Folding surgery and wandering continua

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Holomorphic Dynamics, Around Thurston's Theorem

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§1. Statements

Let f be a rational map.

A **wandering continuum** is a non-degenerate continuum $\boxed{K \subset J_f}$ such that $f^n(K) \cap f^m(K) = \emptyset$ for $0 \leq n < m$.

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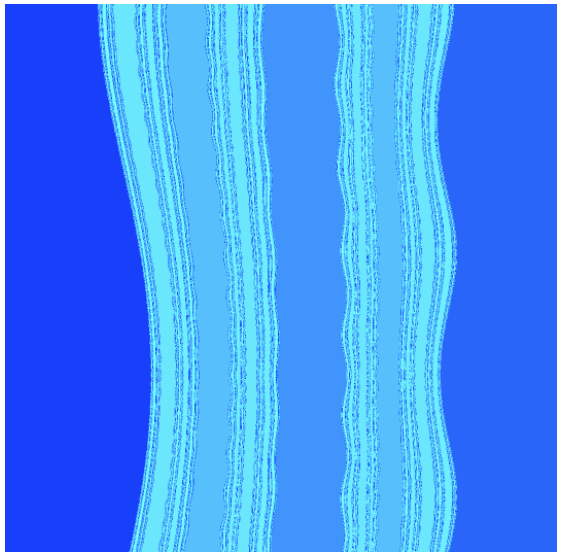
Renormalization

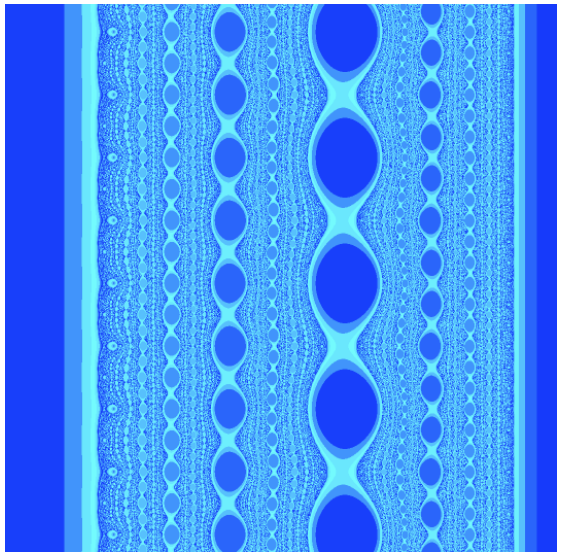
$\boxed{f \text{ polynomial}} \Longrightarrow \boxed{\begin{array}{c} \text{use equipotentials and external rays} \\ \text{to build renormalization pieces} \end{array}}$

Why interesting?

§1. 1 Examples.

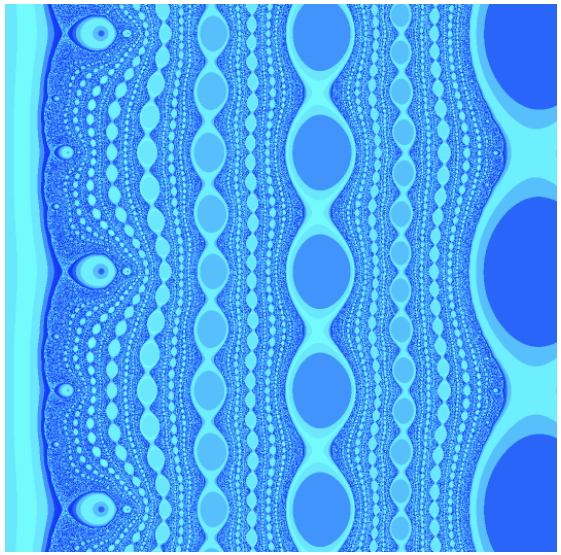


McMullen



Pilgrim-T.

