CODY conference

Holomorphic dynamics, Around Thurston's theorem

On the number of roots of planar harmonic polynomials

report on work of Khavinson-Swiatek and Geyer

Tan Lei, Université d'Angers

Roskilde, 30.09. 2010

We say that $p(z) = a_d z^d + \cdots + a_1 z + a_0$ is a Geyer polynomial if

- p is real (= all coefficients real),
- all critical points are simple,
- ≤ 1 real critical point,
- p sends each critical point c to its complex conjugate $\bar{c}$.

Theorem (Geyer)

For every $d \geq 2$, there is a Geyer polynomial p of degree d.

This result solved a sharpness problem in the study of harmonic polynomials.

Upper bound for $\#\text{roots}(p(z) - q(\overline{z}))$

p, q two polynomials. Assume $d := \deg(p) > \deg(q)$.

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$ (Bezout's theorem).

Upper bound for $\#\text{roots}(p(z) - q(\overline{z}))$

p, q two polynomials. Assume $d := \deg(p) > \deg(q)$.

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$ (Bezout's theorem).

In case $p(z) - \overline{z}$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$ (using Fatou's lemma in complex dynamics).

Upper bound for $\#\text{roots}(p(z) - q(\overline{z}))$

p, q two polynomials. Assume $d := \deg(p) > \deg(q)$.

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$ (Bezout's theorem).

In case $p(z) - \overline{z}$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$ (using Fatou's lemma in complex dynamics).

Remark of Crofoot-Sarason A polynomial p with only simple critical points c and with $p(c) = \overline{c}$ would realize the sharp bound.

Upper bound for $\#\text{roots}(p(z) - q(\overline{z}))$

p, q two polynomials. Assume $d := \deg(p) > \deg(q)$.

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$ (Bezout's theorem).

In case $p(z) - \overline{z}$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$ (using Fatou's lemma in complex dynamics).

Remark of Crofoot-Sarason A polynomial p with only simple critical points c and with $p(c) = \overline{c}$ would realize the sharp bound.

Bshouty-Lyzzaik (2004) Such polynomial exists for $d = 4, 5, 6, 8$ (using algebraic method not generalizable)

Upper bound for $\#\text{roots}(p(z) - q(\overline{z}))$

p, q two polynomials. Assume $d := \deg(p) > \deg(q)$.

Wilmshurst (1998) $\boxed{\#\text{roots} \leq d^2}$ (Bezout's theorem).

In case $p(z) - \overline{z}$,

Khavinson-Swiatek (2003) $\boxed{\#\text{roots} \leq 3d - 2}$ and the bound is sharp for $d = 2, 3$ (using Fatou's lemma in complex dynamics).

Remark of Crofoot-Sarason A polynomial p with only simple critical points c and with $p(c) = \overline{c}$ would realize the sharp bound.

Bshouty-Lyzzaik (2004) Such polynomial exists for $d = 4, 5, 6, 8$ (using algebraic method not generalizable)

Geyer (2008) sharpness for $\forall\, d$.

Counting roots for $f = p - \overline{q}$, $d = \deg(f)$

Lower bound and surjectivity.

$$\deg p \neq \deg q \Longrightarrow \forall\, w \in \mathbb{C}, \quad \#f^{-1}(w) \geq d\ .$$

Geyer (again) $\forall\, d \geq 2$,

$$\left.\begin{array}{l} \exists \text{ degree-}d\ p(z) \in \mathbb{R}[z] \\ \leq 1 \text{ critical points real} \end{array}\right\} \quad \text{s.t.} \quad \exists\, w \in \mathbb{C}, \quad \#\{p(z) - \overline{z} = w\} = 3d - 2.$$

Theorem (El Amrani-Loeb-T. 2010) $\forall\, d \geq 2$,

$$\left.\begin{array}{l} \forall \text{ degree-}d\ p(z) \in \mathbb{R}[z] \\ \text{all critical points real} \end{array}\right\} \forall w \in \mathbb{R},\ \#\{p(z) \pm \overline{z} = w\} \leq \left\{\begin{array}{ll} 2d & d \text{ even} \\ 2d+1 & d \text{ odd.} \end{array}\right.$$

And the bound is sharp for every d.

