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Rational maps as Schwarzian primitives

CUI GuiZhen[1], GAO Yan[2,*], RUGH Hans Henrik[3] & TAN Lei[4]

[1]*Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing* 100190*, China;*
[2]*Department of Mathematics, Sichuan University, Chengdu* 610065*, China;*
[3]*Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud, Paris* 91405*, France;*
[4]*Faculté des sciences, LAREMA, Université d'Angers, Angers* 49045*, France.*
Email: gzcui@math.ac.cn, gyan@scu.edu.cn, Hans-Henrik.Rugh@math.u-psud.fr, tanlei@math.univ-angers.fr

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Abstract We examine when a meromorphic quadratic differential ϕ with prescribed poles is the Schwarzian derivative of a rational map. We give a necessary and sufficient condition: In the Laurent series of ϕ around each pole c, the most singular term should take the form $(1-d^2)/(2(z-c)^2)$, where d is an integer, and then a certain determinant in the next d coefficients should vanish. This condition can be optimized by neglecting some information on one of the poles (i.e. by only requiring it to be a double pole). The case $d = 2$ was treated by Eremenko [4].
We show that a geometric interpretation of our condition is that the complex projective structure induced by ϕ outside the poles has a trivial holonomy group. This statement was suggested to us by W. Thurston in a private communication.
Our work is related to the problem to finding a rational map f with a prescribed set of critical points, since the critical points of f are precisely the poles of its Schwarzian derivative.
Finally, we study the pole-dependency of these Schwarzian derivatives. We show that, in the cubic case with simple critical points, an analytic dependency fails precisely when the poles are displaced at the vertices of a regular ideal tetrahedron of the hyperbolic 3-ball.

Keywords Schwarzian derivatives, rational maps, critical points, meromorphic quadratic differentials

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1 Introduction

Let $U \subset \mathbb{C}$ be a domain. Recall that for a non-constant meromorphic function $f : U \to \widehat{\mathbb{C}}$, its Schwarzian derivative S_f is a meromorphic function defined by

$$S_f(z) = \begin{cases} \dfrac{f'''(z)}{f'(z)} - \dfrac{3}{2}\left(\dfrac{f''(z)}{f'(z)}\right)^2 & \text{if } f(z) \neq \infty \\ \lim\limits_{w\to z} S_f(w) & \text{if } f(z) = \infty \end{cases}.$$

It is easily checked that the Schwarzian derivative of a Möbius transformation is 0. There is a composition formula

$$S_{u\circ v}(z) = S_u(v(z)) \cdot v'(z)^2 + S_v(z). \tag{1.1}$$

*Corresponding author.

Let γ be a Möbius transformation. We have then $S_{\gamma\circ f}(z) \equiv S_f(z)$. On the other hand, by considering $S_f(z)dz^2$ as a quadratic differential and by setting $z = \gamma(w)$, we have $S_{f\circ\gamma}(w)dw^2 = S_f(z)dz^2$. I.e. the quadratic differential $S_f(z)dz^2$ is invariant if we pre-compose f by a Möbius transformation.

If f is a rational map, then the quadratic differential $S_f(z)dz^2$ is meromorphic on the Riemann sphere, whose poles are located precisely at the critical points of f.

We give here a necessary and sufficient condition for a meromorphic quadratic differential on the Riemann sphere to be the Schwarzian derivative of a rational map.

For $d = 1$ set $Y_1 = 0$. For every positive integer $d \geqslant 2$, let $Y_d(x_1, \cdots, x_{d-1})$ be the polynomial so that

$$\begin{vmatrix} x_1 & 2\cdot 1\cdot(1-d) & 0 & \cdots & 0 \\ x_2 & x_1 & 2\cdot 2\cdot(2-d) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x_{d-1} & x_{d-2} & x_{d-3} & \cdots & 2(d-1)(-1) \\ x_d & x_{d-1} & x_{d-2} & \cdots & x_1 \end{vmatrix} = N \cdot \Big(x_d - Y_d(x_1, \cdots, x_{d-1})\Big)$$

for some contant $N \neq 0$.

We will establish:

Theorem 1.1. *Let $\phi(z)dz^2$ be a meromorphic quadratic differential on the Riemann sphere.*

(a) If $\phi(z) = S_f(z)$ for some rational map f, then all poles of the quadratic differential are of order two, and around each critical point $c \in \mathbb{C}$ of f with local degree $d_c \geqslant 2$, the local Laurent series expansion of $\phi(z)$ has the form

$$(z-c)^2\phi(z) = \frac{1-d_c^2}{2} + a_1(z-c) + a_2(z-c)^2 + \cdots, \text{ with } a_{d_c} = Y_{d_c}(a_1, a_2, \cdots, a_{d_c-1}). \tag{1.2}$$

The same relation also holds at ∞ (if it is a critical point of f) after change of coordinates by $1/z$.

(b) Conversely, assume that $\phi(z)dz^2$ is a meromorphic quadratic differential, and for each pole c, except possibly one, it takes the local expression (1.2) for some integer $d_c \geqslant 2$, furthermore the exceptional pole has order 2. Then $\phi(z)dz^2$ is the Schwarzian derivative of some rational function f, having the local expression (1.2) for every pole. The critical points of f are precisely the poles of $\phi(z)dz^2$, with the integer d_c being the local degree of f at c.

We remark that if we require in Theorem 1.1 (b) that $\phi(z)dz^2$ takes the local expression (1.2) around each pole c, without exceptions, then this theorem is equivalent to Theorem 2.2 below. The key point of the above theorem is that as long as $\phi(z)dz^2$ takes the local expression (1.2) around each pole except one, and has order 2 at the exceptional pole, then it automatically takes the local expression (1.2) around the exceptional pole. This point will be explained in Section 4, and we will also see what happens if the exceptional pole is not a double pole of $\phi(z)dz^2$.

In the case that all poles except one have $d_c = 2$, one can translate the conditions (1.2) into more explicit conditions on the coefficients of ϕ. We will give several forms of such results in Section 5. One particular case is (compare with [4]):

Theorem 1.2. *Fix $d \geqslant 2$. Let $c_1, \cdots, c_{2d-2}$ be distinct points of $\mathbb{C}$, and $A_1, \cdots, A_{2d-2}$ be $2d-2$ unknown complex numbers. For $i = 1, \cdots, 2d-2$ and integers $m \geqslant 0$, set*

$$L_i = 3A_i^2 - 4\sum_{j\neq i}\frac{A_j(c_i-c_j)+1}{(c_i-c_j)^2} \text{ and } E_{m+1} = \sum_i^{2d-2}\Big(mc_i^{m-1} + c_i^m A_i\Big).$$

Then the following 4 problems have the identical set of solutions (for A_i's):

$$-\frac{3}{2}\sum_{i=1}^{2d-3}\frac{A_i(z-c_i)+1}{(z-c_i)^2} \text{ is equal to } S_f \text{ with } f \text{ a rational map of degree } d,$$

$$\begin{cases} L_i = 0,\ i = 1, \cdots, 2d-2 \\ E_{m+1} = 0,\ m = 0, 1, 2 \end{cases}, \begin{cases} L_i = 0,\ i = 1, \cdots, 2d-3 \\ E_{m+1} = 0,\ m = 0, 1, 2 \end{cases}, \begin{cases} L_i = 0,\ i = 1, \cdots, 2d-2 \\ E_{m+1} = 0,\ m = 0, 1 \end{cases}.$$

In this case f has simple critical points at c_i, $i = 1, \cdots, 2d-2$ (and a regular point at ∞).

Notice that Eremenko in [4] added one more equivalent problem to the above list, namely

$$\begin{cases} L_i = 0,\ i = 1, \cdots, 2d-2 \\ E_2 = 0 \end{cases}.$$

Since there are more equations than unknowns, it is *a priori* not clear that there are solutions of such systems. Applying known results of L. Goldberg [9] we could see that the number of solutions is finite and non-zero, and depends on the set $c_1, \cdots, c_{2d-2}$. Generically it is the Catalan number u_d, and otherwise it is at most u_d. See Section 7 for a detailed account.

Theorem 1.3. *Fix $k \geqslant 1$. Let $c_1, \cdots, c_k$ be distinct points of $\mathbb{C}$ and let $r_1, \cdots, r_k$ be complex numbers. Define*

$$e_i(z) = \prod_{j \neq i} \frac{z - c_j}{c_i - c_j} \quad \text{and} \quad v_i = \sum_{j \neq i} \left[-\frac{3}{2} \frac{1}{(c_i - c_j)^2} + \frac{r_j}{c_i - c_j} \right].$$

Let G be an entire function and let

$$\psi(z) = \sum_{i=1}^{k} \left[-\frac{3}{2} \frac{1}{(z - c_i)^2} + \frac{r_i}{z - c_i} + (-\frac{1}{2} r_i^2 - v_i) e_i(z) \right] + G(z) \prod_{i=1}^{k} (z - c_i). \tag{1.3}$$

Then $\psi = S_f$ for a function f which is meromorphic in the entire complex plane and has precisely k critical points that are simple and located at $c_1, \cdots, c_k$.

Conversely, any such function f must have a Schwarzian derivative of the form (1.3).

