Cours doctoral, nov.-dec. 2013

Tan Lei

Université d'Angers, November 19, 2014

Plan du cours

1. Dessins d'enfants et application du Thm. de Koebe
2. Longueurs extrémales, modules d'anneaux et Thm. de Koebe
3. Théorèmes de Thurston: fractions rationnelles et homéomorphismes de surfaces
4. Modèles semi-locaux de la dynamique
5. Groupe de monodromie itérée.
6. Espace de Koch
7. Dessins d'enfants et champs de vecteurs
8. z^2 et doublement d'angles

1. Dessins d'enfants, Two key results

Definition. A complex polynomial $P \in \mathbb{C}[X]$ is called Belyi if it has only two critical values, at 0 and 1 (can be chosen to be at other points).

Two such polynomials f, g are covering equivalent if there is an affine map $M(z) = az + b$, $a \neq 0$, such that $\begin{array}{ccc} \overline{\mathbb{C}} & \stackrel{M}{\to} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array}$.

Theorem (Grothendieck). In each equivalence class there is a representative polynomial P with algebraic numbers as coefficients.

Theorem (Belyi) (1) Given any finite set A of algebraic numbers there is a polynomial P with rational coefficients such that $P(A) \cup \{\text{critical values of } P\} \subset \{0, 1\}$.

(2) Given any rational map f with algebraic numbers as critical values (for example $f \in \overline{\mathbb{Q}}[X]$), there is a polynomial P with rational coefficients such that cr. val$(P \circ f) \subset \{0, 1, \infty\}$.

Basic facts

- If $cv(f) \subset A$, then $cv(P \circ f) \subset P(A) \cup cv(P)$.

Proof. Pullback by P then by f a small disc outside $P(A) \cup cv(P)$.

$\mathbb{Q} := \{\text{rational numbers}\}$,

$\overline{\mathbb{Q}} := \{\text{algebraic numbers}\} = \{a \in \mathbb{C} | P(a) = 0 \text{ for some } P \in \mathbb{Q}[X]\}$.

- $\overline{\mathbb{Q}}$ is countable, dense in $\mathbb{C}$, a field, algebraically closed.

Moreover, the roots of a polynomial system in $\overline{\mathbb{Q}}[X_1, \cdots, X_n]$ form either an unbounded set, or a finite set with algebraic coordinates.

Proof of Grothendieck's thm: $\exists P \circ M$ monic and fixing 0

Any P monic, $P(0) = 0$ and $cv(P) \subset \{0, 1\}$ must have algebraic coefficients.

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Proof. Let P be monic. Let $\{a_i, i \in I\} = P^{-1}(0)$, with multiplicity s_i for a_i, and $\{b_j, j \in J\} = P^{-1}(1)$ with multiplicity t_j for b_j, $d = \sum s_i = \sum t_j$. Then

$$\prod_I (z - a_i)^{s_i} = P(z) = \prod_J (z - b_j)^{t_j} + 1 \ .$$

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$$\prod_I (z - a_i)^{s_i} = P(z) = \prod_J (z - b_j)^{t_j} + 1 \ .$$

Equalizing the coefficients on both sides above we get that $\mathbf{a} = \{a_i\}$ and $\mathbf{b} = \{b_j\}$ must satisfy a system of polynomial equations $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}_d$, with $\mathbf{F} = \mathbf{F}_{(s_i, t_j)} \in (\mathbb{Q}[\mathbf{a}, \mathbf{b}])^d$. It is known

$$\textit{algebra} : \{\text{zeros of } \mathbf{F}\} = \begin{cases} \text{either an unbounded algebraic variety} \\ \text{or a finite set with algebraic coordinates} \end{cases} .$$

Choose now the s_i and t_j's so that the critical values of P are in $\{0, 1\}$, in other words, $\sum(s_i - 1) + \sum(t_j - 1) = d - 1$ (s_i and t_j's are called the **passport of** P).

Geometric proof (Buff) of the boundedness of $\mathbf{F}(\mathbf{a},\mathbf{b})=\mathbf{0}$

Let $(\mathbf{a},\mathbf{b})=\{a_i, i\in I; b_j, j\in J\}$ be a solution of $\mathbf{F}(\mathbf{a},\mathbf{b})=\mathbf{0}$.
Let $P(z)=\prod_I (z-a_i)^{s_i}=\prod_J (z-b_j)^{t_j}+1$. Thus the critical values of P are in $\{0,1\}\subset[0,1]$.

$$\begin{array}{ccc} \mathbb{C}\smallsetminus\overline{\mathbb{D}} & \xrightarrow[\text{biholomorphic}]{\exists \frac{1}{4^{1/d}}z+O(1)} & \mathbb{C}\smallsetminus P^{-1}([0,1]) \\ {\scriptstyle z^d}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbb{C}\smallsetminus\overline{\mathbb{D}} & \xrightarrow[\frac{1}{4}(z+\frac{1}{z})+\frac{1}{2}]{\text{Riemann}} & \mathbb{C}\smallsetminus[0,1] \end{array}$$

Conclude using $P(0)=0$ and Koebe 1/4-theorem:

$$\{a_i, b_j \mid i\in I, j\in J\}=P^{-1}(\{0,1\})\subset P^{-1}([0,1])\subset \mathbb{D}(0,4^{1-\frac{1}{d}}).$$

(#{normalized Belyi polynomials with a given passport} =
$\#\{\mathbf{F}=\mathbf{0}\} \overset{\text{Bezout}}{=} \prod \deg(F_i)$ =?? counted with multiplicity.)

Action of the absolute Galois group $\Gamma = Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$

Let $P(X) = \sum a_i X^i$ with $a_i \in \overline{\mathbb{Q}}$. Let $\sigma \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

$$\begin{array}{ccccccccc} a_i & & P & \xrightarrow{\textit{differentiate}} & P' & \xrightarrow{\textit{take zeros}} & \textit{cpts} & \xrightarrow{P} & \textit{cvs} \\ \sigma\downarrow & & \sigma\downarrow & & \sigma\downarrow & & \downarrow\sigma & & \downarrow\sigma \\ \sigma(a_i) & & Q & \xrightarrow{\textit{differentiate}} & Q' & \xrightarrow{\textit{take zeros}} & \textit{cpts} & \xrightarrow{Q} & \textit{cvs} \end{array}$$

In particular if $a_i \in \mathbb{Q}$, $\forall i$, then $Q = P$ and σ permutes $cpts(P)$ and permutes $cvs(P)$.

alg. conjugacy

$$\begin{array}{ccc} \overline{\mathbb{Q}} \ni z & \xrightarrow{P} & P(z) \\ \sigma\downarrow & & \downarrow\sigma \\ \sigma(z) & \xrightarrow{Q} & Q(z) \end{array}$$

$$\sigma(P \circ L) = \sigma(P) \circ \sigma(L).$$

Proof of Belyi's theorem

Thm: Given any finite set A of algebraic numbers there is a polynomial $P \in \mathbb{Q}[X]$ such that $P(A) \cup \{\text{cr. values of } P\} \subset \{0, 1\}$.

Step 1. Reducing one by one the number of points outside $\mathbb{Q}$
Step 2. Reducing one by one the number of points inside $\mathbb{Q}$

Step 1. Need the known (non trivial) fact : $Fix(\Gamma) = \mathbb{Q}$.

$$B_0 = A \smallsetminus \mathbb{Q} \xrightarrow[\text{say } d^o = d]{P_0 \in \mathbb{Q}[X]} 0 \downarrow$$

$$B_1 = \{0\} \cup cv(P_0) \smallsetminus \mathbb{Q} \xrightarrow[d^o \leq d-1]{P_1 \in \mathbb{Q}[X]} 0 \downarrow$$

$$\{0\} \cup cv(P_1) \smallsetminus \mathbb{Q} \quad \cdots$$

with $P_1(z) = \prod_{\delta \in B_1} (z - \delta)$. Any $\sigma \in \Gamma$ must fixe P_0 thus permutes B_1 and fixes coefficients(P_1). So these coefficients are $\in \mathbb{Q}$.

So $\exists P \in \mathbb{Q}[X]$ such that $Y := P(A) \cup \{\text{critical values of } P\} \subset \mathbb{Q}$.

Step 2. Reducing the number of points inside $\mathbb{Q}$

Given any finite set $Y \subset \mathbb{Q}$, we will show that there is $L \in \mathbb{Q}[X]$ such that $L(Y) \cup \{\text{critical values of } L\} \subset \{0, 1\}$.

For any increasing triple $a, b, c \in \mathbb{Q}$,

$$a < b < c \xrightarrow[\alpha,\beta\in\mathbb{Q}]{\alpha z+\beta} 0 < \frac{p}{p+q} < 1 \xrightarrow[\text{magic map}]{z^p(1-z)^q} 0 < s,\ s \in \mathbb{Q}.$$

□

Question. For a polynomial f in $\mathbb{C}[X]$, is this possible to find another polynomial P so that $cvs(P \circ f) \in \{0, 1\}$?

Dessin d'enfants

For any meromorphic map $F : S \to \overline{\mathbb{C}}$ with S a compact Riemann surface such that $cv(F) \subset \{0, 1, \infty\}$. The set $D_F := F^{-1}([0, 1])$ is called the dessin of F.

In the case of a Belyi polynomial P, D_P is an embedded tree in $\mathbb{C}$ whose vertices are bi-colored.

Grothendieck's correspondence:
The set $Belyi(\overline{\mathbb{C}})/\sim_{\mathbb{C}} = Belyi(\overline{\mathbb{Q}})/\sim_{\mathbb{C}} = Belyi(\overline{\mathbb{Q}})/\sim_{\overline{\mathbb{Q}}}$

$\overset{1:1}{\longleftrightarrow}$

the set of embedded bi-colored trees/ $homeo^+$ preserving color.

Any bounded arc linking 0 and 1 will lift to a homeomorphic dessin.

Proof. Need $Aut(\mathbb{D})$, the Theorems of Riemann (for simply connected domains in $\mathbb{C}$), Carathéodory, Schönflies to get a topological polynomial, and then Riemann mapping theorem and branched covering liftings to get a genuine polynomial.