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Important Notice : $\boxed{\text{Every critical point is 2-periodic.}}$

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Important Notice : **Every critical point is 2-periodic.**

Apply Thurston's theorem :

1. Create a topological (C^∞, quasi-regular) polynomial G.

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Important Notice : **Every critical point is 2-periodic.**

Apply Thurston's theorem :

1. Create a topological (C^∞, quasi-regular) polynomial G.
2. Prove that $\nexists$ obstructions (Levy).

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Important Notice : **Every critical point is 2-periodic.**

Apply Thurston's theorem :

1. Create a topological (C^∞, quasi-regular) polynomial G.
2. Prove that $\nexists$ obstructions (Levy).
3. Apply Thurston's theorem to obtain a polynomial P.

Proof of Geyer's theorem

$$\text{Find } p \in \mathbb{R}[z] \text{ such that } p'(c) = 0 \Longrightarrow \begin{cases} p''(c) \neq 0 \\ c \neq \bar{c} \text{ (except possibly one)} \\ p(c) = \bar{c} \end{cases}$$

Important Notice : **Every critical point is 2-periodic.**

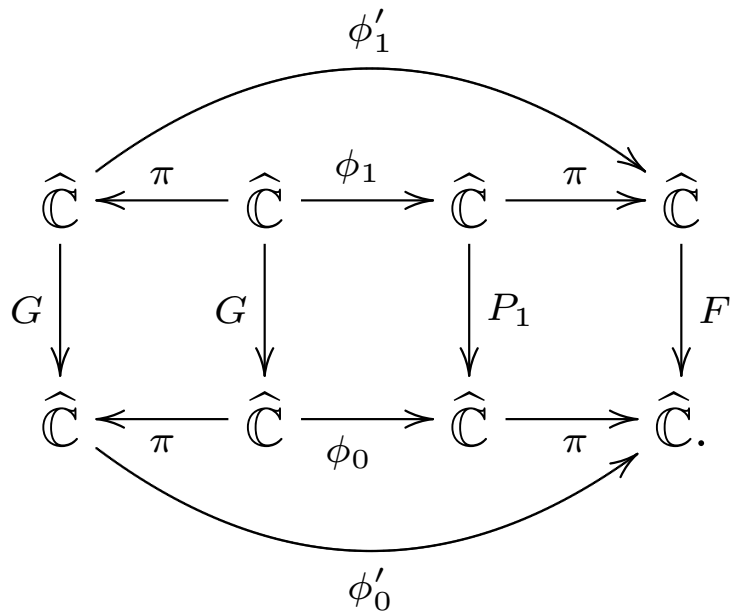
Apply Thurston's theorem :

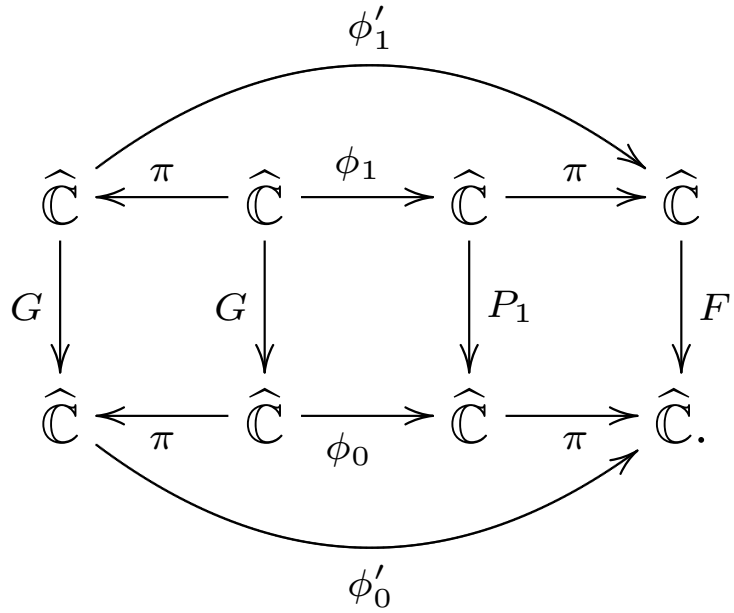
1. Create a topological (C^∞, quasi-regular) polynomial G.
2. Prove that $\nexists$ obstructions (Levy).
3. Apply Thurston's theorem to obtain a polynomial P.
4. Study the symmetry of P.

