Extremal length estimates in perturbations of rational dynamics

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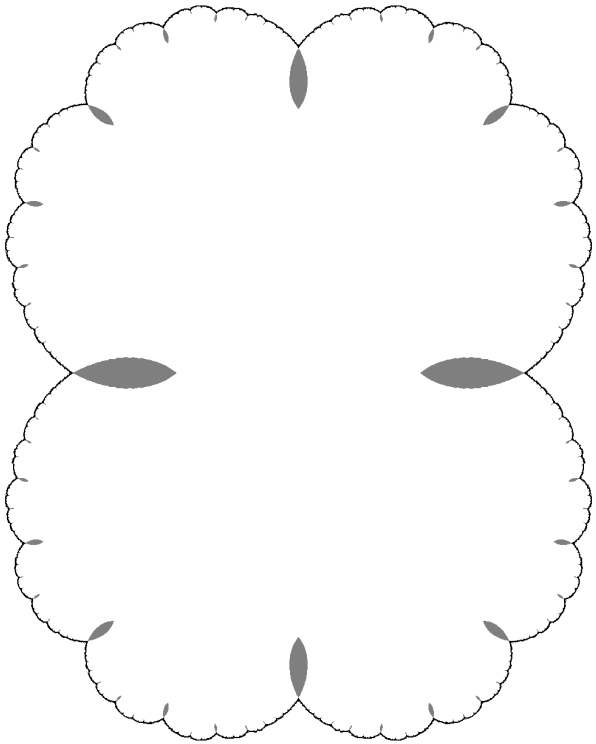
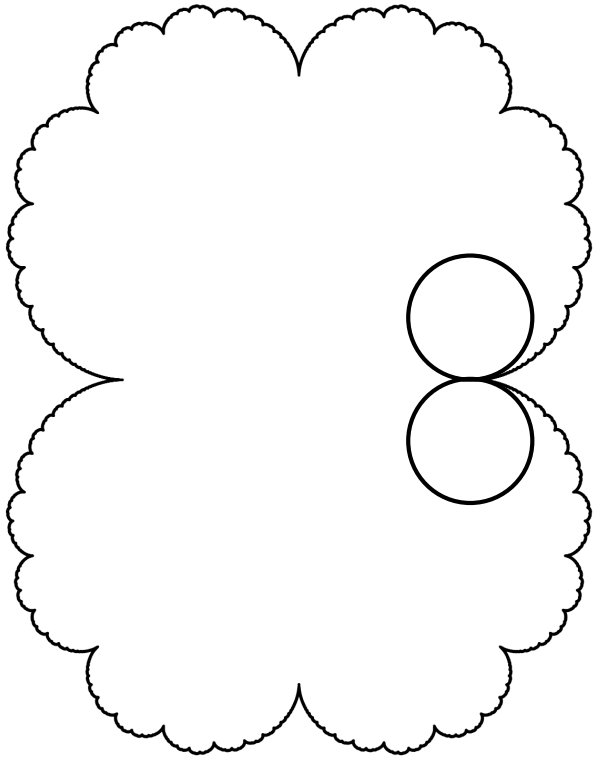
Pinching deformations of iterated rational maps

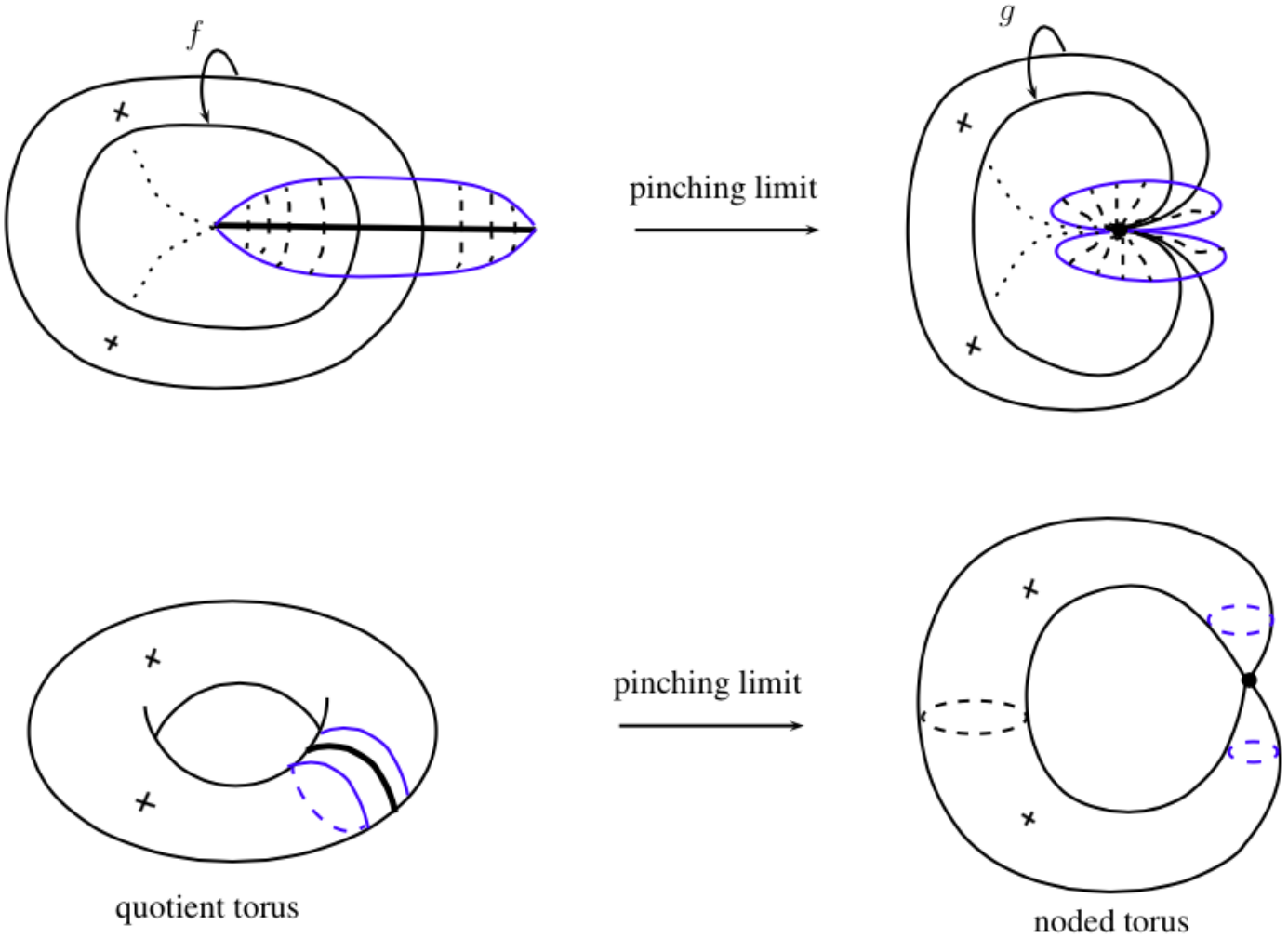




Theorem.

