# Uni-critical Branner-Hubbard conjecture

Tan Lei and Yin Yongcheng

## 1 Introduction

The Branner-Hubbard conjecture has been recently proved in its full generality (see [KS], [QY]):

**Theorem** (Qiu-Yin, Kozlovski-van Strien). *Let $f : \mathbb{C} \to \mathbb{C}$ be a complex polynomial. Denote by $K_f$ its filled Julia set. Assume that every component of $K_f$ containing a critical point is aperiodic under the iteration of $f$. Then $K_f$ is a Cantor set.*

A particular case of this result is the uni-critical one:

**Theorem 1.1** *Assume that $f : \mathbb{C} \to \mathbb{C}$ is a polynomial with all but one critical point $c$ escaping to $\infty$. The non-escaping critical point $c$ might have multiplicity. Assume that the $K_f$-component of $c$ is not periodic. Then $K_f$ is a Cantor set.*

The case $c$ is a simple critical point was originally proved by Branner-Hubbard ( [BH]), which lead them to the conjecture for a general polynomial.

In this short note we will illustrate the new techniques involved in the proof of this conjecture, by proving a key statement (Theorem 1.2 below) that leads easily to the unicritical Theorem 1.1. The proof we present here follows essentially the same line as [QY].

These new techniques, namely Kozlovski-Shen-Strien nest and Kahn-Lyubich covering lemma, when combined with various other techniques, have lead to several interesting results in the field. For a detailed account, see for example [QY].

## 2 Results

We set up a puzzle by taking an equipotential of $f$ so that the bounded pieces contain no other critical points then $c$. These pieces consist of our puzzle of depth 0. A connected component under $f^{-n}$ is a puzzle piece of depth $n$.

The tableau $T(c)$ is by definition a grill indexed by $-\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, \cdots\}$, with the 0-th column representing the nest of the consecutive puzzle pieces containing the critical point $c$. Now let $I$ be a critical puzzle piece of depth, say, $m$. It occupies therefore the $(-m, 0)$th entry of $T(c)$. We use $f^j(I)$, $j = 1, \cdots, m$ to form the north-east diagonal segment starting from $I$, so that $f^j(I)$ occupies the $(-m + j, j)$th entry of $T(c)$.

A position in $T(c)$ is marked by $\circ$ if it is non-critical, by $\bullet$ if it is a critical puzzle piece, and by $\times$ if unknown.

A *child* of a critical puzzle piece $I$ is defined to be a critical puzzle piece $J$ with deeper depth so that, setting $k = |I - J|$, $f^j(J)$, $j = 1, 2, \cdots, k-1$, is not a critical piece, and $f^k(J) = I$. We will use the symbol $J \overset{f^k}{\rightarrowtail} I$. In other words $f : J \to f(J)$ is non-univalent but $f^{k-1} : f(J) \to I$ is univalent. We will use $\delta$ to denote the degree of $f$. Then $\deg(J \overset{f^k}{\rightarrowtail} I) = \delta$.

Viewed from $T(c)$, a child $J$ of a critical piece $I$ (say at depth $m$) is a position on the column zero, say of depth $m'$, so that marching from $J$ diagonally up right until the depth $m$, one meets only non-critical positions, except the last one, which is critical.

The critical tableau $T(c)$ is said *persistently recurrent*, if

- every horizontal line contains infinitely many critical positions,
- every vertical line contains at most finitely many critical positions,
- every critical piece has at least one and at most finitely many children.

The key in proving Theorem 1.1 is the following:

**Theorem 1.2** *Assume that the critical tableau $T(c)$ is persistently recurrent. Then in the critical nest there are disjoint annuli with moduli $\mu_n$, such that $\mu_n \geq C > 0$ for some $C$ independent of $n$.*

The implication Theorem 1.2 $\Longrightarrow$ Theorem 1.1 is quite classical. For details see [QY].

To start with, we recall the three tableaux rules:

Rule 1 (vertical segment): in each column either there is no critical position or the critical positions form a single vertical segment on the top of the column.

An **upper triangle** in a tableau is by definition a triangle with a vertical side on the left, a horizontal side on top and a diagonal segment as the third side. Its **size** is simply the length of any of its sides, and its **depth** is the depth of its lowest vertex.

Rule 2 (double triangle rule): Two upper triangles in $T(c)$ of the same size and depth such that both vertical sides are critical, are identical.

A **puzzle parallelogram** is a parallelogram with two vertical sides and with the top and lower sides made by diagonal segments.

Rule 3 (double parallelogram rule). Given a puzzle parallelogram $D$ in $T(c)$ whose two vertical sides are critical and whose other pieces are non-critical, then for any other puzzle parallelogram $D'$ of the same size and depth, with the two top vertices being critical, then either the lower side of $D'$ is either completely non critical, or $D'$ is identical to $D$.