Carathéodory's theorem (1913)

1. If U is a Jordan domain in $\mathbb{C}$ then a Riemann map f: U $\to \mathbb{D}$ (the unit disk) extends continuously to a homeomorphism of the closures.
2, For such U there is a homeomorphism F : $\overline{U} \to \overline{\mathbb{D}}$, conformal in the interior.
3. For any pair of Jordan domains there is a conformal homeomorphism between them which extends to a homeomorphism of the closures.
4. (more general version) Let $g : \mathbb{D} \to U$ be the inverse of a Riemann map for a simply connected domain $U \subset \overline{\mathbb{C}}$. Then g extends continuously to $G : \overline{\mathbb{D}} \to \overline{U}$ iff ∂U is locally connected.
5. (full form, not needed here) g extends to a homeomorphism $G : \overline{\mathbb{D}} \to$ the prime ends closure of U.

Corollary (Schönflies theorem): Any homeomorphism between two Jordan curves in S^2 extends to a global homeomorphism.

2. Extremal width, moduli of annuli and Koebe 1/4 thm

An **annulus** is an open set $A \subset \overline{\mathbb{C}}$ with exactly two complementary components E, E'.

Theorem 1: If at least one of E, E' contains more than one point, there is a conformal map $f : A \to \{1 < |z| < R\} \approx \{0 < x < \log R, 0 \leq y \leq 2\pi\}/(x, 0) \sim (x, 2\pi)$. The value R is unique, and is $< \infty$ iff both E, E' contain more than one point. We define the modulus: $m(A) = \dfrac{logR}{2\pi} = \dfrac{width}{\text{length(equator)}} = \dfrac{area}{(\text{length(equator)})^2}$. If both E, E' are points, A is conformally $\mathbb{C}^*$. Set $m(A) = \infty$.

Theorem 2. Grötzsch inequality: For A an annulus and $A_i, i \in I$ pairwise disjoint sub-annuli of A, we have $\sum_i m(A_i) \leq m(A)$.

Theorem 3. Koebe 1/4 theorem.

Proof of Theorem 1. Use twice Riemann to send A to an annulus bounded by a S^1 and a smooth curve, uniformize the universal cover strip to a straight strip. Use Carathéodory and unicity in $Aut(\mathbb{D})$ to show the periodicity and go to the quotient.

Proof of Theorems 2 and 3, extremal width techniques.

Main problem The conformal map $f : A \to$ a round annulus is implicit, so the modulus $m(A)$ is difficult to compute. But the extremal width technique helps to make estimates.

Let $A \subset \mathbb{C}$ be open. Let $\rho \geq 0$ be measurable. We say that $\rho(z)|dz|$ is an admissible conformal metric on A if

$$area_A(\rho) := \iint_A \rho(x+iy)^2 dxdy \neq 0, +\infty$$

Given such a ρ, a curve $\gamma : [a, b] \to A$ with absolutely continuous coordinates has a well defined ρ-length

$$\ell_\gamma(\rho) = \int_a^b |\gamma'(t)| \cdot \rho(\gamma(t))\, dt.$$

Let Γ be a curve family contained in A. One defines its extremal width, extremal length, by

$$w(\Gamma) = \frac{1}{\lambda(\Gamma)} := \inf_{\rho \text{ admissible}} \frac{area_A(\rho)}{\inf_\gamma \ell_\gamma(\rho)^2}.$$

$\Gamma \subset A$, $w(\Gamma) = \inf_\rho \frac{area_A(\rho)}{\inf_\gamma \ell_\gamma(\rho)^2}$ for $area_A(\rho) \neq 0, \infty$.

Properties of $w(\Gamma)$: 0. invariance w.r.t. $\rho \to \lambda\rho$ and $A \subset A'$.

1. conformal invariance
2. If Γ has more curves than Γ_1, then $w(\Gamma) \geq w(\Gamma_1)$
3. Assume Γ_1 and Γ_2 lay on disjoint open sets.
   Γ has more curves than $\Gamma_1 \cup \Gamma_2 \implies w(\Gamma) \geq w(\Gamma_1) + w(\Gamma_2)$.
4. For a round annulus A, $m(A) = w$(essential curves in A). And $\exists$ extremal metric.
5. Grötzsch inequality on moduli of annuli (Theorem 2)
6. Grötzsch extremal problem: For an annulus A separating a disc E and a continuum E' linking a point u to ∞, and $A_{symmetric}$ with complement E and E'' a ray from u to ∞ normal to E, we have $m(A) \leq m(A_{symmetric})$.

Proof. 3. Let $\Gamma_i \subset A_i \subset A$ with $A_1 \cap A_2 = \emptyset$. Normalize ρ so that $\inf_\gamma \ell_\gamma(\rho) = 1$. 4. Use Cauchy-Schwarz $\leq$. 6. Use the (symmetric) extremal metric ρ of $A_{symmetric}$ and set $\rho = \infty$ on the ray.

Proof of 4. Let $A = \{1 < |z| < R\}$. Let Γ be the family of essential curves in A. To show $\boxed{w(\Gamma) := \inf_\rho \frac{area_A(\rho)}{\inf_\gamma \ell_\gamma(\rho)^2} = \frac{\log R}{2\pi}}$.

For $\mu(z) = \frac{1}{|z|}$ on A, and 0 elsewhere, we have $\inf_\gamma \ell_\gamma(\mu) = 2\pi$ and $area_A(\mu) = 2\pi \log R$. So $w(\Gamma) \leq \frac{area_A(\mu)}{\inf_\gamma \ell_\gamma(\mu)^2} = \frac{\log R}{2\pi} < \infty$.

Now for any admissible $\rho(z)$ on A we may assume $0 < \inf_\gamma \ell_\gamma(\rho) < \infty$ (otherwise either area=∞ or it does not contribute to sup.) We may then normalize so that $\inf_\gamma \ell_\gamma(\rho) = 2\pi$.

Now $area(\rho) = \int_1^R \int_0^{2\pi} \rho^2(re^{i\theta}) r dr d\theta$, and

$$(2\pi)^2 \overset{inf}{\leq} \left(\int_0^{2\pi} \rho(re^{i\theta}) r d\theta \right)^2 \overset{Cauchy-Sch}{\leq} \int_0^{2\pi} \rho^2(re^{i\theta}) r d\theta \cdot \int_0^{2\pi} r d\theta.$$

Hence $2\pi \log R = \int_1^R \frac{2\pi}{r} dr \overset{by\ above}{\leq} \int_1^R \int_0^{2\pi} \rho^2(re^{i\theta}) r d\theta dr = area$. So

$$\frac{area}{\inf_\gamma \ell_\gamma(\rho)^2} = \frac{area}{(2\pi)^2} \underset{by\ above}{\geq} \frac{\log R}{2\pi} \text{ and } w(\Gamma) = \inf_\rho \geq \frac{\log R}{2\pi}.$$

$\Gamma \subset A$, $w(\Gamma) = \inf_\rho \frac{area_A(\rho)}{\inf_\gamma \ell_\gamma(\rho)^2}$ for $area_A(\rho) \neq 0, \infty$.

Proof of 5 (and 2,3). Let $\{A_i, i \in I\}$ be disjoint sub annuli of A. Let $\Gamma_i \subset A_i$, $\Gamma \subset A$ be the families of essential curves. Clearly Γ contains Γ_i. To show $\sum_i m(A_i) = \sum_i w(\Gamma_i) \leq w(\Gamma) = m(A)$ (the two '=' come from conformal invariance).

Take any ρ s.t. $area_A(\rho) \neq 0, \infty$ and $\inf_{\Gamma_i} \ell_\gamma(\rho) \geq \inf_\Gamma \ell_\gamma(\rho) > 0$. Then $\infty > area_{A_i}(\rho) > 0$ (Cauchy-Schwarz), and

$$\sum_i w(\Gamma_i) \leq \sum_i \frac{area_{A_i}(\rho)}{\inf_{\Gamma_i} \ell_\gamma(\rho)} \leq \sum_i \frac{area_{A_i}(\rho)}{\inf_\Gamma \ell_\gamma(\rho)} \leq \frac{area_A(\rho)}{\inf_\Gamma \ell_\gamma(\rho)}.$$

Proof of 6. Let Γ be the set of Jordan curves separating the disc E and the set $\{u, \infty\}$. Take the (symmetric) extremal metric ρ of $A_{symmetric}$ and set $\rho = \infty$ on the ray. Any curve γ in Γ admits, by reflection along the ray if necessary, an essential curve γ' in $A_{symmetric}$ with shorter length. So

$$m(A) \overset{more\ curves}{\leq} w(\Gamma) \leq \frac{area(\rho)}{(length\ in\ A_{symmetric})^2} \overset{extremal}{=} m(A_{symmetric}).$$

Koebe function and Koebe 1/4 theorem

Koebe 1/4 theorem. Let $f : \mathbb{D} \to \mathbb{C}$ univalent with $f(0) = 0$ and $f'(0) = 1$. Let $\delta = \inf\{|w|, w \notin f(\mathbb{D})\}$. Then $\delta \geq 1/4$.

The Koebe function is : $K(z) = \dfrac{z}{(1-z)^2} = \sum_{n\geq 1} nz^n$. The map $(4\delta K)^{-1}$ maps $\mathbb{C} \setminus]-\infty, -\delta]$ univalently onto $\mathbb{D}$.
Variants : z^2, $z + 1/z$, $z + 1/z - 2$...

Proof. Idea : estimate the modulus of $f(\{\epsilon < |z| < 1\})$ as $\varepsilon \to 0$.

$$\text{Set} \quad \min_{|z|=\epsilon} |f(z)| := \epsilon_*, \quad \min_{|w|=\epsilon_*} |(4\delta K)^{-1}(w)| =: \eta.$$

Then $\dfrac{1}{2\pi} \log \dfrac{1}{\epsilon} = m\{\epsilon < |z| < 1\} = m(f(\{\epsilon < |z| < 1\})) \overset{\text{Grötzsch}}{\leq}$

$m(\mathbb{C} \setminus (\overline{D}(0, \epsilon_*) \cup [-\delta, -\infty[)) \overset{(4\delta K)^{-1}}{\leq} m\{\eta < |z| < 1\} = \dfrac{1}{2\pi} \log \dfrac{1}{\eta}$

$\underset{\eta \sim \frac{\epsilon_*}{4\delta} \sim \frac{\epsilon}{4\delta}}{=} \dfrac{1}{2\pi} \log \dfrac{4\delta}{\epsilon + o(\epsilon)}$. Letting $\varepsilon \to 0$, we get $1 \leq 4\delta$. □

(see Rudin or wikipedia for a proof without extremal length.)