$\exists$ a path $\{\phi_t, t \geq 0\}$ of qc-homeomorphisms, conformal outside the bands, such that for $f_t := \phi_t \circ f \circ \phi_t^{-1}$, the pair (ϕ_t, f_t) converges uniformly to limits (φ, g), and ϕ_t pinches gradually a tree linking several fixed points into one point, and makes no other identifications.

Bands, pinching and expected limits



(see Torus.avi)

Warning : other paths may lead to **discontinuity !** (see mandel animation).

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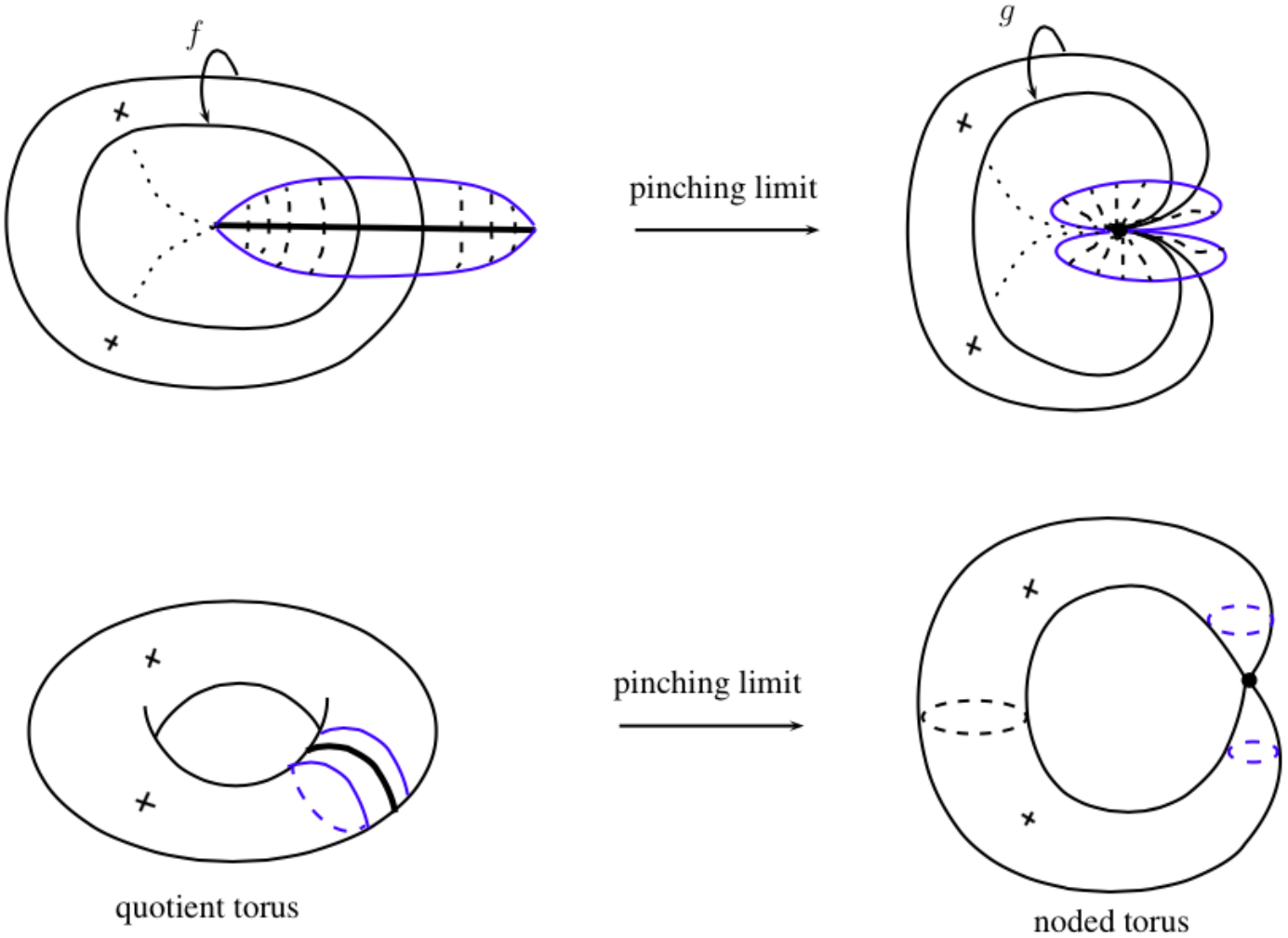

