On disc-annulus surgery of rational maps

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1 Introduction

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map, regarded as a holomorphic dynamical system on the Riemann sphere $\overline{\mathbb{C}}$. The sphere decomposes naturally into two invariant subsets: the open Fatou set Ω_f and the closed Julia set J_f. Loosely, the Fatou set is the locus of stable, non-recurrent dynamical behavior, while its complementary Julia set is the locus of chaotic, recurrent dynamical behavior. In this note, we introduce some new, flexible surgery operations supported on the Fatou set of f. We define our operations in such a way so that the new map has dynamics which is closely related to that of f. The non-recurrent nature of the dynamics on the Fatou set implies that the delicate analytical arguments (cf. e.g. [BF]) needed to describe surgery near the Julia set are not needed. For background material, properties of quasi-conformal mappings, and Fatou and Julia sets of rational maps, we refer the reader to [Bea] and [CG].

Our construction is similar in many ways to a related construction in the setting of Kleinian groups. The "Klein-Maskit combination of type I using trivial discs" (cf. [Ma]) takes as input two Kleinian groups Γ_1, Γ_2 and two round discs $\tilde{D}_1, \tilde{D}_2$ with trivial stabilizer contained in the domains of discontinuity Ω_1, Ω_2. The output is a Kleinian group $\Gamma = \Gamma_1 * \Gamma_2$ containing Möbius conjugate copies of Γ_1 and Γ_2. That is, the dynamics of Γ contains that of Γ_1 and Γ_2. Moreover, it can easily be shown that the connected components of the limit set Λ of Γ are either translates of connected components of the limit sets Λ_i, or points. This operations also has a natural combinatorial analog in the setting of three-manifolds, namely *boundary connect sum.* Let $M_i = (\mathbb{H}^3 \cup \Omega_i)/\Gamma_i, i = 1, 2$ be the quotient three-manifolds. Since Ω_i is assumed nonempty, M_i has nonempty boundary, and the discs $\tilde{D}_i$ descend to discs D_i on ∂M_i. The manifold $M = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ is obtained from the manifolds M_i by identifying D_1 and D_2 via an orientation-reversing homeomorphism.

In the setting of rational maps, however, the fact that maps with interesting dynamics have degree strictly greater than one makes the formulation of combination theorems taking as input two (or several) rational maps rather subtle. One difficulty is the lack of a natural, simple combinatorial analog to describe the gluing data, which can be quite complicated to describe. We therefore content ourselves with the more modest program of beginning with a single map f and modifying it (in a more-or-less arbitrary fashion) inside a smoothly bounded disc $V \subset \Omega_f$ to obtain a new map F.

Contents. In Section 2 we define precisely our first surgery operation, which we call *disc-annulus surgery.* In Section 3 we give applications and relate the Julia set of F to that of f. In Section 4 we discuss the dependence of the map F on the various choices made in the construction and prove a uniqueness theorem (Theorem 4.1). In Section 5 we discuss the inverse operation to disc-annulus surgery. Finally, in Section 6 we give a related construction in which the modification procedure is slightly more involved, but which is useful for the construction of examples such as those found in [PT].

2 The disc-annulus surgery

We say that $V \subset \overline{\mathbb{C}}$ is a *smooth disc* if ∂V is a real-analytic Jordan curve. By a *branched covering* we mean a proper C^1 map between smooth, oriented (real) 2-manifolds, possibly with boundary, such that the boundary map is a covering map of (real) 1-manifolds, and such that on the interior, the map is given in appropriate local (complex) coordinates by $z \mapsto z^d$ for some d.

The following two lemmas are the technical ingredients for the definition of our disc-annulus surgery.

Lemma 2.1. *Let $A \subset \overline{\mathbb{C}}$ be an open annulus bounded by two C^1 Jordan curves $\gamma^\pm$, and let W be an open disc bounded by a C^1 Jordan curve η. Give orientations to the curves such that A and W lie to the left of their boundaries. Let $f^\pm : \gamma^\pm \to \eta$ be two orientation-preserving C^1-coverings with degree $d^\pm \geq 1$. Then there exists a branched covering $a : \overline{A} \to \overline{W}$ satisfying the following properties:*

1. *$a|_{\gamma^\pm} = f^\pm$*
2. *$a(A) = W$ and the degree of a is $d^+ + d^-$*
3. *a can be chosen to be C^1 in $\overline{A}$ and holomorphic and proper in a union of any collection of finitely many disjoint smooth discs.*

We call the map a a *covering extension* of the boundary maps $f^\pm$. In practice, we will take a to be holomorphic in a neighborhood of its critical points.