We want to prove that P is real.

Let $\pi : z \mapsto \bar{z}$.

Set $\phi_1' = \pi \circ \phi_1 \circ \pi^{-1}$, $\phi_0' = \pi \circ \phi_0 \circ \pi^{-1}$, and $F = \pi \circ P \circ \pi^{-1}$. Then $F(z) = \overline{P(\bar{z})}$ is again a polynomial and we have the following chains of commutative diagrams:





Due the unicity of ψ making $\phi_0 \circ G \circ \psi^{-1}$ holomorphic,

$$\phi_0 \text{ is real} \Longrightarrow \left\{ \begin{array}{l} \phi_1 \\ P_1 \end{array} \right. \text{ are real} \Longrightarrow \left\{ \begin{array}{l} \phi_n \\ P_n \end{array} \right. \text{ are real ,}$$

where (ϕ_n, P_n) are the pairs that appear in the Thurston algorithm.

But $P_n \to P$, so is P.

Geyer:

$$\#\{\text{degree } d \text{ Geyer polynomials}\}/(\text{real conj.})$$

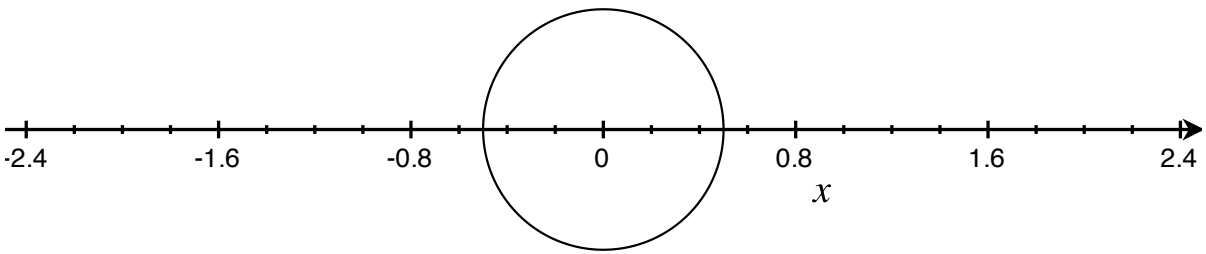
$$= m - \text{th Catalan number} = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} \text{ if } \begin{cases} d = 2(m+1) \\ d = 2m+1 \end{cases}$$

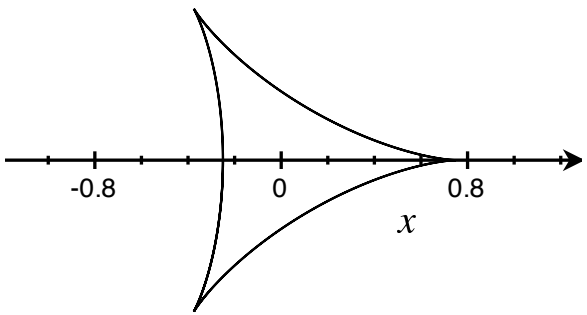
Thurston's theorem $\longrightarrow$ Classification results.

Why studying $p(z) - q(\bar{z})$? $p(z) - \bar{z}$?