Exercises (2 and 3 for understanding Thurston's theorem)

1. Give the variants of Koebe 1/4 theorem in the case $f'(0) \neq 1$, and in the case that f is univalent outside the unit disc fixing ∞. Prove Grothendieck's theorem for dessin d'enfants.

2. Let $f : A \to A'$ be a degree d holomorphic covering between two annuli A, A'. Give a relation between $m(A)$ and $m(A')$.

3. Let $I_1, \cdots I_k$ be disjoint intervals. Let $I_{ij\delta}$ be finitely many disjoint intervals so that :

$$I_i \supset I_{ij\delta} \xrightarrow[\text{of slope } d_{ij\delta}]{f, \text{affine homeo}} I_j$$

Find an inequality between $(d_{ij\delta})$ and $(|I_i|)$ (try first $k = 1$).

4. Prove, for a round annulus A,
$m(A) = \lambda$(a full measure set of transverse rays).

5. Prove that the inverse of a Riemann mapping has radial limit along almost all rays.

3. Return to the theorems

A complex polynomial $P \in \mathbb{C}[X]$ is Belyi if it has only two finite critical values, which are at 0 and 1.

We say $f \sim_{affine} g$ if there is an affine map $M(z) = az + b$, $a \neq 0$, such that

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{M} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array} .$$

With Koebe 1/4 theorem we have completed the proof of

Theorem (Grothendieck). In each equivalence class of Belyi polynomials there is a representative polynomial P with algebraic numbers as coefficients.

A priori, it's not clear that the number of equivalence classes in a given degree is $\neq 0$, $\neq \infty$.

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the set of embedded bi-colored trees/ $homeo^+$ preserving color.

Any bounded arc linking 0 and 1 will lift to a homeomorphic dessin.

Proof. Need $Aut(\mathbb{D})$, the Theorems of Riemann (for simply connected domains in $\mathbb{C}$), Carathéodory, Schönflies to get a topological polynomial, and then Riemann mapping theorem and branched covering liftings to get a genuine polynomial.

Corollary $\#\{\text{equivalence classes in degree } d\} < \infty$.

dessin $\rightarrow$ topological polynomial (by Carathéodory, Schönflies)

Proof of : topological polynomial $\rightarrow$ genuine complex polynomial.
We need

Riemann mapping theorem. Any simply connected Riemann surface is biholomorphic to either $\mathbb{D}$, $\mathbb{C}$ or $\overline{\mathbb{C}}$. Any doubly connected Riemann surface ($\pi_1 \approx \mathbb{Z}$) is biholomorphic to either $\{1 < |z| < R\}$, $\mathbb{D}^*$ or $\mathbb{C}^*$, has then a well defined modulus.

Lifting-structure lemma. Any orientation preserving topological branched covering π from a compact topological surface S onto $\overline{\mathbb{C}}$ induces a complex structure on S making it a compact Riemann surface.

So

$$\begin{array}{ccc} S & \xrightarrow{\text{by Riemann } \exists\varphi} & \overline{\mathbb{C}} \\ \text{top. poly.} \searrow & & \swarrow \text{complex poly.} \\ & \overline{\mathbb{C}} & \end{array} .$$

□

Lifting-structure lemma. Any orientation preserving branched covering π from a compact topological surface S onto $\overline{\mathbb{C}}$ induces a complex structure on S making it a compact Riemann surface.

Proof (following Douady-Douady) :

Key step. Given any $U \subset \overline{\mathbb{C}}$ open and simply connected containing a unique critical value b, and any connected component V of $\pi^{-1}(U)$, then $V(b) := V \cap \pi^{-1}(b)$ reduces to a single point, V is simply connected, and there are homeos φ, ψ such that

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & D_r \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle z^d} \\ U & \xrightarrow[\psi]{} & D_{r'} \end{array}$$

(Corollary: Any dessin d'enfant is connected).

Proof of the key step.

Since there is only one subgroup of $\mathbb{Z}$ of index d, the d-fold covering $\pi : V^* \to U^*$ lifts along z^d to a homeo $\psi : V^* \to D_r^*$.

Montrons que $\mathrm{V}(b)$ est réduit à un point. Notons x_i les points de $\mathrm{V}(b)$, avec $i \in \mathrm{I}$ fini. L'application $\pi : \mathrm{V} \to \mathrm{U}$ est propre, donc on peut trouver un voisinage U' de b dans U et une partition $(\mathrm{V}'_i)_{i\in\mathrm{I}}$ de $\mathrm{V}' = \mathrm{V} \cap \pi^{-1}(\mathrm{U}')$ en ouverts non vides telle que $x_i \in \mathrm{V}'_i$ pour tout i. Et on peut supposer que U' est de la forme $\varphi^{-1}(\mathrm{D}_\rho)$ avec $\rho > 0$. Alors les $\psi(\mathrm{V}'_i \cap \mathrm{V}^*)$ forment une partition de $\mathrm{D}^*_{\rho'}$, où $\rho' = \rho^{1/d}$, en ouverts non vides. Il en résulte que $\mathrm{Card}\,\mathrm{V}(b) = 1$.

Notons x l'unique point de $\mathrm{V}(b)$, et prolongeons ψ en une bijection, encore notée ψ, de V sur D' par $\psi(x) = 0$. Montrons que c'est un homéomorphisme. Si $x' \in \mathrm{V}^*$ tend vers x, $\pi(x')$ tend vers b, $\varphi(\pi(x')) = (\psi(x'))^d$ tend vers 0, donc $\psi(x')$ tend vers 0 . Par suite ψ est continue. Si $\psi(x')$ tend vers 0, $\pi(x') = \varphi^{-1}((\psi(x'))^d)$ tend vers b, donc x' tend vers x puisque $\pi; \mathrm{V} \to \mathrm{U}$ est propre et $\mathrm{V}(b) = \{x\}$. Par suite ψ^{-1} est continue.

Définition. *Soit* B *une surface topologique. Un* **revêtement ramifié fini** *de* B *est une surface topologique* X *munie d'une application continue propre* $\pi : X \to B$ *vérifiant la condition suivante :*

(RR) *Pour tout point* $x \in X$, *il existe une carte complexe de* X *centrée en* x *et une carte complexe de* B *centrée en* $\pi(x)$ *telles que l'expression de* π *dans ces cartes soit de la forme* $z \mapsto z^d$.

Exemples. 1) Tout revêtement fini de B est un revêtement ramifié fini de B. Par contre, un revêtement ramifié fini de B n'est pas en général un revêtement de B.

2) Soient B une surface de Riemann et X une surface de Riemann munie d'une application analytique propre $\pi : X \to B$. On suppose que π n'est constante sur aucune composante connexe de X. Alors X est un revêtement ramifié (analytique) fini.

Proof of the Lifting-structure lemma

6.1.10.

Proposition. *Soit* X *un revêtement ramifié fini de* B. *Pour toute structure* $\mathbb{C}$*-analytique sur* B, *il existe une structure* $\mathbb{C}$*-analytique sur* X *et une seule telle que* $\pi : \mathrm{X} \to \mathrm{B}$ *soit analytique.*

Démonstration. Notons δ l'ensemble de ramification de π. Soit $(\varphi_i)_{i\in \mathrm{I}}$ un atlas de B formé de cartes centrées tel que, pour tout i, φ_i soit une carte de domaine U_i centrée en b_i avec $(\mathrm{U}_i - \{b_i\}) \cap \Delta = \varnothing$. Pour chaque i, soit $(\mathrm{V}_{i,j})_{j\in \mathrm{J}_i}$ la famille des composantes connexes de $\pi^{-1}(\mathrm{U}_i)$, et posons $\mathrm{J} = \{(i,j) \mid j \in \mathrm{J}_i\}$. Pour chaque $(i,j) \in \mathrm{J}$, soit $\psi_{i,j}$ une carte de X telle que l'expression de π dans les cartes $\psi_{i,j}$ et φ_i soit de la forme $z \mapsto z^{d_{i,j}}$ (6.1.8, proposition). Les $\psi_{i,j}$ forment un atlas de X, et il résulte de (6.1.5) que les changements de cartes sont analytiques. Cet atlas définit une structure analytique sur X qui répond à la question. L'unicité résulte de (6.1.5). cqfd

4. Faithful action of $\Gamma = Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$

Let $P(X) = \sum a_i X^i$ with $a_i \in \overline{\mathbb{Q}}$. Let $\sigma \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

$$\begin{array}{ccccccc} a_i & P & \xrightarrow{differentiate} & P' & \xrightarrow{take\ \ zeros} & cpts & \xrightarrow{P} & cvs \\ \sigma\downarrow & \sigma\downarrow & & \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma \\ \sigma(a_i) & Q & \xrightarrow{differentiate} & Q' & \xrightarrow{take\ \ zeros} & cpts & \xrightarrow{Q} & cvs \end{array}$$

alg. conjugacy
$$\begin{array}{ccc} \overline{\mathbb{Q}} \ni z & \xrightarrow{P} & P(z) \\ \sigma\downarrow & & \downarrow\sigma \\ \sigma(z) & \xrightarrow{Q} & Q(z) \end{array} \qquad \sigma(P \circ L) = \sigma(P) \circ \sigma(L).$$

Theorem. The group Γ acts faithfully : 1. on $Belyi(\overline{\mathbb{Q}})$, $Belyi(\overline{\mathbb{Q}})/\sim$ and $clean\text{-}Belyi(\overline{\mathbb{Q}})/\sim$ (Lenstra-Schneps); 2. on the set of dynamical Belyi polynomials and commutes with iterations (Pilgrim). The pull-back dessins Hausdorff-converge to the Julia set, which contains a dense set of algebraic numbers (Pilgrim).

Open problem. See the action on trees.