Godillon's example



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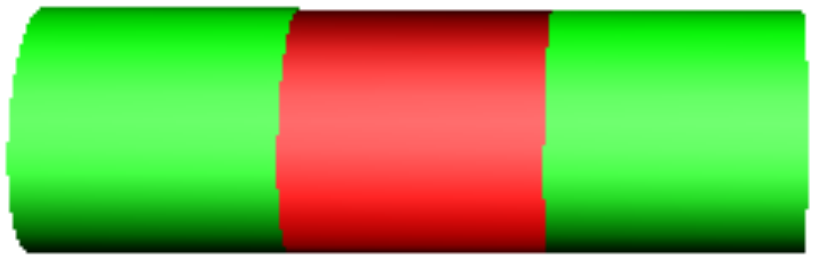
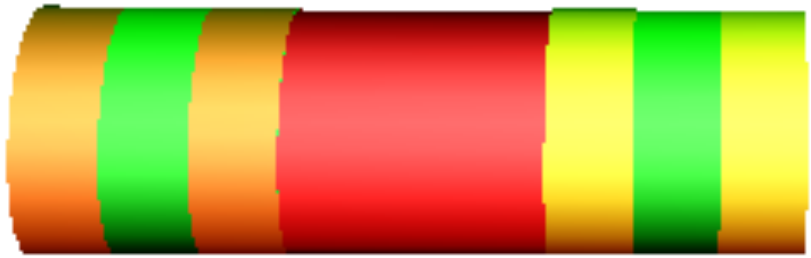
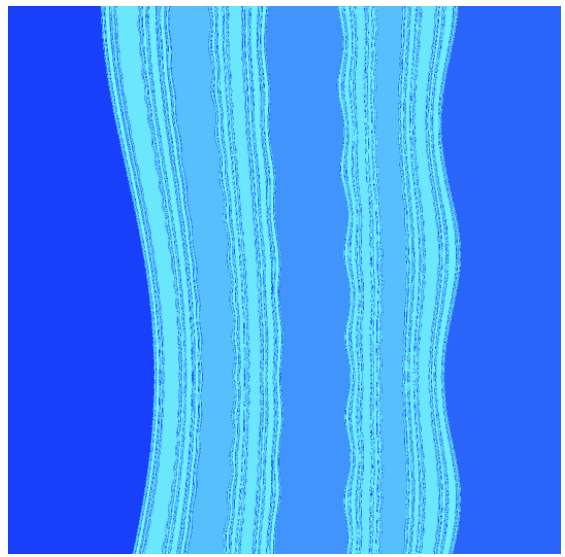
Theorem 2 (Cui-Peng-T.) Combinatorial criterion, $\#P_f < \infty$

∃ Jordan curve wandering continua $\iff$ ∃ Cantor multicurve.

renormalizable $\impliedby$ ∃ Cantor multicurve.







§2. Cantor multicurve $\Gamma = \{\gamma_1, \cdots, \gamma_n\}$

Assume P_f, the postcritical set of f, is finite.

Γ is an irreducible (ergodic) multicurve in $\widehat{\mathbb{C}} \setminus P_f$, if

- every γ_i is a non-peripheral Jordan curve in $\widehat{\mathbb{C}} \setminus P_f$,
- they are disjoint and non-homotopic in $\widehat{\mathbb{C}} \setminus P_f$,
- The following unweighted transition matrix is irreducible.

$$f_{\#,\boldsymbol{\Gamma}} = (b_{ij}) \quad b_{ij} = \#\{\text{curves in } f^{-1}(\gamma_j) \text{ homotopic to } \gamma_i\}.$$

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- $f_{\#,\boldsymbol{\Gamma}}$ is not a permutation matrix,
- The leading eigenvalue $\lambda(f_{\#,\boldsymbol{\Gamma}}) > 1$,
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- $\#\{\text{curves in } f^{-n}(\boldsymbol{\Gamma}) \text{ homotopic to curves in } \boldsymbol{\Gamma}\} \longrightarrow \infty$ (exponentially).

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Folding provides a cantor multicurve with one curve.