Finally we are interested in the pole-dependency of the Schwarzian derivatives of rational maps of a given degree. We have obtained a complete answer on the cubic case:

Theorem 1.4. *Let $\frac{z^3 + a_1 z + a_0}{z^2 + b_1 z + b_0}$ be a rational function with 4 distinct critical points. Denote by Δ the cross ratio of these four points. Then the associated Schwarzian derivative depends locally holomorphically on the 4 critical points if and only if $a_1 + 3b_0 \neq 0$, and if and only if $\Delta \notin \{(1 \pm i\sqrt{3})/2\}$.*

2 Local solution

Let $U \subset \mathbb{C}$ be a domain. Recall that for a holomorphic map $f : U \to \widehat{\mathbb{C}}$, its Schwarzian derivative S_f can be expressed as

$$S_f(z) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2.$$

It is easy to check that $z_0 \in U$ is a pole of $S_f(z)$ if and only if z_0 is a critical point of $f(z)$. In that case the pole is always a double pole. More precisely, if $\deg_{z_0} f = d > 1$, then

$$S_f = \frac{1 - d^2}{2(z - z_0)^2} + \frac{a_1}{z - z_0} + a_2 + \cdots \qquad \text{as } z \to z_0.$$

We address the inverse problem: Let $U \subset \mathbb{C}$ be a domain. Given a meromorphic function $\phi(z)$ on U, find a holomorphic map $f : U \to \widehat{\mathbb{C}}$ such that $S_f(z) = \phi(z)$.

The following result is well known:

Lemma 2.1 (see e.g. Theorem 1.1 in [11], pp. 53)**.** . *If the domain U is simply-connected and the meromorphic function $\phi(z)$ has no poles in U, then the equation $S_f(z) = \phi(z)$ has solutions. Moreover, if $f(z)$ and $g(z)$ are two solutions, there is a Möbius transformation γ of $\widehat{\mathbb{C}}$ such that $g(z) = \gamma \circ f(z)$.*

We consider first the case that U is simply-connected, $\phi(z)$ has only finitely many poles in U and each pole of $\phi(z)$ is a double pole. Due to the above statements, we only need to consider the following local problem:

Inverse problem. Let $d \geqslant 2$ be an integer and $\phi(z)$ be a meromorphic function defined in a neighborhood of the origin, with a double pole at 0 and leading coefficient $\frac{1-d^2}{2}$, in other words, with the following local expression

$$\phi(z) = \frac{1-d^2}{2z^2} + \frac{a_1}{z} + a_2 + \cdots \quad \text{as } z \to 0. \tag{2.1}$$

The question is: Under which conditions is there a holomorphic function $f(z)$ defined in a neighborhood of the origin such that $S_f(z) = \phi(z)$?

Theorem 2.2. *Let $d \geqslant 1$ be an integer and $\phi(z)$ be a meromorphic function defined in a neighborhood of the origin such that*

$$\phi(z) = \frac{1-d^2}{2z^2} + \frac{a_1}{z} + a_2 + \cdots \quad \textit{as } z \to 0.$$

Then the equation $S_f = \phi$ has a meromorphic solution if and only if

$$\det \begin{pmatrix} a_1 & k_1 & 0 & \cdots & 0 \\ a_2 & a_1 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{d-1} & a_{d-2} & a_{d-3} & \cdots & k_{d-1} \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} = 0 \ \textit{where } k_j = 2j(j-d),\ j = 1, \cdots, d-1. \tag{2.2}$$

Proof. **Step 1. Reduction to a linear differential equation**. We may look for solutions of the form

$$f(z) = \frac{z^d}{d}(1 + b_1 z + b_2 z^2 + \cdots) \text{ as } z \to 0. \tag{2.3}$$

Note that if such a solution f exists the function $f'(z)/z^{d-1}$ is holomorphic in a neighborhood of the origin and takes the value 1 at the origin. Thus there is a local holomorphic map $g(z)$ such that

$$f'(z) = \frac{z^{d-1}}{g^2(z)}, \text{ and } g(z) = 1 + c_1 z + c_2 z^2 + \cdots \text{ as } z \to 0.$$

This motivates us to set $f(z) = \int_0^z \frac{\zeta^{d-1}}{g^2(\zeta)} \mathrm{d}\zeta$, and express S_f in terms of g. By a direct computation, we have:

$$\frac{f''}{f'} = \frac{d-1}{z} - \frac{2g'}{g} \text{ and } \left(\frac{f''}{f'}\right)' = \frac{1-d}{z^2} - \frac{2(g''g - g'^2)}{g^2}.$$

Thus

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{1-d^2}{2z^2} + \frac{2(d-1)g'}{zg} - \frac{2g''}{g}.$$

On the other hand, set

$$q(z) = z\left(\phi(z) - \frac{1-d^2}{2z^2}\right) = a_1 + a_2 z + \cdots. \tag{2.4}$$

Then the equation $S_f(z) = \phi(z)$ takes the following form:

$$2(d-1)g' - 2zg'' = qg. \tag{2.5}$$

This is a linear differential equation.

Step 2. Solving the differential equation.

For later purpose, we solve the equation (2.5) with d replaced by an arbitrary complex number δ.

Lemma 2.3. *Let $q(z) = a_1 + a_2 z + \cdots$ be a convergent power series in a disk $\{|z| < R\}$ and δ an arbitrary complex number. Then the equation*

$$2(\delta - 1)g' - 2zg'' = qg \tag{2.6}$$

has a formal power series solution in the form

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots \text{ as } z \to 0 \tag{2.7}$$

if and only if: either δ is not a strictly positive integer, or $\delta = d \geqslant 1$ is an integer and the coefficient a_d of $q(z)$ is related to the previous coefficients by the condition (2.2). In both cases the power series of g converges in the disk $\{|z| < R\}$.

Proof. Set $c_0 = 1$. The right hand side of (2.6) is equal to

$$qg = \sum_{n=1}^{\infty} \left(\sum_{j=0}^{n-1} a_{n-j} c_j \right) z^{n-1},$$

whereas the left hand side takes the expansion, by setting $k_n = 2n(n - \delta)$,

$$2(\delta - 1)g' - 2zg'' = \sum_{n=1}^{\infty} [2n(\delta - n)] c_n z^{n-1} = \sum_{n=1}^{\infty} -k_n c_n z^{n-1}.$$

Comparing the coefficients, we get the recursive formula:

$$\forall\, n \geqslant 1, \quad -k_n c_n = \sum_{j=0}^{n-1} a_{n-j} c_j = a_n + a_{n-1} c_1 + \cdots a_1 c_{n-1}. \tag{2.8}$$

One may rewrite it in matrix form:

$$\forall\, n \geqslant 1, \quad \begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & k_n & 0 & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \\ c_{n+1} \\ \vdots \end{pmatrix} = 0. \tag{2.9}$$

As $k_n = 2n(n - \delta)$, we have $k_n = 0$ if and only if δ is a strictly positive integer and $n = \delta$.

Case 1. Assume that δ is a complex number that is not a strictly positive integer. Then $k_n \neq 0$ for all $n \geqslant 1$. So, using $c_0 = 1 \neq 0$, the recursive formula (2.8) determines a unique sequence c_n, $n \geqslant 0$ so that its generating function (2.7) is a formal power series solution of of linear equation (2.6).

Case 2. Assume that δ is a strictly positive integer. Set $\delta = d$. In this case $k_d = 0$ and $k_n \neq 0$ for any $n \neq d$. It is then easy to see that the following statements are equivalent:

1. the equation (2.6) has a power series solution in the form of (2.7)

2. the following linear system has a solution with $c_0 = 1$,

$$\begin{pmatrix} a_1 & k_1 & 0 & \cdots & 0 \\ a_2 & a_1 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{d-1} & a_{d-2} & a_{d-3} & \cdots & k_{d-1} \\ a_d & a_{d-1} & a_{d-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-2} \\ c_{d-1} \end{pmatrix} = 0,$$

3. the same system has a non-zero vector solution, and

4. the determinant of the square matrix vanishes, that is, the condition (2.2).

Convergence of the formal solutions in all cases. Let $R > 0$ be the convergence radius of $q(z) = \sum_{n\geqslant 1} a_n z^{n-1}$. Let δ be any complex number. Let $(c_0, c_1, \cdots)$ be a solution of the system (2.8). We want to show that for $|z| < R$ the power series $\sum_{m=0}^{\infty} c_m z^m$ converges.

Fix $0 < \rho < R$, and $0 < S < +\infty$ such that

$$\sum_{m=1}^{\infty} |a_m|\rho^{m-1} \leqslant S.$$

Now fix an integer m_0 such that for $m > m_0$, we have $|k_m| = 2m|m-\delta| \geqslant 2S\rho$. By (2.8), for $m > m_0$,

$$|c_m|\rho^m \leqslant \frac{|a_m|\rho^m(|c_0|\rho^0) + \cdots + |a_1|\rho^1(|c_{m-1}|\rho^{m-1})}{|k_m|} \leqslant \frac{S\rho}{|k_m|} \cdot \max_{l<m} |c_l|\rho^l \leqslant \frac{1}{2} \max_{l<m} |c_l|\rho^l.$$

It follows that the sequence $|c_m|\rho^m$ is bounded. Therefore $\sum_{m=0}^{\infty} c_m z^m$ converges for any $|z| < \rho < R$. As $\rho < R$ was arbitrary, we conclude that the series $\sum_{m=0}^{\infty} c_m z^m$ converges on the disk of convergence of the power series of $q(z)$. □

Step 3. Conclusion. Let now consider to the case that $\delta = d \geqslant 1$ is an integer.