Lemma 2.2. *Let D, D' be two smooths discs in $\overline{\mathbb{C}}$. Then a holomorphic proper mapping $F : D \to D'$ extends to a holomorphic map in a neighborhood of $\overline{D}$. In particular $F : \partial D \to \partial D'$ is a C^1 covering.*

Proof. This is an application of Schwarz reflection principle. □

A branched covering $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is *quasi-regular* if $F = h \circ f \circ g$ where f a rational map and h, g are quasi-conformal homeomorphisms. A branched covering F is *quasi-rational* if it is quasi-conformally conjugate to a rational map. The Julia set of a quasi-rational map is thus well defined, and has the same qualitative metric and measure-theoretic properties as the Julia set of a rational map. The following lemma is an industry standard in the construction of new rational maps via surgery procedures:

Lemma 2.3. Shishikura's principle. *Let F be a C^1 branched covering which is holomorphic a.e. in $\overline{\mathbb{C}} - B$, holomorphic in a neighborhood of critical points in B, and for some integer k, $F^j(B) \cap B = \emptyset$ for all $j \geq k$. Then F is quasi-rational.*

Proof. Pull back the standard complex structure under the full collection $F^{\circ j}$ of iterates of F to obtain an F-invariant measurable conformal structure μ. The finite-recurrence condition implies that μ is bounded. Then the Ahlfors-Bers Measurable Riemann Mapping Theorem gives a quasiconformal conjugacy h^μ between F and a rational map. □

We now describe disc-annulus surgery.

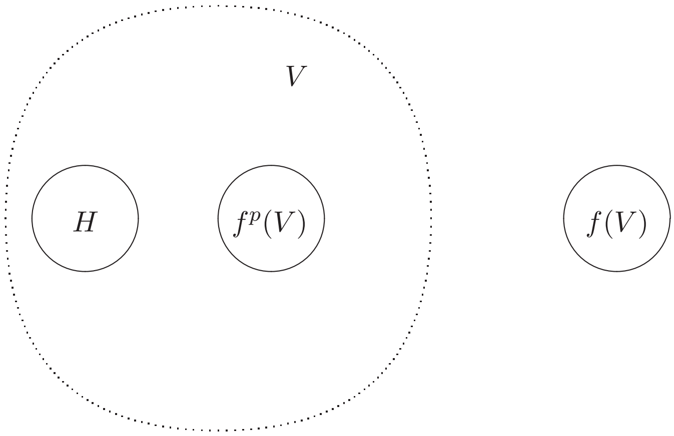

Figure 1 : The domains $V, f^p(V)$ and H

Theorem 2.4. Disc-annulus surgery. *Let (f, V, H, h, a) satisfy the following conditions:*

- *f is a rational map;*
- *V is a smooth disc such that*
  - *∂V contains no critical points;*
  - *$f : V \to f(V)$ is proper;*
  - *there is $1 \le p \le \infty$ such that*

$$f^j(\overline{V}) \cap V = \emptyset \text{ for } 0 < j < p,$$

and, in case $p < \infty$,

$$f^p(\overline{V}) \subset V \text{ and } f^p : V \to f^p(V) \text{ is proper;}$$

- *H is a smooth disc with $\overline{H} \subset V - f^p(\overline{V})$;*
- *$h : H \to \overline{\mathbb{C}} - f(V)$ is a holomorphic proper map, and*
- *$a : V - \overline{H} \to f(V)$ is a covering extension of the boundary maps such that a is holomorphic and proper on $f^p(V)$ and near the critical points.*

Then the map

$$F = \begin{cases} f & \text{on } \overline{\mathbb{C}} - V \\ h & \text{on } H \\ a & \text{on } V - \overline{H} \end{cases}$$

is quasi-rational.

Thus, f is the original map; V is the disc with controlled recurrence on which the modification is supported; $h : H \to \overline{\mathbb{C}}$ is the new, added dynamics; and a is an interpolating map gluing f off of V to h on H. We refer to F as a *disc-annlus extension of f supported on V*. If h is univalent we call the extension F *univalent* or *degree one.* Note that $F(V) = \overline{\mathbb{C}}$ and $\deg(F) = \deg(f) + \deg(h)$.

Proof. We extend at first h on $\overline{H}$ by Lemma 2.2. Set $f^p(V) = \emptyset$ when $p = \infty$. Then F satisfies the conditions of Lemma 2.3 for $B = V - (\overline{H} \cup \overline{f^p(V)})$ and $k = 1$. So F is quasi-rational. □

3 Applications and properties

Here is an instance of Theorem 2.4 with $p = \infty$.

