>closed potent. domain | $\kappa_{\mathbf{a}} : B_{\mathbf{a}} \to [-\infty, +\infty[$<br>$U^0_{\mathbf{a}}(t) =$c.c. of $\kappa^{-1}_{\mathbf{a}}[-\infty, t[$<br>$L_{\mathbf{a}}(t) =$c.c. of $\kappa^{-1}_{\mathbf{a}}[-\infty, t]$ | $\kappa_\lambda : B_\lambda \to [-\infty, +\infty[$<br>$U^0_\lambda(t) =$c.c. of $\kappa^{-1}_\lambda[-\infty, t[$<br>$L_\lambda(t) =$c.c. of $\kappa^{-1}_\lambda[-\infty, t]$ |
| **case** $\lambda = 0$**,** $a \neq 0$ | | |
| Böttcher coordinates | $\phi_{\mathbf{a}}$ | $\phi_0 = id$ |
| max. domain mapped univalently onto a round disc by $\phi$ | $U^0_{\mathbf{a}}$ | $B_0 = \mathbb{D}$ |
| critical points<br>potential function | $c^0_{\mathbf{a}} = 0$, $c^1_{\mathbf{a}} \in \partial U^0_{\mathbf{a}}$<br>$g_{\mathbf{a}} : B_{\mathbf{a}} \to [-\infty, 0[$ | $c^0_0 = 0$<br>$g_\lambda : B_\lambda \to [-\infty, 0[$ |
| quotient | | $\Gamma_\lambda = \partial U^0_\lambda / \sim, \mathcal{Y}_\lambda = B_\lambda \setminus U^0_\lambda / \sim$<br>$\mathcal{X}_\lambda = \mathbb{C} \setminus U^0_\lambda / \sim$<br>where $z_1, z_2 \in \partial U^0_\lambda$, $z_1 \sim z_2$<br>$\Longleftrightarrow \phi_\lambda(z_1) = \overline{\phi_\lambda(z_2)}$ |
| parameter space | $\mathcal{H}$ the hyperbolic component of $z^3$<br>$\eta_{\mathbf{a}} =$ extension of $\phi^{-1}_\lambda \circ \phi_{\mathbf{a}}$: a subset of $B_{\mathbf{a}} \longrightarrow B_\lambda$<br>$\mathcal{X} := \bigsqcup_\lambda \mathcal{X}_\lambda$, $\mathcal{Y} := \bigsqcup_\lambda \mathcal{Y}_\lambda$, $\Gamma = \bigsqcup_\lambda \Gamma_\lambda$<br>$\Phi : \mathbf{a} \mapsto [\eta_{\mathbf{a}}(c^1_{\mathbf{a}})]$, $\mathcal{H} \to \mathcal{Y} \subset \mathcal{X}$, $\mathcal{F} \to \Gamma$ | |

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