If the coefficients of $q(z)$ do not satisfy (2.2), then the equation (2.6) does not have a solution $g(z)$ in the form $1 + c_1 z + \cdots$. So the equation $S_f = \phi$ does not have a solution f normalized so that $f(z) = \frac{z^d}{d}(1 + O(z))$. Suppose that f_1 is a meromorphic solution of $S_f = \phi$ in a neighborhood of the origin. Then there is a Möbius transformation γ of $\widehat{\mathbb{C}}$ such that $\widetilde{f_1} := \gamma \circ f_1$ has the form $\frac{z^m}{m}(1 + O(z))$, where $m \geqslant 1$ denotes the local degree of f_1 at the origin. Note that the holomorphic function $\widetilde{f_1}$ is also a solution of $S_f = \phi$ by (1.1). We then get a contradiction.

Assume now that the coefficients of $q(z)$ do satisfy (2.2). Then the equation (2.6) has a solution $g(z)$ in the form $1 + c_1 z + \cdots$, whose convergence disk contains the convergence disk of $q(z)$. Thus, the holomorphic function $f(z) = \int_0^z \frac{\zeta^{d-1}}{g^2(\zeta)} \mathrm{d}\zeta$ is a Schwarzian primitive of ϕ in the convergence disk the power series of $q(z) = z\left(\phi(z) - \frac{1-d^2}{2z^2}\right)$. □

Notice that in the case that the leading coefficient of ϕ is $\frac{1-d^2}{2}$, one may choose $\delta = -d$ and obtain a solution of (2.6) in the form $g(z) = 1 + c_1 z + \cdots$, independent of the determinant in (2.2) being zero or not. There is therefore $r > 0$ a constant so that $g(z) \neq 0$ for every point z with $|z| < r$. Fix $0 < r_0 < r$. Set $z_0 = r_0$ and

$$\widehat{f}(z) = \int_{z_0}^{z} \frac{\zeta^{\delta-1}}{g^2(\zeta)} \mathrm{d}\zeta = \int_{z_0}^{z} \frac{g^{-2}(\zeta)}{\zeta^{d+1}} \mathrm{d}\zeta, \tag{2.10}$$

where z belongs to a disk neighborhood of z_0 disjoint from 0. Denote by $\widehat{b}_d$ the ζ^d coefficient of the power series expansion of $g^{-2}(\zeta)$. Let $\log z$ be the branch of logarithm in a neighborhood of z_0 so that $\log z_0 \in \mathbb{R}$. Then

$$\widehat{f}(z) = \frac{1}{-d \cdot z^d(1 + H(z))} + \widehat{b}_d(\log z - \log z_0)$$

for some holomorphic function H defined in a neighborhood of z_0, and $\widehat{f}$ is locally a Schwarzian primitive of ϕ around z_0.

This $\widehat{f}$ has an analytic continuation and when making a full turn around 0 its value has a phase shift of $\widehat{b}_d \cdot 2\pi i$, with H extending to a holomorphic function in a neighborhood of 0 and $H(0) = 0$.

We have proved:

Corollary 2.4 (second criterion)**.** *Assume that ϕ is a meromorphic function in a neighborhood of the origin with the local expression*

$$\phi(z) = \frac{1-d^2}{2z^2} + \frac{a_1}{z} + a_2 + \cdots \qquad \text{as } z \to 0$$

for some integer $d \geqslant 1$. Then the equation $S_f = \phi$ admits always an eventually multivalued solution $\widehat{f}$ in the form

$$\widehat{f}(z) = \frac{1}{-d \cdot z^d(1+H(z))} + \widehat{b}_d \log z \ ,$$

where H is a holomorphic map (with value in $\mathbb{C}$) around 0 with $H(0) = 0$. And $S_f = \phi$ admits a holomorphic solution in a neighborhood of 0 if and only if $\widehat{b}_d = 0$.

This criterion has the advantage that it tells the form of an eventual obstruction (the log term, see Theorem 3.1 below for an application), but has the disadvantage that the condition $\widehat{b}_d = 0$ can not be explicitly expressed as the determinant in (2.2).

Remark 2.5. It is known by a direct calculation that if $S_f = \phi$, then $v = \frac{1}{\sqrt{f'}}$ satisfies the Sturm-Liouville equation

$$z^2 v'' + \frac{z^2\phi(z)}{2} v = 0. \tag{2.11}$$

Looking for solutions in the form

$$v(z) = z^{\frac{1-d}{2}} \sum_{k=0}^{\infty} v_k z^k \tag{2.12}$$

will lead to the same criteria. This approach was used by Eremenko [4], with a somewhat different presentation. For instance only the case $d = 2$ was worked out explicitly there.

3 Developing maps and holonomy around a puncture

In this section, we give a geometric interpretation of the conditions (2.1) and (2.2).

We may consider the Schwarzian equation on any multiply-connected domain $U \subset \widehat{\mathbb{C}}$. Let $\phi(z)$ be a holomorphic function on U. Then $S_f = \phi$ has a solution on any simply-connected sub-domain in U. Start from a fixed disk D in U, we have a solution f_0 in D which is unique up to a Möbius transformation. The function element (f_0, D) admits an analytic continuation along any paths in U starting from a point of D. For any path $p : [0,1] \to U$ with $p(0) = p(1) \in D$, by following an analytic continuation of (f_0, D) along p, we get another solution f_p on D and thus there is a Möbius transformation γ_p such that $f_p = \gamma_p \circ f_0$. Note that γ_p is determined by the homotopy class of p and the choice of f_0. Thus γ_p is unique up to a Möbius transformation conjugacy.

Denote by $\mathcal{M}$ the group of Möbius transformations of $\widehat{\mathbb{C}}$. Then we have defined a group homomorphism from the fundamental group $\pi_1(U)$ to a sub-group G_ϕ in $\mathcal{M}$ up to a Möbius transformation conjugacy. The group G_ϕ is called the **holonomy group** of the complex projective structure on U induced by ϕ. The equation $S_f = \phi$ has a global solution if and only if G_ϕ is trivial (i.e. it contains only the identity).

When U is an annulus, for example a punctured disk, the fundamental group of U is isomorphic to $\mathbb{Z}$. Therefore, G_ϕ is generated by a single transformation $\gamma_\phi \in \mathcal{M}$. Either γ_ϕ is hyperbolic or elliptic and its conjugate class is determined by its eigenvalue at fixed points, or γ_ϕ is conjugated with the parabolic transformation $z \to z+1$, or γ_ϕ is the identity.

Theorem 3.1. *Assume that ϕ is a meromorphic function in a neighborhood of the origin with the local expression*

$$\phi(z) = \frac{1-\delta^2}{2z^2} + \frac{a_1}{z} + a_2 + \cdots \qquad \text{as } z \to 0, \quad \delta \in \mathbb{C}.$$

Then, for U a punctured neighborhood of 0,

(a) *if δ is not an integer, the generator γ_ϕ is a non-identity elliptic transformation;*

(b) *if $\delta = 0$, the generator γ_ϕ is a parabolic transformation;*

(c) *if $|\delta|$ is an integer greater than or equal to 1, the generator γ_ϕ is a parabolic transformation or the identity, and furthermore, the generator is equal to the identity if and only if $\phi(z)$ satisfies additionally the condition (2.2).*

Proof. For simplicity we assume that $\phi(z)$ is a holomorphic map on the punctured disk $\mathbb{D}^*$ with values in $\mathbb{C}$ (in particular $\phi(z)$ has no poles on $\mathbb{D}^*$). In this case the power series $q(z) = z(\phi(z) - \frac{1-\delta^2}{2z^2}) = a_1 + a_2 z + \cdots$ converges in $\mathbb{D}$.

For a non-integer complex number δ, we define $z^\delta = e^{\delta \log z}$ for some determination of $\log z$.

I. Assume that either δ is not an integer or $\delta = 0$. We see from Lemma 2.3 that equation (2.6) always has a holomorphic solution $g(z) = 1 + c_1 z + \cdots$ in $\mathbb{D}$. Therefore, we get a Schwarzian primitive $\widetilde{f}$ of ϕ in a small disk D close but disjoint from the origin such that

$$\tilde{f}(z) = \begin{cases} \displaystyle\int_{z_0}^{z} \frac{\zeta^{\delta-1}}{g^2(\zeta)} d\zeta = \frac{z^\delta}{\delta}(1 + h(z)) + C_0 & \text{if } \delta \text{ is not an integer} \\ \displaystyle\int_{z_0}^{z} \frac{1}{\zeta g^2(\zeta)} d\zeta = \log z + v(z) + C_1 & \text{if } \delta = 0 \end{cases},$$

where h, v are holomorphic maps near 0 with $h(0) = 0 = v(0)$ and C_0, C_1 are constant. Post composing $\tilde{f}$ by a translation to get rid of the constants, we get a Schwarzian primitive of the form

$$f(z) = \begin{cases} \displaystyle\int_{z_0}^{z} \frac{\zeta^{\delta-1}}{g^2(\zeta)} d\zeta = \frac{z^\delta}{\delta}(1 + h(z)) & \text{if } \delta \text{ is not an integer} \\ \displaystyle\int_{z_0}^{z} \frac{1}{\zeta g^2(\zeta)} d\zeta = \log z + v(z) & \text{if } \delta = 0 \end{cases},$$

where h, v are holomorphic maps near 0 with $h(0) = 0 = v(0)$, and $z^\delta = e^{\delta \log z}$ for some fixed determination of $\log z$.