Corollary 3.1. *Let f be a rational map. Let z_0 be a point in the Fatou set such that z_0 is neither a periodic point nor contained in a rotation domain. Then there is a disc-annulus extension F of f supported on a neighborhood V of z_0.*

Proof. Choose V a smooth disc containing z_0 such that $\overline{V} - \{z_0\}$ contains no critical points, $f : V \to f(V)$ is proper, and $f^n(\overline{V}) \cap V = \emptyset$ for all $n > 0$. Then any choice of (H, h, a) satisfying the conditions of Theorem 2.4 (for $p = \infty$) will do. □

Thus, for any map f and any *nonrecurrent* point z_0 in its Fatou set, one can modify f via disc-annulus surgery supported in a small neighborhood V of z_0 so that on a disc H in this neighborhood, the new dynamics is more or less completely arbitrary.

Theorem 2.4 also applies to recurrent points in the Fatou set:

Corollary 3.2. *Let f be a rational map. Let z_0 be a (super)attracting periodic point of period p. Then there is a disc-annulus extension F of f supported on a neighborhood V of z_0.*

Proof. Choose V to be a smooth disc containing z_0 such that $\overline{V} - \{z_0\}$ contains no critical points, $f^p : V \to f^p(V)$ is proper with $f^p(\overline{V}) \subset V$ and $f^n(\overline{V}) \cap V = \emptyset$ for $0 < n < p$. Then any choice of (H, h, a) satisfying the conditions of Theorem 2.4 (for $p < \infty$) will do. □

We now examine how the dynamics of f and F in the above two Corollaries are related. Given a rational map or branched covering f, let

$$P_f = \text{the } \textit{postcritical set} = \overline{\bigcup_{n>0} f^{\circ n}(C_f)}$$

where C_f is the set of critical points of f.

Corollary 3.3. Properties of disc-annulus extensions. *Let f and F be as in either of Corollaries 3.1 or 3.2. Then*

1. *$J_f \subset J_F$, J_F is disconnected, and every connected component of J_f is a connected component of J_F.*
2. *if $H \cap P_f = \emptyset$ and h is univalent, then also $H \cap P_F = \emptyset$.*
3. *if $H \cap P_F = \emptyset$, then*
   - *every Julia component of F passing through H infinitely many times is a point, and*
   - *every other Julia component of F is conformally homeomorphic to a Julia component of f.*

Proof. **(1)** The set J_f may be described as the closure of the set of repelling periodic points of f. Since V is contained in the Fatou set of f and $F = f$ outside V, repelling periodic points of f stay repelling periodic points of F. So $J_f \subset J_F$. Since also $F(H) = h(H) = \overline{\mathbb{C}} - f(V) \supset J_f$ and J_F is totally invariant, $J_F \cap H \neq \emptyset$. Moreover, since $f = F$ on ∂V and $\overline{V}$ is contained in the Fatou set of f, it is easy to show that ∂V and hence ∂H are in fact also in the Fatou set of F. So $J_F \cap H \neq \emptyset, J_F \cap (\overline{\mathbb{C}} - H) \neq \emptyset$, and $\partial H \subset \Omega_F$. Hence J_F is disconnected. That each component of J_f is also a component of J_F follows from the fact that a rational map sends connected components of the Julia set onto such components; see e.g. [Bea] for details.

(2) The critical points of F are the union of the critical points of $f|_{\overline{\mathbb{C}}-V}$, a and h. The F-orbit of the a-critical points do not intersect H. So if h is conformal and $H \cap P_f = \emptyset$, $H \cap P_F = \emptyset$.

(3) Assume now $H \cap P_F = \emptyset$. Take a closed disc $H' \subset H$ such that $J_F \cap H \subset H'$. By a Lemma of Fatou, the diameters of the components of $F^{-n}(H')$ tend to zero as $n \to \infty$. A Julia component J' such that $F^n(J') \subset H$ for some n is contained in a component of $F^{-n}(H')$. Therefore if $F^n(J') \subset H$ for infinitely many n, J' is a point.

For example if h is conformal, it has a unique repelling fixed point in H, which is a point component of J_F.

Now let J' be a component of J_F such that $F^l(J') \subset H$ and $F^n(J') \cap H = \emptyset$ for $n > l$. Then $F^{l+1}(J')$ is a component of J_f, and F^{l+1} is conformal in the component of $F^{-l}(H)$ containing J'. □

Corollary 3.4. Connectivity of Fatou components. *Denote by W_f, W_F the Fatou components of f and F containing ∂V. If W_f is periodic, then W_F is periodic and infinitely connected. If W_f is strictly preperiodic, and h is univalent, then the connectivity of W_F is equal to $m_0 + m_1$, where m_0 and m_1 are the connectivities of W_f and $f(W_f)$.*

Corollary 3.5. (Baker) *There exists rational maps with a non periodic Fatou component of a given number of connectivity.*

These corollaries provide many examples of maps with disconnected Julia sets. Explicit examples of rational maps with similar properties are given in [Bea], Ch. 11. The following examples use the freedom of $a|_{f^pV}$ and $h|_H$ to assign interesting dynamics to F.