Proof of the faithfulness of Γ on $Belyi(\overline{\mathbb{Q}})/\sim_{affine}$

Let $id \neq \gamma \in \Gamma$. Then $\exists \alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ such that $\gamma(\alpha) \neq \alpha$. Set $f(z) = \int^z z(z-1)^2(z-\alpha)^3 dz$. By Belyi $\exists g \in \mathbb{Q}[X]$ such that $g \circ f \in$ *clean-Belyi*$(\overline{\mathbb{Q}})$. To prove: $\gamma.(g \circ f) \not\sim_{affine} g \circ f$.

Assume the contrary: $\gamma.(g \circ f) = g \circ f \circ M$, with M affine. Using $g \in \mathbb{Q}[X]$, we get $g \circ \gamma.f = \gamma.(g \circ f) = g \circ f \circ M$. By the decomposition theorem below, $\gamma.f = affine \circ f \circ M$. This implies that M transports the critical points of $\gamma.f$: $\{0, 1, \gamma.\alpha\}$ to those of f: $\{0, 1, \alpha\}$, preserving the multiplicities. So M must fixe 0 and 1 and must be the identity. But $M(\alpha) = \gamma.\alpha \neq \alpha$. A contradiction. □

Decomposition theorem. If 4 polynomials in $\mathbb{C}[X]$ satisfy $g_1 \circ f_1 = g_2 \circ f_2$ with the same degree decomposition, then $f_2 = affine \circ f_1$.

Question. True in other fields? For rational maps?

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Let $id \neq \gamma \in \Gamma$. Then $\exists \alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ such that $\gamma(\alpha) \neq \alpha$. Set $f(z) = \int^z z(z-1)^2(z-\alpha)^3 dz$. By Belyi $\exists g \in \mathbb{Q}[X]$ such that $g \circ f \in$ *clean-Belyi*$(\overline{\mathbb{Q}})$. To prove: $\gamma.(g \circ f) \not\sim_{affine} g \circ f$.

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Question. True in other fields? For rational maps?

Geometric proof. Look at the preimage of a line segment linking the critical values.

5. Pilgrim's theory on dynamical Belyi polynomials

It is a polynomial $P \in \mathbb{C}[X]$ with critical values at 0,1, with all preimages of 1 simple critical points (i.e. two-valenced in the dessin), and with 0,1 as two extremities of the dessin.

Properties:

- The iterates of P are all dynamical Belyi polynomials.
- For two such polynomials P, L, if $P = B \circ L \circ A$ with A, B affine. Then $B = id$.

Proof. B preserves the set $\{0, 1\}$ of critical values. But 0 et 1 are distinguished since all preimages of 1 simple critical points, so B fixes 0 and 1, and $B = id$.

- In each class of $\textit{clean-Belyi}(\mathbb{C})/\sim_{\textit{affine}}$ there are (finitely many) dynamical representatives.
- Exo. (by Grothendieck), dynamical Belyi polynomials are in $\overline{\mathbb{Q}}[X]$.
- Γ acts on the tower of iterates, and the action is faithful.

Theorem. Let $P \neq L$ be two distinct dynamical Belyi polynomials of the same degree.

1. P, L are not conformally conjugate. And P has no auto-conjugacy (except id).
2. $n \geq 2 \Longrightarrow P^{\circ n} \not\sim_{affine} L^{\circ n}$, and $P^{\circ n}, L^{\circ n}$ are not conformally conjugate, furthermore $P^{\circ n}, L^{\circ n}$ are not Thurston equivalent.

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Proof. 1. Recall that $P = B \circ L \circ A$ with A, B affine implies $B = id$. So if $P = A^{-1} \circ L \circ A$, then $A = id$ and $P = L$.

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$P^{\circ n} = L^{\circ n} \circ A \Longrightarrow P \circ P^{\circ(n-1)} = L \circ (L^{\circ(n-1)} \circ A)$

$\overset{\text{decomposition}}{\Longrightarrow} P^{\circ(n-1)} = affine \circ L^{\circ(n-1)} \circ A \Longrightarrow P^{\circ(n-1)} = L^{\circ(n-1)} \circ A$

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Let z be a fixed point of P, i.e. $z = P(z) = P^{\circ 2}(z)$. Then $L(z) = L \circ P(z) = L \circ L \circ A(z) = P^{\circ 2}(z) = z$. So $Fix(P) \subset Fix(L)$. By symmetry $Fix(P) = Fix(L)$. So $\exists\ \lambda \neq 0$ s.t. $P(x) - x = \lambda(L(x) - x)$. For $x = 1$, we have $0 - 1 = \lambda(0 - 1)$. So $\lambda = 1$ and $P = L$. $\square$

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Theorem. Let $P \neq L$ be two distinct dynamical Belyi polynomials of the same degree.

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6. Thurston's equivalence and theorems

Consider pairs (F, Z) such that $F : S^2 \to S^2$ is an orientation preserving branched covering of a topological sphere S^2 of degree $\deg(F) \geq 2$ and $Z := \cup$ iterated-orbit(critical points) is finite. [1]

We say that (F_0, Z_0) is **combinatorially equivalent** to (F_1, Z_1) if there is a pair of homeomorphisms $\phi, \theta : S^2 \to S^2$ such that $\phi(Z_0) = Z_1$, $\theta(Z_0) = Z_0$, $\theta \sim_{isotopic} id$ rel Z_0 and $F_1 \circ \phi \circ \theta = \phi \circ F_0$.

$$\begin{array}{ccc} S^2, Z_0 & \xrightarrow{\theta \text{ then } \phi} & S^2, Z_1 \\ \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_1} \\ S^2, Z_0 & \xrightarrow{\phi} & S^2, Z_1 \end{array} \qquad \begin{array}{ccc} S^2, Z & \xrightarrow{\theta \text{ then } \phi} & \overline{\mathbb{C}}, X \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f \in \mathrm{Rat}_d} \\ S^2, Z & \xrightarrow{\phi} & \overline{\mathbb{C}}, X. \end{array}$$

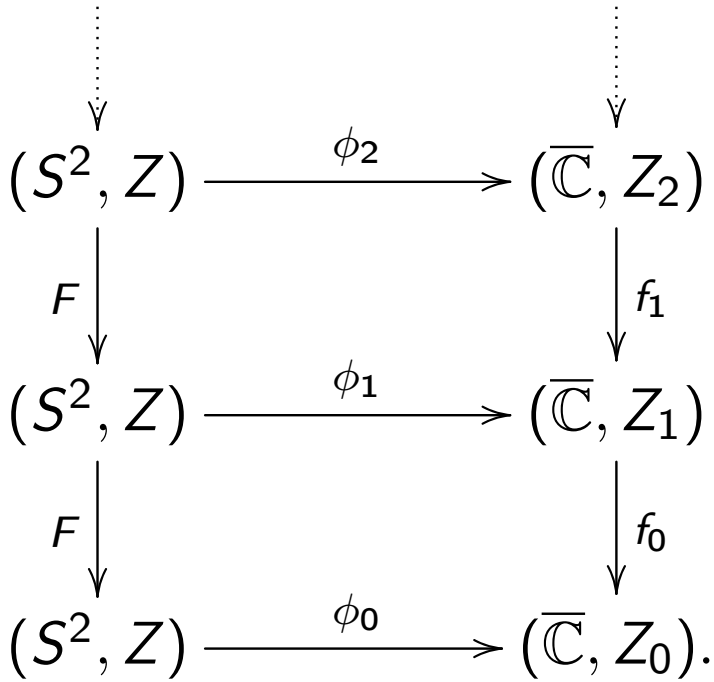
Thurston's characterization theorem provides a necessary and sufficient condition for the class of (F, Z) to contain a **rational map representative** (f, X). We will now present the condition.

[1] one can take a larger finite set Z satisfying $F(Z) \subseteq Z$.

A Jordan curve γ is said *non-peripheral* rel Z if it is disjoint from Z, and each of its two complementary components contains at least two points of Z. We say that $\Gamma = \{\gamma_1, \cdots, \gamma_k\}$ is a Z-*multicurve*, if each γ_i is non-peripheral rel Z, and the γ_j's are mutually disjoint and mutually non-homotopic rel Z.

Each such Γ induces a (F, Z)-*transition matrix* F_Γ together with its *leading eigenvalue* λ_Γ as follows: Let $(\gamma_{i,j,\delta})_\delta$ denote the components of $F^{-1}(\gamma_j)$ homotopic to γ_i rel Z (there might be no such components). Then $F : \gamma_{i,j,\delta} \to \gamma_j$ is a topological covering of a certain degree $d_{i,j,\delta}$. The transition matrix is defined to be $F_\Gamma = (\sum_\delta 1/d_{i,j,\delta})$. This is a non-negative matrix. By Perron-Frobenius Theorem there is a non-negative eigenvalue λ_Γ that coincides with the spectral radius of F_Γ.

Thurston's theorem (branched cover version) Assume that (F, Z) is not in some exceptional cases. Then the equivalence class of (F, Z) contains a rational representative (f, X) iff $\lambda_\Gamma < 1$ for every Γ. Furthermore (f, X) is unique up to Möbius conjugacy, and there is an algorithm approximating f.



By normalizing three points, f_n converges uniformly to f.

Corollary. For any two dynamical Belyi polynomials P, L, if $P^{\circ n}$, $L^{\circ n}$ are combinatorially equivalent for some $n \geq 1$, then $P = L$.

Relation with Exercises 2 and 3

Lemma. Let $A = (a_{ij})$ be a non-negative square matrix. Denote by λ its leading eigenvalue. Then $\lambda < 1$ iff there is a vector $v > 0$ such that $Av < v$.

Proof (provided by H.H. Rugh). Necessity: Assume $v > 0$ and $Av < v$. Then $Av \leq av$ for some $0 \leq a < 1$. Define a norm on the underlying vector space by $\|x\| = \sum_i (v_i \cdot |x_i|)$. Then, writing $|x|$ as the vector whose i-th entry is $|x_i|$, we have

$$\|A^t x\| = v^t A^t |x| = (Av)^t |x| \leq a v^t |x| = a\|x\|,$$

where A^t and v^t denote the transposes. Therefore, $\lambda = \|A^t\| \leq a < 1$.