**Lemma 1.3** *Every critical puzzle piece $I$ has at least two children.*

*Proof:* This is similar to Lemma 1.3 (b) in [M]. Since the convention there is slightly different from ours (we look at the pieces rather than the annuli), we include a proof here.

Start from a piece $I$ on the column zero of $T(c)$, say of depth $d$. March to the right until we first meet a critical position, say at column $k \geq 1$. Denote this position by $I'$ (we know that $I$ and $I'$ represent the same critical piece). Now marching diagonally south-west (left-down) to reach a position $S$ on the column zero. By Rule 1, every position on the open diagonal $diag]S, I'[$ is non critical. Therefore $S$ is a child of $I$, and is in fact the first child.

By hypothesis of persistent recurrence of $T(c)$ and by Rule 1, the top of the $k$th-column consists of a segment of finitely critical positions (containing $I'$) and then becomes non-critical from some depth, say $d'$ (with $d' > d$). Denote by $P$ this first non-critical position.

Start now from $P$. Following first the left down diagonal to reach a piece $L$ on the column zero. By Rule 1 every position on $diag]L, P[$ is non-critical. Now start from $P$ again and proceed up right along the diagonal until the depth of $I$. There one hits a position, denoted by $W$. Consider the closed diagonal segment $diag[P, W]$. Apply repeatedly Rule 2 and Rule 3 to each block of $k$ steps starting from $P$ (by comparing with a left most triangle/parallelogram), one concludes that the entire diagonal from $P$ to $W$, except possibly the last block (of length $< k$), is non-critical. Using then Rule 2 (by comparing with a left most triangle) and then Rule 1 to conclude that every position on the last block, except possibly $W$, is non-critical. Therefore $diag]L, W[$ is entirely non-critical.

If $W$ is critical then it represents the critical piece $I$ and $L$ is a second child of $I$. If not march right from $W$ until the first hit of a critical position $I''$ (representing again the critical piece $I$). This position exists by assumption. Then by Rule 1 $I''$ leads to a second child $Z$ of $I$ by following its left down diagonal until the 0th column.

**Lemma 1.4** *For any two nested pieces $L \subset\subset K$ in the 0-column of $T(c)$, then $(K \setminus L) \cap P_f = \emptyset$ iff the horizontal strip in $T(c)$ from $K$ to $L$ does not contain semi-critical positions, i.e. no critical vertical segment ends at a depth $m$ with $depth(L) > m \geq depth(K)$.*