Example 1: Surgery of a polynomial to get a higher degree polynomial.

Let f be a polynomial of degree ≥ 1 such that ∞ is a (super)attracting fixed point. Let $V = \{|z| > R\}$. Choose R large enough so that $\overline{V} - \{\infty\}$ contains no critical point of f and $f(\overline{V}) \subset V$. Choose any smooth disc H with $\overline{H} \subset V - \overline{f(V)}$ and any holomorphic proper map $h : H \to \overline{\mathbb{C}} - \overline{f(V)}$. Finally choose a covering extension a on $V - H$ so that a is holomorphic proper from $f(V)$ onto $a(f(V))$ with ∞ as a critical fixed point of local degree $\deg(f) + \deg(h)$; a map of the form $z^{\deg(f)}(z-1)^{\deg(g)}$ provides the model showing such a covering extension exists. This provides a quasi-rational map F with $F^{-1}(\infty) = \infty$. So F is quasi conformally conjugate to a polynomial of degree $\deg(f) + \deg(h)$.

Example 2: capturing a critical point.

In the setting of Example 1, assume that a critical point c of f satisfies $f^l(c) \in V - f(\overline{V})$ for some $l > 0$. We take H a smooth disc containing $f^l(c)$ and $h : H \to \overline{\mathbb{C}} - f(V)$ conformal such that $h(f^l(c)) = c$. Therefore the critical point c, escaping to ∞ under f, is "captured" back and becomes periodic (hence superattracting) under F.

Similarly, one can exploit the same idea to send a critical point c to e.g. a point $x \in J_f$ whose orbit is dense in J_f, thereby obtaining a map F whose postcritical set has complicated topology. Note, however, that in such an example the altered critical point lying in Julia set is nonrecurrent.

Example 3: case for $1 < p < \infty$.

We describe a surgery on $f : z \to z^2 - 1$ to get a cubic rational map with disconnected Julia set, and with 0 a double critical point. Let V be a smooth disc containing 0 whose closure is contained in the basin of 0 so that the second iterate $f^2(V)$ is relatively compact in V, and

$f : V \to f(V)$ is a degree two covering ramified at 0. Now take H a smooth disc whose closure is contained in $V - f^2(\overline{V})$. Define $h : H \to \overline{\mathbb{C}} - \overline{f(V)}$ to be conformal, $a : V - \overline{H} \to f(V)$ to be a covering extension which is a holomorphic branched covering of degree 3 in $f^2(V)$, with 0 as a double critical point and -1 as the critical value. Now the joining of $f|_{\overline{\mathbb{C}}-V}$, h and a gives a quasi-rational map F. This map has a simple critical point at ∞, a double critical point at 0 with orbit $0 \mapsto -1 \mapsto 0$, and another simple critical point in $V - (H \cup f^2(V))$ whose orbit is attracted by the cycle $\{0, -1\}$.

It is clear that the Julia set of F contains a fixed copy J_0 of the Julia set of z^2-1, a countable collection of homeomorphic preimages of J_0, and a Cantor set's worth of point components.

4 Uniqueness of disc-annulus extensions

The construction of a disc-annulus extension of F depends on the non-canonical choices of V, a, H, and h. Clearly, the flexibility of choice in the map a implies that one cannot expect the qc conjugacy class on the whole sphere to be independent of such choices. With this in mind, we say two maps F, F' are *J-equivalent* if there is a qc homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which conjugates F on a neighborhood of J_F to F' on a neighborhood of $J_{F'}$.

In this section, we establish a prototype uniqueness theorem:

Theorem 4.1. *Let f be a rational map and Ω a connected component of the Fatou set of f which is not a parabolic basin. Then the J-conjugacy class of a univalent disc-annulus extension F supported on $V \subset \Omega - P_f$ depends only on Ω and not on V, a, h, H.*

This fails without the assumption of univalence, or the fact that $H \cap P_f = \emptyset$ (which is a consequence of the assumption $V \cap P_f = \emptyset$). Assume on the contrary there is a point $z \in H$ which is either a point of P_f or a critical point of h. Let $h' : H \to \overline{\mathbb{C}} - f(V)$ be another holomorphic map satisfying $h'(z) \neq h(z)$ and $h'(z) \in J_f$. Then V and H are identical but the two resulting extensions F, F' are distinct on their Julia sets.