An analytic continuation of $f(z)$ along the curve $[0, 1] \ni \alpha \to e^{2\pi i \alpha} z_0$ for some $z_0 \in D$ gives another Schwarzian primitive f_1 of ϕ on D such that

$$f_1(z) = \begin{cases} e^{2\pi i \delta} \dfrac{z^\delta}{\delta}(1 + h(z)) = \gamma_\phi \circ f(z), \quad \gamma_\phi(w) = e^{2\pi i \delta} w & \text{if } \delta \text{ is not an integer} \\ \log z + 2\pi i + v(z) = \gamma_\phi \circ f(z), \quad \gamma_\phi(w) = w + 2\pi i & \text{if } \delta = 0 \end{cases}.$$

II. Assume now that $d := |\delta| \geqslant 1$ is an integer. For D a small disk close but disjoint to 0, the holomorphic map

$$\widehat{f}(z) = \frac{z}{-d \cdot (1 + H(z))} + \widehat{b}_d \log z : D \to \widehat{\mathbb{C}}$$

given by Corollary 2.4 is a solution of the equation $S_f = \phi$ on D. By following an analytic continuation along the same curve as above we get another solution on D

$$\widehat{f}_1(z) = \frac{z}{-d \cdot (1 + H(z))} + \widehat{b}_d \log z + \widehat{b}_d \cdot 2\pi i = \gamma_\phi \circ \widehat{f}(z), \ \gamma_\phi(w) = w + 2\pi i \widehat{b}_d.$$

The map γ_ϕ is a Möbius map having a parabolic fixed point at ∞. It is the identity map if and only if $\widehat{b}_d = 0$. □

Applying analytic extension, we have:

Corollary 3.2. *Let $U \subset \mathbb{C}$ be a simply-connected domain and $\phi(z)$ be a meromorphic function on U with only finitely many poles in U. Suppose that at each pole $z_0 \in U$, the map ϕ takes a local expansion of the form*

$$\phi(z) = \frac{1-d^2}{2(z-z_0)^2} + \frac{a_1}{z-z_0} + a_2 + \cdots, \quad \text{as } z \to z_0, \quad d \in \mathbb{N}, d \geqslant 2, a_i \in \mathbb{C}.$$

Then the holonomy group of ϕ is generated by finitely many parabolic Möbius transformations. Furthermore, there is a holomorphic map $f: U \to \widehat{\mathbb{C}}$ such that $S_f(z) = \phi(z)$ if and only if ϕ satisfies the condition (2.2) at each pole, and if and only if the holonomy group is reduced to the identity. In this case f is unique up to post composition by Möbius transformations.

4 Schwarzian derivative near singular points

Following Corollary 3.2, we know that if $\phi(z)$ is a meromorphic function on $\mathbb{C}$ with only finitely many poles, and takes a local expansion (1.2) around each pole, then there exists a holomorphic map $f: \mathbb{C} \to \widehat{\mathbb{C}}$ fulfilling $S_f = \phi$. Thus, to prove Theorem 1.1, we need to study the behavior of S_f near ∞.

Theorem 4.1. *Set $\mathbb{D}^* = \{z: 0 < |z| < 1\}$. Let $f: \mathbb{D}^* \to \widehat{\mathbb{C}}$ be a locally injective holomorphic map and $S_f(z)$ be its Schwarzian derivative. Then we have the following:*
(i) The origin is never a simple pole of $S_f(z)$.
(ii) If the origin is a double pole of $S_f(z)$, then f extends to a holomorphic function on $\mathbb{D}$ with a critical point at 0. In this case S_f has a double pole at 0 with leading coefficient $\frac{1-d^2}{2}$ for some integer $d \geqslant 2$, and S_f satisfies the condition (2.2).

Proof. Set $\phi(z) = S_f(z)$. We just consider the case that the origin is either a double pole, a simple pole or a regular point of ϕ. There are therefore complex numbers δ, a_n $(n \geqslant 1)$ such that

$$\phi(z) = \frac{1-\delta^2}{2z^2} + \frac{a_1}{z} + a_2 + \cdots.$$

We may choose δ so that $\Re\delta \geqslant 0$.

If either δ is not an integer or $\delta = 0$, we know from Theorem 3.1 (a), (b) that ϕ can not have a Schwarzian primitive in a punctured neighborhood of 0.

If $\delta = d \geqslant 1$ is an integer and the condition (2.2) is not satisfied, by Corollary 2.4 and Theorem 3.1 (c), the meromorphic function ϕ can not have a Schwarzian primitive in a punctured neighborhood of 0.

If $\delta = d = 1$, note that the condition (2.2) is equivalent to $a_1 = 0$, so that ϕ has no pole at 0.

Therefore, in order to have $\phi = S_f$ in a punctured neighborhood of 0 we must have either $\delta = 1$ and ϕ has no pole at 0, or $\delta = d$ is an integer greater than or equal to 2 and ϕ satisfies the condition (2.2). In both cases, the map f has an analytic continuation on $\mathbb{D}$ according to the discussion in section 2. □

Note that the above proof gives the nature of Schwarzian primitives for a meromorphic map ϕ in a neighborhood of 0 with at worse double pole at 0.

Proof of Theorem 1.1. Point (a) follows directly from Theorem 2.2. To prove point (b), without loss of generality, we assume that the exceptional pole of ϕ is ∞. By Corollary 3.2, we obtain a Schwarzian primitive f of ϕ on $\mathbb{C}$. Since ∞ is a double pole of ϕ, by Theorem 4.1 (ii) f extends to a rational map and $S_f = \phi$ takes the local expression (1) around ∞ after change of coordinates by $1/z$. □

5 Schwarzian derivatives of rational maps, a global study

In this section, we give a necessary and sufficient condition for a meromorphic quadratic differential to be the Schwarzian derivative of a polynomial, or a rational map, in the case that all except one poles have $d_c = 2$.

Let $c_1, \cdots, c_k$ ($k \geqslant 1$) be distinct points of $\mathbb{C}$. Set

$$\phi(z) := -\frac{3}{2}\sum_{i=1}^{k}\frac{A_i(z-c_i)+1}{(z-c_i)^2}, \tag{5.1}$$

and

$$L_i := 3A_i^2 - 4\sum_{j\neq i}\frac{A_j(c_i-c_j)+1}{(c_i-c_j)^2},\ (i=1,\cdots,k).$$

Recall that if a meromorphic function has a double pole at c, the number d_c is defined by the expression

$$\frac{1-d_c^2}{2(z-c)^2}+\frac{a_1}{z-c}+a_2+\cdots \quad \text{as } z\to c.$$

Lemma 5.1. *For each c_i, the condition (2.2) holds at c_i for ϕ of form (5.1) if and only if $L_i = 0$.*

Proof. Fix a pole c_i. The map $\phi(z)$ has the local expression

$$\phi(z) = \frac{1-2^2}{2(z-c_i)^2}+\frac{-\frac{3}{2}A_i}{z-c_i}-\frac{3}{2}\sum_{j\neq i}\frac{A_j(z-c_j)+1}{(z-c_j)^2}.$$

As a double pole, we have $d_{c_i} = 2$, $a_1 = -3A_i/2$ and $a_2 = -\frac{3}{2}\sum_{j\neq i}\frac{A_j(c_i-c_j)+1}{(c_i-c_j)^2}$. In this case, condition (2.2) holds if and only if $a_1^2 - k_1a_2 = 0$, if and only if $L_i = 0$. □

In order to study the behavior at ∞ we set

$$E_{m+1} = \sum_{i=1}^{k}(mc_i^{m-1}+c_i^mA_i),\ m\geqslant 0.$$

Then

$$\begin{aligned}\phi(z) &= -\frac{3}{2}\sum_i\left(\frac{1}{z^2}\left(1-c_i\frac{1}{z}\right)^{-2}+A_i\frac{1}{z}\left(1-c_i\frac{1}{z}\right)^{-1}\right) &(5.2)\\ &= -\frac{3}{2}\left(\frac{1}{z}E_1+\frac{1}{z^2}E_2+\frac{1}{z^3}E_3+\frac{1}{z^4}E_4+\cdots\right). &(5.3)\end{aligned}$$

Changing coordinates $w = 1/z$, the quadratic differential becomes:

$$\phi(z)dz^2 = \phi(w^{-1})w^{-4}dw^2 =: \Phi(w)dw^2$$

with

$$\Phi(w) = -\frac{3}{2}\left(\frac{E_1}{w^3}+\frac{E_2}{w^2}+\frac{E_3}{w^1}\right)+O(w^0).$$

Clearly,

Lemma 5.2.

$$\begin{aligned} E_1 \neq 0 &\iff \infty \textit{ is a triple pole of } \phi(z)dz^2 &(5.4)\\ E_1 = 0 \neq E_2 &\iff \infty \textit{ is a double pole of } \phi(z)dz^2 \textit{ with leading coef. } \frac{3E_2}{-2} &(5.5)\\ E_1 = E_2 = 0 \neq E_3 &\iff \infty \textit{ is a simple pole of } \phi(z)dz^2 &(5.6)\\ E_1 = E_2 = E_3 = 0 &\iff \infty \textit{ is a regular point of } \phi(z)dz^2\ . &(5.7)\end{aligned}$$

Note that for $\frac{W\cdot(z^m+l.o.t.)}{z^{m+2}+l.o.t}dz^2$, with $W\neq 0$, the point ∞ is a double pole and after change of coordinates, one may check that the leading coefficient of the quadratic differential at ∞ is W.