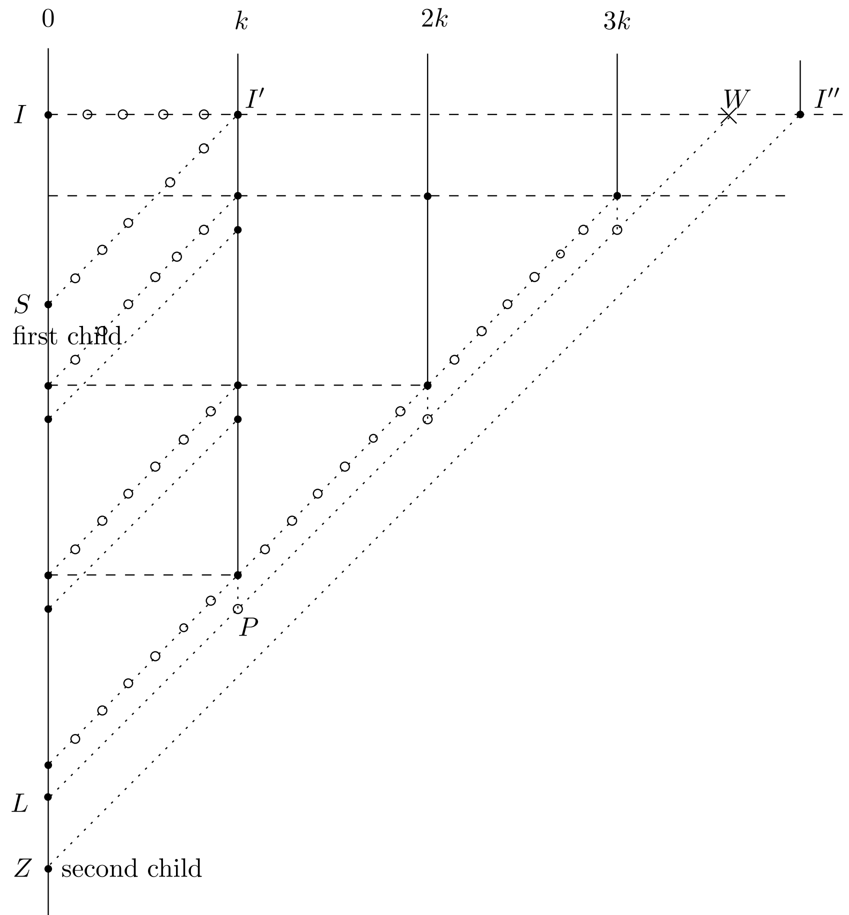



**Figure 2.1.** The existence of two children (or more).

## 3 KSS nest

Definition. Let $I_0$ be a critical puzzle piece. We will define inductively a nested sequence of critical puzzle pieces

$$I_0 \supsetneqq \cdots \supsetneqq I_n \supsetneqq L_n \supsetneqq K_n' \supsetneqq K_n \supsetneqq I_{n+1} \supsetneqq \cdots ,$$

all descendants of $I_0$, as follows: Assume $I_n$ is already defined. Then its first and last child are named respectively $L_n, K_n'$. The piece $K_n$ is the critical puzzle piece contained in $K_n'$ such that the depth difference $|K_n - K_n'|$ is equal to $|L_n - I_n|$. Finally $I_{n+1}$ is the last child of $K_n$. By Lemma 1.3 all

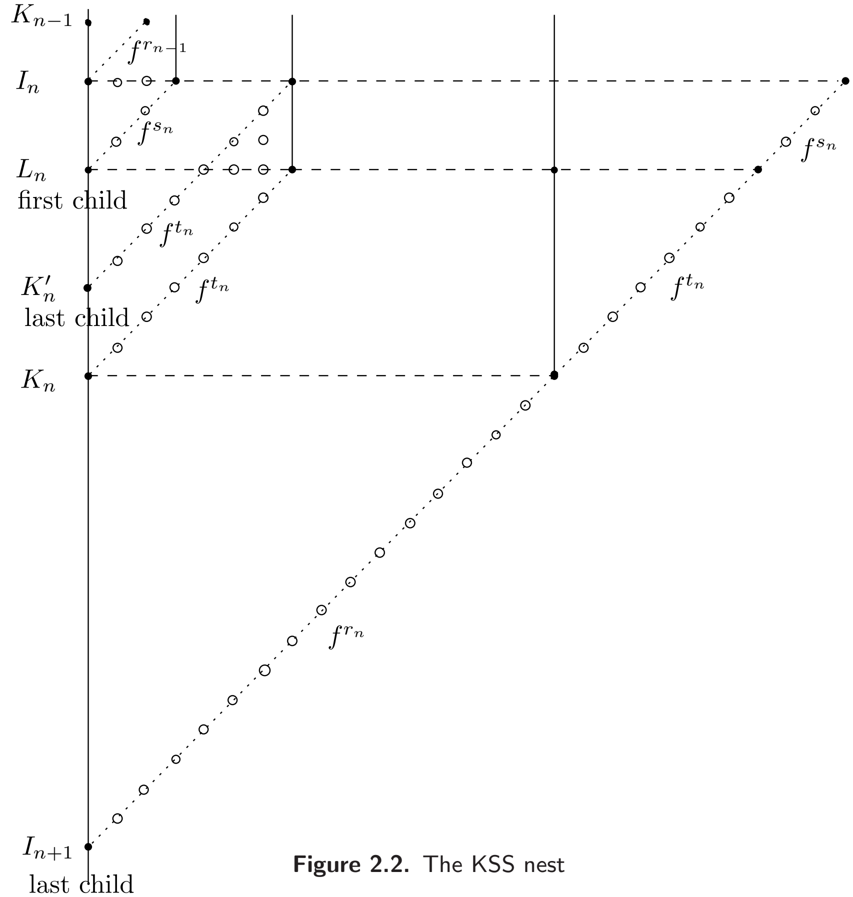
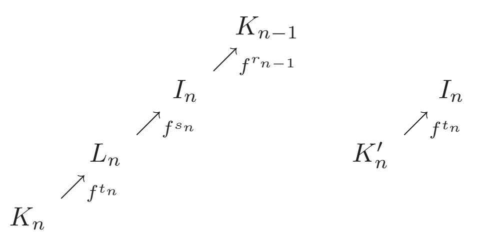



**Figure 2.2.** The KSS nest

these pieces exist and are mutually distinct.




This nested sequence of puzzle pieces is called *KSS nest* (Kozlovski-Shen-Strien nest). See [KSS] the its original construction.

Here are some basic combinatorial properties of the KSS nest. Set $s_n := |L_n - I_n| = |K_n - K'_n|,\ t_n := |K'_n - I_n|,\ r_n := |I_{n+1} - K_n|,\ p_n := |K_n - K_{n-1}|$ .

