Hurwicz numbers and intersection numbers of algebraic curves,
the polynomial case
following Zvonkin-Lando and Buff-Epstein-Koch
talk at Angers, 11 march 2014

$$\mathcal{LL} : f(z) \mapsto D(t) := \prod_{v \text{ crv of } f, \text{ counted with mul.}} (t - v). \quad m = d - 1.$$

$$\begin{array}{ccc} \{z^d + O(z^{d-2}), \text{crv-centered}\} & \xrightarrow{\mathcal{LL}} & \{t^{d-1} + O(t^{d-3})\} \\ \uparrow \alpha \text{ (generic } d-1! : 1) & & \uparrow \tau \text{ (generic } d-1! : 1) \\ \mathbb{C}^m \supset \{\mathbf{c} \text{ ordered-centered}\} & \xrightarrow{H} & \{\mathbf{v} \text{ ordered-centered}\} \subset \mathbb{C}^m \end{array}$$

Theorem

- H is polynomial, homogenous, of degree d per coordinate, proper, open, surjective, of multiplicity $d^{\dim} = d^{d-2}$.
- Let $\mathcal{J}$ be a partition of $I = \{1, \cdots, m\}$, each part J is equipped with the weight $m_J = |J|$.
- Stratum $L_{\mathcal{J}} = \{\mathbf{c} \text{ constant on each part}\}$, the restriction $H_{\mathcal{J}}$ satisfies $H_{\mathcal{J}}(L_{\mathcal{J}}) = L_{\mathcal{J}}$, multiplicity$=d^{\dim} = d^{|\mathcal{J}|-1}$,
- critical locus $= \Delta_{\mathcal{J}}$, of multiplicity $m_K + 1 = m_{J_1} + m_{J_2} + 1$ on a contraction $K = \{J_1, J_2\}$.
- $cv(H_{\mathcal{J}}) = \Delta_{\mathcal{J}}$, $co - cr(H_{\mathcal{J}}) = \{\mathbf{c} \in L_{\mathcal{J}} - \Delta_{\mathcal{J}}, H(\mathbf{c}) \in \Delta_{\mathcal{J}}\}$.

Degree 4

$$H : \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} (average = 0) \mapsto \Big(P' \mapsto (\text{crv-centered }) P \mapsto \begin{pmatrix} P(c_1) \\ P(c_2) \\ P(c_3) \end{pmatrix} = \Big)$$

$$- \begin{pmatrix} c_1^4 + 2c_1(c_1c_2c_3) \\ c_2^4 + 2c_2(c_1c_2c_3) \\ c_3^4 + 2c_3(c_1c_2c_3) \end{pmatrix} + \frac{1}{3} \begin{pmatrix} c_1^4 + c_2^4 + c_3^4 \\ c_1^4 + c_2^4 + c_3^4 \\ c_1^4 + c_2^4 + c_3^4 \end{pmatrix} \text{ (average=0)}$$

$$\mathcal{LL}(z^4 + p_2z^2 + p_3z + p_4) =$$

$$\text{discriminant}(z^4 + p_2z^2 + p_3z + p_4 - t, z)/(-256) =$$

$$t^3 + \frac{1}{2}(p_2^2 - 6p_4)t^2 + \frac{1}{16}(p_2^4 + 9p_2p_3^2 - 16p_2^2p_4 + 48p_4^2)t +$$

$$\frac{1}{4^4}(4p_2^3p_3^2 + 27p_3^4 - 16p_2^4p_4 - 144p_2p_3^2p_4 + 128p_2^2p_4^2 - 256p_4^3)$$

H is homogenous and proper.

$I =$ a finite index set of cardinality $m \geq 1$.

$$E = \{c : I \to \mathbb{C} \text{ with 0 average}\}.$$

$$\Phi_c(w) = (m+1)\prod_{i\in I}(w - c_i),\ P_c(t) = \frac{1}{m}\sum_{j\in I}\int_{c_j}^{t} \Phi_c(w)\,\mathrm{d}w$$

and $H(c) = P_c \circ c$. induces : $\mathbb{C}^m \to \mathbb{C}^m$ poly. of rational coefficients.

$$P_{\lambda c}(\lambda t) = \frac{m+1}{m}\sum_{j\in I}\int_{\lambda c_j}^{\lambda t}\prod_{i\in I}(w - \lambda c_i)\,\mathrm{d}w$$

$$\underset{w=\lambda v}{=} \frac{m+1}{m}\sum_{j\in I}\int_{c_j}^{t}\prod_{i\in I}(\lambda v - \lambda c_i)\,\mathrm{d}(\lambda v) = \lambda^{m+1}P_c(t),$$

so $H(\lambda c) = \lambda^d H(c)$, i.e. homogeneous of degree d.

As $H^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ (one counts the preimages of the unique crv), H is proper. By Bezout H is of multiplicity d^{d-2}. q.e.d.