Theorem 5.3. *(1) If $\phi(z) = S_P(z)$ for P a polynomial with $P'(z) = \prod_i (z - c_i)$, then*

$$A_i = \frac{2}{3}\sum_{j\neq i}\frac{1}{c_i - c_j},\ i = 1, \cdots, k\ .$$

They verify: $L_i = E_1 = 0$ for $1 \leqslant i \leqslant k$, and $-\frac{3}{2}E_2 = \frac{1-(k+1)^2}{2}$.

(2) Conversely, if $L_i = E_1 = 0$ for $1 \leqslant i \leqslant k$, and $-\frac{3}{2}E_2 = \frac{1-(k+1)^2}{2}$, then $\phi(z) = S_P(z)$ with $P'(z) = \prod_i (z - c_i)$.

Proof. (1) The formula for A_i can be checked using the formula of S_P. The fact $L_i = 0$ for $1 \leqslant i \leqslant k$ is due to Theorem 2.2 and Lemma 5.1.

Note that each c_i is a simple critical point of P and ∞ is a critical point of P of local degree $k+1$. It follows that ∞ is a double pole of $S_P(z)dz^2$ with leading coefficient $\frac{1-(k+1)^2}{2}$.

It follows in our case that $E_1 = 0$ and $-\frac{3}{2}E_2 = \frac{1-(k+1)^2}{2}$.

(2) By Lemma 5.1 and Theorem 2.2, if $L_i = 0$ for $1 \leqslant i \leqslant k$, then there is a holomorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $S_f(z) = \phi(z)$. Moreover, $E_1 = 0$ and $-\frac{3}{2}E_2 = \frac{1-(k+1)^2}{2}$ implies that $\phi(z)dz^2$ has a double pole at ∞. By Theorem 4.1 our map f entends to a holomorphic map at the puncture ∞, so f is a rational map. Furthermore, ∞ is a critical point of local degree $k+1$. So f is a polynomial. □

The following result is to be compared with results in [4].

Theorem 5.4. *(a) Assume $k = 2d - 2$. If there is a rational map f with $\deg(f) = d$ such that $S_f(z) = \phi(z)$, then $L_i = E_j = 0$ for $1 \leqslant i \leqslant 2d-2$ and $j = 1, 2, 3$. Conversely*

1. *if $L_i = E_j = 0$ for $1 \leqslant i \leqslant 2d-3$ and $j = 1, 2, 3$ then automatically $L_{2d-2} = 0$ and there is a rational map f with $\deg(f) = d$ such that $S_f(z) = \phi(z)$.*

2. *Or if $L_i = E_j = 0$ for $1 \leqslant i \leqslant 2d-2$ and $j = 1, 2$ then automatically $E_3 = 0$ and there is a rational map f with $\deg(f) = d$ such that $S_f(z) = \phi(z)$.*

(b) If $L_i = 0$ for $1 \leqslant i \leqslant k$, then there is a holomorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $S_f(z) = \phi(z)$. Moreover,

1. *if $E_1 \neq 0$, then f has an essential singularity at ∞.*

2. *if $E_1 = 0$ and $E_2 \neq 0$, the f is a rational map with critical set $\{c_1, \cdots, c_k, \infty\}$. And $-\frac{3}{2}E_2 = \frac{1-(m+1)^2}{2}$ for some integer $1 \leqslant m \leqslant k$. The map f has local degree $m+1$ at ∞. In particular $k+m$ is an even number and is equal to $2\deg(f) - 2$.*

3. *if $E_1 = E_2 = 0$, then automatically $E_3 = 0$ and f is a rational map with $\deg(f) = d = \frac{k+2}{2}$, with critical set $\{c_1, \cdots, c_k = c_{2d-2}\}$.*

(c) If $k = 2d-3$, $L_i = 0$ for $1 \leqslant i \leqslant 2d-3$, $E_1 = 0$ and $E_2 \neq 0$, Then $E_2 = 1$ and $\phi = S_f$ for a rational map f with simple critical points at c_i, $i = 1, \cdots, 2d-3$ and a simple critical point at ∞. In particular f is of degree d.

Proof. (a) Let $k = 2d-2$. If there is a rational map f with $\deg(f) = d$ such that $S_f(z) = \phi(z)$, then $L_i = 0$ for $1 \leqslant i \leqslant 2d-2$ by Theorem 2.2 and Lemma 5.1.

Note that each c_i is a simple critical point of f and thus the infinity is not a critical point of f (recall that f has $2d-2$ critical points, counted with multiplicity). Thus the infinity is a regular point of the quadratic differential $\phi(z)dz^2$. Therefore $E_i = 0$ for $i = 1, 2, 3$.

Conversely, for Case 1, one may assume that the exceptional pole c_{2d-2} is located at 0. By Lemma 5.1 and the conditions $L_i = 0$, $i = 1, \cdots, 2d-3$, the condition (2.2) is satisfied at c_i, $i = 1, \cdots, 2d-3$. By Lemma 5.2, the conditions $E_j = 0$, $j = 1, 2, 3$ tell that ∞ is a regular point of the quadratic differential $\phi(z)dz^2$. Thus there is a Schwarzian primitive f of $\phi(z)dz^2$ defined on the simply connected domain $\widehat{\mathbb{C}} \setminus \{0\}$. Now we may apply Theorem 4.1 to show that f extends meromorphically to 0. So f is a rational map of degree d.

Case 2 will be covered by the case (b).3 below.

(b) By Lemma 5.1 and Theorem 2.2, if $L_i = 0$ for $1 \leqslant i \leqslant k$, then there is a holomorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $S_f(z) = \phi(z)$ and each c_i is a simple critical point of f.

1. If $E_1 \neq 0$, then f can not be extended to a rational map at ∞. So ∞ must be an essential singularity.

2. If $E_1 = 0$ and $E_2 \neq 0$. Then ∞ is a double pole of $\phi(z)dz^2$. By Theorem 4.1 there is an integer $d_\infty \geqslant 2$ and f extends to a rational map with local degree d_∞ at ∞.

Note that at most half of the critical points of a rational map can be merged into a single one. So $d_\infty - 1 \leqslant k$.

3. If $E_1 = E_2 = 0$, then the infinity is either a regular point or a simple pole of the quadratic differential $\phi(z)dz^2$. According to Theorem 4.1 (i), the latter case never happens, so necessarily $E_3 = 0$. Since ∞ is a regular point of $\phi(z)dz^2$, then f extends to a rational map with critical points $c_1, \dots, c_k$.

(c) is a particular case of (b).2 with $m = 1$. It follows that $E_2 = 1$. □

Example 5.5. Let us choose two critical points $c_1 = 1$, $c_2 = 0$. We get $(A_{c_1}, A_{c_2}) = (3, -3)$ or $(-1, 1)$. In the first case we get

$$\phi_1(z) = -\frac{3}{2z^2(z-1)^2}, \quad f_1(z) = \frac{z^2}{(z-1)^2}.$$

In the second case,

$$\phi_2(z) = -\frac{8z^2 - 8z + 3}{2z^2(z-1)^2}, \quad f_2(z) = 2z^3 - 3z^2.$$

Proof of Theorem 1.2. It follows directly from Theorem 5.4 (a). □

Proof of Theorem 1.3. Note that a meromorphism ψ on $\mathbb{C}$ has a global expression (1.3) if and only if its poles are exactly $c_1, \dots, c_k$ and it takes the form

$$\psi(z) = \frac{1 - 2^2}{2(z-c_i)^2} + \frac{r_i}{z - c_i} - \frac{r_i^2}{2} + \cdots, \text{ as } z \to c_i$$

around each c_i. It follows that each $d_{c_i} = 2$ and ψ satisfies condition (2.2) at each pole. Then, by Corollary 3.2, we get $\psi = S_f$ for a holomorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$ which has precisely k critical points that are simple and located at $c_1, \cdots, c_k$, and vice versa. □

6 Parameter dependence

Given a set $\mathcal{C}$ of $2d-2$ district points in $\widehat{\mathbb{C}}$, we know that there are finitely many choices of quadratic differentials ϕ_i with double poles at $\mathcal{C}$ that are Schwarzian derivatives of degree-d rational maps. One may ask the question whether each ϕ_i depends holomorphically on the set $\mathcal{C}$.

The answer is no. Here is a counter example, using a family studied by [2]. Set $j := e^{2\pi i/3}$ and let

$$h_\alpha(z) = \frac{\alpha(z^3 + 2) + 3z^2}{3\alpha z + 2z^3 + 1},$$

critical points	1	j	j^2	α^2
critical values	1	j^2	j	$\dfrac{\alpha^4 + 2\alpha}{2\alpha^3 + 1}$

When $\alpha^6 \neq 1$, we have a degree 3 rational map with four simple critical points.

One can check easily that if $M \circ h_\alpha = h_\beta$ for some pair α, β and a Möbius map M, then $\alpha = \beta$ and $M = id$. Thus the Schwarzian derivative map $\alpha \mapsto S_{h_\alpha}$ is injective. Now

$$S_{h_\alpha}(z) = -\frac{3}{2} \cdot \frac{1 + 4\alpha^3 - 4\alpha z + 8\alpha^4 z - 18\alpha^2 z^2 + 8z^3 - 4\alpha^3 z^3 + 4\alpha z^4 + \alpha^4 z^4}{(z - \alpha^2)^2 (z^3 - 1)^2}$$

$$= -\frac{3}{2(z-1)^2} + \frac{-2+2\alpha+\alpha^2}{(\alpha^2-1)(z-1)} - \frac{3}{2(z-\alpha^2)^2} + \frac{3(-2\alpha+\alpha^4)}{(\alpha^6-1)(z-\alpha^2)} +$$

$$+ \frac{9}{2(1+z+z^2)^2} + \frac{-1-2\alpha-3\alpha^2-4\alpha^3-2\alpha^4-2z-4\alpha z-3\alpha^2 z-2\alpha^3 z-\alpha^4 z}{(1+\alpha^2+\alpha^4)(1+z+z^2)}$$

Thus $\alpha \mapsto S_{h_\alpha}$ holomorphically embeds $\mathbb{C} \setminus \{\pm 1, \pm j, \pm j^2\}$ into the space of degree-8 rational maps.

