7.6 Disc-annulus surgery on rational maps

By *Kevin M. Pilgrim* and *Tan Lei*

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. In this section, we introduce a flexible cut and paste surgery supported on the Fatou set of f. The resulting new map has dynamics which is closely related to that of f, although the connectivity of some of the Fatou components may increase.

The construction is in many ways similar to a related construction in the context of Kleinian groups. The "Klein-Maskit combination of type I using trivial discs" (cf. [Mar]) takes as input two Kleinian groups Γ_1, Γ_2 and two round discs $\widetilde{D}_1, \widetilde{D}_2$ with trivial stabilizer contained in the ordinary sets Ω_1, Ω_2. The output is a Kleinian group $\Gamma = \Gamma_1 * \Gamma_2$ containing Möbius conjugate copies of Γ_1 and Γ_2. That is, the dynamics of Γ contains that of Γ_1 and Γ_2. Moreover, it can easily be shown that the connected components of the limit set Λ of Γ are either translates of connected components of the limit sets Λ_j, or points. This operation also has a natural combinatorial analog in the setting of three-manifolds, namely *boundary connect sum.* Let $M_j = (\mathbb{H}^3 \cup \Omega_j)/\Gamma_j, j = 1, 2$, be the quotient three-manifolds. Since Ω_j is assumed non-empty, M_j has non-empty boundary, and the discs $\widetilde{D}_j$ descend to discs D_j on ∂M_j. The manifold $M = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$ is obtained from the manifolds M_j by identifying D_1 and D_2 via an orientation-reversing homeomorphism.

In the setting of rational maps, however, the fact that maps with interesting dynamics have degree strictly greater than one makes the formulation of combination theorems taking as input two (or several) rational maps rather subtle. One difficulty is the lack of a natural, simple combinatorial analog to describe the gluing data, which can be quite complicated to describe. We therefore content ourselves with the more modest program of beginning with a single map f and modifying it (in a more-or-less arbitrary fashion) inside a smoothly bounded disc $V \subset \mathcal{F}_f$ to obtain a new map F.

Contents. In Section 7.6.1 we define precisely the first surgery, which we call *disc-annulus surgery.* In Section 7.6.2 we give applications and relate the Julia set of F to that of f. In Section 7.6.3 we discuss the dependence of the map F on the various choices made in the construction and prove a uniqueness theorem (Theorem 7.60). In Section 7.6.4 we discuss the inverse operation to disc-annulus surgery. Finally, in Section 7.6.5 we give a related construction in which the modification procedure is slightly more involved. Nevertheless it is useful for the construction of examples such as those found in [PT]. One such example is a hyperbolic rational maps whose Julia set contains a wandering component which is a Jordan curve but not a quasicircle.

7.6.1 The disc-annulus surgery

We say that $V \subset \widehat{\mathbb{C}}$ is a *smooth disc* if ∂V is a real-analytic Jordan curve. By a *branched covering* we mean a proper $\mathcal{C}^1-$map between smooth, oriented (real) 2-manifolds, possibly with boundary, such that the boundary map is a covering map of (real) 1-manifolds, and such that on the interior, the map is given in appropriate local (complex) coordinates by $z \mapsto z^d$ for some d.

The following two lemmas are the technical ingredients for the definition of the disc-annulus surgery.

Lemma 7.47 (**Key Lemma**). *Let $A \subset \widehat{\mathbb{C}}$ be an open annulus bounded by two $\mathcal{C}^1$ Jordan curves $\gamma^\pm$, and let W be an open disc bounded by a $\mathcal{C}^1$ Jordan curve η. Give orientations to the curves such that A and W lie to the left of their boundaries. Let $f^\pm : \gamma^\pm \to \eta$ be two orientation-preserving $\mathcal{C}^1-$coverings of degree $d^\pm \geqslant 1$. Then there exists a branched covering $g : \overline{A} \to \overline{W}$ satisfying the following properties:*

(1) *$g|_{\gamma^\pm} = f^\pm$;*

(2) *$g(A) = W$ and the degree of g is $d^+ + d^-$;*

(3) *g can be chosen to be $\mathcal{C}^1$ in $\overline{A}$ and holomorphic and proper in a union of any collection of finitely many smooth discs compactly contained in A with pairwise disjoint closures.*

See Figure 7.21.

We call the map g a *covering extension* of the boundary maps $f^\pm$. In practice, we will take g to be holomorphic in a neighborhood of its critical points.

We recall the next lemma which was already mentioned in Remark 2.11 of Chapter 2, as an application of the Schwarz Reflection Principle.

Lemma 7.48 (**Extension Lemma**). *Let D, D' be two smooth discs in $\widehat{\mathbb{C}}$. Then a holomorphic proper mapping $F : D \to D'$ extends to a holomorphic map in a neighborhood of $\overline{D}$. In particular $F : \partial D \to \partial D'$ is a $\mathcal{C}^1-$covering.*

A branched covering F is *quasirational* if it is quasiconformally conjugate to a rational map. The Julia set of a quasirational map is thus well defined, and has the same qualitative metric and measure theoretic properties as the Julia set of a rational map. The following lemma is standard in the construction of new rational maps via surgeries. It is a restatement of a special case of the Second Shishikura Principle (Proposition 5.5).