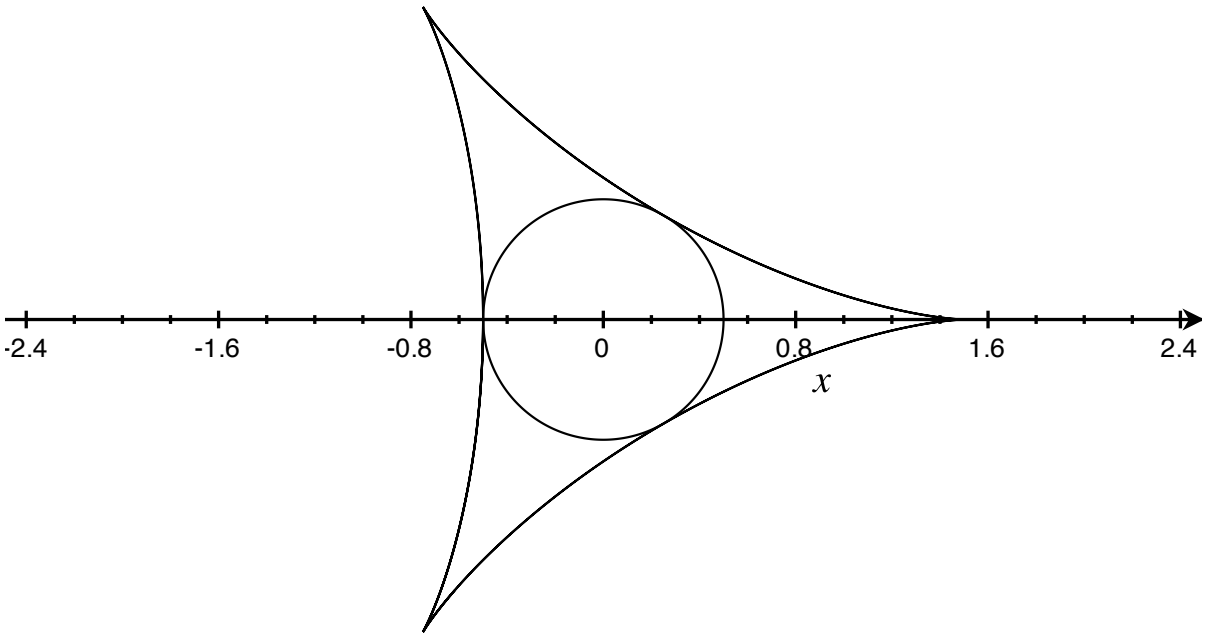
- real algebraic geometry
- global singularity theory of planar maps (local study by Wittney)
- holomorphic dynamics applies
- gravitational lensing (astrophysics)

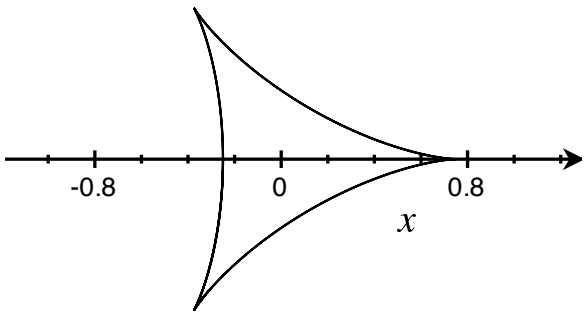
The map $z \mapsto z^2 - \overline{z}$, cr. set and cr. value set





Case $z^2 - \overline{z}$, co-critical set





Cubic polynomials

El Amrani-Loeb-T. (work in progress)

- $\boxed{f(z) = q(z) - \overline{z},\ \deg q = 3} \sim \boxed{\frac{z^3}{3} + pz - \overline{z}}$.

  - $\mathcal{C}_f = \begin{cases} \text{a Jordan curve} & \text{if } |p| < 1 \\ \text{figure 8} & \text{if } |p| = 1 \\ \partial(\Omega^1 \sqcup \Omega^2 \text{Jordan domains}) & \text{if } |p| > 1 \end{cases}$

- $|p| > 1 \Longrightarrow$ universality of $z^2 - \overline{z}$), and

  - $\#f^{-1}(c) = \begin{cases} 7 & | \ c \in f(\Omega^1) \cap f(\Omega^2) \\ 5 & | \ c \in f(\Omega^1) \ \triangle \ f(\Omega^2) \\ 3 & | \ c \in \widehat{\mathbb{C}} \setminus f(\Omega^1) \cup f(\Omega^2) \end{cases}$

- #{Whitney's cusp singularities}=8,4,6

Upperbound $\#\{\overline{p(z)} - z\} \leq 3d - 2\}$

Set $K(z) = \overline{p(z)}$, $f(z) = K(z) - z$.

$$
\begin{array}{rcccc}
 & & \{K(z) = z, |K'| > 1\} & & \{K(z) = z, |K'| < 1\} \\
 & & \| & & \| \\
\#\{f(z) = 0\} = & \#\{f(z) = 0, \deg_z = -1\} & + & \#\{f(z) = 0, \deg_z = 1\} \\
 & \overset{\text{degree}}{=} d + \#\{\deg_z = 1\} & & \\
 & = d + 2 \cdot \#\{\text{attracting fixed pts of } K\} & & \\
 & \overset{\text{Fatou}}{\leq} d + 2(d-1) = 3d - 2 & &
\end{array}
$$