**Lemma 1.5** *We have* $f^{r_{n-1}}(I_n) = K_{n-1}$, $f^{s_n}(L_n) = I_n$, $f^{t_n}(K'_n) = I_n$, $f^{t_n}(K_n) = L_n$, *each is of degree* $\delta$, *where* $\delta = deg(f)$, *and* $f^{p_n}(K_n) = K_{n-1}$, $deg(f^{p_n} : K_n \to K_{n-1}) = \delta^3$ .

*Proof:* The first equalities $f^{r_{n-1}}(I_n) = K_{n-1}$, $f^{s_n}(L_n) = I_n$, $f^{t_n}(K'_n) = I_n$ are almost trivial from the construction: a child $S$ of $I$ is in particular a pullback of $I$. And the number of pullback iterates is equal to the depth difference.

However the equality $f^{t_n}(K_n) = L_n$ is a lot less trivial. We know that $f^{t_n}(K_n)$ is a piece at the same depth as $L_n$.

Looking in the tableau $T(c)$, by definition of a child, there is no critical position on the diagonal between $K'_n$ and $f^{t_n}(K'_n)$, except the ends. By Rule 1 there is no critical position on the diagonal between $K_n$ and $f^{t_n}(K_n)$, except probably the north-east end.

Assume by contradiction that $f^{t_n}(K_n)$ is not critical.

Starting from $f^{t_n}(K_n)$ and go to the west (left) until one reaches the diagonal between $K'_n$ and $f^{t_n}(K'_n)$, this horizontal segment has no critical position by Rule 1 and the fact that $K'_n$ is a child.

One goes now from $f^{t_n}(K_n)$ to the east (right) until one hits for the first time a critical position and then turn to going north until the depth of $I_n$, to reach a piece, which we name $I'$. Now $I'$ must be critical by Rule 1, i.e. is equal to $I_n$. Now follow the south-west diagonal from $I'$ until the 0th column, we will find another child of $I_n$: by Rule 2 and the fact that $L_n$ is the first child the pieces of the diagonal from $I'$ until the depth above that of $L_n$ has no critical positions. When it reaches the depth of $L_n$ it is again non-critical. And then it continues to avoid critical positions all the way to the 0th-column due to Rule 1.

This contradicts the choice of $K'_n$ as the last child. Consequently $f^{t_n}(K_n) = L_n$.

Finally the equality $f^{p_n}(K_n) = K_{n-1}$, i.e. $K_n$ is a pullback of $K_{n-1}$ follows: as a child $I_n$ is a pullback of $K_{n-1}$, and $L_n$ is a pullback of $I_n$ and finally $K_n$ is a pullback of $L_n$ by above.

The degree estimates are simple consequences.

The following properties are essential:

**Lemma 1.6** *For any* $n \geq 1$,

(a) *we have* $t_n > s_n \geq r_{n-1}$, $p_n = t_n + s_n + r_{n-1}$, $(K'_n \smallsetminus K_n) \cap P_f = \emptyset$;

(b) *any two consecutive critical positions on the horizontal line of* $K_n$ *have a width difference* $\geq t_n$ *and* $\leq r_n$;

(c) *we have* $r_n \geq 2t_n$, $3t_n \geq p_n \geq 2p_{n-1}$;

(d) *for any* $m > m^-$, *and for* $M := |K_m - K_{m^-}|$, *we have* $f^M(K_m) = K_{m^-}$ *of degree* $\delta^{3(m-m^-)}$, *and* $M < 2p_m$.

*Proof:* (a). The relations $t_n > s_n \geq r_{n-1},\ p_n = t_n + s_n + r_{n-1}$ are trivial.

To prove $(K'_n \setminus K_n) \cap P_f = \emptyset$, we apply Lemma 1.4. Assume by contradiction that there is a semi-critical position $Q \supset E$ on the horizontal strip of $K'_n$ and $K_n$. As $Q$ is critical, we have $Q = K'_n$. As $E$ is not critical, we have $E \neq K_n$. By Rule 2 $f^{t_n}(Q) =: I'$ is critical therefore is equal to $I_n$, and there is not other critical positions on the diagonal. By Rule 3 $f^{t_n}(E)$ is not critical. Now follow the argument as in the proof of $f^{t_n}(K_n) = L_n$, by going at first to the right of $f^{t_n}(E)$ until the first critical position, then upwards until the depth of $I_n$ and then follow the left-down diagonal until the first critical position. We then find a child of $I_n$ with depth deeper than $depth(Q) = depth(K'_n)$, contradicting the fact that $K'_n$ is the last child.