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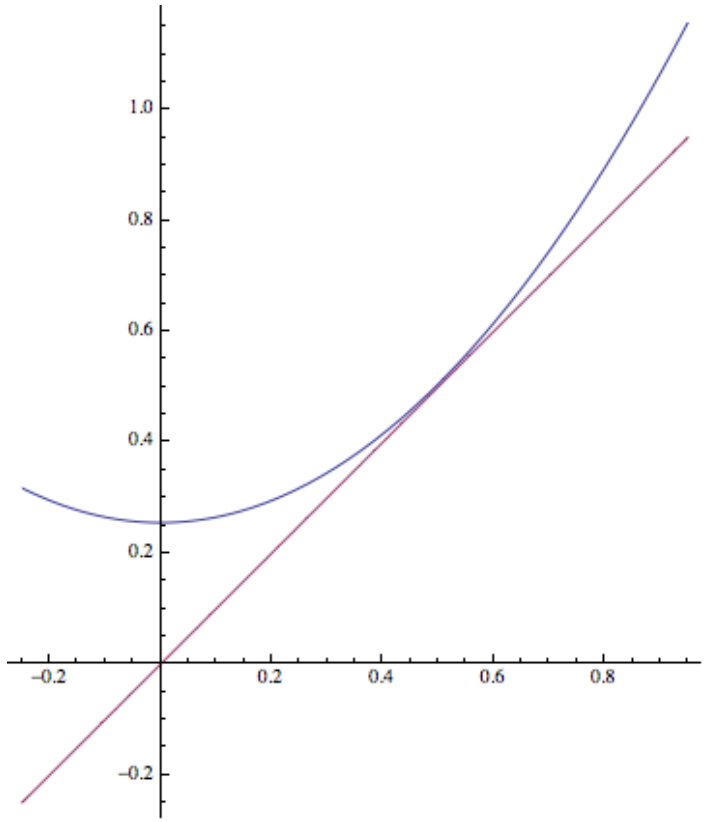
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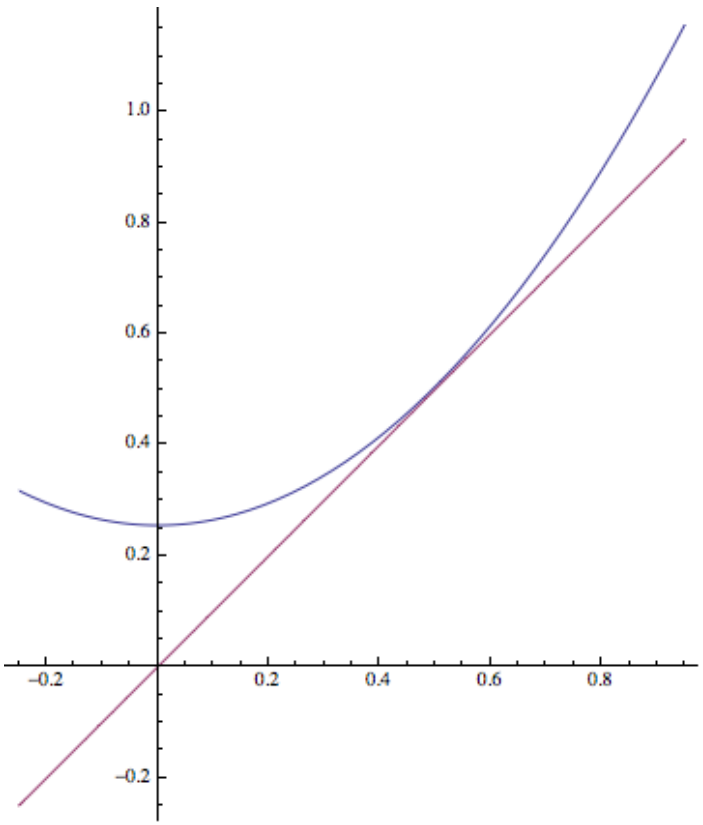
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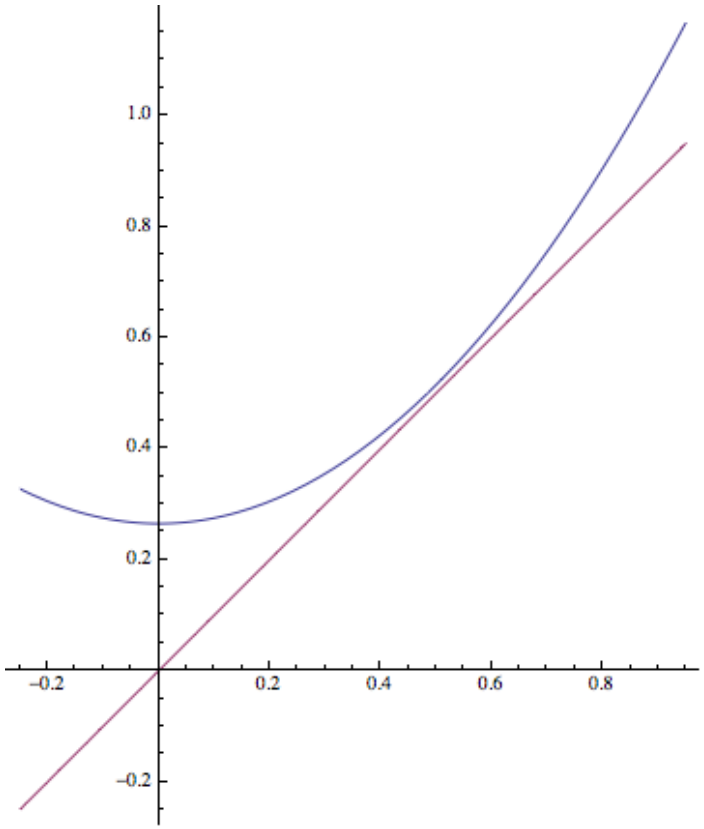
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Accessing parabolic and Misiurewicz maps

We also generalize the following results to more general settings

In the quadratic family $\{z \mapsto z^2 + c\}$,

1. the fat rabbit parameter is accessible from both the main cardioid and the rabbit hyperbolic component. In the main cardioid direction, the parabolic point 'breaks' into an attracting fixed point and a period-3 repelling cycle, whereas in the rabbit direction, the parabolic point 'splits' into a repelling fixed point and a period-3 attracting cycle.

2. a Misiurewicz parameter is accessible from the escape hyperbolic component.

Moduli of annuli and extremal lengths

If $A = (\text{a horiz. strip})/(z \mapsto z + r)$, $\mathrm{mod} A = \dfrac{height}{r}$ a conformal invariant.

Let $A \subset \mathbb{C}$ be an annulus. Let $\rho \geq 0$ be measurable. We say that $\rho(z)|dz|$ is an admissible conformal metric on A if

$$\mathrm{area}_\rho := \iint_A \rho(x+iy)^2 dxdy \text{ is } > 0, \text{ and } \neq \infty.$$

Such a metric defines lengths on curves $\gamma : [a, b] \to A$:

$$\ell_\gamma(\rho) = \int_a^b |\gamma'(t)| \cdot \rho(\gamma(t))\, dt.$$

Then for

$$\ell_{\mathrm{arc}}(\rho) := \inf\{\ell_\gamma(\rho) \mid \gamma \text{ arc connecting } \partial^\pm A\}$$

$$\ell_{\mathrm{loop}}(\rho) := \inf\{\ell_\gamma(\rho) \mid \gamma \text{ loop separating } \partial^\pm A\}$$

we have

$$\frac{\ell_{\mathrm{arc}}(\rho)^2}{\mathrm{area}_\rho} \leq \mathrm{mod} A \leq \frac{\mathrm{area}_\rho}{\ell_{\mathrm{loop}}(\rho)^2}.$$

An extremal length estimate: A annulus.

$\rho_0 =$ **extremal metric** if $\quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$

$\wedge\!\backslash \qquad\qquad \wedge\!\backslash$

ρ is **length increasing** if $\quad \ell_{\text{loop}}(\rho) \qquad \ell_{\text{arc}}(\rho)$

Key lemma

Let ϕ conformal near ∂A. Let A_ϕ bounded by $\phi(\partial A)$. Then

$$\frac{\text{area}_{\rho_0}}{\text{area}_\rho} \leq \frac{\text{mod}A_\phi}{\text{mod}A} \leq \frac{\text{area}_\rho}{\text{area}_{\rho_0}}, \quad \forall\, \rho \begin{cases} = 0 \text{ where } \phi \text{ is not conformal} \\ \text{length increasing.} \end{cases}$$

Proof. $\dfrac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} = \dfrac{\ell_{\text{arc}}(\phi_*\rho)^2}{\text{area}_{\phi_*\rho}} \leq \text{mod}A_\phi \leq \dfrac{\text{area}_{\phi_*\rho}}{\ell_{\text{loop}}(\phi_*\rho)^2} = \dfrac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$

$$\frac{\text{mod}A_\phi}{\text{mod}A} \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} \frac{\ell_{\text{loop}}(\rho_0)^2}{\text{area}_{\rho_0}} \overset{\text{length}\nearrow}{\leq} \frac{\text{area}_\rho}{\text{area}_{\rho_0}}.$$