Lemma 7.49 (**Shishikura Principle**). *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a $\mathcal{C}^1$ branched covering which is holomorphic a.e. in $\widehat{\mathbb{C}} \backslash B$, holomorphic in a neighborhood of the critical points in B, and for some integer k, $F^j(B) \cap B = \varnothing$ for all $j \geqslant k$. Then F is quasirational.*

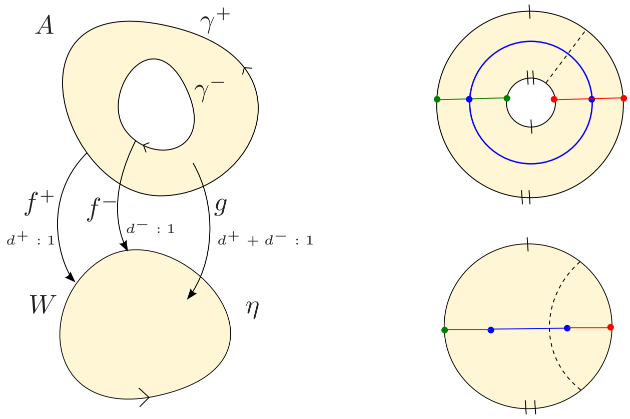

Figure 7.21: Left: The setup for Lemma 7.47. Right: Example of a branched covering of degree two from an annulus onto a disc, such that each of the boundaries of the annulus is mapped onto the boundary of the disc by degree one. The blue curve is mapped to the blue segment two to one. The outer and inner boundaries of the annulus are mapped to the boundary of the disc as indicated in the figure. The red and green segments are respectively mapped to the red and green slits in the disc two to one. The dotted line is mapped bijectively to the dotted line. Each of the connected domains in the annulus minus the segments are mapped bijectively to the open disc minus the two slits. See Exercises at the end of the section for generalizations.

We now describe disc-annulus surgery.

Theorem 7.50 (Disc-annulus surgery). *Let (f, V, H, h, g) satisfy the following conditions:*

- *f is a rational map;*
- *V is a smooth disc such that*
  - *∂V contains no critical points;*
  - *$f : V \to f(V)$ is proper;*
  - *there exists $1 \leqslant p \leqslant \infty$ such that*

$$f^j(\overline{V}) \cap V = \varnothing \text{ for } 0 < j < p,$$

and, in case $p < \infty$,

$$f^p(\overline{V}) \subset V \text{ and } f^p : V \to f^p(V) \text{ is proper;}$$

- *H is a smooth disc with $\overline{H} \subset V \setminus \overline{f^p(V)}$;*
- *$h : H \to \widehat{\mathbb{C}} \setminus \overline{f(V)}$ is a holomorphic proper map, and*

- *$g : V \setminus \overline{H} \to f(V)$ is a covering extension of the boundary maps such that g is holomorphic and proper on $f^p(V)$ and near the critical points.*

Then the map

$$F := \begin{cases} f \text{ on } \widehat{\mathbb{C}} \setminus V \\ h \text{ on } H \\ g \text{ on } V \setminus \overline{H} \end{cases}$$

is quasirational.

Remark 7.51 (V in $\mathcal{F}_f$). Observe that the hypotheses imply that V is a subset of the Fatou set of f. If $p < \infty$, by Schwarz Lemma there is an attracting periodic point in $f^p(V)$, and therefore V is part of a basin of attraction of f (and of F). If $p = \infty$ then either (i) V belongs to a strictly preperiodic component of f or (ii) since V is non-recurrent, V is part of a basin of attraction of an attracting or parabolic cycle. In none of the cases V can be part of a periodic rotation domain.

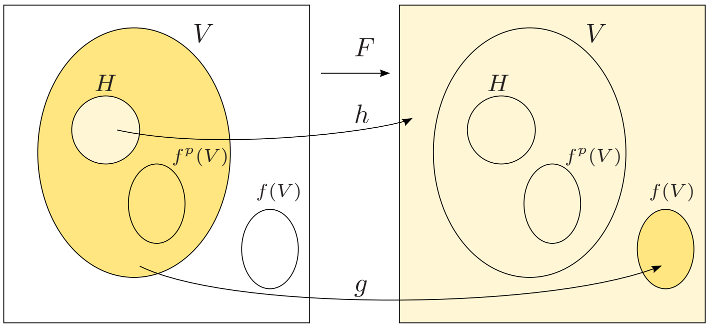

Figure 7.22: The domains $V, f^p(V)$ and H in the case $2 \leqslant p < \infty$. The new map F sends H to $\widehat{\mathbb{C}} \setminus \overline{f(V)}$ via h, and $V \setminus \overline{H}$ to $f(V)$ via g, using Lemma 7.47. On the white region, $F = f$.

See Figure 7.22. Summarizing, f is the original map; V is the disc with controlled recurrence on which the modification is supported; $h : H \to \widehat{\mathbb{C}} \setminus \overline{f(V)}$ is the new, added dynamics; and g is an interpolating map gluing f outside of V to h on H. We refer to F as a *disc-annulus extension of f supported on V*. If h is univalent we call the extension F *univalent* or *degree one*. Note that $F(V) = \widehat{\mathbb{C}}$ and $\deg(F) = \deg(f) + \deg(h)$.

Proof. We first extend h to $\overline{H}$ by Lemma 7.48. If $p = \infty$, set $f^p(V) = \varnothing$. Then F satisfies the conditions of Lemma 7.49 for $B = V \setminus (\overline{H \cup f^p(V)})$ and $k = 1$. So F is quasirational. □

7.6.2 Applications and properties

Here is an application of Theorem 7.50 with $p = \infty$.