(b) At first we look at the distance between $K_n$ and the first critical position $E'$ on the east (right) of $K_n$. As the diagonal between $K_n$ and $f^{t_n}(K_n)$ has no critical positions (except the ends), by Rule 1 the width difference $|E' - K_n|$ is at least $t_n$.

Let now $E'', E'''$ be two consecutive critical positions on the depth of $K_n$. As $E''$ is critical it is equal to $K_n$. By Rule 2 the diagonal between $E''$ and $f^{t_n}(E'')$ has no critical positions (except the ends), therefore the width difference from $E''$ to $E'''$ is at least $t_n$.

Assume that the width difference from $E'', E'''$ is $> r_n$, and that $E'''$ is on the right of $E''$. Then starting from $E'''$ and follow the south-west diagonal until we reach a critical position for the first time, we find a child of $K_n$ that is deeper than $I_{n+1}$, contradicting the choice of $I_{n+1}$.

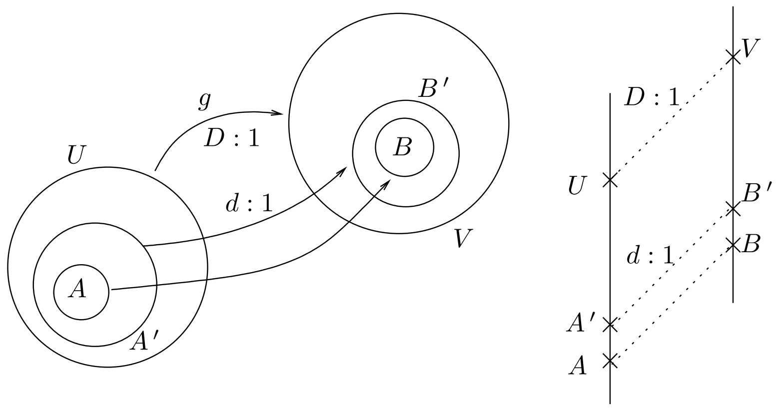
(c) Proof of $r_n \geq 2t_n$. As $I_{n+1}$ is a child of $K_n$, but is not the first child, we have that $f^{r_n}(I_{n+1})$ is some $k$'th critical position to the east (right) of $K_n$ with $k \geq 2$. So $r_n \geq 2t_n$. The fact that $p_n = t_n + s_n + r_{n-1} \leq 3t_n$ is due to (a). Now $t_n \geq s_n \geq r_{n-1} \geq 2t_{n-1} \geq 2s_{n-1} \geq 2r_{n-2}$, so
$p_n = t_n + s_n + r_{n-1} \geq 2t_{n-1} + 2s_{n-1} + 2r_{n-2} = 2p_{n-1}$.

(d) $M = p_m + p_{m-1} + \cdots + p_{m^-+1}$. So $M + p_{m^-+1} \leq 2p_m$. The rest follows from Lemma 1.5.

# 4 From KSS nest to the conditions of KL Lemma

**Lemma 1.7** *(Kahn-Lyubich Covering Lemma)( [KL]) Fix $\eta > 0$ and $D \in \mathbb{N}$. There is $\varepsilon(\eta, D) > 0$ such that:*
*given any two nests of three hyperbolic discs $A \subset\subset A' \subset\subset U$ and $B \subset\subset B' \subset\subset V$, and any proper holomorphic map $g : U \to V$ of degree $D$ such that $g|_{A'} : A' \to B'$ and $g|_A : A \to B$ are both proper and that $mod(B' \setminus B) \geq \eta \cdot mod(U \setminus A)$, then, setting $d := deg(g|_{A'})$,*

$$\text{either} \quad mod(U \setminus A) > \varepsilon(\eta, D) \quad \text{or} \quad mod(U \setminus A) > \frac{\eta}{2d^2} mod(V \setminus B)\ .$$




$$mod(B' \setminus B) > \eta \cdot mod(U \setminus A) \implies mod(U \setminus A) \begin{cases} \overset{\text{either}}{>} & \epsilon(\eta, D) > 0 \\ \overset{\text{or}}{>} & \frac{\eta}{2d^2} mod(V \setminus B) \end{cases}$$

**Figure 4.3.** Kahn-Lyubich Covering Lemma

This lemma will be our main analytic tool (along side with Grötzsch inequality) to estimate moduli of annuli.

Fix $N > 0$ (which will be $2\delta^{34} + 1$ later on). Fix $m > N$. We will construct a KL-map $g : (U, A', A) \to (V, B', B)$ depending on the pair $(N, m)$. Set $U = K_m$ and $V = K_{m-N}$.