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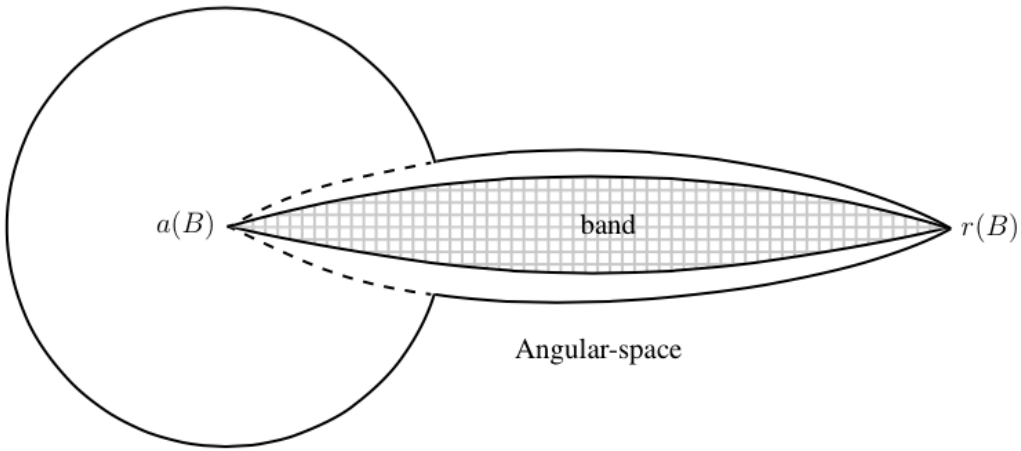
Conformal maps off bands with equi-angluar spaces

A Jordan domain D is called an K**-angular space** of B if $D \cup r(B) \supset B$ and for any point $z \in \partial D \setminus r(B)$

$$K \cdot \mathrm{dist}_{\mathrm{euclidean}}(z, B) \geq \text{arc-length}(\partial D, z, r(D)).$$



Theorem (a priori bound). Let $K > 1$.

A annulus containing K-angular spaces of finitely many disjoint $\{B_j\}_j$

$$\forall \text{ conformal map } \phi \text{ outside } \bigcup_j B_j, \quad \frac{1}{K^2} \leq \frac{\mathrm{mod} A_\phi}{\mathrm{mod} A} \leq K^2.$$

Proof using Key lemma

Theorem (a priori bound). Let $K > 1$.

For a flat annulus A containing K-angular spaces of finitely many disjoint (except at $r(B)$'s) bands $\{B_j\}_j$

$$\forall \text{ conformal map } \phi \text{ outside } \bigcup_j B_j, \quad \frac{1}{K^2} \leq \frac{\mathrm{mod} A_\phi}{\mathrm{mod} A} \leq K^2.$$

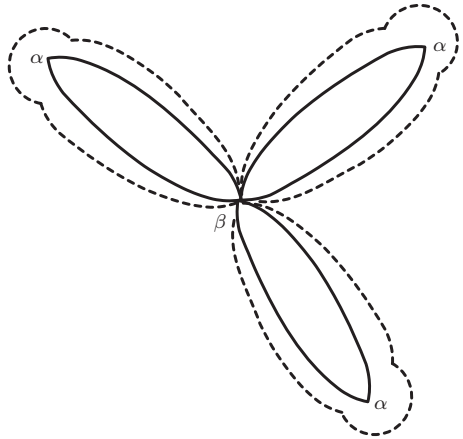
$$\text{Define} \quad \rho(z) = \begin{cases} K \cdot \rho_0(z) & \text{on } A - \bigcup_j B_j, \\ 0 & \text{on } \bigcup_j B_j. \end{cases}$$

Let γ be an arc or a loop in A. Then

$$\ell_\gamma(\rho) \geq \ell_{\widehat{\gamma}}(\rho_0), \quad \frac{\mathrm{area}_\rho(A)}{\mathrm{area}_{\rho_0}(A)} \leq K^2.$$

So ρ is length increasing and we can apply Key-lemma.

Application to star-like pinchings



Put a path of Beltrami differentials only on the quotient annuli pinching the core curves to points (and doing no other identifications), and then lift to a path of Beltrami differentials $\{\mu_t\}_{t\geq 0}$ supported on the bands.

1. Get angular spaces at periodic bands using repelling periodic points, and then at pre-repelling ones by pullback and Koebe distortion.
2. Truncate μ_t up to n-level bands and integrate to get $\phi_{t,n}$.
3. Use the a priori bound to control modulus distortions of $\phi_{t,n}$.
4. Get control of modulus distortions of ϕ_t by letting $n \to \infty$.
5. Use normalization to show equicontinuity of ϕ_t.
6. Use unicity of get convergence of ϕ_t, and then of $f_t = \phi_t \circ f \circ \phi_t^{-1}$.

Accessing Misiurewicz polynomials

Douady-Hubbard (Kiwi for higher degree)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

We give a new (intrinsic) proof that works for more general settings.
Set $f_r : z \mapsto z^2 + c(r)$. Construct via surgery $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ uniformly on } n \\ & & r & \longrightarrow & 0 \end{array}$$

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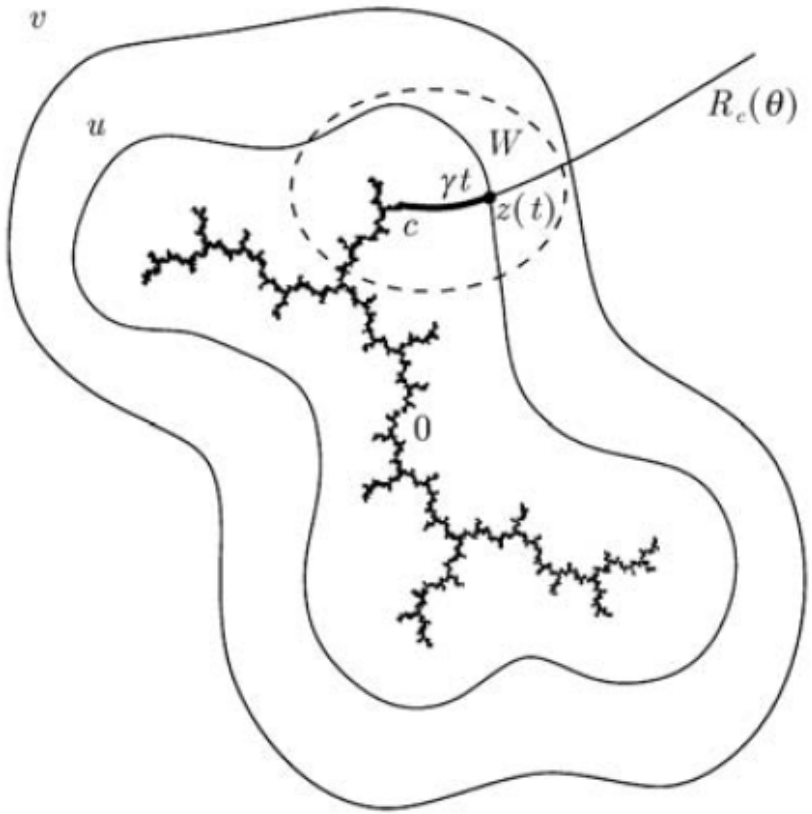
The vertical convergence is due to Thurston's algorithm (no need of Thurston's theorem).

The horizontal one is due to a distortion control on pseudo-conjugacies $f_{r,n} \xrightarrow[p_{r,n-1}]{p_{r,n}} g$, with $p_{r,n}$ mostly conformal.

Surgery on g



1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. Pullback the standard complex structure infinitely many times to get an F_r-invariant structure and integrate using Measurable Riemann Mapping Theorem.