Corollary 7.52. *Let f be a rational map. Let z_0 be a point in the Fatou set such that z_0 is neither a periodic point nor contained in a rotation domain. Then there is a disc-annulus extension F of f supported on a neighborhood V of z_0.*

Proof. Choose V a smooth disc containing z_0 such that $\overline{V}\setminus\{z_0\}$ contains no critical points, $f : V \to f(V)$ is proper, and $\overline{f^j(V)} \cap V = \varnothing$ for all $j > 0$. Then any choice of (H, h, g) satisfying the conditions of Theorem 7.50 (for $p = \infty$) will do. ☐

Thus, for any map f and any *non-recurrent* point z_0 in its Fatou set, one can modify f via a disc-annulus surgery supported in a small neighborhood V of z_0 so that on a disc H in this neighborhood, the new dynamics is more or less completely arbitrary.

Theorem 7.50 also applies to recurrent points in the Fatou set.

Corollary 7.53. *Let f be a rational map. Let z_0 be a (super)attracting periodic point of period p. Then there is a disc-annulus extension F of f supported on a neighborhood V of z_0.*

Proof. Choose V to be a smooth disc containing z_0 such that $\overline{V}\setminus\{z_0\}$ contains no critical points, $f^p : V \to f^p(V)$ is proper with $f^p(\overline{V}) \subset V$ and $f^j(\overline{V}) \cap V = \varnothing$ for $0 < j < p$. Then any choice of (H, h, g) satisfying the conditions of Theorem 7.50 (for $p < \infty$) will do. ☐

We now examine how the dynamics of f and F in the above two Corollaries are related. Given a rational map or branched covering f, recall that the *postcritical set* of f is defined as

$$P_f = \overline{\bigcup_{n>0} f^n(C_f)}$$

where C_f is the set of critical points of f. We say that two compact sets K_1 and K_2 are *conformally homeomorphic* if there is a conformal map from a neighborhood of K_1 to a neighborhood of K_2 which sends K_1 onto K_2.

Corollary 7.54 (Properties of disc-annulus extensions). *Let f and F be as in Corollaries 7.52 or 7.53. Then*

(1) *$\mathcal{J}_f \subset \mathcal{J}_F$, $\mathcal{J}_F$ is disconnected, and every connected component of $\mathcal{J}_f$ is a connected component of $\mathcal{J}_F$.*

(2) *If $H \cap P_f = \varnothing$ and h is univalent, then $H \cap P_F = \varnothing$.*

(3) *If $H \cap P_F = \varnothing$, then*

- *every Julia component of F passing through H infinitely many times is a point, and*
- *every other Julia component of F is conformally homeomorphic to a Julia component of f.*

Proof. (1) The set $\mathcal{J}_f$ is equal to the closure of the set of repelling periodic points of f. Since V is contained in the Fatou set of f and $F = f$ outside V, repelling periodic points of f stay repelling periodic points of F. So $\mathcal{J}_f \subset \mathcal{J}_F$. Since also $F(H) = h(H) = \widehat{\mathbb{C}} \setminus \overline{f(V)} \supset \mathcal{J}_f$ and $\mathcal{J}_F$ is totally invariant, $\mathcal{J}_F \cap H \neq \varnothing$. Moreover, since $f = F$ on ∂V and $\overline{V}$ is contained in the Fatou set of f, it is easy to show that ∂V and hence ∂H are in fact also in the Fatou set of F. So $\mathcal{J}_F \cap H \neq \varnothing, \mathcal{J}_F \cap (\widehat{\mathbb{C}} \setminus H) \neq \varnothing$, and $\partial H \subset \mathcal{F}_F$. Hence $\mathcal{J}_F$ is disconnected. That each component of $\mathcal{J}_f$ is also a component of $\mathcal{J}_F$ follows from the fact that a rational map sends connected components of the Julia set onto such components; see e.g. [Bea] for details.

(2) The critical points of F are the union of the critical points of $f|_{\widehat{\mathbb{C}} \setminus V}$, g and h. The F-orbit of the g-critical points do not intersect H. So if h is conformal and $H \cap P_f = \varnothing$, $H \cap P_F = \varnothing$.

(3) Assume now $H \cap P_F = \varnothing$. Take a closed disc $H' \subset H$ such that $(\mathcal{J}_F \cap H) \subset H'$. By a lemma of Fatou, the diameters of the components of $F^{-n}(H')$ tend to zero as $n \to \infty$ (see Lemma 3.40). A Julia component $\mathcal{J}'$ such that $F^n(\mathcal{J}') \subset H$ for some n is contained in a component of $F^{-n}(H')$. Therefore if $F^n(\mathcal{J}') \subset H$ for infinitely many n, $\mathcal{J}'$ is a point.

For example, if h is conformal, it has a unique repelling fixed point in H, which is a point component of $\mathcal{J}_F$.

Now let $\mathcal{J}'$ be a component of $\mathcal{J}_F$ such that $F^{\ell}(\mathcal{J}') \subset H$ and $F^n(\mathcal{J}') \cap H = \varnothing$ for $n > \ell$. Then $F^{\ell+1}(\mathcal{J}')$ is a component of $\mathcal{J}_f$, and $F^{\ell+1}$ is conformal in the component of $F^{-\ell}(H)$ containing J'. □

Corollary 7.55 (**Connectivity of Fatou components**). *Denote by W_f, W_F the Fatou components of f and F containing ∂V. If W_f is periodic, then W_F is periodic and infinitely connected. If W_f is strictly preperiodic, and h is univalent, then the connectivity of W_F is equal to $m_0 + m_1$, where m_0 and m_1 are the connectivities of W_f and $f(W_f)$.*

Proof. By assumption $V \subset W_f$ and $V \setminus H \subset W_F$. If W_f is periodic with $p < \infty$ then W_F is also periodic since $F^p(V \setminus H) = f^p(V) \subset V$. If $p = \infty$ the periodicity is not changed either since $F = f$ outside V. Since ∂V is in the Fatou set, but H contains points of the Julia set, it follows that W_F is not simply connected. But any invariant component of the Fatou set which is not a Herman ring is either simply connected or infinitely connected ([Bea]

Section 7.5). By Remark 7.51, W_F is not a Herman ring, hence W_F is infinitely connected.