Let $\sigma, M$ be the integers such that $f^\sigma(K'_{m+4}) = K_m = U$, $f^M(K_m) = K_{m-N} = V$. View $U, V$ as to sit on the $\sigma$th and the $(\sigma + M)$th column of $T(c)$. Set $g = f^M|_U : U \to V$.

Note that $c \in K'_{m+4}$. Set $x = f^\sigma(c)$ and $y = f^M(x)$. As $c$ is persistently recurrent, there is a minimal $s \geq 0$ such that $f^s(y) \in K_m$. This corresponds in $T(c)$ to start from the column of $y$ at $depth(K_m)$ going to the right $s \geq 0$ steps until the first hit of a critical position. If $s = 0$, i.e. $y \in K_m$, set $B = K_m$ and $B' = K'_m$. If $s > 0$, the $f^{-s}(K_m)$ has a component containing $y$, denoted by $B$. As $B$ never visits a critical piece before reaching $K_m$, the map $f^s : B \to K_m$ is actually conformal. Now $f^{-s}(K'_m)$ has a component containing $y$, denoted by $B'$. As $(K'_m \setminus K_m) \cap P_f = \emptyset$ by Lemma 1.6, we know that $f^s : B' \to K'_m$ is also conformal. As $B, B'$ are puzzle pieces of depth deeper than that of $K_{m-N} = V$, we have $B \subset\subset B' \subset\subset V$.

Pull back $B, B'$ by $g = f^M$ to get components $A, A'$ containing $x$.

In order to apply KL-lemma we need to control $D := deg(g|_U)$ and $d := deg(g|_{A'})$.

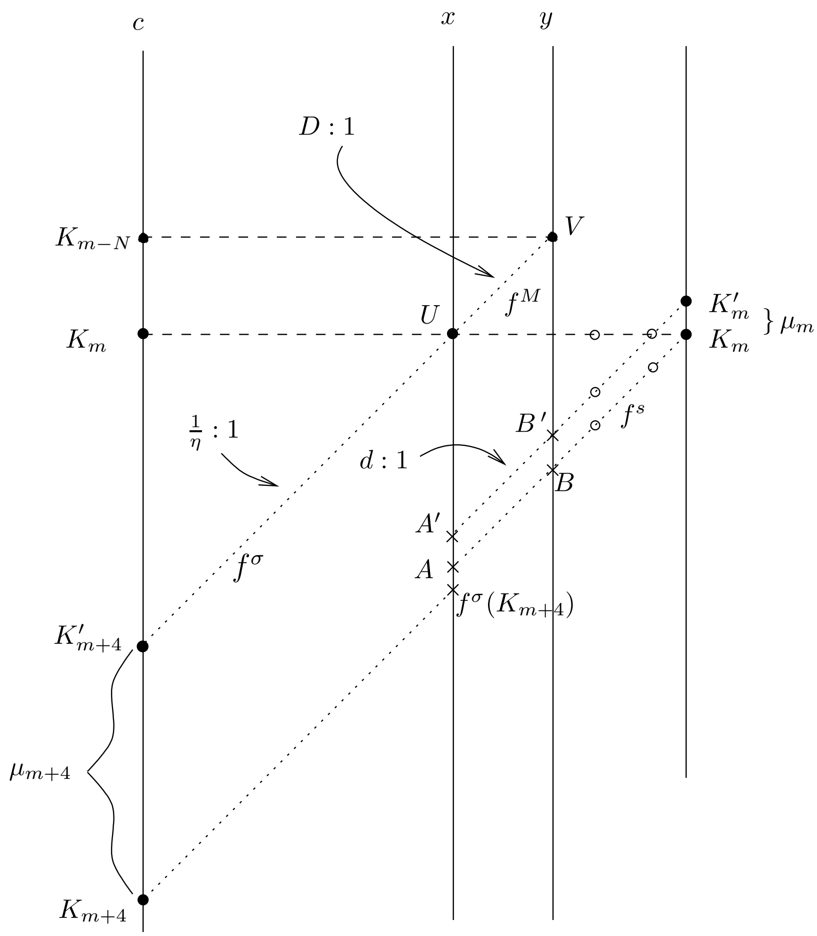



**Figure 4.4.** Applying Kahn-Lyubich Covering Lemma

**Lemma 1.8** *Set* $\mu_j := mod(K'_j \smallsetminus K_j)$.

a) $\sigma = t_{m+4} + r_{m+3} + p_{m+3} + p_{m+2} + p_{m+1}$ *and* $deg(K'_{m+4} \xrightarrow{f^\sigma} K_m) = \delta^{11}$;

b) $D := deg(g : U \to V) = \delta^{3N}$ *(independent of* $m$*) and* $M < 2p_m$;

c) $\#\{0 \leq j < M + s, f^j(A) \subset K_m\} \leq 6$;

d) $d := deg(g : A \to B) = deg(g : A' \to B')$ *is at most* $\delta^6$;

e) $f^\sigma(K_{m+4}) \subset A$ *and* $mod(U \smallsetminus A) \leq \delta^{11} \cdot \mu_{m+4}$;

f) $mod(V \smallsetminus B) \geq \sum_{j=m-N+1}^{m} \mu_j$;

g) $mod(B' \smallsetminus B) = \mu_m$.