Analogy with Kleinian groups. The univalent disc-annulus extensions constructed in Theorem 4.1 are analogous to *adding a handle* to a three-manifold with boundary. Let $M_1 = (\mathbb{H}^3 \cup \Omega_{\Gamma_1})/\Gamma_1$ be the three-manifold with boundary associated to a Kleinian group Γ_1. Let D, D' be disjoint round discs in ∂M_1, and consider the three-manifold M obtained by gluing D to D' via an orientation-reversing homeomorphism (equivalently, join D to D' with a solid tube). The resulting manifold admits a hyperbolic structure inherited from the quotient of $\mathbb{H}^3$ by a new group $\Gamma = \Gamma_1 *_{\mathbb{Z}}$ which is an HNN-extension of Γ. The discs D, D' yield a *compressing disc* in M, i.e. its boundary is an essential curve on ∂M. Indeed, if Ω is an attracting basin, then V descends to a closed disc on the quotient punctured torus associated to Ω, which is like the boundary of the quotient three-manifold.

Lemma 4.2. *Let f be a rational map, Ω a component of the Fatou set of f which is not a parabolic basin. Suppose (see Figure 2)*

1. *$W \subset \Omega$ is an open subset such that there is $1 \leq p \leq \infty$ with*
   - *(a)* *$f : W \to f(W)$ is proper,*
   - *(b)*
   $$f^j(\overline{W}) \cap W = \emptyset \text{ for } 0 < j < p,$$

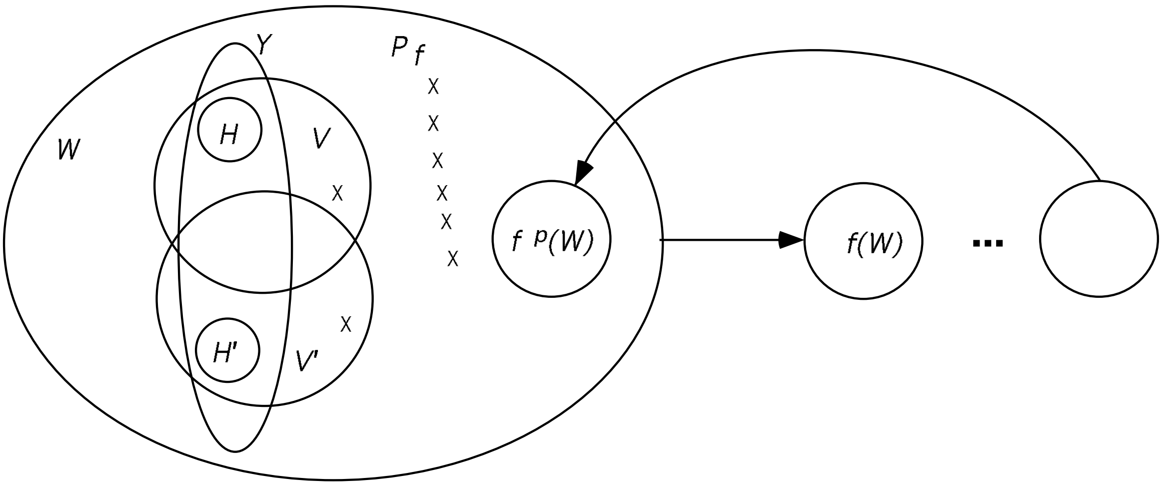

Figure 2

and in case $p < \infty$,

$$\overline{f^p(W)} \subset W \quad \textit{and } f^p : W \to f^p(W) \textit{ is proper.}$$

2. *(V, H) and (V', H') are two pairs of smooth discs in W satisfying the conditions of Theorem 2.4 and the additional condition that $\overline{H \cup H'}$ is contained in a disc or annulus $Y \subset W - \overline{f^p(W)} \cup P_f$.*

Then any two univalent extensions $F = F(V, a, h, H), F' = F'(V', a', h', H')$ are J-equivalent.

Proof of Lemma 4.2. We will construct a combinatorial equivalence (in the sense of McMullen [McM]) between the holomorphic dynamical systems $F : X_1 \to X_0, F' : X_1' \to X_0'$ where X_0, X_0' are neighborhoods of the respective Julia sets. From this, a standard pullback argument implies the existence of a conjugacy $\phi : X_0 \to X_0'$, yielding the result.