Now suppose W_f is strictly preperiodic and h is univalent, and let $B = f(W_f)\backslash f(V)$, which has $m_1 + 1$ boundary components. Observe that points in $h^{-1}(B)$ are in H and belong to W_F, since they never come back to W_f under f. In particular, every boundary component of B except $\partial f(V)$, is a boundary component of W_F, and H contains no other. All those original boundary components of W_f are also so in W_F, and the counting follows. □

The following corollary has previously been obtained by Baker (cf. [Bea, §11.7]).

Corollary 7.56. *There exists rational maps with a non-periodic Fatou component of any given number of connectivity.*

These corollaries provide many examples of maps with disconnected Julia sets. Explicit examples of rational maps with similar properties are given in [Bea, Chapter 11]. The following examples use the freedom of $g|_{f^p(V)}$ and $h|_H$ to assign interesting dynamics to F.

Example 7.57 (Surgery of a polynomial to obtain a higher degree polynomial). Let f be a polynomial of degree greater than 1, so that ∞ is a superattracting fixed point. Let $V' = \{|z| > R\}$ where R is large enough so that $V := f^{-1}(V') \supset \overline{V'}$ and $\overline{V}\backslash\{\infty\}$ contains no critical point of f. Choose any smooth disc H with $\overline{H} \subset V\backslash\overline{f(V)}$ and any holomorphic proper map $h : H \to \widehat{\mathbb{C}}\backslash\overline{f(V)}$. Finally, choose a covering extension g on $V\backslash\overline{H}$ so that g is holomorphic and proper from $f(V)$ onto $g(f(V))$ with ∞ as a critical fixed point of local degree $\deg(f) + \deg(h)$; the preimage of the map $z^{\deg(f)}(z-1)^{\deg(h)}$ outside a small neighbourhood of 0 provides a model, showing that such a covering extension exists. This provides a quasirational map F with $F^{-1}(\infty) = \infty$. So F is quasiconformally conjugate to a polynomial of degree $\deg(f) + \deg(h)$.

Example 7.58 (Capturing a critical point). In the setting of Example 7.57, assume that a critical point c of f satisfies $f^\ell(c) \in V\backslash\overline{f(V)}$ for some $\ell > 0$. We take H to be a smooth disc containing $f^\ell(c)$ and $h : H \to \widehat{\mathbb{C}}\backslash\overline{f(V)}$ conformal such that $h(f^\ell(c)) = c$. Therefore the critical point c, escaping to ∞ under f, is "captured" back and becomes periodic (hence superattracting) under F.

Similarly, one can exploit the same idea to send a critical point c to e.g. a point $x \in \mathcal{J}_f$ whose orbit is dense in $\mathcal{J}_f$, thereby obtaining a map F whose postcritical set has complicated topology. Note, however, that in such an example the altered critical point lies in the Julia set of F and is non-recurrent.

Example 7.59 (Case $1 < p < \infty$). We describe a surgery on the quadratic polynomial $f(z) = z^2 - 1$ to obtain a cubic rational map with disconnected Julia set, and with 0 a double critical point. Let V be a smooth disc containing 0 whose closure is contained in the basin of attraction of 0 so that the second iterate $f^2(V)$ is relatively compact in V, and $f : V \to f(V)$ is a degree two covering ramified at 0. Now take H to be a smooth disc whose closure is contained in $V \setminus \overline{f^2(V)}$. Define $h : H \to \overline{\mathbb{C}} \setminus \overline{f(V)}$ to be conformal, $g : V \setminus \overline{H} \to f(V)$ to be a covering extension which is a holomorphic branched covering of degree 3 in $f^2(V)$, with 0 as a double critical point and -1 as the critical value. Now pasting together $f|_{\widehat{\mathbb{C}} \setminus V}$, h and g gives a quasirational map F. This map has a simple critical point at ∞, a double critical point at 0 with orbit $0 \mapsto -1 \mapsto 0$, and another simple critical point in $V \setminus (H \cup f^2(V))$ whose orbit is attracted to the cycle $\{0, -1\}$.

The map F is of degree 3 and the Julia set of F contains a fixed copy $\mathcal{J}_0$ of the Julia set $\mathcal{J}_f$, a countable collection of homeomorphic preimages of $\mathcal{J}_0$, and a Cantor set worth of point components.

7.6.3 Uniqueness of disc-annulus extensions

The construction of a disc-annulus extension of f depends on the non-canonical choices of V, g, H, and h. Clearly, the flexibility of choice in the map g implies that one cannot expect the quasiconformal conjugacy class on the whole sphere to be independent of such choices. With this in mind, we say that two maps F, F' are *$\mathcal{J}-$conjugate* if there is a quasiconformal homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which conjugates F on a neighborhood of $\mathcal{J}_F$ to F' on a neighborhood of $\mathcal{J}_{F'}$. Note that this notion of $\mathcal{J}-$conjugacy is stronger than the one used elsewhere in the book.