*Proof:* a). Note that the number of iterates from $K_{m+4}$ to $K_m$ is $p_{m+4} + p_{m+3} + p_{m+2} + p_{m+1}$. But $|K_{m+4} - K'_{m+4}| = s_{m+4}$. So $\sigma = -s_{m+4} + p_{m+4} + p_{m+3} + p_{m+2} + p_{m+1} = t_{m+4} + r_{m+3} + p_{m+3} + p_{m+2} + p_{m+1}$. Now the total degree is $\delta^{1+1+3+3+3} = \delta^{11}$ by Lemma 1.5.

b). This is due to Lemma 1.6.

c) and d). Note that $A$ is on the $\sigma$-th column of $T(c)$, and $f^M(A) = B$. As $f^s(B) = K_m$, we have $depth(B) - depth(K_m) = s$ and $depth(A) - depth(B) = M$. As $A \subset K_m$, the critical segment on the column of $A$ reaches at least to the depth of $K_m$.

By Lemma 1.6 two adjacent critical positions on the depth $K_m$ are at least $t_m$ apart. Each visit of $A, f(A), \cdots f^{M-1}(A)$ to $K_m$ corresponds to a critical position. Hence

$$\#\{0 \le j < M, f^j(A) \subset K_m\} \le \frac{M}{t_m} + 1 \overset{b)}{<} \frac{2p_m}{t_m} + 1 \overset{Le.1.6}{\le} \frac{2 \cdot 3t_m}{t_m} + 1 = 7 \ .$$

Note that $f^M(A)$ (which is $B$) first visits $K_m$ after $s$-iterates. So

$$\#\{0 \le j < M + s, f^j(A) \subset K_m\} = \#\{0 \le j < M, f^j(A) \subset K_m\} \le 6.$$

Now for $j \in \{0, 1, \cdots, M + s - 1\}$, if $f^j(A)$ is a critical piece then $f^j(A) \subset K_m$. It follows that there are at most six such $j$ and $deg(f^{M+s} : A \to K_m) \le \delta^6$. But $deg(f^{M+s} : A \to K_m) = deg(f^M : A \to B) = d$. We have $d \le \delta^6$.

Now $B'$ is a pullback by $f^s$ of $K'_m$ and $B$ is a pullback by $f^s$ of $K_m$. As $(K'_m \smallsetminus K_m) \cap P_f = \emptyset$, we have that $B' \smallsetminus B$ is a pullback by $f^s$ of $K'_m \smallsetminus K_m$ and is also disjoint from $P_f$. This implies that $A' \smallsetminus A$ is a pullback by $f^M$ of $B' \smallsetminus B$ and $deg(f^M|_{A'}) = deg(f^M|_A)$.

e). Note that $f^\sigma(K_{m+4})$ and $A$ are both puzzle pieces containing $x$. So one just needs to estimate the depths to see which is contained in which. As they are both pullbacks of $K_m$ we will estimate the number of visits to $K_m$ through iteration:

$$K_{m+4} \xrightarrow{f^\sigma} f^\sigma(K_{m+4}) \xrightarrow[a)]{f^{s_{m+4}}} K_m.$$

By Lemma 1.6 two adjacent critical positions on the depth $K_m$ are at most $r_m$ apart. So, denoting by $[x]$ the integer part of $x$,

$$\#\{0 \le j < s_{m+4}, f^{j+\sigma}(K_{m+4}) \subset K_m\} \ge \left[\frac{s_{m+4}}{r_m}\right] \overset{Le.1.6}{\ge} \left[\frac{r_{m+3}}{r_m}\right] \overset{Le.1.6}{\ge} 8 \ .$$

Therefore $f^\sigma(K_{m+4}) \subset A$. Now

$$mod(U \smallsetminus A) \overset{\text{Grötzsch}}{\le} mod(U \smallsetminus f^\sigma(K_{m+4})) = mod(f^\sigma(K'_{m+4}) \smallsetminus f^\sigma(K_{m+4}))$$
$$\overset{*}{\le} deg(f^\sigma|_{K'_{m+4}}) \cdot mod(K'_{m+4} \smallsetminus K_{m+4}) = deg(f^\sigma|_{K'_{m+4}}) \cdot \mu_{m+4} \overset{\text{a)}}{=} \delta^{11} \mu_{m+4},$$

where the inequality marked by * is an equality if $f^\sigma$ has no critical point on $K'_{m+4}\smallsetminus K_{m+4}$, and a strict inequality otherwise (this can be proved by Grötzsch inequality, see [M], Problem 1-b).