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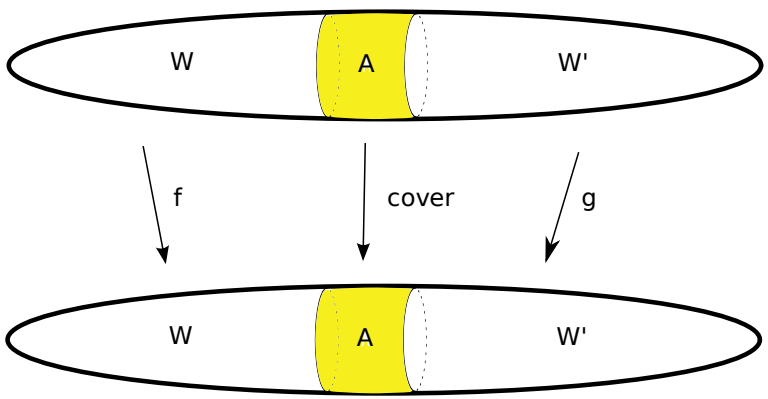
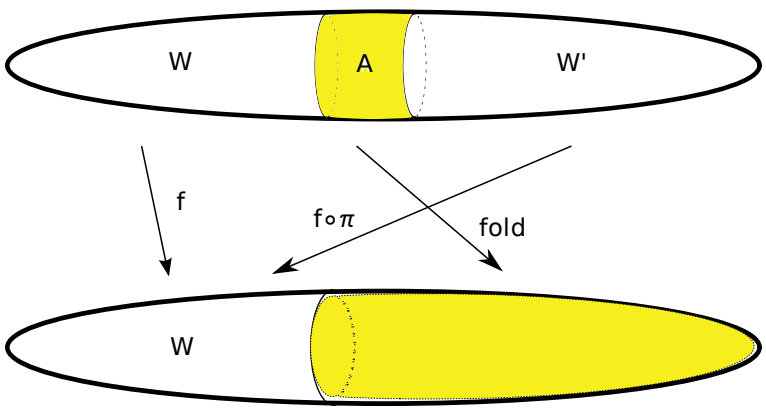
Let's practice this to find a postcritically finite R with

- a Cantor multicurve,
- a wandering continuum,
- a renormalization.

§3.1. Topological folding

- $\overline{W} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta}, \theta \in \mathbb{R}\}, \quad \overline{W'} = \mathbb{C}' \cup \{(\infty \cdot e^{2i\pi\theta})', \theta \in \mathbb{R}\},$
- $A = [-1, 1] \times S^1,$
- $S = \overline{W} \sqcup A \sqcup \overline{W'} / \sim,$
  with $\infty \cdot e^{2\pi i\theta} \sim (-1, e^{2\pi i\theta})$ and $(+1, e^{2\pi i\theta}) \sim (\infty \cdot e^{-2\pi i\theta})',$
- $\pi = id : \overline{W'} \to \overline{W}.$

f, g polynomials, degree=d, d', postcritically finite,

$d = d'$, f, g monic mating(f, g) : M :	folding$(f,$ fold$)$: F : $F^2(\mathcal{C}^A) \subset K_f$, (pre)periodic
	

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Partial answers :

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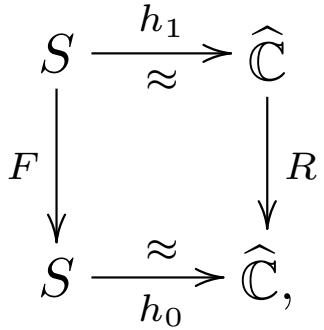
- (harder) is unobstructed if $\begin{cases} d \geq 3 \\ F(\mathcal{C}_F \cap A) \subset \text{ a star-like of Böttcher rays.} \end{cases}$
- (not trivial) $\exists$ obstructed examples with $d = 4$.

We will postpone the proof of this.

An unobstructed folding F is

combinatorially equivalent to a rational map R:



$$h_1|_{P_F} = h_0|_{P_F}, \quad h_1 \sim_{P_F} h_0$$

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Proposition. For an unobstructed folding F

$h(A) \supset$ non empty annulus $\mathcal{A}$ s.t.