In this section, we establish a prototype uniqueness theorem.

Theorem 7.60 (Uniqueness up to $\mathcal{J}-$conjugacy). *Let f be a rational map and U a Fatou component of $\mathcal{F}_f$. Then the $\mathcal{J}-$conjugacy class of a univalent disc-annulus extension F supported on $V \subset (U \setminus P_f)$ depends only on U and not on V, g, h, H.*

This fails without the assumption of the univalence of h, or the fact that $H \cap P_f = \emptyset$ (which is a consequence of the assumption $V \cap P_f = \emptyset$). Assume on the contrary that there is a point $z \in H$ which is either a point of P_f or a critical point of h. Let $h' : H \to \widehat{\mathbb{C}} \setminus \overline{f(V)}$ be another holomorphic map satisfying $h'(z) \neq h(z)$ and $h'(z) \in \mathcal{J}_f$. Then V and H are identical but the two resulting extensions F, F' are distinct on their Julia sets.

Analogy with Kleinian groups. The univalent disc-annulus extensions constructed in Theorem 7.60 are analogous to *adding a handle* to a three-manifold with boundary. Let $M_1 = (\mathbb{H}^3 \cup \Omega_{\Gamma_1})/\Gamma_1$ be the three-manifold with boundary associated to a Kleinian group Γ_1. Let D, D' be disjoint round discs in ∂M_1, and consider the three-manifold M obtained by gluing D to D' via an orientation-reversing homeomorphism (equivalently, join D to D' with a solid tube). The resulting manifold admits a hyperbolic structure inherited from the quotient of $\mathbb{H}^3$ by a new group $\Gamma = \Gamma_1 *_{\mathbb{Z}}$ which is an HNN-extension of Γ. The discs D, D' yield a *compressing disc* in M, i.e. its boundary is an essential curve on ∂M. Indeed, if Ω is a basin of attraction, then V descends to a closed disc on the quotient punctured torus associated to U, which is like the boundary of the quotient three-manifold.

The following lemma is the key step in the proof of Theorem 7.60.

Lemma 7.61. *Let f be a rational map, U a Fatou component of $\mathcal{F}_f$. Suppose (see Figure 7.23)*

1. *$W \subset U$ is an open subset such that there is $1 \leqslant p \leqslant \infty$ with*
   a) *$f : W \to f(W)$ proper,*
   b) *$f^j(\overline{W}) \cap W = \varnothing$ for $0 < j < p$, and in case $p < \infty$, $f^p(\overline{W}) \subset W$ and $f^p : W \to f^p(W)$ is proper.*
2. *(V, H) and (V', H') are two pairs of smooth discs in W satisfying the conditions of Theorem 7.50 and the additional condition that $\overline{H \cup H'}$ is contained in a disc or annulus $Y \subset W \backslash \left(\overline{f^p(W)} \cup P_f\right)$.*

Then any two univalent extensions $F = F(V, g, h, H), F' = F'(V', g', h', H')$ are $\mathcal{J}-$conjugate.

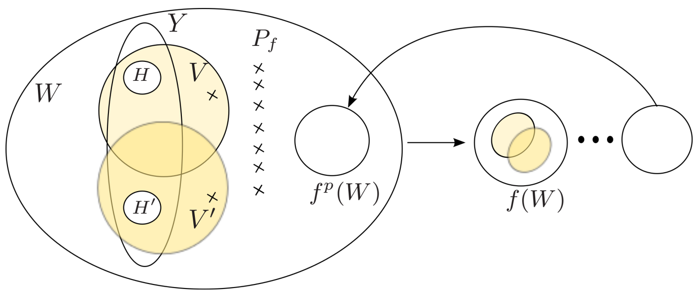

Figure 7.23: Setup of Lemma 7.61. Two univalent extensions with the modified domains close enough to each other and avoiding the postcritical set, give rise to two $\mathcal{J}-$conjugate rational maps.

Proof of Lemma 7.61. We will construct a combinatorial equivalence (in the sense of McMullen [McM4], Appendix) between the holomorphic coverings $F : X_1 \to X_0, F' : X_1' \to X_0'$ where X_0, X_0' are neighborhoods of the respective Julia sets and X_1 and X_1' are respective subsets of X_0 and X_0'.

Such an equivalence consists of a pair of quasiconformal homeomorphisms $\phi_j : X_j \to X'_j$, $j = 0, 1$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_0 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_0} \\ X'_1 & \xrightarrow{F'} & X'_0 \end{array}$$

commutes and such that ϕ_0 extends to a quasiconformal homeomorphism from X_0 to X'_0 and ϕ_0 is isotopic to ϕ_1 rel. the ideal boundary of X_0.

From this, a standard pullback argument (Theorem A.1 in [McM4]) implies the existence of a conjugacy $\phi : X_0 \to X'_0$, yielding the result.

We now proceed to define the combinatorial equivalence. In case $p < \infty$, let $L' = \overline{f(W) \cup \cdots \cup f^p(W)}$. In case $p = \infty$, choose L' to be a union of finitely many closed discs or annuli contained in $\mathcal{F}_f$ such that $L' \cap W = \varnothing$, $L' \supset \bigcup_{n>0} f^n(W)$ and $f(L') \subset L'$. Denote by P' the union of finitely many disjoint smooth open discs containing P_f such that $f(\overline{P'}) \subset P'$ and $\partial P' \cap \partial L' = \varnothing$.