f). Note that $B$ is a piece on the $(\sigma+M)$th column of $T(c)$. Its critical vertical segment goes down at least to the depth of $K_{m-N}$. For each $j=m-N+1,\cdots,m$, denote by $l_j\ge 0$ the integer so that $f^{l_j}(y)$ visits $K_j$ for the first time. This corresponds in $T(c)$ to start from the column of $y$ at $depth(K_j)$ and go to the right $l_j\ge 0$ steps until the first hit of a critical position. We have $l_j\le s$ by Rule 1. Use $f^{l_j}$ to pullback the annulus $K'_j\smallsetminus K_j$ to get an annulus $C_j$ surrounding $B$. It is quite easy, using Rule 1 and the fact that $(K'_j\smallsetminus L_j)\cap P_f=\emptyset$, to see that the $C_j$'s are mutually disjoint, and $mod(C_j)=\mu_j$. Therefore $mod(V\smallsetminus B)\ge \sum_{j=m-N+1}^{m}\mu_j$.

g). $mod(B'\smallsetminus B)=mod(f^s(B')\smallsetminus f^s(B))=mod(K'_m\smallsetminus K_m)=\mu_m$.

## 5 Final arguments

Proof of Theorem 1.2. Fix a critical puzzle piece $I_0$. Construct the corresponding KSS nest. In particular the annuli $K'_n\smallsetminus K_n$ are mutually disjoint and are nested. Set $\mu_n=mod(K'_n\smallsetminus K_n)$. We want to show that $\liminf\mu_n>0$.

Set $N=2\delta^{34}+1$. $\forall\ m>N$, construct $(U,V,\sigma,M,x,y,g,B',B,A',A)$ as above. Then by Lemma 1.8,

$$(*)\begin{cases}\eta:=\dfrac{1}{deg(f^\sigma|_{K'_{m+4}})} & =\dfrac{1}{\delta^{11}},\\ D:=deg(f^M|_U) & =\delta^{3N},\\ d:=deg(f^M|_{A'}) & \le\delta^6 \quad\text{and}\quad f^\sigma(K_{m+4})\subset A\ .\end{cases}$$

Assume there is a sequence $k_n\to\infty$ such that $\mu_{m'}\overset{m'<k_n}{\ge}\mu_{k_n}$ (otherwise $\liminf\mu_n=\mu_{n_0}>0$). Fix $n$ such that $k_n-4>N$. Set $m=k_n-4$. Then $\mu_{m'}\ge\mu_{m+4}$ for any $m'\le m+4$. Hence,

$$mod(B'\smallsetminus B)\overset{g)}{=}\mu_m\overset{\text{choice of }m}{\ge}\mu_{m+4}\ge\frac{mod(U\smallsetminus f^\sigma(K_{m+4}))}{deg(f^\sigma)}\overset{e)}{\ge}\eta\cdot mod(U\smallsetminus A).$$

We can then apply Kahn-Lyubich Lemma to $g=f^M\colon\ (U,A',A)\to(V,B',B)$, to obtain:

$$\frac{1}{\eta}\mu_{m+4}\ge mod(U\smallsetminus A)\begin{cases}\overset{\text{either}}{>}\varepsilon(\eta,D)>0\ ;\\ \overset{\text{or}}{>}\dfrac{\eta}{2d^2}mod(V\smallsetminus B)\overset{f)}{\ge}\dfrac{\eta}{2d^2}(\mu_m+\cdots+\mu_{m-N+1})\ .\end{cases}$$

But $\dfrac{\eta}{2d^2}(\mu_m+\cdots+\mu_{m-N+1})\overset{\text{choice of }m}{\ge}\dfrac{\eta N}{2d^2}\mu_{m+4}$. As $N=2\delta^{34}+1>\frac{2d^2}{\eta^2}$, we get $\mu_{m+4}>\eta\cdot\varepsilon(\eta,D)>0$.

Therefore $\liminf \mu_n = \lim \mu_{k_n} \geq \eta \cdot \varepsilon(\eta, D) > 0$. (As one can see from the proof, the actual constants in (*) are not important, one needs the following: $D$ is independent of $m$, the constant $\eta$ is independent of $m, N$, and the degree $d$ is bounded by a number independent of $m, N$.)

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