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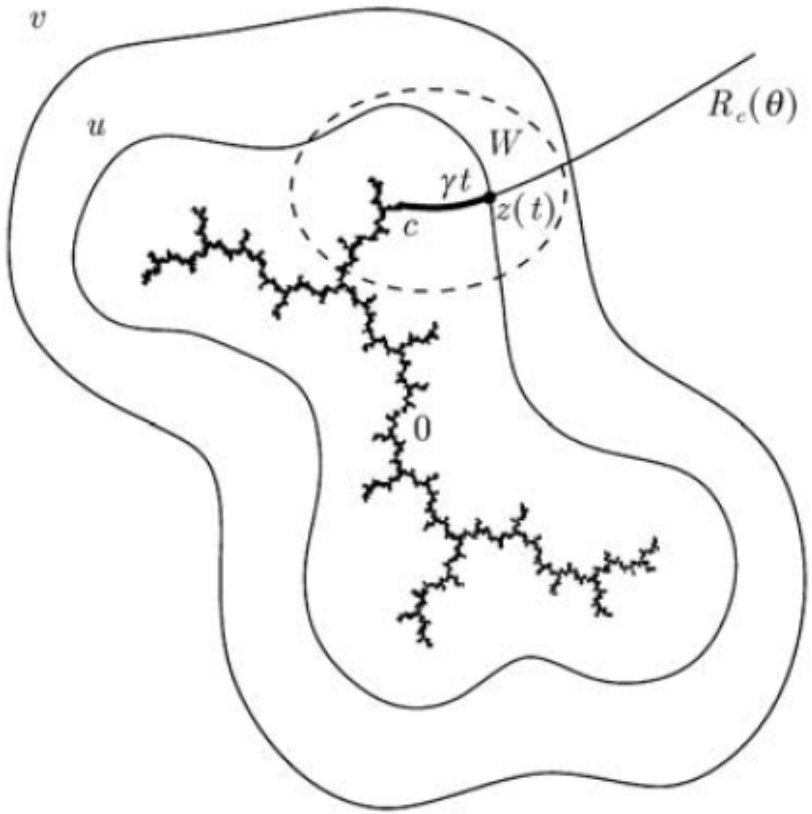

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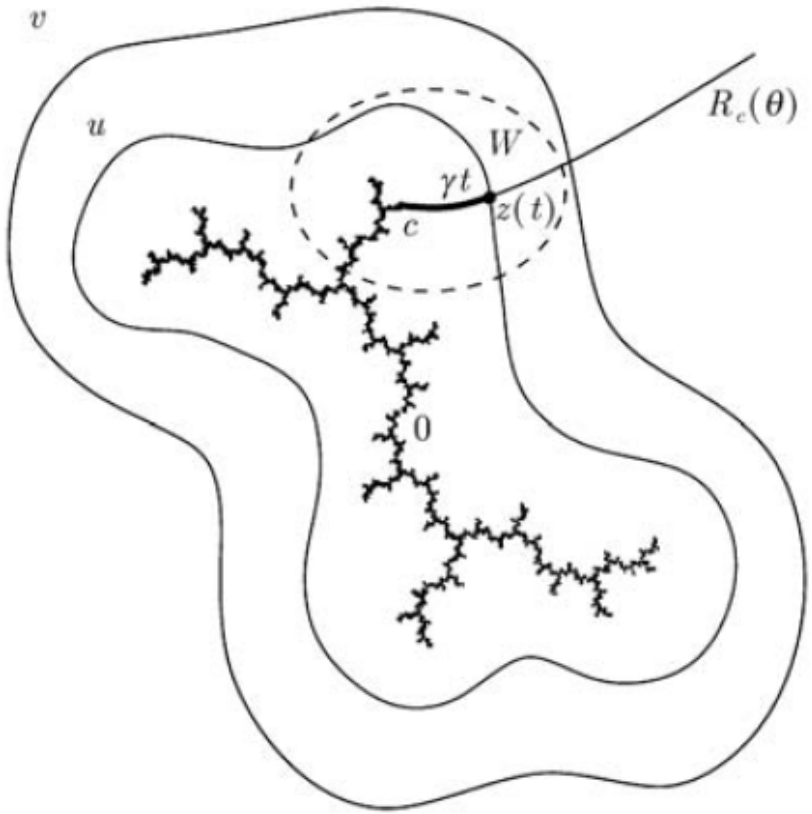

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2. pullback step by step the standard complex structure and integrate, to get

$$\begin{array}{ccc} & \xrightarrow{p_{r,n}} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow[p_{r,n-1}]{} & \end{array}$$

with $p_{r,n}$ conformal outside

$$\bigcup_{j=0\ldots n} F_r^{-j} U_0(r) = \bigcup_{j=0\ldots n} g^{-j} U_0(r).$$

3. So for $V =$ exterior of a large equipotential,

$$g = F_r \Big| \begin{array}{ccc} V & \xrightarrow{p_{r,n},\text{conformal}} & \\ & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1},\text{conformal}]{} & \end{array}$$

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$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ unif. on } n, \\ & & r & \longrightarrow & 0 \end{array}$$

with $f_r : z \mapsto z^2 + c(r)$ and $c(r) \in R_{c(r)}(\theta)$ (of potential $e^{t(r)}$).

Horizontal convergence: control first $p_{r,n}$

Theorem

If $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n, then $f_{r,n}|_{\widehat{\mathbb{C}}} \underset{r\to 0}{\longrightarrow} g|_{\widehat{\mathbb{C}}}$ uniformly on n.

Proof : $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n implies $f_{r,n}|_V \underset{r\to 0}{\longrightarrow} g|_V$ uniformly on n.

Same degree $\Longrightarrow$ $f_{r,n}|_{\widehat{\mathbb{C}}} \underset{r\to 0}{\longrightarrow} g_{\widehat{\mathbb{C}}}$ uniformly on n. □

So we should only prove $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

We control first the distortion of $p_{r,n}$ from being globally conformal (i.e. Möbius transformation). We then use the normalization to control the spherical distance.

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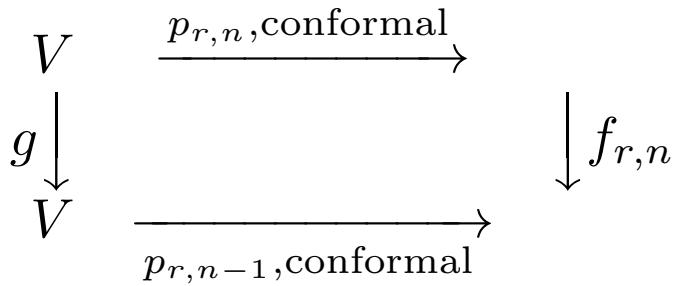