Fixing a critical set $\mathcal{C} = \{1, j, j^2, c = \alpha^2\}$, there are generically two α's and thus two Schwarzians realizing this set, except when $\alpha = 0$.

The dependence of each branch S_{h_α}, and thus of the parameter α, on the critical set $\mathcal{C} = \{1, j, j^2, c\}$ is locally holomorphic except when $c = 0$.

From this example one may guess that generically each branch of the Schwarzian depends holomorphically on the critical set. We have made this idea precise in the following setting.

Fix an integer $\mu \geqslant 1$, and set $d = \mu + 1$. Choose a vector $\mathbf{a} \in \mathbb{C}^{2\mu}$, and express it with two multicoordinates $\mathbf{a} = (\mathbf{a}_p, \mathbf{a}_q) \in \mathbb{C}^\mu \times \mathbb{C}^\mu$. Consider each $\mathbf{a}_p$, $\mathbf{a}_q$ as a column vector of length μ. Let $B_0(z)$ denote the line vector $\begin{pmatrix} 1 & z & \cdots & z^{\mu-1} \end{pmatrix}$. Consider a pair of normalized polynomials, the quotient rational map and its derivative

$$p_{\mathbf{a}}(z) = B_0(z)\mathbf{a}_p + 0 \cdot z^\mu + z^{\mu+1} \tag{6.1}$$

$$q_{\mathbf{a}}(z) = B_0(z)\mathbf{a}_q + z^\mu \tag{6.2}$$

$$f_{\mathbf{a}}(z) = \frac{p_{\mathbf{a}}(z)}{q_{\mathbf{a}}(z)} = \frac{B_0(z)\mathbf{a}_p + 0 \cdot z^\mu + z^{\mu+1}}{B_0(z)\mathbf{a}_q + z^\mu} \tag{6.3}$$

$$g_{\mathbf{a}}(z) := \frac{\mathrm{d}f_{\mathbf{a}}}{\mathrm{d}z}(z) = \frac{p'_{\mathbf{a}}q_{\mathbf{a}} - q'_{\mathbf{a}}p_{\mathbf{a}}}{q^2_{\mathbf{a}}} \tag{6.4}$$

$$w_{\mathbf{a}}(z) := p'_{\mathbf{a}}q_{\mathbf{a}} - q'_{\mathbf{a}}p_{\mathbf{a}} = \mathrm{Wronskian}(q_{\mathbf{a}}, p_{\mathbf{a}}) \tag{6.5}$$

Notice that $w_{\mathbf{a}}$ is a monic polynomial of degree 2μ. Thus

$$w_{\mathbf{a}}(z) = (z - c_1) \cdots (z - c_{2\mu}).$$

Consider the sets

$$\Omega := \{\mathbf{a} \in \mathbb{C}^{2\mu} \mid resultant(p_{\mathbf{a}}, q_{\mathbf{a}}) \neq 0\} = \{\mathbf{a} \mid \deg(f_{\mathbf{a}}) = \mu + 1\}$$

$$\Omega' := \{\mathbf{a} \in \mathbb{C}^{2\mu} \mid resultant(p_{\mathbf{a}}, q_{\mathbf{a}}) \cdot discri(w_{\mathbf{a}}) \neq 0\}.$$

We have:

Lemma 6.1. *1) For $\mathbf{a} \in \Omega$, the point ∞ is not a critical point for every $f_{\mathbf{a}}$, i.e. all critical points of $f_{\mathbf{a}}$ are in the finite plane.*
2) For any rational map f of degree $\mu + 1$ such that the point ∞ is not a critical point, there is a unique Möbius transformation M such that $M \circ f = f_{\mathbf{a}}$ for some $\mathbf{a}$. Consequently the Schwarzian derivative map $\Omega \ni \mathbf{a} \mapsto S_{f_{\mathbf{a}}}$ is injective.
3)

$$\Omega' = \{\mathbf{a} \in \mathbb{C}^{2\mu} \mid discri(w_{\mathbf{a}}) \neq 0\}.$$

Proof. 1) and 2) follow from an easy algebraic manipulation. For 3) it is easy to check that if the polynomial $p'_{\mathbf{a}}q_{\mathbf{a}} - q'_{\mathbf{a}}p_{\mathbf{a}}$ has only simple roots then $p_{\mathbf{a}}$ and $q_{\mathbf{a}}$ are co-prime (the converse is not true). □

Denote by $Poly_{2\mu}$ the 2μ-dimensional space of monic polynomials of degree at most 2μ. Set

$$\mathcal{W} : \mathbb{C}^{2\mu} \ni \mathbf{a} \mapsto w_{\mathbf{a}} \in Poly_{2\mu},$$

it is called the *Wronskian operator*.

$$V := \{\mathbf{a} \in \mathbb{C}^{2\mu} \mid \mathcal{W} \text{ is locally invertible}\} = \{\mathbf{a} \in \mathbb{C}^{2\mu} \mid Jac_{\mathbf{a}}\mathcal{W} \neq 0\}$$

To every vector in $\mathbb{C}^{2\mu}$ we associate a monic polynomial

$$\mathbb{C}^{2\mu} \ni \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_{2\mu} \end{pmatrix} \mapsto R_{\mathbf{c}}(z) = (z - c_1) \cdots (z - c_{2\mu}) \in Poly_{2\mu}.$$

Permutations of the c_i's will give the same polynomial. By the implicit function theorem

$$\mathbf{c} \mapsto R_{\mathbf{c}}$$

is locally invertible if and only if $c_i \neq c_j$ whenever $i \neq j$.

On the other hand, for any $\mathbf{a} \in \Omega'$, the roots of $w_{\mathbf{a}}$ are pairwise distinct. We may choose an ordering to turn the set of roots into a vector $\mathbf{c} \in \mathbb{C}^{2\mu}$. There are $(2\mu)!$ choices of such an ordering.

Lemma 6.2. *The multivalued map $\Omega' \ni \mathbf{a} \mapsto \mathbf{c} \in \mathbb{C}^{2\mu}$ is locally invertible if and only if the single valued map $\Omega' \ni \mathbf{a} \mapsto w_{\mathbf{a}} \in Poly_{2\mu}$ is locally invertible, i.e., $\mathbf{a} \in \Omega' \cap V$.*

Example 6.3. For $\mu = 2$, $\mathbf{a} = (a_0, a_1, b_0, b_1)$, we have

$$w_{\mathbf{a}}(z) = a_1 b_0 - a_0 b_1 - 2a_0 z + (3b_0 - a_1)\, z^2 + 2b_1 z^3 + z^4, \quad Jac_{\mathbf{a}}\mathcal{W} = 4a_1 + 12b_0$$

$$V = \{\mathbf{a} \mid 4a_1 + 12b_0 \neq 0\}$$

$$\Omega' = \{\mathbf{a} = (a_0, a_1, b_0, b_1) \mid discri(w_{\mathbf{a}}) \neq 0\}$$

Let's us choose $a_1 = b_1 = b_0 = 0$ and $a_0 = 1$. In this case $p_{\mathbf{a}}(z) = 1 + z^3$ and $q_{\mathbf{a}}(z) = z^2$. They are clearly coprime. So $\mathbf{a} \in \Omega$. We have $w_{\mathbf{a}}(z) = z(z^3 - 2)$. All roots are simple. So $\mathbf{a} \in \Omega'$. As $Jac_{\mathbf{a}}\mathcal{W} = 0$, we have $\mathbf{a} \notin V$ and $\mathbf{a} \to \mathbf{c}$ is not locally invertible. One checks easily that $f_{\mathbf{a}}$ has four distinct critical values. This example is actually equal to

$$M(h_0\Big(\frac{z}{2^{1/3}}\Big))$$

where the map h_0 is obtained at the beginning of this section with critical set $\{1, j, j^2, 0\}$ and M is a suitable Möbius transformation.

7 Counting

For f a rational map, denote by $\mathcal{C}(f)$ the set of critical points of f.