Sufficiency: Now assume $\lambda < 1$. By the continuity of the spectral radius, there is $\epsilon > 0$ such that the spectral radius λ_ϵ of $A + \epsilon := (a_{ij} + \epsilon)$ satisfies $\lambda_\epsilon < 1$. Now the Perron-Frobenius Theorem assures that λ_ϵ is the leading eigenvalue and it has a strictly positive eigenvector $v > 0$. So $Av \leq (A + \epsilon)v = \lambda_\epsilon v < v$.

Comparing with the surface homeomorphism version

Consider pairs (F, Z) such that $F : S \to S$ is an orientation preserving homeomorphism of a topological compact surface S and Z a finite set satisfying $F(Z) = Z$.

We say that (F_0, Z_0) is **combinatorially equivalent** to (F_1, Z_1) if there is a pair of homeomorphisms $\phi, \theta : S \to S$ such that $\phi(Z_0) = Z_1$, $\theta(Z_0) = Z_0$, $\theta \sim_{isotopic} id$ rel Z_0 and $F_1 \circ \phi \circ \theta = \phi \circ F_0$.

$$\begin{array}{ccc} S, Z_0 & \xrightarrow{\theta \text{ then } \phi} & S, Z_1 \\ \Big\downarrow{\scriptstyle F_0} & & \Big\downarrow{\scriptstyle F_1} \\ S, Z_0 & \xrightarrow[\phi]{} & S, Z_1 \end{array}$$

Thurston's classification theorem provides a simple model for each class.

A Jordan curve γ is said *non-peripheral* rel Z if it is disjoint from Z, and each of its two complementary components contains at least two points of Z. We say that $\Gamma = \{\gamma_1, \cdots, \gamma_k\}$ is a *Z-multicurve*, if each γ_i is non-peripheral rel Z, and the γ_j's are mutually disjoint and mutually non-homotopic rel Z.

Each such Γ induces a *(F, Z)-transition matrix* F_Γ together with its *leading eigenvalue* λ_Γ as follows: Let $\gamma_{i,j}$ denote the component (if exists) of $F^{-1}(\gamma_j)$ homotopic to γ_i rel Z. Then $F : \gamma_{i,j} \to \gamma_j$ is a homeomorphism. The transition matrix is defined to be $F_\Gamma = (1_{ij})$, where $1_{ij} = 1$ if such $\gamma_{i,j}$ exists, and $= 0$ otherwise. In this case the spectral radius $\lambda_\Gamma = 0$ or 1. It is 1 iff Γ contains an invariant multicurve. And $\max_\Gamma \lambda_\Gamma = 1$ iff f has an invariant multicurve.

Thurston's theorem (homeomorphism version) The class $[(F, Z)]$ contains a representative (f, X) in one of the three cases:
1. $f^{\circ m} = id$ for some m (periodic, or m-volutive).
2. f not periodic but $\lambda_\Gamma = 1$ for some Γ (reducible).
3. pseudo-Anosov (expand/contract the horizontal/vertical foliation of a quadratic differential with at most simple poles at X-points).

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3. pseudo-Anosov (expand/contract the horizontal/vertical foliation of a quadratic differential with at most simple poles at X-points).

Examples can be found in the pillow with 4 corners as Z.

• The dihedral group D_4 induces periodic examples, some have an invariant multicurve, some have none.

• (Post composing a flip involution along a core curve with) a Dehn twist along the curve gives a reducible map.

• Composing one horizontal and one vertical Dehn twist gives a Pseudo-Anosov map: like $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, or $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ acting on $\mathbb{C}$ as a branched cover of the pillow, with $\mathbb{Z}^2$ as the critical set. The quadratic differential is the quotient of $(dz/u)^2$ on $\mathbb{C}$ with u pointing an eigen-direction.

Ideas of the proof

$\mathcal{T}_Z = \mathrm{Teich}(S^2, Z)$, *the Teichmüller space of the marked sphere* (S^2, Z), is defined to be the space of equivalence classes of homeomorphisms $\phi : S^2 \to \overline{\mathbb{C}}$ with $\phi \sim \varphi$, if there is a Möbius transformation M such that $M \circ \varphi|_Z = \phi|_Z$ and $M \circ \varphi = \phi \circ h$, with h a homeomorphism isotopic to the identity rel Z. See the left diagram:

$$\begin{array}{ccc} (S^2, Z) & \xrightarrow{\varphi} & \left(\overline{\mathbb{C}}, \varphi(Z)\right) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle M} \\ (S^2, Z) & \xrightarrow[\phi]{} & \left(\overline{\mathbb{C}}, \phi(Z)\right). \end{array} \qquad \begin{array}{ccc} S^2 & \xrightarrow{\exists\, \psi \in \mathcal{T}_Z} & \overline{\mathbb{C}} \\ \text{given } F \downarrow & & \downarrow \text{such that } f \in \mathrm{Rat}_d \\ S^2 & \xrightarrow[\forall\, \phi \in \mathcal{T}_Z]{} & \overline{\mathbb{C}}. \end{array}$$

The map $\sigma_F : [\phi] \to [\psi]$ is well defined, is a contraction (resp. an isometry in the homeomorphism case). It is called the **Thurston's pullback map**.

7. Semi-local dynamical models

Recall that dynamical Belyi polynomials are postcritically finite, i.e. each critical point has a finite iterated orbit. In order to complete Pilgrim's theorem we needed to show that the fixed points of these polynomials are all simple.

0 is a multiple fixed point of f iff $f(z) - z = O(z^2)$ iff $f(0) = 0$ and $f'(0) = 1$.

Lemma. For a rational map f with $f(0) = 0$ and $f'(0) = 1$, there is a critical point c with infinite orbit such that the orbit converges to 0. In particular f is postcritically infinite.

Proof. From $f(z) = z(1 + az^n + \textit{higher order terms})$, $a \neq 0$, one can see that if $az^n \in \mathbb{R}^-$ and $|z|$ small then $|f(z)| < |z|$.

Local model : Leau-Fatou flower theorem : there are n attracting petals.

Semi-local model : There are n attracting basins. Each basin contains a critical point c with the required properties.

8. Monodromy groups and iterated monodromy groups

For a Belyi polynomial f, the passport of a dessin is a Galois invariant. The group $\pi_1(\mathbb{C} \smallsetminus \{0,1\}, \frac{1}{2})$ acts on the right on the set of edges E_f of its dessin. The monodromy group $Mon(f)$, is the subgroup of the symmetric group $Sym(E_f)$ (or $Sym(f^{-1}(1/2))$) generated by $\ell_0(f), \ell_1(f)$, the lift of a loop around 0, 1. As a group, $Mon(f)$ is isomorphic to $\pi_1(\mathbb{C} \smallsetminus \{0,1\}, \frac{1}{2})/Kernel$.

Theorem (monodromy)
A (combinatorics, Hurwitz). $f, g \in Bel(\mathbb{C})$ are $\sim_{affine}$ iff there is a bijection $\tau : E_f \to E_g$ conjugating the actions of ℓ_0 and ℓ_1.

Conversely, any pair of permutations in $Sym(d)$ whose product is a cyclic permutation gives rise to a Belyi polynomial of degree d.

B (arithmetic, Jones and Streit). If $P, Q \in Bel(\overline{\mathbb{Q}})$ are galois conjugate, then $Mon(P)$ and $Mon(Q)$ are conjugate permutation groups.

Proof of A.

Label E_f by $1, ..., d$. Make a ray to infinity from each access of each vertex of the dessin. Cut along these rays to get d strips.

The cyclic decompositions of each ℓ_0 and ℓ_1 tell exactly how to glue back these strips along their half boundaries.

Question. From the pair of generating permutations, can one compute $Mon(f)$ or at least its cardinality?

Ideas of the proof of B.

First make a normal covering above P: $\tilde{X} \xrightarrow{s} X \xrightarrow{P} \overline{\mathbb{C}}$ so that $\tilde{P} = P \circ s$ is the minimal normal cover. Every loop $\gamma \in \pi_1(\overline{\mathbb{C}} \setminus \{0, 1, \infty\}, 1/2)$ induces a transformation in $Deck_{\tilde{P}}$, which in turn induces by precomposition an automorphism φ_γ of the meromorphic functions field, fixing any function that is $\tilde{P}$-symmetric. This Galois group G admits a subgroup K of automorphisms fixing functions that are s-symmetric, of index $deg(P)$. And $(Mon(P), E_P) \sim_{isomorphic} (G, \text{right-cosets}(K))$.

$$\begin{array}{ccccccc}
\tilde{X} & \xrightarrow{\gamma} & \tilde{X} & & & & \\
 & & \Big| & & & & \\
\mathcal{M}_{\overline{\mathbb{Q}}}(\tilde{X}) & \xrightarrow{\varphi_\gamma} & \mathcal{M}_{\overline{\mathbb{Q}}}(\tilde{X}) & \xrightarrow{s^*} & \mathcal{M}_{\overline{\mathbb{Q}}}(X) & \xrightarrow{P^*} & \mathcal{M}_{\overline{\mathbb{Q}}}(\overline{\mathbb{C}}) \\
\sigma \Big\downarrow & & \Big\downarrow \sigma & & \Big\downarrow \sigma & & \Big\downarrow \sigma \\
\mathcal{M}_{\overline{\mathbb{Q}}}(\tilde{X}^\sigma) & \xrightarrow{\sigma\varphi_\gamma\sigma^{-1}} & \mathcal{M}_{\overline{\mathbb{Q}}}(\tilde{X}^\sigma) & \xrightarrow{(\sigma.s)^*} & \mathcal{M}_{\overline{\mathbb{Q}}}(X^\sigma) & \xrightarrow{(\sigma.P)^*} & \mathcal{M}_{\overline{\mathbb{Q}}}(\overline{\mathbb{C}})
\end{array}$$

(By Belyi the new, galois cover is defined over $\overline{\mathbb{Q}}$.)

Examples

1. (D-D) There are only two trees T, T' with passport $(3,2,2),(2,2,1,1,1)$, with $\ell_0 = (123)(45)(67)$ and $\ell_1(T) = (24)(36)$, $\ell_1(T') = (14)(56)$. D-D proves that $\sigma \in \Gamma$ exchanges them iff it exchanges $\pm\sqrt{21}$. The monodromy groups are both Alt_7, of order 2520.