In case $p < \infty$, let $U = f(\overline{W}) \cup \cdots \cup f^p(\overline{W})$. In case $p = \infty$, choose U to be an union of finitely many closed discs or annuli contained in the Fatou set of f such that $U \cap W = \emptyset$, $U \supset \bigcup_{n>0} f^n(W)$ and $f(U) \subset U$ (this is possible because by hypothesis and the Classification Theorem $f^n(W)$ is contained in a attracting basin or a rotation domain). Denote by P' the set of $z \in P_f$ such that $f^n(z) \in U$ for some n.

Let $X_0 = \overline{\mathbb{C}} - U \cup P'$; note that $X_0 \supset J_F, J_{F'}$, and that X_0 has finitely many boundary components.

Lifting step. Define $X_1 = F^{-1}X_0$, $X_1' = F'^{-1}X_0$ and $\phi_0|_{X_0} = id$. Extend ϕ_0 to a qc-homeomorphism from $\overline{\mathbb{C}} - f(V)$ to $\overline{\mathbb{C}} - f(V')$ (this is possible since both $f(V)$ and $f(V')$ are compactly contained in $f(W)$). Define $\phi_1 = id$ on $X_0 - W$, and on H define ϕ_1 as the lift of ϕ_0 under h, h'; this is possible since h, h' are univalent and indeed we have $\phi_1|H = (h')^{-1} \circ \phi_0 \circ h$. Then $X_1 \subset (X_0 - W) \cup H$, $X_1' \subset (X_0 - W) \cup H'$ and the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & X_1' \\ F \downarrow & & \downarrow F' \\ X_0 & \xrightarrow{\phi_0} & X_0 \end{array}$$

Isotopy step. Moreover there is a C^1 extension of ϕ_1 on X_0 such that ϕ_1 is isotopic to ϕ_0 relative to $\partial X_0 \cup (P_F \cap X_0)$. To see this, extend ϕ_1 to W such that $\phi_1 = id$ on $W - Y$ and that $\phi_1|_Y$ is isotopic to the identity rel. ∂Y. Then ϕ_1 is isotopic to id relative to $\overline{\mathbb{C}} - Y \supset \partial X_0 \cup (P_F \cap X_0)$.

The pair ϕ_0, ϕ_1 gives the desired combinatorial equivalence, and so we obtain a qc-mapping $\varphi : X_0 \to X_0$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_1' \\ F \downarrow & & \downarrow F' \\ X_0 & \xrightarrow{\varphi} & X_0 \end{array}$$

Extending φ arbitrarily to the whole sphere shows that F and F' are J-equivalent. □

Remarks: In Theorem 4.1, the global qc conjugacy class depends on the critical orbit relations of F, which are difficult to control since the gluing map a necessarily introduces new critical values in $f(V) \subset \Omega_F$.

Also, in the non-univalent case, it may not be possible to carry out the lifting step. Or the lifting step may be possible but the isotopy step impossible.

Proof of Theorem 4.1. Let $V, V' \subset \Omega - P_f$ be any two smooth discs. Set $V_0 = V, V_1 = V'$, and let V_t be a path of smooth discs joining V_0 to V_1. Any two univalent extensions supported on sufficiently close V_s, V_t are J-equivalent, by Lemma 4.2; the Theorem follows by compactness of the interval $[0, 1]$. □

5 Inverse of disc-annulus extensions: simplification

The next result offers a converse procedure.

Theorem 5.1. *Let F be a rational map. Let A be a smooth annulus satisfying the following conditions:*

- *F maps A properly onto a disc;*
- *∂A contains no critical points;*
- *there is $1 \le p \le \infty$ such that*

$$F^j(\overline{A}) \cap A = \emptyset \text{ for } 0 < j < p,$$

and, in case $p < \infty$,

$$F^p(\overline{A}) \subset A \text{ and } F^p : A \to F^p(A) \text{ is proper;}$$

- *there is a component H of $\overline{\mathbb{C}} - \overline{A}$ such that $H \cap \bigcup_{1<j<p} F^j(A) = \emptyset$.*

Then there is a quasi-rational map f which coincides with F on $\overline{\mathbb{C}} - (A \cup \overline{H})$ and $f(A \cup \overline{H}) = F(A)$.

Note that $\deg(f) < \deg(F)$. In the analogy with three-manifolds, one may think of this operation as a special case of cutting along a compressing disc.