Given any subset $\mathcal{C}$ of $\mathbb{C}$ consisting of $2(d-1)$ distinct points, and for $\mathcal{A} = (A_c)_{c \in \mathcal{C}}$ a collection of complex numbers, set

$$E_{m+1}(\mathcal{A}) = \sum_{c \in \mathcal{C}} (mc^{m-1} + c^m A_c), \ m \geqslant 0.$$

We have proved the following:

Lemma 7.1. *Given any subset $\mathcal{C}$ of $\mathbb{C}$ consisting of $2(d-1)$ distinct points, the following sets are mutually bijective:*

1. $\mathcal{W}^{-1}(\{\prod_{c \in \mathcal{C}}(z - c)\}),\ \Omega \cap \mathcal{W}^{-1}(\{\prod_{c \in \mathcal{C}}(z - c)\}),\ \Omega' \cap \mathcal{W}^{-1}(\{\prod_{c \in \mathcal{C}}(z - c)\})$
2. $\{$ *rational maps of degree d with critical set $\mathcal{C}$* $\}/_{f \sim \text{Möbius} \circ f}$
3. $\{\mathbf{a} \in \Omega \mid \mathcal{C}(f_{\mathbf{a}}) = \mathcal{C}\}$
4. *the set of quadratic differentials with doubles poles at $\mathcal{C}$, leading coefficients $\dfrac{-3}{2}$ and trivial holonomy group.*

5. $\{\mathcal{A} = (A_c)_{c\in\mathcal{C}} \mid 3A_c^2 = 4 \sum_{c'\in\mathcal{C},c'\neq c} \left(\frac{1}{(c-c')^2} + \frac{A_{c'}}{c-c'}\right), \forall\, c \in \mathcal{C},$ *and* $E_j(\mathcal{A}) = 0, j = 1, 2, 3\}$

6. $\{\mathbf{a} \in \Omega \mid \mathcal{C}(f_{\mathbf{a}}) = M(\mathcal{C})\}$, *where* M *is a Möbius map with* $M(\mathcal{C}) \subset \mathbb{C}$

7. $\{$ *rational maps of degree* d *with critical set* $M(\mathcal{C})$ $\}/_{f\sim \text{Möbius}\circ f}$.

For example the sets 7 and 2 are related by $f \mapsto f \circ M$.

Using a highly non trivial result of Goldberg, we know that for a generic choice of $\mathcal{C}$ the cardinality of the above sets is the Catalan number $u_d = \frac{1}{d}\binom{2(d-1)}{d-1}$. There are more elementary methods to prove that for a generic choice of $\mathcal{C}$, a lower bound of the cardinality is u_d. For instance, A. Eremenko and A. Gabrielov [6] provided an elementary way to construct u_d different real rational maps with prescribed simple real critical points. Using this result, it is not difficult to prove that for a generic choice of $\mathcal{C}$, the sets in Lemma 7.1 are non-empty and have cardinality at least u_d.

For further discussions on this counting problem, we refer to [5–7, 9, 13].

The case $d = 3$ with $u_3 = 2$ can be computed explicitly:

$$\mathcal{W}(\{a_0, a_1, b_0, b_1\}) = \{a_1b_0 - a_0b_1, -2a_0, 3b_0 - a_1, 2b_1\}\,.$$

When we try to solve

$$\{a_1b_0 - a_0b_1, -2a_0, 3b_0 - a_1, 2b_1\} = \{w_0, w_1, w_2, w_3\},$$

we get

$$b_1 = \frac{w_3}{2},\ a_0 = -\frac{w_1}{2},\ a_1 = 3b_0 - w_2,\ (3b_0 - w_2)b_0 + \frac{w_1w_3}{4} = w_0.$$

So

$$a_1 = \frac{1}{2}\left(-w_2 \pm \sqrt{w_2^2 + 12w_0 - 3w_1w_3}\right),\ b_0 = \frac{1}{6}\left(w_2 \pm \sqrt{w_2^2 + 12w_0 - 3w_1w_3}\right).$$

So generically $\mathcal{W}^{-1}(\{w_0, w_1, w_2, w_3\})$ has two simple solutions, and it has a double solution if and only if $w_2^2 + 12w_0 - 3w_1w_3 = 0$, if and only if $a_1 + 3b_0 = 0$.

Thus

Lemma 7.2. *The set* $\{\mathbf{a} \mid a_1 + 3b_0 = 0\}$ *is the critical and the co-critical set of* $\mathcal{W}$ *and* $\{\mathbf{w} \mid w_2^2 + 12w_0 - 3w_1w_3 = 0\}$ *is the critical value set of* $\mathcal{W}$. *The map*

$$\mathcal{W}: \begin{cases} \mathbb{C}^4 \smallsetminus \{\mathbf{a} \mid a_1 + 3b_0 = 0\} \to \mathbb{C}^4 \smallsetminus \{\mathbf{w} \mid w_2^2 + 12w_0 - 3w_1w_3 = 0\} & \textit{is a double covering} \\ \{\mathbf{a} \mid a_1 + 3b_0 = 0\} \to \{\mathbf{w} \mid w_2^2 + 12w_0 - 3w_1w_3 = 0\} & \textit{is a homeomorphism.} \end{cases}$$

We want to give a geometric interpretation of these sets in terms of the critical points of the rational map $f_{\mathbf{a}}$, or the zeros of the polynomial $w_{\mathbf{a}}(z)$.

8 Geometry in the cubics

For $w \neq 0$ with $\Im w > 0$, or $\Im w = 0$ and $\Re w > 0$, we consider $w, 0, 1, 1 + w$ as the 4 corners in cyclic order of a (eventually degenerate) parallelogram P_w.

Lemma 8.1. *Given an ordered 4 distinct points* $a, b, c, d \in \mathbb{C}$, *we define its cross ratio by*

$$[a, b, c, d] = \frac{(a - c)(b - d)}{(c - b)(d - a)}\ .$$

Then there is a unique Möbius transformation M *mapping* a, c, b, d *to the corners* $w, 0, 1, 1 + w$ *of* P_w, *in order.*

Proof. Choose w so that $w^2 = [a,b,c,d]$ and either $\Im w > 0$ or $\Im w = 0$ and $\Re w \geqslant 0$.

Let M be the unique Möbius transformation sending a, b, c to $w, 1, 0$. Then

$$[M(a), M(b), M(c), 1+w] = [w, 1, 0, 1+w] = \frac{(w-0)(1-1-w)}{(0-1)\cdot(1+w-w)} = w^2 = [a,b,c,d] \ .$$

It follows that $M(d) = 1+w$. □

If $t = [a,b,c,d]$, then any permutation of the four points will give a cross ratio in the set

$$R(t) = \{t, \frac{1}{t}, 1-t, \frac{1}{1-t}, \frac{t}{t-1}, \frac{t-1}{t}\} \ .$$

And any number in $R(t)$ is realized this way. For example, set $j := e^{2\pi i/3}$, then $R(-j^2) = \{-j, -j^2\}$.

Definition 8.2. We say that a set $\mathcal{C}$ of four distinct points in $\widehat{\mathbb{C}}$ forms a **regular tetrahedron** if there is a Möbius map sending $\mathcal{C}$ to $\{1, j, j^2, 0\}$, where $j := e^{2\pi i/3}$.

The motivation behind this definition is that the four points $1, j, j^2, 0$ are located at the vertices of a regular ideal tetrahedron of the hyperbolic 3-ball. Any four distinct points on the Riemann sphere form the vertices of an ideal tetrahedron (maybe degenerate). The collection of the angles between adjacent faces of the tetrahedron, or the 'shape' of the tetrahedron in W. Thurston's language, is an invariant of the Möbius action. These angles are all equal precisely when the four points are Möbius images of $\{1, j, j^2, 0\}$.

Lemma 8.3. *A set $\mathcal{C}$ of four distinct points in $\widehat{\mathbb{C}}$ forms a* **regular tetrahedron** *if and only the cross ratio of an ordering of the set $\mathcal{C}$ belongs to $R(-j^2)$.*

Proof. Note that $[1, j, j^2, 0] = \dfrac{(1-j^2)j}{(j^2-j)(0-1)} = 1 + j = -j^2$ and Möbius maps preserve cross ratios. □

Lemma 8.4. *Given any set V of four distinct points on $\widehat{\mathbb{C}}$ there is a unique 4-group $\mathcal{M}_V$ of Möbius transformations, isomorphic of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, such that every non-identity element of $\mathcal{M}_V$ acts on V as an order 2 permutation without fixed points. The map $V \mapsto \mathcal{M}_V$ is not injective. The group of Möbius transformations preserving the set V may be strictly larger than $\mathcal{M}_V$.*

Proof. Assume $V = \{a, b, c, d\}$. Let $M_{a,b}$ be the unique Möbius transformation mapping a, b, c to b, a, d respectively. As $[b, a, d, c] = [a, b, c, d] = [M(a), M(b), M(c), M(d)]$ we have necessarily $M(d) = c$.

Here is a geometric proof. Choose w so that $w^2 = [a,b,c,d]$ and either $\Im w > 0$ or $\Im w = 0$ and $\Re w \geqslant 0$.

Let N be the unique Möbius transformation sending a, b, c, d to $w, 1, 0, 1+w$. The vertical line from $\dfrac{1+w}{2}$ to ∞ in the upper 3-space is the unique common perpendicular of the two diagonal geodesics $w \leftrightarrow 1$ and $0 \leftrightarrow 1+w$. A rotation of 180 degree switches w and 1, and switches 0 and $1+w$. There is therefore a unique common perpendicular of the two geodesics $a \leftrightarrow b$ and $c \leftrightarrow d$. A rotation of 180 degree with respect to this perpendicular is our desired map $M_{a,b}$.

Now the 4-group is simply $\{id, M_{a,b}, M_{a,c}, M_{a,d}\}$. □

Lemma 8.5 (Buff)**.** *Two rational maps f and g having the same set of critical points and the same images for each of them, are in fact equal (except maybe in the bicritical case).*

Proof. Let $\mathcal{C}$ be the common critical set and d be the common degree of f and g.

On the one hand, each point in $\mathcal{C}$ is a critical point of f/g with multiplicity $\geqslant$ to the multiplicity as a critical point of f. The image of such a point is 1. Thus, 1 has at least $2d - 2 + |\mathcal{C}|$ preimages counting multiplicities.