2. (J-S) There are only two trees T, T' with passport $(4,1,1),(2,2,1,1)$, with $\ell_0 = (1234)$ and $\ell_1(T) = (45)(16)$, $\ell_1(T') = (16)(53)$. The monodromy groups have orders $120/48$.

3. (D-D) Two trees T, T' each is a 5 branches star plus 1,2,3,4,5 fingers of T, and 1,2,5,4,3 fingers for T'. The monodromy groups are both S_{20} but D-D proves that they are not in the same Galois orbit. So the invariant is not complete.

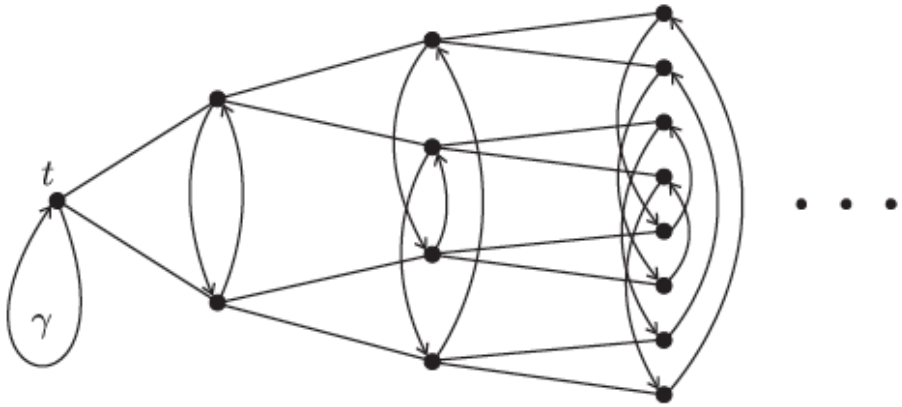
4. (J-S) Four trees T, T', L, L' with $\ell_1 = (1234)(56)$. $\ell_0 = (15)(67),(35)(17),(15)(27),(47)(15)$ resp. $\sigma \in \Gamma$ exchanges T, T' (resp. L, L') iff it exchanges $\pm\sqrt{21}$ (resp. $\pm i\sqrt{7}$). The monodromy groups are of order 2520 for T, T' and 168 for L, L'.

Iterated monodromy action

The theory works for any partial self covering $\mathcal{M} \supset \mathcal{M}_1 \xrightarrow{p \text{ covering}} \mathcal{M}$.

For a dynamical Belyi polynomial, or more generally a postcritically finite branched covering f, one may take $\mathcal{M} = \overline{\mathbb{C}} \smallsetminus P_f$ (which is $\overline{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ in the Belyi case). Fix $t \in \mathcal{M}$ a base point.

The fundamental group $\pi_1(\mathcal{M}, t)$ acts on $p^{-n}(t)$ for every $n \geqslant 0$. Consider the set $p^{-n}(t)$ to live in the nth copy of $\mathcal{M}$ and we form a preimage tree $T(p, t)$ by assigning an edge from every $x \in p^{-n}(t)$ to its image $p(x)$ in the $(n-1)$th copy, for every $n \geq 1$.

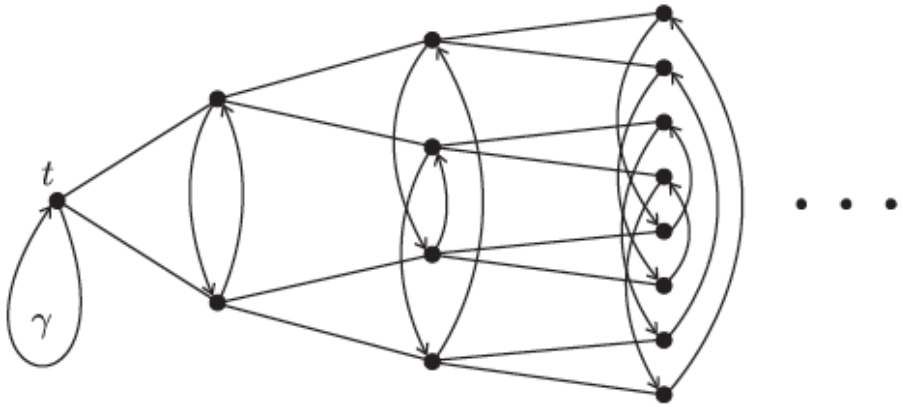


The action of $\pi_1(\mathcal{M}, t)$ on the set of all vertices in the tree of preimages $T(p, t)$ is called the iterated monodromy action. It may be seen as the following group homomorphism.

$$\Phi : \pi_1(\mathcal{M}, t) \to \prod_{n \geq 0} Sym\left(p^{-n}(t)\right), \ [\gamma] \mapsto \left(\Phi_{[\gamma]} : x \mapsto [\gamma]x\right)$$

Theorem (Nekrashevych). $\forall \gamma$, $\Phi_{[\gamma]}$ preserves the edge relation. So

$$\Phi : \pi_1(\mathcal{M}, t) \to Aut\left(T(p, t)\right), \ [\gamma] \mapsto \left(\Phi_{[\gamma]} : x \mapsto [\gamma]x\right)$$



Proof. Let $\{p(x), x\}$ be an edge of $T(p,t)$ where $x \in p^{-(n+1)}(t)$ for some $n \geqslant 0$. Let $\Gamma_{p(x)}$ be the p^n-lift of γ from $p(x)$ (whose terminal point is $[\gamma]p(x)$) and Γ_x be the p^{n+1}-lift of γ from x (whose terminal point is $[\gamma]x$). By unicity of path-lifting Γ_x is also a p-lift of $\Gamma_{p(x)}$. It follows that $p([\gamma]x) = [\gamma]p(x)$.

$$\begin{array}{ccccc} t & \xleftarrow{\quad p^n \quad} & p(x) & \xleftarrow{\quad p \quad} & x \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\Gamma_{p(x)}} & & \downarrow{\scriptstyle\Gamma_x} \\ t & \xleftarrow[\quad p^n \quad]{} & [\gamma]p(x) = p([\gamma]x) & \xleftarrow[\quad p \quad]{} & [\gamma]x \end{array}$$

Iterated monodromy action as a partial invariant

Theorem (A-dyn1) If f and g are two combinatorially equivalent post-critically finite branched coverings on the topological sphere S^2, say $\psi_0 \circ f = g \circ \psi_1$, with $\psi_0 \sim_{isotopic} \psi_1$ rel P_f, then for any base point $t \in S^2 \backslash P_f$, and for every loop γ in $S^2 \backslash P_f$ with base point t, the monodromy action induced by $[\psi_0 \circ \gamma]$ on the tree of preimages $T(g, \psi_0(t))$ is the same as the one induced by $[\gamma]$ on the tree of preimages $T(f, t)$.

Proof. We may assume $\psi_0 = id$ and $\psi_1 \sim_{isotopic} id$ rel P_f. Then ψ_1 can be consecutively lifted to ψ_n giving a canonical bijection between the two preimage trees and transporting the action of γ on one tree to the other. □

The action of π_1 is in general not injective or equivalently, in terms of group action, the monodromy action is in general not faithful. So one defines the iterated monodromy group $IMG(p, t) := \pi_1 / Kernel$.

IMG is in general not a complete invariant.

Biset as a complete invariant

Let $f : S^2 \to S^2$ be a post-critically finite branched covering, i.e. the forward orbits of f's critical points form a finite set, denoted by P_f. Set $\mathcal{M} = S^2 \setminus P_f$. Choose a base point t and let $G = \pi_1(\mathcal{M}, t)$. It is a free group with $|P_f| - 1$ generators. The biset is:

$$\mathfrak{B}_f = \{\text{homotopy classes of paths } \gamma : [0,1] \to \mathcal{M} | f(\gamma(0)) = \gamma(1) = t\}$$

It is equipped with two commuting actions of G (called a biset). The right action appends a path at the end of γ, and the left action prepends at the beginning of γ the unique lift of a path that ends at $\gamma(0)$.

Theorem (Kameyama, Nekrashevych). (A-dyn2) Two maps f and g are combinatorially equivalent (via a pair (ψ_0, ψ_1)) if and only if their bisets are conjugate, namely if and only if there exists a homeomorphism $\phi : \mathcal{M}_f \to \mathcal{M}_g$ and a bijection $\tau : \mathfrak{B}_f \to \mathfrak{B}_g$ such that $\tau(gbh) = (\phi_* g)\tau(b)(\phi_* h)$ for all $b \in \mathfrak{B}_f$ and all $g, h \in G_f$. ($\phi = id$ if $\psi_0 = id$).

Arithmetic invariants

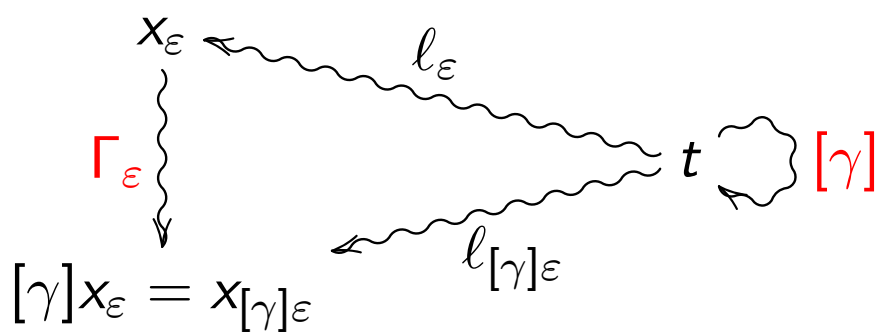
Corollary of Jones and Streit (Pilgrim) (B-dyn1) For dynamical Belyi polynomials f, g that are galois conjugate, $IMG(f)$ and $IMG(g)$ as product to permutations groups, or as preimage tree automorphisms, are conjugate. But this conjugacy does not preserve the generators, nor the kernels if we consider π_1.

Question. How to see the biset version of the invariant?

How to compute these invariants?

Labeling each point x in $p^{-n}(t)$ by a word of length n : $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}\varepsilon_n$ with $\varepsilon_i \in \{1,...,d\}$ inductively so that $p(x)$ has the label $\varepsilon_1\varepsilon_2\cdots\varepsilon_{n-1}$, will turn the tree to T_d and turn the monodromy actions into elements of $Aut(T_d)$.