Proof. Define f to be F on $\overline{\mathbb{C}} - (A \cup \overline{H})$ and extend f so that $f : A \cup \overline{H} \to F(A)$ is a C^1 proper map which is holomorphic in $F^p(A)$. By assumption, $f^n(B) \cap B = \emptyset$ for $B = (A \cup \overline{H}) - F^p(A)$ and $n > 0$. So f is quasi-rational by Lemma 2.3. □

6 Surgery with two gluing regions

Here, we describe another, similar surgery construction. In the disc-annulus surgery, the new dynamics h is holomorphic on a disc. Here, we allow for the more flexible setting of adding new dynamics h which is holomorphic on an annulus. Since the annulus has two boundary components, the gluing map a will be defined on two disjoint pieces. Controlling the recurrence of the additional region where conformal distortion occurs requires our introduction of an additional smooth disc G; see Figure 3.

Theorem 6.1. *Let (f, V, H, G, h, a) satisfy the following conditions:*

- *f is a rational map,*
- *V a smooth disc contained in a Fatou component Ω such that:*
  - *∂V contains no critical points of f;*
  - *there is $p < \infty$ such that*

$$f^j(\overline{V}) \cap V = \emptyset \text{ for } 0 < j < p, \quad f^p(\overline{V}) \subset V \text{ and } f^p : V \to f^p(V) \text{ is proper;}$$

- *H is a smooth annulus such that $\overline{H} \subset V - f^p(\overline{V})$ and $\partial f^p(V)$ is a boundary component of H;*
- *G is a smooth disc contained in a Fatou compoent Σ such that Ω and Σ have disjoint orbit and $\mathrm{mod}(\overline{\mathbb{C}} - (\overline{G} \cup \overline{f(V)})) = d' \,\mathrm{mod}\, H$ for some integer $d' > 0$;*
- *$h : H \to \overline{\mathbb{C}} - (\overline{G} \cup \overline{f(V)})$ a holomorphic covering of degree d' with $h(f^p(\partial V)) = \partial G$ and $a : V - \overline{H \cup f^p(V)} \to f(V)$, $f^p(V) \to G$ a covering extension holomorphic in a neighborhood of the critical points.*

Then

$$F = \begin{cases} f & \text{on } \overline{\mathbb{C}} - V \\ h & \text{on } H \\ a & \text{on } V - H \end{cases}$$

is quasi-rational. Moreover in $\overline{H} \cap P_f = \emptyset$ the J-conjugacy class of F depends only on Ω, Σ, d' and the homotopy class of ∂H relative to P_f.

Proof. Set $B = V - \overline{H}$. We have $F^j(B) \cap B = \emptyset$ for $j > p$. So by Lemma 2.3 F is quasi-rational.

To prove the unicity, we first establish a local result. Assume (f, V, H, G, h, a) and (f, V', H', G', h', a') are two set of choices of the theorem such that:

1. $\mathrm{mod}(\overline{\mathbb{C}} - \overline{f(V) \cup G} / \mathrm{mod}\, A = \mathrm{mod}(\overline{\mathbb{C}} - \overline{f(V') \cup G'} / \mathrm{mod}\, A' = d'$ is an integer, there is a set U which is the union of finitely many disjoint closed discs such that $\overline{G \cup G'} \subset int(U) \subset U \subset \Sigma$ and $f(U) \subset U$;

2. there is a smooth annulus $\hat{H}$ containing $\overline{H \cup H'}$ such that $f^{-p}(\partial\hat{H}) \cap \hat{H} = \emptyset$.

Then the two maps F and F' are J-conjugate.

To prove this, set W to be the $hatH \cup overline f^p(V)$. It is a disc containing $H \cup H' \cup f^p(V) \cup f^p(V')$. By assumption 2, $f^p(W)$ is contained in $f^p(V) \cap f^p(V')$.

Let $X_0 = \overline{\mathbb{C}} - (U \cup \overline{f(W) \cup \cdots \cup f^p(W)} \cup P')$, where P' is the set of $z \in P_f$ such that $f^n(z) \in U \cup \overline{f(W) \cup \cdots \cup f^p(W)}$ for some n. Note that $X_0 \supset J_F, J_{F'}$ and that X_0 has finitely many boundary components.

Lifting step. Define $X_1 = F^{-1}(X_0)$, $X'_1 = F'^{-1}(X_0)$ and $\phi|_{X_0}$ is a C^1 diffeomorphism isotopic to the identity rel $(P_f - P') \cup \partial X_0$, mapping $\partial f(V)$ to $\partial f(V')$. This is possible since $\partial f(V)$ and $\partial f(V')$ are isotopic rel P_f.

Now extend ϕ_0 to a C^1 diffeomorphism from $\overline{\mathbb{C}} - (f(V) \cup G)$ to $\overline{\mathbb{C}} - (f(V') \cup G')$.