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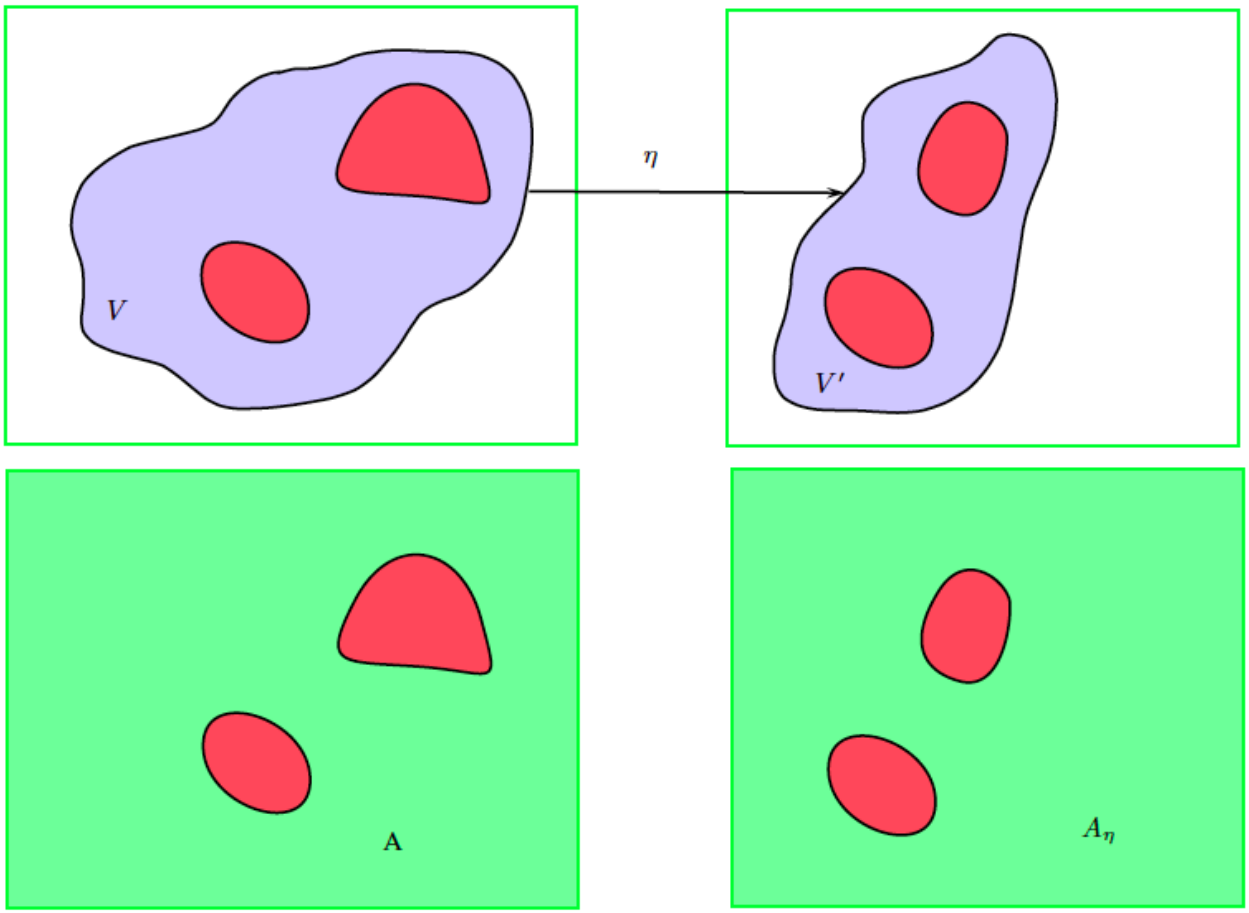
Distortion of conformal maps

Given $V \subset \widehat{\mathbb{C}}$ open and $\eta : V \xrightarrow{\text{conformal}} \widehat{\mathbb{C}}$,
how far is η being globally conformal?



Define $\mathcal{D}(\eta, V) = \sup_{A \text{ with } \partial A \subset V} \left| \mathrm{mod} A - \mathrm{mod} A_\eta \right|$. Example: $\mathcal{D}(z^2, \mathbb{H}) = \infty$.

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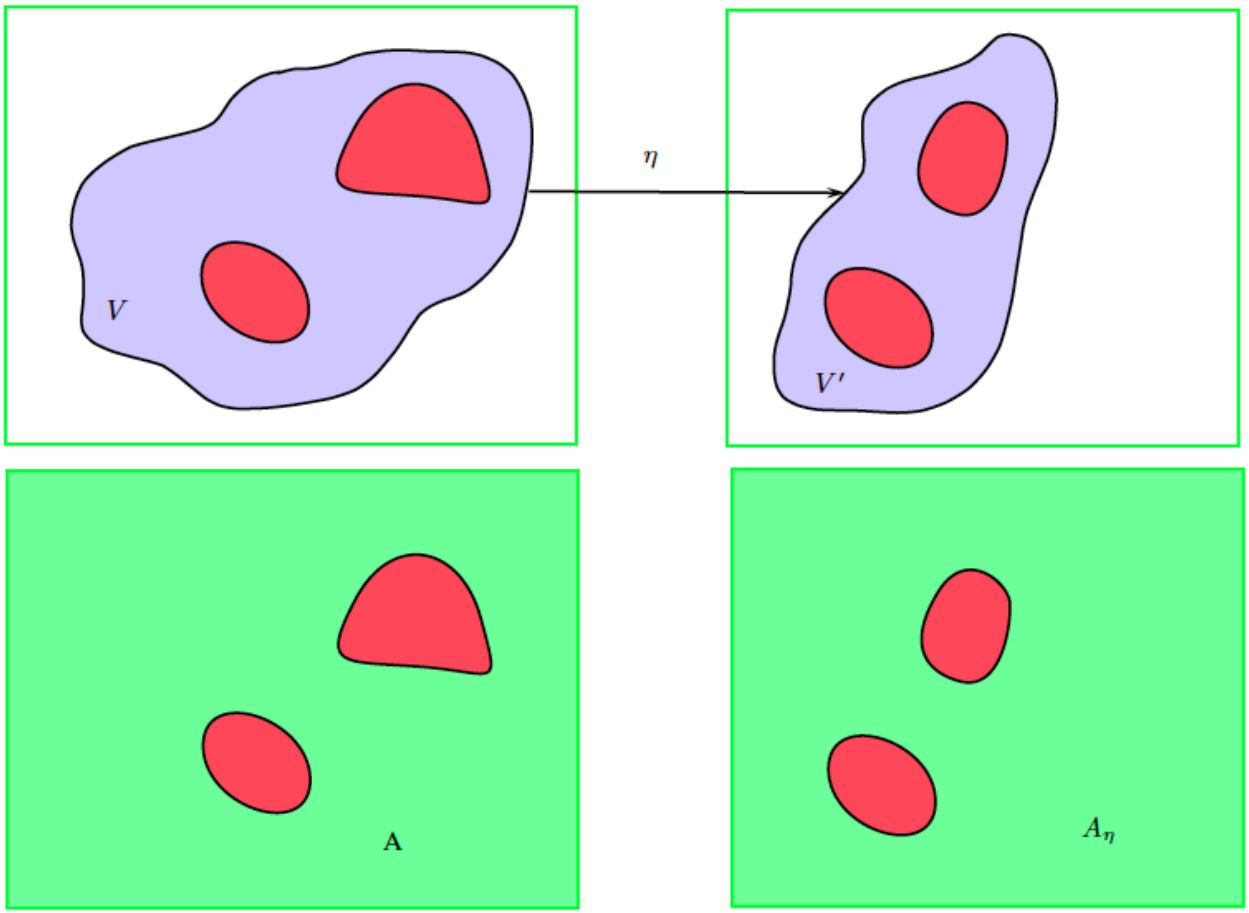