Theorem (Nekrashevych) One can give a labeling choice using a collection of paths from t to each point in $p^{-1}(t)$ and their consecutive lifts. In this case, one can compute recursively the monodromy action by $[\gamma]\varepsilon word = \Big([\gamma]\varepsilon\Big)\Big(\Big[\ell_\varepsilon.\Gamma_\varepsilon.\ell^{-1}_{[\gamma]\varepsilon}\Big]\, word\Big)$ where Γ_ε is the lift of γ from x_ε. Here $\ell_\varepsilon.\Gamma_\varepsilon.\ell^{-1}_{[\gamma]\varepsilon}$ is a loop with base point t so its homotopy class is well defined in $\pi_1(\mathcal{M}, t)$.

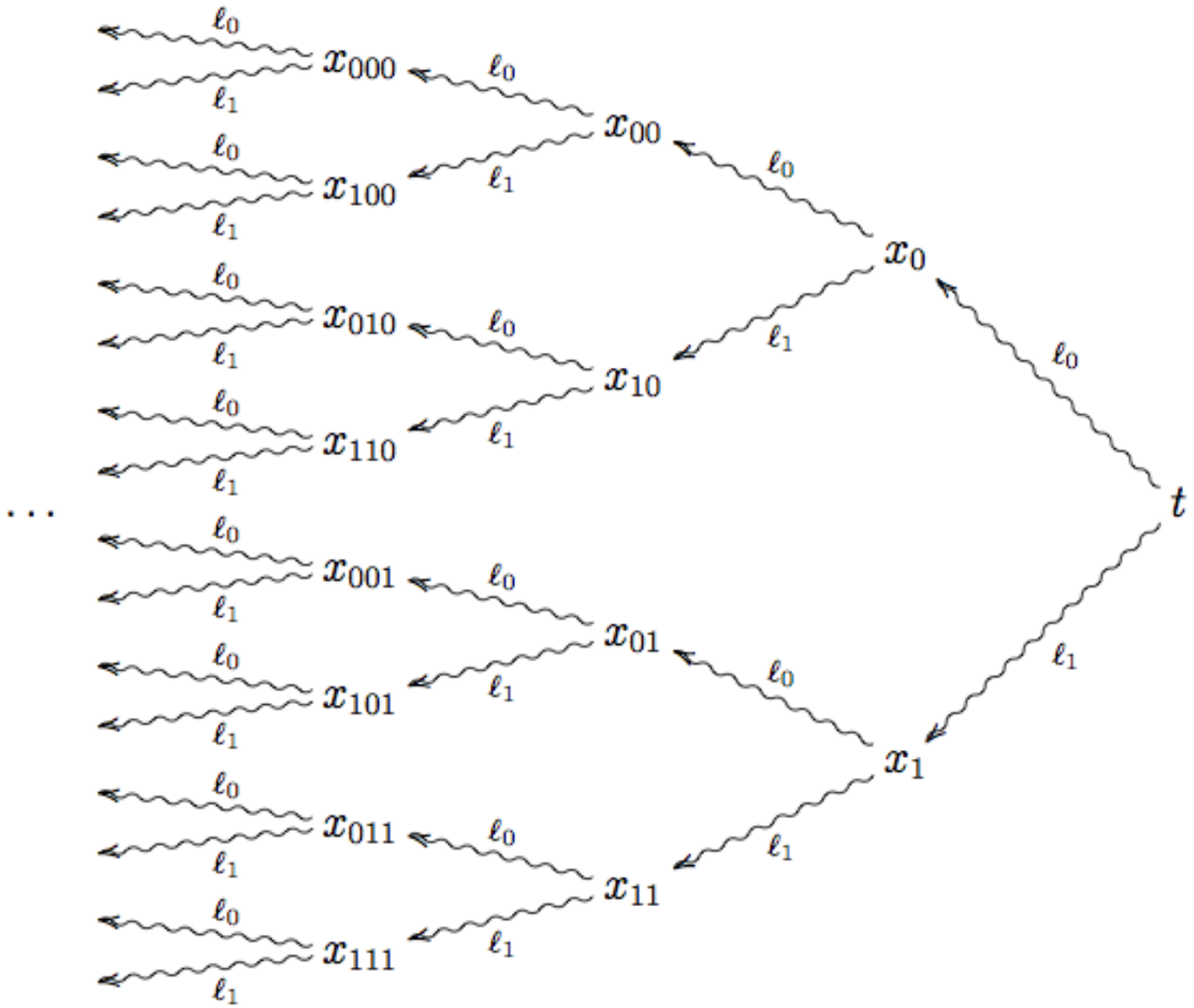


or in wreath product presentation (see below)

$$\frac{\forall\gamma, \qquad \gamma}{\forall\,\varepsilon,\ \varepsilon \xrightarrow[\ell_\varepsilon.\Gamma_\varepsilon.\ell^{-1}_{[\gamma]\varepsilon}]{} [\gamma]\varepsilon}.$$

The (finite) table with a generating set $\{\gamma_i\}$ determines the action.

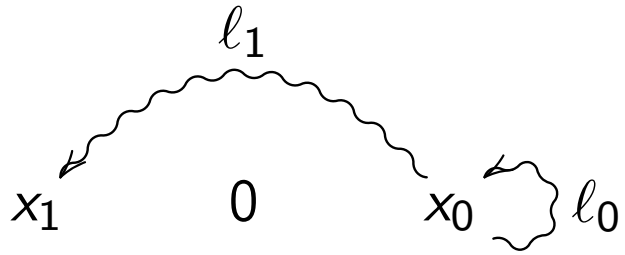
A labeling choice induced by paths. Choose $\ell_i : t \rightsquigarrow x_i$ and lift. These paths live in the same plane as concatenations. And the p-image of a trail is the trail with the same indices except the rightmost path (it becomes a loop on t).



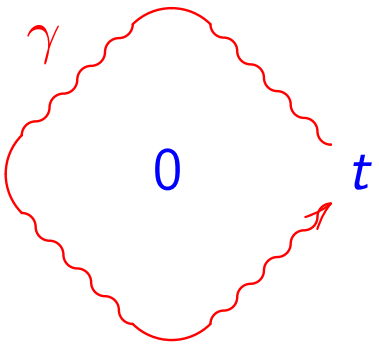
Such a collection X of paths, seen now as one from each point of $p^{-1}(t)$ to t, forms a basis of $\mathfrak{B}_p$. Then $\mathfrak{B}_p$ is bijective to $X \times G$ so that (x, g) represents the (homotopy class of the) path $x.g$. Now the right action is natural: $(x, g)h = (x, gh)$. The left action can be represented by the (finite) table of multiplication on the generators of G : $g_i.x_j = (x_k, h)$ for some $h \in G$. In good cases, for example when h are among g_k and its inverse, the table of multiplication can be presented by a finite automaton.

Example

For $Q_0 : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\},\ z \mapsto z^2$ with $t = 1$, $x_0 = 1$ and $x_1 = -1$, choose the paths ℓ_0, ℓ_1 as follows

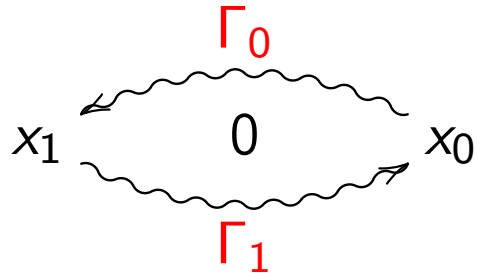


The fundamental group $\pi_1(\mathbb{C}\backslash\{0\}, 1)$ may be described as the infinite cyclic group generated by the homotopy class $[\gamma]$ coming from the following loop

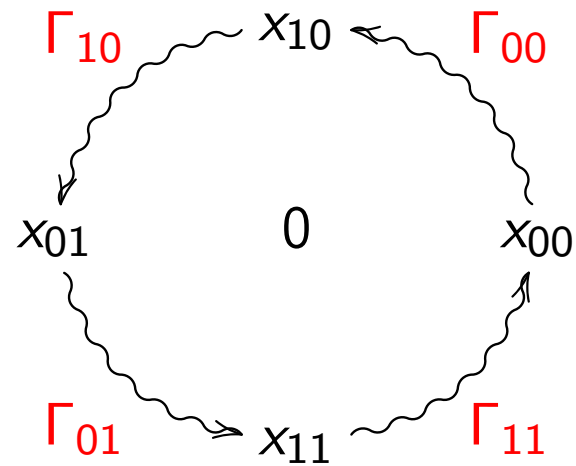


where γ surrounds the point 0 in a counterclockwise motion.

Lifting this loop gives



One can deduce the monodromy action of $[\gamma]$ on the first level $Q_0^{-1}(t)$, that is $[\gamma]x_0 = x_1$ and $[\gamma]x_1 = x_0$. Equivalently the action on $\mathcal{E}^1 = \{0, 1\}$ is given by $[\gamma]0 = 1$ and $[\gamma]1 = 0$. Lifting the loop γ by $Q_0^2 : z \mapsto z^4$ gives



One can deduce the action of $[\gamma]$ on the second level $Q_0^{-2}(t)$ or equivalently on $\mathcal{E}^2 = \{00, 10, 01, 11\}$, that is $[\gamma]00 = 10$, $[\gamma]01 = 01$, $[\gamma]01 = 11$ and $[\gamma]11 = 00$.