Define $\phi_1|_{X_0 - W} = id$ and $H \to H'$ to be a lifting of $\phi_0 : \overline{\mathbb{C}} - (f(V) \cup G) \to \overline{\mathbb{C}} - (f(V') \cup G')$. Extend ϕ_1 to $\hat{H}$ so that it's isotopic to the identity rel $\partial \hat{H}$. Then ϕ_1 is isotopic to id rel ∂X_0. The only problem is that it is not exactly a lift of ϕ_0 (near $f^{-1}(\partial f(V))$.

The pair ϕ_0, ϕ_1 gives the desired combinatorial equivalence and hence a qc-conjugacy of $F : X_1 \to X_0$ to $F' : X'_1 \to X_0$. Therefore F and F' are J-equivalent.

From this one can get easily the globle unicity result. □

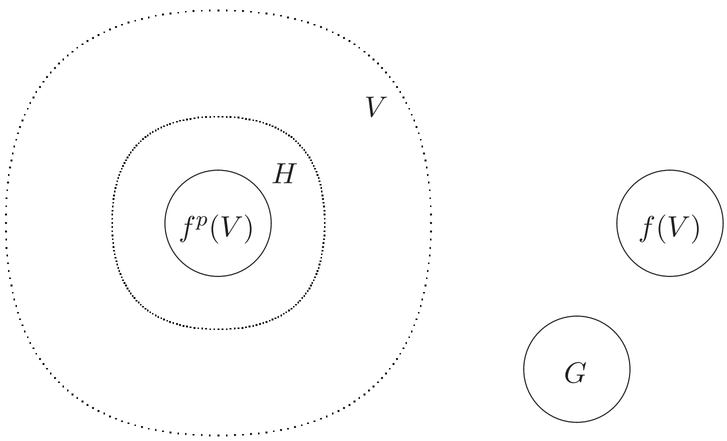

Figure 3: The domains $V, f^p(V), H$ and G

Lemma 6.2. *Let (f, V) be as in Theorem 6.1. If there is a Fatou component U whose orbit under f is disjoint from V, then the discs G, H exist.*

Proof. One can either choose G first or H first. Assume that H and hence mod$H$ are given. Choose $z_0 \in U$. Since the modulus of the annulus between z_0 and $\partial f(V)$ is infinite, one can find a smooth disc G containing z_0 such that $\mathrm{mod}(\overline{\mathbb{C}} - (\overline{G} \cup \overline{f(V)})) = d' \bmod H$ for some (probably very large) integer $d' > 0$. Of course, we can also start from G. Let G be any smooth disc whose closure is contained in U. There is a minimal integer d' such that $\mathrm{mod}(\overline{\mathbb{C}} - (\overline{G} \cup \overline{f(V)})) < d' \bmod H$. Then one can find H in $V - f^p(\overline{V})$ such that $\mathrm{mod}(\overline{\mathbb{C}} - (\overline{G} \cup \overline{f(V)})) = d' \bmod H$. □

In practice we may want d' to be as small as possible. We will show that in case $p = 1$, one can choose V, H, G so that $d' = 2$ if $\deg(f) \geq 3$ and $d' = 3$ if $\deg(f) = 2$.

Surgery on the basin of infinity of a polynomial. Given a polynomial f, its *filled Julia set* K_f is the set of points which do not escape to infinity under iteration. Let $f(z) = z^2 + c, K_c = K_f$ be such that $int(K_c) \neq \emptyset$ (for example $c = -1$). Fix a smooth disc G with closure in $int(K_c)$. Denote by ϕ the Böttcher coordinate of f in the basin of infinity. Let R large enough such that

$$\log R > \frac{1}{3} \bmod (\overline{\mathbb{C}} - (\overline{G} \cup \phi(\{|z| = R^2\}))) \ ;$$

this is possible since the right hand side is comparable to $\frac{2}{3}\log R$. Now take $V = \phi(\{|z| > R\})$. One can choose H as in Theorem 6.1 with $d' = 3$. See [PT] for a computer generated picture of the Julia set of such a map with $c = -1$ and $d' = 3$.

A similar surgery can be done for any degree d polynomial f with $int(K_f) \neq \emptyset$, or with rational maps with multiple superattracting cycles. Moreover if $d \geq 3$ one can choose V, G, H so that $d' = 2$. Finally, note that if $d' \geq 2$ then J_F may contain preimages of components of J_f under covering maps of positive (indeed, arbitrarily large) degree.

We conclude with an easy consequence of Theorem 6.1.

Corollary 6.3. *Let F be the quasi-rational map given by Theorem 6.1. Then J_F is not connected, $J_f \subset J_F$, $\deg(F) = \deg(f) + \deg(h) = \deg(f) + d'$ and $J_F - J_f$ contains uncountably many wandering components which are not points.*

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