Let $X_0 = X'_0 = \widehat{\mathbb{C}} \backslash (L' \cup \overline{P'})$; note that $X_0 \supset \mathcal{J}_F, \mathcal{J}_{F'}$, because $f(L') \subset L'$ so $f^n(L') \cap (H \cup H') = \varnothing$. Moreover X_0 has finitely many boundary components.

Next we lift ϕ_0 to obtain ϕ_1. We call this the lifting step.

Lifting step. Define $X_1 = F^{-1}X_0$, $X'_1 = (F')^{-1}X_0$ and $\phi_0|_{X_0} = \mathrm{Id}$. Observe that $(L' \cup P') \subset F^{-1}(L' \cup P')$ and therefore $X_1 \subset X_0$. Also $X'_1 \subset X_0$. Extend ϕ_0 to a quasiconformal homeomorphism from $\widehat{\mathbb{C}} \backslash f(V)$ to $\widehat{\mathbb{C}} \backslash f(V')$ (this is possible since both $f(V)$ and $f(V')$ are compactly contained in $f(W)$). Define $\phi_1 = \mathrm{Id}$ on $X_0 \backslash W$, and on H define ϕ_1 as the lift of ϕ_0 under h, h'; this is possible since h, h' are univalent and indeed we have $\phi_1|_H = (h')^{-1} \circ \phi_0 \circ h$. Observe that $X_1 \subset (X_0 \backslash W) \cup H$, $X'_1 \subset (X_0 \backslash W) \cup H'$ and the following diagram commutes, with the top and the bottom maps being holomorphic coverings.

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_0 \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_{X_0}} \\ X'_1 & \xrightarrow{F'} & X_0 \end{array}$$

Finally we show that ϕ_0 is isotopic to ϕ_1 rel. the ideal boundary of X_0. We call this the Isotopy step.

Isotopy step. There is a $\mathcal{C}^1-$extension of ϕ_1 on X_0 such that ϕ_1 is isotopic to ϕ_0 rel. ∂X_0. To see this, extend ϕ_1 to W such that $\phi_1 = \mathrm{Id}$ on $W \backslash Y$ and $\phi_1|_Y$ is isotopic to the identity rel. ∂Y. Then ϕ_1 is isotopic to Id rel. $\widehat{\mathbb{C}} \backslash Y \supset \partial X_0$.

The pair ϕ_0, ϕ_1 gives the desired combinatorial equivalence, and so we obtain a quasiconformal mapping $\phi : X_0 \to X_0$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{F} & X_0 \\ \phi \downarrow & & \downarrow \phi \\ X_1' & \xrightarrow{F'} & X_0 \end{array}$$

Extending ϕ arbitrarily to the whole sphere shows that F and F' are $\mathcal{J}-$conjugate. □

Remark 7.62. In Theorem 7.60, the global quasiconformal conjugacy class depends on the critical orbit relations of F, which are difficult to control since the gluing map g necessarily introduces new critical values in $f(V) \subset \mathcal{F}_F$.

Also, in the non-univalent case, it may not be possible to carry out the lifting step. Or the lifting step may be possible but the isotopy step impossible.

Proof of Theorem 7.60. Let $V, V' \subset U \backslash P_f$ be any two smooth discs. Set $V_0 = V, V_1 = V'$, and let V_t be a path of smooth discs joining V_0 to V_1 through $U \backslash P_f$. Any two univalent extensions supported on sufficiently close V_s, V_t are $\mathcal{J}-$conjugate, by Lemma 7.61; the theorem follows by compactness of the interval $[0, 1]$. □

7.6.4 Inverse of disc-annulus extensions: simplification

The next result offers an inverse procedure.

Theorem 7.63 (**Inverse disc-annulus surgery**)**.** *Let F be a rational map. Let A be a smooth annulus satisfying the following conditions:*

- *F maps A properly onto a disc;*
- *∂A contains no critical points;*
- *there is $1 \leqslant p \leqslant \infty$ such that*

$$F^j(\overline{A}) \cap A = \varnothing \text{ for } 0 < j < p,$$

and, in case $p < \infty$,

$$F^p(\overline{A}) \subset A \text{ and } F^p : A \to F^p(A) \text{ is proper;}$$

- *there is a component H of $\widehat{\mathbb{C}} \backslash \overline{A}$ such that $H \cap \bigcup_{1<j<p} F^j(A) = \varnothing$.*

Then there is a quasirational map f which coincides with F on $\widehat{\mathbb{C}}\backslash(A \cup \overline{H})$ and $f(A \cup \overline{H}) = F(A)$.

Note that $\deg(f) < \deg(F)$. In the analogy with three-manifolds, one may think of this operation as a special case of cutting along a compressing disc.

Proof. Define f to be F on $\widehat{\mathbb{C}}\backslash(A \cup \overline{H})$ and extend f so that $f : A \cup \overline{H} \to F(A)$ is a $\mathcal{C}^1$ proper map which is holomorphic in $F^p(A)$. By assumption, $f^n(B) \cap B = \varnothing$ for $B = (A \cup \overline{H})\backslash F^p(A)$ and $n > 0$. So f is quasirational by Lemma 7.49. □

7.6.5 Surgery with two gluing regions

Here, we describe another, similar surgery construction. In the disc-annulus surgery, the new dynamics h is holomorphic on a disc. Here, we allow for the more flexible setting of adding new dynamics h which is holomorphic on an annulus. Since the annulus has two boundary components, the gluing map g will be defined on two disjoint pieces. Controlling the recurrence of the additional region where conformal distortion occurs requires our introduction of an additional smooth disc G; see Figure 7.24.