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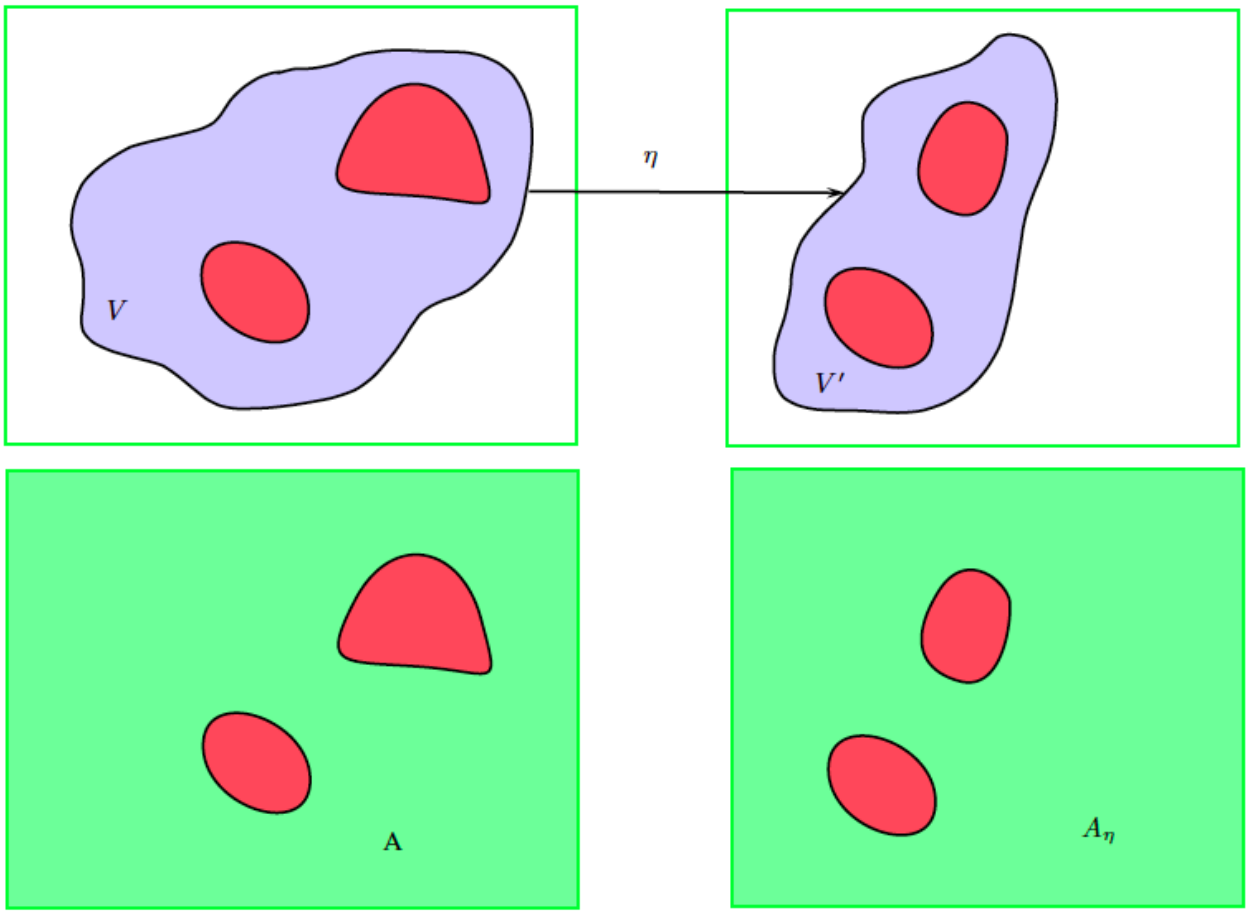

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An extremal length estimate: A annulus.

$$\rho_0 = \textbf{extremal metric} \text{ if } \quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$$

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$$\rho \text{ is } \textbf{length increasing} \text{ if } \quad \ell_{\text{loop}}(\rho) \qquad \ell_{\text{arc}}(\rho)$$

Key lemma

Let ϕ conformal near ∂A. Let A_ϕ bounded by $\phi(\partial A)$. Then

$$|\text{mod}A_\phi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0}, \quad \forall\, \rho \begin{cases} = 0 \text{ where } \phi \text{ is not conformal} \\ \text{length increasing.} \end{cases}$$

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A typical application

Take a flat annulus $A \supset \bigcup_{i=1}^{m} D(x_i, 1)$ disjoint discs. Fix $r < 1$. Assume ϕ conformal outside the r-core $\bigcup_i D(x_i, r)$. Set

$$\rho = \begin{cases} 0 & \text{in the } r\text{-core } \bigcup_i D(x_i, r) \\ \dfrac{1}{1-r} & \text{in the rings } r \le |z - x_i| < 1 \\ 1 & \text{otherwise} \end{cases}$$

For a horizontal or vertical line γ,

$$\ell_\gamma(\rho) \ge \ell_\gamma(\rho_0), \ \text{area}_\rho - \text{area}_{\rho_0} = m \cdot \left(\frac{1+r}{1-r}\pi - \pi \right) \le 3m\pi r$$

$\forall\, \phi$, by Key lemma

$$|\mathrm{mod} A_\phi - \mathrm{mod} A| \le 3m\pi r \xrightarrow[r \searrow 0]{} 0.$$

If the marked discs are not round, when r is small enough, one can insert round annuli in the rings. Replacing $\dfrac{1}{1-r}$ by $\dfrac{1}{1-\sqrt{r}}$ in ρ would still do the job. But we want a control independent of m and nesting neighborhoods. Need extra conditions.

Conformal maps outside nested or disjoint disks

Let $0 < \lambda < 1$ a factor, and $1 \geq \mathbf{s}(r) > 0$ a function on [0,1[with $\mathbf{s}(r) \underset{r \searrow 0}{\longrightarrow} 0$.

Let $\{D_y\}_{y \in Y}$ finitely many marked discs. They are $(\lambda, \mathbf{s})$-**nested** if :
(1) Any two are either disjoint or nested (but $\neq$).
(2) If $D_y \subset D_x$ and D_y touches $D_x(r)$, then $D_y \subset D_x(\mathbf{s}(r))$ (near the center they are not too long)
(3) $\text{area}\Big(\bigcup_{D_y \subsetneq D_x} D_y \Big) \leq \lambda \, \text{area} D_x$.

Theorem (a priori bound)

$\exists \infty \geq C(r) \underset{r \searrow 0}{\longrightarrow} 0$ depending only on $(\lambda, \mathbf{s})$, such that for any set $\{D_y\}_{y \in Y}$ of $(\lambda, \mathbf{s})$-nested discs and any disjoint $\overline{V}$,

$$\forall \text{ conformal map } \phi \text{ outside } \bigcup \overline{D_y(r)}, \quad \mathcal{D}(\phi, V) \leq C(r) \cdot M(\cup D_y, V),$$

where $M(\cup D_y, V)$ is a constant depending only on $\cup D_y$ and V.