On the other hand, f/g is a rational map of degree $\leqslant 2d$. Thus, $2d - 2 + |\mathcal{C}| \leqslant 2d$ and thus, $|\mathcal{C}| \leqslant 2$. □

For a rational map f, denote by $\mathcal{C}(f)$, resp. $\mathcal{V}(f)$, the set of critical points, resp. critical values, of f. The following was a remark of W. Thurston.

Lemma 8.6. *For any cubic rational map f with four distinct critical points and four distinct critical values, there is a unique bijection $M \leftrightarrow N$ between $\mathcal{M}_{\mathcal{C}(f)}$ and $\mathcal{M}_{\mathcal{V}(f)}$ such that $f \circ M = N \circ f$.*

Proof. Let $M \in \mathcal{M}_{\mathcal{C}(f)}$, $N \in \mathcal{M}_{\mathcal{V}(f)}$ be two maps so that M permutes the critical points in the same way as N on the corresponding critical values. Then $f \circ M|_{\mathcal{C}(f)} = N \circ f|_{\mathcal{C}(f)}$. So the two rational maps $f \circ M$ and $N \circ f$ have the same action on a identical critical set. By Lemma 8.5 this implies that they are equal. □

The following statement was first suggested to us by W. Thurston in 2011. Actually it was already proved by Lisa Goldberg in [G], Theorem 1.4, as a consequence of her deep counting result. Here we provide an elementary proof using Lemma 7.2.

Theorem 8.7 (Goldberg)**.** *A vector* $\mathbf{a} \in \Omega'$ *is a critical point of the Wronskian operator* $\mathcal{W}$ *if and only if the set* $\mathcal{C}(f_{\mathbf{a}})$ *of critical points of* $f_{\mathbf{a}}$ *forms a regular tetrahedron. A polynomial in* $Poly_4$ *with 4 distinct roots is a critical value of the Wronskian operator* $\mathcal{W}$ *if and only if the set of its roots forms a regular tetrahedron.*

Proof. Recall the family defined above:

$$h_\alpha(z) = \frac{\alpha(z^3+2)+3z^2}{3\alpha z+2z^3+1}.$$

Set $\mathcal{C} = \{1, j, j^2, 0\}$. Let $\mathcal{C}' = \{s, t, u, v\}$ be a set of 4 distinct points in $\mathbb{C}$.

If $[s, t, u, v] \in R(-j^2)$, there is a Möbius map M such that $M(\mathcal{C}) = \mathcal{C}'$. By Lemma 7.1.6

$$1 = \#\mathcal{W}^{-1}(\{\prod_{c\in\mathcal{C}}(z-c)\}) = \#\mathcal{W}^{-1}(\{\prod_{c\in\mathcal{C}'}(z-c)\}).$$

It follows that the polynomial $\prod_{c\in\mathcal{C}'}(z-c)$ is a critical value of the operator $\mathcal{W}$.

If $[s, t, u, v] \notin R(-j^2)$, then for the unique Möbius map N sending s, t, u to $1, j, j^2$, we have $N(v) \in \mathbb{C} \smallsetminus \{0, 1, j, j^2\}$. Set $\alpha^2 = N(v)$. There are two maps h_α and $h_{-\alpha}$ realizing $\{1, j, j^2, \alpha^2\}$ as critical set, and h_α and $h_{-\alpha}$ do not differ by a post composition of a Möbius map. We have

$$2 \leqslant \#\{\text{cubic rat. maps } f \text{ with } \mathcal{C}(f) = \{1, j, j^2, \alpha^2\}\}/_{f\sim \text{Möbius}\circ f}$$

$$\overset{\text{Lemma 7.1.2}}{=} \#\mathcal{W}^{-1}(\{\prod_{c\in\{1,j,j^2,\alpha^2\}}(z-c)\}) \overset{\text{Lemma 7.1.6}}{=} \#\mathcal{W}^{-1}(\{\prod_{c\in\mathcal{C}'}(z-c)\}) \overset{\text{Lemma 7.2}}{\leqslant} 2$$

It follows that $\#\mathcal{W}^{-1}(\{\prod_{c\in\mathcal{C}'}(z-c)\}) = 2$ and the polynomial $\prod_{c\in\mathcal{C}'}(z-c)$ is not a critical value of the operator $\mathcal{W}$.

Similarly, for $\mathbf{a} \in \Omega'$ so that $\mathcal{C}(f_{\mathbf{a}})$ has 4 distinct critical points. Then

$$\mathcal{W}^{-1}(\{\prod_{c\in\mathcal{C}(f_{\mathbf{a}})}(z-c)\})$$

has 1 or 2 points depending whether $\mathcal{C}(f_{\mathbf{a}})$ forms a regular tetrahedron or not. In the former case, $\mathbf{a}$ is a critical point of the operator $\mathcal{W}$. Otherwise $\mathbf{a}$ is a regular point of $\mathcal{W}$. □

We may express the result in algebraic terms as follows.

Corollary 8.8. *Let* $z^4+w_3z^3+w_2z^2+w_1z+w_0$ *be a quartic polynomial with 4 distinct roots. Denote by* Δ *their cross ratio. Then* $\Delta \in R(-j^2)$ *if and only if* $w_2^2+12w_0-3w_1w_3 = 0$*. Let* $(z^3+a_1z+a_0, z^2+b_1z+b_0)$ *be a pair of polynomials whose wronskian has 4 distinct roots, or equivalently whose ratio as a rational function has 4 distinct critical points. Denote by* Δ *the cross ratio of these four points. Then* $\Delta \in R(-j^2)$ *if and only if* $a_1 + 3b_0 = 0$.

The corollary together with Lemma 7.2 gives Theorem 1.4.

Note that the Möbius symmetric group of any four distinct points is non-trivial (Lemma 8.4). Thus for any cubic rational map f with four distinct critical values (hence with four distinct critical points), there exist non-trivial Möbius maps M and N such that $f \circ M = N \circ f$ (see Lemma 8.6).

In the case $d \geqslant 4$, a generic set of $2d-2$ distinct points has a trivial Möbius symmetric group. We suspect that the critical points of $\mathcal{W}$ occur precisely at maps $f_{\mathbf{a}}$ such that there are non-trivial Möbius maps M, N with $f_{\mathbf{a}} \circ M = N \circ f_{\mathbf{a}}$.

For a holomorphic map from an open set of $\mathbb{C}^k$ into $\mathbb{C}^k$, its Jacobian is non zero if and only if it is locally injective (see [8, Thm. 7.1]). One may use this fact to prove that the critical set and and critical value set of $\mathcal{W}$ are both Möbius invariant, in the following sense.

Lemma 8.9. *Let $\mu \geqslant 2$. Consider two sets of 2μ distinct points $\mathcal{C}$ and $\mathcal{C}'$ in $\mathbb{C}$ such that there is a Möbius map M satisfying $M(\mathcal{C}) = \mathcal{C}'$. Then the polynomial with roots $\mathcal{C}$ is a critical value of $\mathcal{W}$ if and only if the polynomial with roots $\mathcal{C}'$ is a critical value of $\mathcal{W}$.*

Let $\mathbf{a} \in \mathbb{C}^{2\mu}$ such that the rational map $f_{\mathbf{a}}$ has critical point set $\mathcal{C}$. Then there is a unique Möbius map $N = N_{\mathbf{a}}$ and a unique $\mathbf{a}'$ such that $N \circ f_{\mathbf{a}} \circ M^{-1} = f_{\mathbf{a}'}$. And $\mathbf{a}$ is a critical point of $\mathcal{W}$ if and only if $\mathbf{a}'$ is.

One can use this lemma to normalize three critical points and three critical values, and thus reduce our study of $\mathcal{W}$ to a subvariety of dimension $2\mu - 3$, just as what we did in the cubic case.

9 Problems for future developments

In sections 6, 7, 8, we studied how the Schwarzian derivative S_f of a rational map f with only simple critical points depends upon the poles of S_f itself. We showed that S_f does not depend holomorphically on its poles if and only if the poles are the zeros of a critical value polynomial of the Wronskian operator (see Section 6). Furthermore, we gave both algebraic and geometric characterizations of the critical points and critical value polynomials of the Wronskian operator in the case of $d = 3$.

For a future research, a natural question is to extend this study of the Wronskian operator to higher degrees. More precisely, in the algebraic aspect, we want to find an explicit formula (using the coefficients of $p_{\mathbf{a}}$ and $q_{\mathbf{a}}$) defining the critical locus of the Wronskian operator; while in the geometric aspect, we wish to characterize the critical points of the Wronskian operator by the symmetry of the rational maps: we conjecture that the critical points of $\mathcal{W}$ occur precisely at rational maps f such that there are non-trivial Möbius maps M, N satisfying $f \circ M = N \circ f$.

In early sections, we gave necessary and sufficient conditions for a local meromorphic function ϕ to admit a local meromorphic Schwarzian primitive f. This map f may have a critical point of any multiplicity. We then gave, in section 5, a global description of the Schwarzian derivatives of the rational maps with only simple critical points. A natural sequel would be to carry out the global study to rational maps with multiple critical points. In this more general setting, questions about holomorphic dependencies on poles may still be addressed.

Another fascinating direction of further research is to study the trajectory structure of Schwarzian derivatives, as a particular class of quadratic differentials. During autumn 2010, encouraged by W. Thurston, the last two authors of this present article did quite a lot of numerical experiences on the cubic case. Part of their observations have been proved in the sub sequel group discussion led by Thurston. But many things remain to be proved and higher degree cases remain to be exploited.

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