Theorem 7.64 (Surgery with two gluing regions). *Let (f, V, H, G, h, g) satisfy the following conditions:*

- *f is a rational map,*
- *V a smooth disc contained in a Fatou component U such that:*
  - *∂V contains no critical points of f;*
  - *there is $p < \infty$ such that $f^j(\overline{V}) \cap V = \varnothing$ for $0 < j < p$,*

$$f^p(\overline{V}) \subset V, \text{ and } f^p : V \to f^p(V) \text{ is proper;}$$

- *H is a smooth annulus such that $\overline{H} \subset V\backslash f^p(\overline{V})$ and $\partial f^p(V)$ is a boundary component of H;*
- *G is a smooth disc contained in a Fatou component U' such that U and U' have disjoint orbit and $\operatorname{mod}\left(\widehat{\mathbb{C}}\backslash(\overline{G \cup f(V)})\right) = d' \operatorname{mod} H$ for some integer $d' > 0$;*
- *$h : H \to \widehat{\mathbb{C}}\backslash(\overline{G \cup f(V)})$ a holomorphic covering of degree d' with $h(f^p(\partial V)) = \partial G$ and $g : V\backslash(\overline{H \cup f^p(V)}) \to f(V)$, $g : f^p(V) \to G$ a covering extension holomorphic in a neighborhood of the critical points.*

Then

$$F := \begin{cases} f \text{ on } \widehat{\mathbb{C}}\setminus V \\ h \text{ on } H \\ g \text{ on } V\setminus H \end{cases}$$

is quasirational. Moreover, if $H \cap P_f \neq \emptyset$ *and* f *is hyperbolic, the* $\mathcal{J}-$*conjugacy class of* F *depends only on* U, U', d' *and the homotopy class of* ∂H *relative to* P_f.

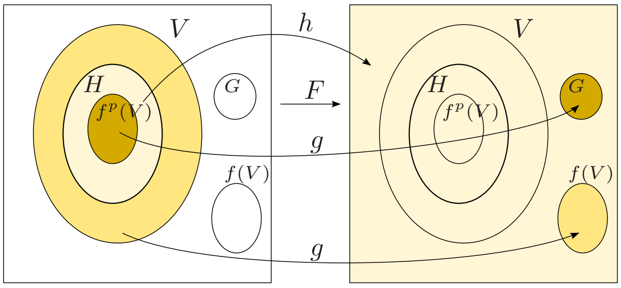

Figure 7.24: The domains $V, f^p(V), H$, and G in the setup of Theorem 7.64.

Proof. Set $B = V\setminus\overline{H}$. We have $F^j(B) \cap B = \emptyset$ for $j > p$. So by Lemma 7.49 F is quasirational.

To prove the unicity, we first establish a local result. Assume (f, V, H, G, h, g) and (f, V', H', G', h', g') are two sets of choices of the theorem such that:

1. $\text{mod}\left(\widehat{\mathbb{C}}\setminus\overline{G \cup f(V)}\right) / \text{ mod } H = \text{mod}\left(\widehat{\mathbb{C}}\setminus\overline{G' \cup f(V')}\right) / \text{ mod } H' = d'$ is an integer, there is a set U' which is the union of finitely many disjoint closed discs such that $\overline{G \cup G'} \subset \text{int}(U') \subset U' \subset \widetilde{U}$ and $f(U') \subset U'$;

2. there is a smooth annulus $\widehat{H}$ containing $\overline{H \cup H'}$ such that $f^{-p}(\partial\widehat{H}) \cap \widehat{H} = \emptyset$.

Then the two maps F and F' are $\mathcal{J}-$conjugate.

To prove this, set $W := \widehat{H} \cup \overline{f^p(V)}$. It is a disc containing $H \cup H' \cup f^p(V) \cup f^p(V')$. By assumption 2, $f^p(W)$ is contained in $f^p(V) \cap f^p(V')$.

Let $X_0 = \widehat{\mathbb{C}}\setminus(\overline{U' \cup f(W) \cup \cdots \cup f^p(W) \cup P'})$, where P' is the union of finitely many disjoint smooth open discs containing P_f such that $f(\overline{P'}) \subset P'$ and $\partial P' \cap \partial(U' \cup f(W) \cup \cdots \cup f^p(W)) = \emptyset$. Note that $X_0 \supset \mathcal{J}_F, \mathcal{J}_{F'}$ and that X_0 has finitely many boundary components.

Lifting step. Define $X_1 = F^{-1}(X_0)$, $X_1' = F'^{-1}(X_0)$, and a $\mathcal{C}^1-$diffeomorphism $\phi_0|_{X_0}$ isotopic to the identity rel. ∂X_0, mapping $\partial f(V)$ to $\partial f(V')$.

Now extend ϕ_0 to a $\mathcal{C}^1-$diffeomorphism from $\widehat{\mathbb{C}}\setminus(\overline{G \cup f(V)})$ to $\widehat{\mathbb{C}}\setminus(\overline{G' \cup f(V')})$.

Define $\phi_1|_{X_0\backslash W} = \mathrm{Id}$ and $H \to H'$ to be a lifting of $\phi_0 : \widehat{\mathbb{C}}\backslash(f(V) \cup G) \to \widehat{\mathbb{C}}\backslash(f(V') \cup G')$. Extend ϕ_1 to $\widehat{H}$ so that it is isotopic to the identity rel. $\partial\widehat{H}$. Then ϕ_1 is isotopic to Id rel. ∂X_0. The only problem is that it is not exactly a lift of ϕ_0 (near $f^{-1}(\partial f(V))$.