Proof using Key lemma

Fix $r << 1$ (so that $\mathbf{s}(r) < 1$) and $A \supset \cup D_y$ with $\partial A \subset V$.

For $k \geq 1$, among the discs D_x of nesting depth k, those touching some $D_y(r)$ are contained in $D_y(\mathbf{s}(r))$, the others form a union J_k. Define $J_k(\mathbf{s}(r))$ by reducing the discs' sizes.

Define $\rho_k(z)$ inductively on k by

$$\rho_k(z) = \begin{cases} 0 & \text{on } J_k(\mathbf{s}(r)) \\ \rho_0(z)(1-\sqrt{\mathbf{s}(r)})^{-k} & \text{on } J_k - J_k(\mathbf{s}(r)), \\ \rho_{k-1}(z) & \text{on } A - J_k. \end{cases}$$

When reaching the deepest nest, the metric stabilizes to ρ, which increases length, $= 0$ on $\cup D_x(r)$ and

$$0 \leq \text{area}_\rho(A) - \text{area}_{\rho_0}(A) \leq C(r) \cdot M(D, V)$$

Our ϕ is conformal outside the r-core: $\cup D_x(r)$.

$$|\text{mod}A - \text{mod}A_\phi)| \overset{\text{Key lemma}}{\leq} \text{area}_\rho A - \text{area}_{\rho_0} A \leq C(r) \cdot M(D, V).$$

So $\mathcal{D}(\phi, V) \leq C(r) \cdot M(D, V)$. □

In dynamical settings, we run into overlapped discs rather than nested discs

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Vertical convergence

$$\begin{array}{cc} \infty & f_r \\ \uparrow & \uparrow \\ n & f_{r,n} \end{array}$$: (forgetting r) F is Thurston equivalent to f through a pair ϕ_0, ϕ_1 (they can be constructed by hand in this case, no need to apply Thurston's theorem).

Use Thurston's pullback map, get f_n. The sequence converges to $f = f_r$.

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ & \xleftarrow{\phi_3} & & \xrightarrow{p_3} & \\ f\downarrow & & F\downarrow & & \downarrow f_3 \\ & \xleftarrow{\phi_2} & & \xrightarrow{p_2} & \\ f\downarrow & & F\downarrow & & \downarrow f_2 \\ & \xleftarrow{\phi_1} & & \xrightarrow{p_1} & \\ f\downarrow & & F\downarrow & & \downarrow f_1 \\ & \xleftarrow{\phi_0} & & \xrightarrow{p_0} & \end{array}$$

$p_n \circ \phi_n^{-1}$ are more and more conformal and remain K-qc., any limit map must be conformal thus must be the identity. Thus f_n converges to $f = f_r$.

Accessing parabolic maps from stable directions

Same procedure: Construct via surgery and Thurston's algorithm $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ uniformly on } n \\ & & r & \longrightarrow & 0 \end{array}$$

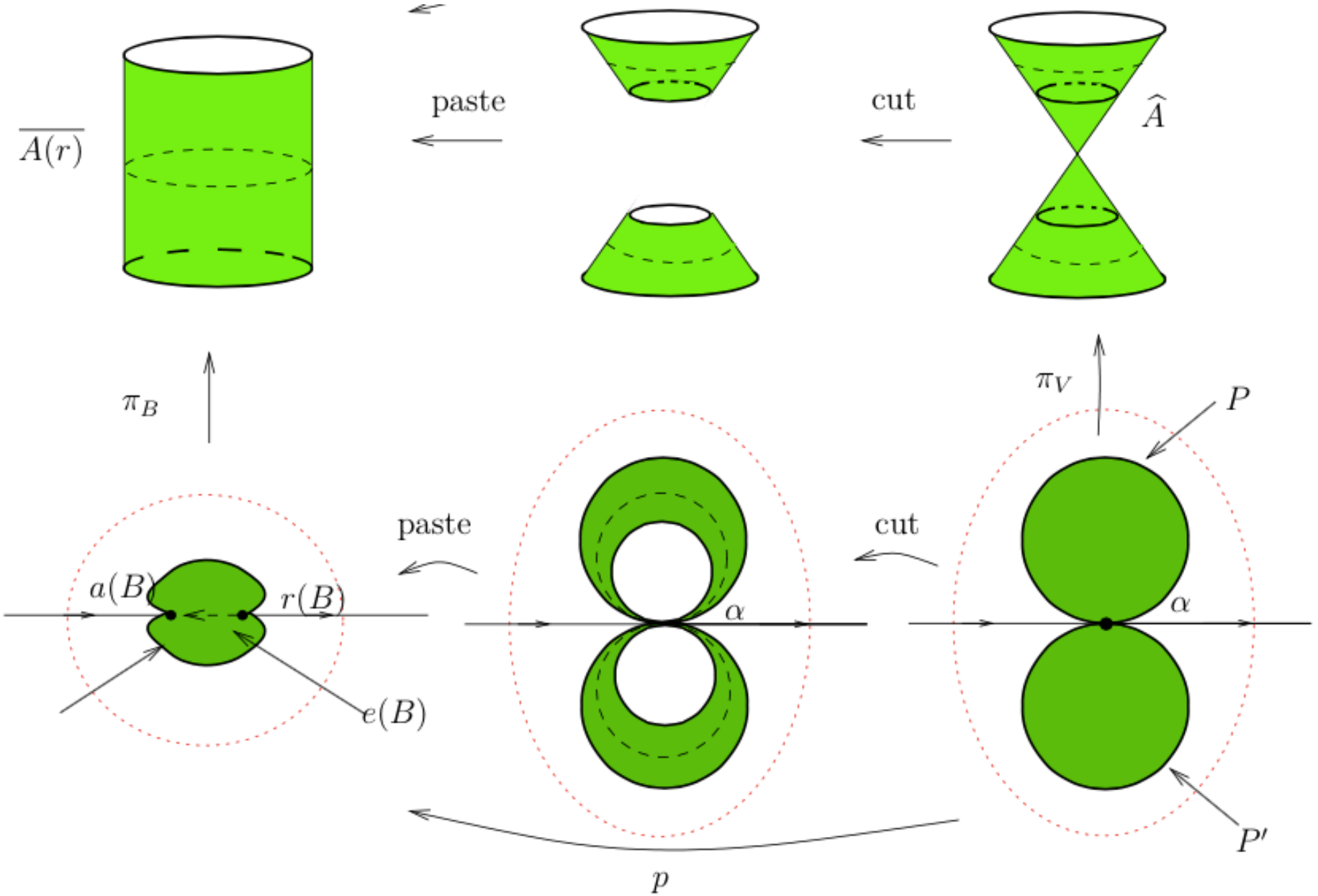
Extra difficulties:

Surgery on the space and the map

The vertical convergence (DO need Thurston's theorem).

(The horizontal one is identical.)

The plumbing surgery



Thank you !