(Pilgrim) There is a canonical labeling of $p^{-n}(t)$ for a dynamical Belyi polynomial p. Take the unique path in the Julia set connecting 0 and 1. Pull it back to get the dessin (a tree). Label the edges by $0, ..., d-1$ (arbitrarily). For convenience we may label the edge adjacent to the vertex 0, 1 by 0 and 1. By Caratheodory there is a unique external ray landing at each access of each vertex of the dessin. The union of these rays partitions the plane into d open labeled strips $S_0, \cdots, S_{d-1}$, with a common image U containing each S_i. Further pullbacks consist of consecutive refinements of the partition, and a strip S of depth n can be labeled by $\varepsilon_1 \cdots \varepsilon_n$ so that its itinerary with respect to the initial partition is $\varepsilon_n, \varepsilon_{n-1}, \cdots$, that is, $S \subset S_{\varepsilon_n}$, $p(S) \subset S_{\varepsilon_{n-1}}$, etc. In particular $p(S)$ has label $\varepsilon_1 \cdots \varepsilon_{n-1}$. Pilgrim computed the multiplication table:

	$0w$	$1w$	εw
ℓ_0	$0\ell_0(w)$	$1\ell_1(w)$	$\ell_0(\varepsilon)w$
ℓ_1	$\ell_1(0)w$	$\ell_1(1)w$	$\ell_1(\varepsilon)w$

,

ℓ_0

$0 \xrightarrow{\ell_0} 0$

$1 \xrightarrow{\ell_1} 1$

$\varepsilon \longrightarrow \ell_0(\varepsilon)$

ℓ_1

$\varepsilon \longrightarrow \ell_1(\varepsilon)$

It can also be represented by a 3-states $\{\ell_0, \ell_1, id\}$ automaton.

Wreath product presentation

Let G be a group and $\Sigma \leq Sym(\mathbb{Z}_d)$ (acting on the right). Denote by

$$G^d \rtimes \Sigma = \left\{ g := \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \sigma_g \middle| g_i \in G, \sigma_g \in \Sigma \right\}$$

The multiplication (as a right action) is defined by

$$g \cdot h = \begin{pmatrix} g_1 \\ \vdots \\ g_d \end{pmatrix} \sigma_g \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} \sigma_h = \begin{pmatrix} g_1 \cdot h_{\sigma_g(1)} \\ \vdots \\ g_d \cdot h_{\sigma_g(d)} \end{pmatrix} \sigma_g \cdot \sigma_h.$$

Under this multiplicity, $G^d \rtimes \Sigma$ is a group with unit element

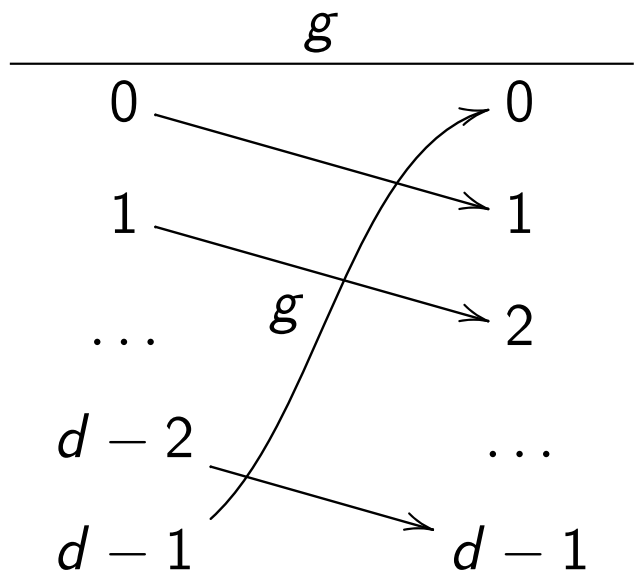
$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} id \text{ and } g^{-1} = \begin{pmatrix} g^{-1}_{\sigma_g^{-1}(1)} \\ g^{-1}_{\sigma_g^{-1}(2)} \\ \vdots \\ g^{-1}_{\sigma_g^{-1}(d)} \end{pmatrix} \sigma_g^{-1}.$$

A more explicit way to present an element of $G^d \rtimes \Sigma$ is to present the permutation σ with arrows as usual and label each arrow $i \longrightarrow \sigma(i)$ by $i \xrightarrow{g_i} \sigma(i)$ (ignoring the label if it is identity). For example, the d-aray adding machine (adding 1 on the left most digit and carry over to the right) g, which is also the IMG of z^d, is recursively defined by: $\forall$ letter ε, $\forall$ word w,

$$g(\varepsilon w) = \begin{cases} (\varepsilon + 1)w & \text{if } \varepsilon \in \{0, 1, \cdots, d-2\} \\ 0g(w) & \text{if } \varepsilon = d-1 \end{cases}$$

Then g may be seen as the wreath recursion

$$g = \begin{pmatrix} id \\ \vdots \\ id \\ g \end{pmatrix} (0, 1, \ldots, d-1), \text{ or}$$



9. Moduli space of marked spheres and Koch space

- I is a finite set containing at least three points.
- Two injective maps $x_1 : I \to \mathbb{S}$ and $x_2 : I \to \mathbb{S}$ are equivalent when there is a Möbius transformation $M : \mathbb{S} \to \mathbb{S}$ such that $x_2 = M \circ x_1$.
- $\mathrm{Mod}(I)$ is the set of equivalence classes of injective maps $x : I \to \mathbb{S}$.
- $\mathrm{Mod}(I)$ may be endowed with the structure of a quasiprojective variety of dimension $|B| - 3$.

(here $B = I$).

- $f \in \mathrm{Rat}_D$, $\mathcal{V}(f)$ is the set of critical values of f.
- $B, C \subset \mathbb{S}$ are finite sets with $\mathcal{V}(f) \subseteq B$, $|B| \geq 3$ and $C = f^{-1}(B)$.
- $\mathcal{K}_B(f)$ is the set of pairs $(\mathbf{y}, \mathbf{z}) \in \mathrm{Mod}(B) \times \mathrm{Mod}(C)$ for which there are triples $(F, y, z) \in \mathrm{Rat}_D \times \mathbb{S}^B \times \mathbb{S}^C$ with $[y] = \mathbf{y}$, $[z] = \mathbf{z}$ and:

$$\begin{array}{ccc} C & \xrightarrow{z} & \mathbb{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow[y]{} & \mathbb{S} \end{array} \quad \text{with} \quad \deg_{z(c)} F = \deg_c f.$$

Theorem

- *The set $\mathcal{K}_B(f)$ is a smooth submanifold of $\mathrm{Mod}(B) \times \mathrm{Mod}(C)$.*
- *The dimension of $\mathcal{K}_B(f)$ is $|B| - 3$.*
- *The projection $\mathcal{K}_B(f) \to \mathrm{Mod}(B)$ is a finite cover.*
- *The projection $\mathcal{K}_B(f) \to \mathrm{Mod}(C)$ is a proper embedding.*

Question: Compute the degree of the covering. It is related to the Hurwitz counting problem, which is still open in many cases.
Maybe one can express the degree in terms of the Hurwitz numbers.

10. Dessin as trajectories of polynomial vector fields

A Belyi passport (s_i, t_j) is called clean if $t_j \equiv 2$. Any Belyi polynomial Q can be turned into a clean B by post composing $z \mapsto 4z(1-z)$. In terms of the dessin, one just needs to add a vertex in each edge to be the new colored 1 vertices and color the old vertices by 0. So $\boxed{d \text{ old vertices}} \Longrightarrow \boxed{d-1 \text{ added vertices}}$.

Theorem (Pilgrim). Any clean Belyi polynomial is isomorphic to a $B(z)$ of the form $-\prod_{I,|I|=d}(z-a_i)^{s_i}$, with a_i pairwise distinct, $\sum_I a_i = 0$ and $\sum_I s_i = deg(B) = \sum_{j=1}^{d-1} t_j = 2(d-1)$. Any B induces a monic centered polynomial $P_B(z) = \prod_I (z-a_i)$ and

$$\dot{z} = P_B(z) \underset{pullsback}{\overset{B^*}{\longleftarrow}} \dot{z}^2 = 4(z^2 - z^3)$$

with separatrices $B^{-1}([-\infty, 0])$. The edges of the dessin $B^{-1}([0,1])$ are trajectories. Each a_i receives s_i separatrices. The vector field $\dot{z} = P_B(z)$ is generic, monic centered with constant analytic invariant $(= \dfrac{\pi i}{d-1})$.

Bijection: from vector fields to Belyi polynomials

Let $P(z) = \prod_{I,|I|=d}(z - a_i)$ be a monic centered polynomial such that the vector field $P(z)\dfrac{\partial}{\partial z}$ is generic: all roots are simple and no homoclinic connections of the $2(d-1)$ separatrix germs at ∞ (in particular each a_i is either a source or a sink)

Each a_i receives $s_i > 0$ separatrices (as a_i is not a center), and $\sum_I s_i = 2(d-1)$ (every separatrix germ lands at some a_i).

So $(s_1, \cdots, s_d)$ is an integer partition of $2(d-1)$.

Within the same landing pattern there is a unique choice of $\mathbf{a} = (a_i)_I$ such that the analytic invariant of P is $\{\dfrac{\pi i}{d-1}, \cdots, \dfrac{\pi i}{d-1}\}$. For this P, set $B_P(z) = -\prod_I (z - a_i)^{s_i}$.

Then (by Pilgrim) B_P is a clean Belyi polynomial. This is obtained by uniformizing and then squaring the zones of $\dot{z} = P(z)$ to get a global holomorphic map B which turns out to be B_P.

Counting result The number of normalized clean Belyi polynomial of given degree is the catalan number.

(Tomasini) There is a closed form for the number of equivalence classes of clean Belyi polynomials of degree 2p as well as the number of combinatorial invariants of generic polynomial vector fields.

Question. Given a degree d, the number $P(d)$ of partitions of d into sum of (unordered) integers can be computed by some recursive formulae and generating functions. But no closed form is known. How many passports are there? Recall that a passport is a pair of partitions $(s_i), (t_j)$ of d satisfying the Hurwitz condition: $\sum(s_i - 1) + \sum(t_j - 1) = d - 1$. A passport is a Galois invariant.

11. z^2 and angle doubling

Every angle of the form $s/(2^k - 1)$ is periodic. How to describe the set of all periodic angles?

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Let p, q be two co-prime integers. Then $x \mapsto p \cdot x \mod q$, $\mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ is bijective. So every n is periodic. In particular 1. So for some k, $p^k \equiv 1 \mod q$ and $p^k - 1 = qm$ for some m. □

What are the preperiodic angles? The wandering angles? Is the orbit of a wandering angle dense?

What are the angles periodic for the angle-tripling map?

What is the period distribution of the map on $\mathbb{Z}/q\mathbb{Z}$?

Mega theorem. z^2 encodes all quadratic dynamics.