The pair ϕ_0, ϕ_1 gives the desired combinatorial equivalence and hence a quasiconformal-conjugacy of $F : X_1 \to X_0$ to $F' : X_1' \to X_0$. Therefore F and F' are $\mathcal{J}-$conjugate.

From this one can easily obtain the global unicity result. □

Lemma 7.65. *Let (f, V) be as in Theorem 7.64. If there is a Fatou component U whose orbit under f is disjoint from V, then the disc G and the annulus H exist.*

Proof. One can first choose either G or H first. Assume that H and hence $\mathrm{mod}\, H$ are given. Choose $z_0 \in U$. Since the modulus of the annulus between z_0 and $\partial f(V)$ is infinite, one can find a smooth disc G containing z_0 such that $\mathrm{mod}\ (\widehat{\mathbb{C}}\backslash(\overline{G \cup f(V)})) = d' \mathrm{mod}\ H$ for some (probably very large) integer $d' > 0$. Of course, we can also start from G. Let G be any smooth disc whose closure is contained in U. There is a minimal integer d' such that mod $(\widehat{\mathbb{C}}\backslash(\overline{G \cup f(V)})) < d' \mathrm{mod}\ H$. Then one can find H in $V\backslash f^p(\overline{V})$ such that $\mathrm{mod}(\widehat{\mathbb{C}}\backslash(\overline{G \cup f(V)})) = d' \mathrm{mod}\ H$. □

In practice we may want d' to be as small as possible. We will show that in case $p = 1$, one can choose V, H, G so that $d' = 2$ if $\deg(f) \geqslant 3$ and $d' = 3$ if $\deg(f) = 2$.

Surgery on the basin of infinity of a polynomial. Let $f(z) = z^2 + c$ and let $\mathcal{K}_c = \mathcal{K}_f$ be its filled Julia set, which we assume to have non-empty interior (for example $c = -1$). Fix a smooth disc G with closure in $\mathrm{int}(\mathcal{K}_c)$. Denote by φ the Böttcher coordinate of f in the basin of infinity. Let R be chosen sufficiently large such that

$$\log R > \tfrac{1}{3} \mathrm{mod} \left(\widehat{\mathbb{C}}\backslash(\overline{G} \cup \varphi^{-1}(\{|z| \geqslant R^2\}))\right) \ ;$$

this is possible since the right hand side is comparable to $\frac{2}{3}\log R$. Now take $V = \varphi^{-1}(\{|z| > R\})$. One can choose H as in Theorem 7.64 with $d' = 3$. See [PT] for a computer generated picture of the Julia set of such a map with $c = -1$ and $d' = 3$.

A similar surgery can be done for any degree d polynomial f with $\mathrm{int}(\mathcal{K}_f) \neq \varnothing$, or with rational maps with multiple superattracting cycles. Moreover, if $d \geqslant 3$ one can choose V, G, H so that $d' = 2$. Finally, note that if $d' \geqslant 2$ then $\mathcal{J}_F$ may contain preimages of components of $\mathcal{J}_f$ under covering maps of positive (indeed, arbitrarily large) degree.

We conclude with an easy consequence of Theorem 7.64.

Corollary 7.66 (Properties of the extensions). *Let F be the quasirational map given by Theorem 7.64. Then $\mathcal{J}_F$ is not connected, $\mathcal{J}_f \subset \mathcal{J}_F$, $\deg(F) = \deg(f) + \deg(h) = \deg(f) + d'$ and $\mathcal{J}_F \backslash \mathcal{J}_f$ contains uncountably many wandering components which are not points.*

Exercises Section 7.6

7.6.1 It is shown in Figure 7.21 how one can construct a branched covering of degree two of an annulus onto a disc such that each of the boundaries of the annulus is mapped onto the boundary of the disc by degree one. Generalize this idea to construct a degree $2d$ branched covering, mapping each of the boundaries of the annulus onto the boundary of the disc by degree d.
Hint: Draw $2d$ segments in the annulus similar to the two drawn in the figure. Map each connected component of the annulus minus the segments bijectively onto the disc minus the two slits, and map each segment onto one of the two critical slits, alternately. Alternatively, change coordinates so that the annulus separates 0 and ∞ and precompose the degree 2 map by z^d.

7.6.2 Give a (topological) example of a degree three branched covering map from an annulus onto a disc such that the outer boundary of the annulus is mapped onto the boundary of the disc by degree two while the inner boundary of the annulus is mapped by degree one.
Hint: Construct first a branched covering of degree two as in Figure 7.21. Afterwards, make a slit in the annulus from the outer boundary to an interior point. Open up along the slit and glue in a copy of a slitted disc, as shown in Figure 7.25. The resulting space is still an annulus. Explain how to map the open annulus 3 : 1 onto the open disc.

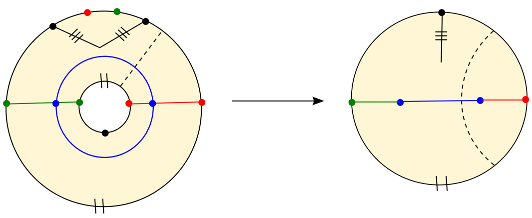

Figure 7.25: Sketch of the construction in Exercise 7.6.2.

7.6.3 For any natural numbers $d^+ > 1$ and $d^- > 1$, generalize the constructions in the exercises above to give an example of a branched covering of degree $d^+ + d^-$ from an annulus onto a disc, such that the outer respectively inner boundary of the annulus is mapped onto the boundary of the disc by degree d^+ respectively d^-.