- $\mathcal{A}$ separates $h(\overline{W})$ and $h(\overline{W'})$,
- $\mathcal{A} \supset$ two essential annuli A_1, A_2 with $R : A_1 \to \mathcal{A}$ and $R : A_2 \to \mathcal{A}$ coverings, and $\partial\mathcal{A} \subset \partial(A_1 \cup A_2)$.

$\Longrightarrow R$ has a polynomial renormalization and wandering continua, and they are all Jordan curves due to uniform expansion (as in Pilgrim-T.).

§3.4 Survival of the central annulus

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Proof. Assume $T := h(\overline{W}) \cap h(\overline{W'}) \neq \emptyset$.

Set $T^* = \cap_{k>0} R^k(T)$. Then $R(T) \subset T \Longrightarrow R(T^*) = T^*$.

(1) $\exists$ an backword orbit $\{w_k\}_{k\geq 0} \subset T^*$ s.t. $R(w_{k+1}) = w_k$.

(2) We may assume that $F : h^{-1}(w_1) \to h^{-1}(w_0)$ is a homeomorphism.

(3) If $\#h^{-1}(w_0) \cap \partial W = 1$, then

$$\#h^{-1}(w_1) \cap F^{-1}(\partial W) = \#h^{-1}(w_1) \cap (\partial W \cap \partial W') = \begin{cases} 1 \text{ (homeo)} \\ 2 \text{ (homotopy)} \end{cases}.$$

A contradiction.

(4) If not, iterate backward a few times... $\square$

§3.5 No obstructions

The idea is similar.

§4. Decomposition along a Cantor multicurve

Theorem.

For a postcritically finite rational map g,

$$\exists \text{ a Cantor multicurve } \mathbf{\Gamma} = \{\gamma_1, \cdots, \gamma_n\} \Longrightarrow \begin{cases} \exists \text{ renormalization} \\ \exists \text{ wandering continua.} \end{cases}$$

Key proposition (homotopy inv. $\longrightarrow$ invariance)

Each γ_i can be promoted to an annulus $A_i \subset \widehat{\mathbb{C}} \setminus P_g$ such that

- core-curve$(A_i) \sim \gamma_i$,
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Proof. Deform g into a branched covering G s.t.

- G is combinatorially equivalent to g,
- the annuli survive under Rees-Shishikura's semi-conjugacy (same proof).

§5. An obstructed folding

§6. Decomposition of hyperbolic rational maps

J_R disconn. $\Longrightarrow$

$$\begin{cases} \text{a canonical multicurve} \\ \text{renormalizations with connected } J \Rightarrow \begin{cases} \text{cantor multicurve} \\ \text{renormalizations} \Rightarrow \ldots \end{cases} \end{cases}$$

(For non-hyperbolic ones, see next talk)

Thanks for your attention !

§7. Questions.

If f is a polynomial with connected but non-locally connected Julia set, does f have a wandering continuum in J ? Parital answers: Yes by Inou-Shishikura if: $deg(f) = 2$, with a periodic indifferent point, with high-type rotation number and no critical point on the boundary of the Siegel disc (if exists);

Kiwi: f poly. no irrationnally neutral periodic point. Then 'no wandering continuum' is equivalent to local connectivity.

Yes if deg(f)=2 and infinitely renormalizable and $J_n \not\to$ point.

Otherwise ?

- Is there a $\left\{ \begin{array}{l} \text{postcritically finite} \\ \text{and non-Lattès} \end{array} \right.$ rational map having a full wandering continuum?
- How to characterize the dynamics of f having no multicurves? Or having no Cantor multicurves?
- Is there a wandering Julia component K such that $f^n(K)$ is complex type for all $n \geq 0$?
- Are there infinitely many complex type Julia components?
- Let f be a subhyperbolic rational map. Is the number of cycles of complex Julia components bounded by $C(\deg(f))$?