Define $\Delta := \{c \in E : \exists\, i \neq j \text{ with } c_i = c_j\}$
$= \{c : I \to \mathbb{C} \text{ non injective, average=0}\}$.

Theorem.

Δ is a stratified space, each stratum is a vector subspace, $H(\Delta) = \Delta$ and Δ coincides with the critical locus of H.

Stratification. Denote by $\mathrm{Part}(I)$ the set of partitions of I. Let $\mathcal{I} \in \mathrm{Part}(I)$ be the singleton partition of I : that is

$$\mathcal{I} := \big\{\{i\} : i \in I\big\}.$$

Given $\mathcal{J} \in \mathrm{Part}(I)$, let $L_{\mathcal{J}} \subseteq E$ be the linear space

$L_{\mathcal{J}} := \{c \in E : c \text{ is constant on each element of } \mathcal{J}\}$, dim= $|\mathcal{J}|-1$.

Note that $E = L_{\mathcal{I}}$. We say that $\mathcal{K} \in \mathrm{Part}(I)$ is a contraction of $\mathcal{J} \in \mathrm{Part}(I)$ if all elements of $\mathcal{K}$ are unions of elements of $\mathcal{J}$. We denote this as $\mathcal{K} \preceq \mathcal{J}$, and if $\mathcal{K} \neq \mathcal{J}$, we use the notation $\mathcal{K} \prec \mathcal{J}$. When $\mathcal{K}$ is a contraction of $\mathcal{J}$, the linear space $L_{\mathcal{K}}$ is contained in $L_{\mathcal{J}}$, and the codimension of $L_{\mathcal{K}}$ in $L_{\mathcal{J}}$ is $|\mathcal{J}| - |\mathcal{K}|$.

The stratification of Δ is given by

$$\Delta = \bigcup_{\mathcal{J} \prec \mathcal{I}} L_{\mathcal{J}}.$$

If $c \in L_{\mathcal{J}}$, then $H(c) = P_c \circ c$ is constant on each element of $\mathcal{J}$. Thus $H(L_{\mathcal{J}}) \subseteq L_{\mathcal{J}}$. For $\mathcal{J} \in \mathrm{Part}(I)$, define the set

$$\Delta_{\mathcal{J}} := \bigcup_{\mathcal{K} \prec \mathcal{J}} L_{\mathcal{K}}.$$

Clearly

- $H(L_{\mathcal{J}}) = L_{\mathcal{J}}$ (since $H(L_{\mathcal{J}}) \subset L_{\mathcal{J}}$ and H is proper, so surjective)
- $H(\Delta_{\mathcal{J}}) = \Delta_{\mathcal{J}}$ (since for each $\mathcal{K} \prec \mathcal{J}$, $H(L_{\mathcal{K}}) \subset L_{\mathcal{K}}$ and H is proper, so surjective)
  We now show that
- (critical locus of $H : L_{\mathcal{J}} \to L_{\mathcal{J}}$)$= \Delta_{\mathcal{J}}$.

Let $c \in L_{\mathcal{J}}$. Let $b : \mathcal{J} \to \mathbb{C}$ so that b_J is the common value c_i for $i \in J$, and $J \in \mathcal{J}$.
More generally let $\mathcal{J}$ be a finite set and each element J is equipped with a positive integer β_J as weight (i.e. $|J|$). Set $m = \sum_{J \in \mathcal{J}} \beta_J$.

Let $b : \mathcal{J} \to \mathbb{C}, J \mapsto b_J$ be any function.

$$\text{Set } \Phi_b(y) := (m+1) \prod_{J \in \mathcal{J}} (y - b_J)^{\beta_J}$$

$$P_b(z) := \frac{1}{m} \sum_{M \in \mathcal{J}} \beta_M \int_{b_M}^{z} \Phi_b(y)\, \mathrm{d}y =: \operatorname*{average}_{M} \int_{b_M}^{z} \Phi_b(y)\, \mathrm{d}y$$

Lemma 1. $Jac(H)_{JK} = \operatorname*{average}_{M} \int_{b_M}^{b_J} \omega_K$, with $\omega_K = \dfrac{\partial}{\partial b_K} \Phi_b(y)$.

Proof. The J-th entry is $H_J(b) = P_b(b_J) = \operatorname*{average}_{M} \int_{b_M}^{b_J} \Phi_b(y)\, \mathrm{d}y$.

Using $\Phi_b(b_K) = 0$, one gets

$$\frac{\partial H_J}{\partial b_K} = \underset{M}{\text{average}} \int_{b_M}^{b_J} \frac{\partial}{\partial b_K} \Phi_b(y)\, \mathrm{d}y = \underset{M}{\text{average}} \int_{b_M}^{b_J} \omega_K\, \mathrm{d}y.$$

Lemma 2. If $b_J \neq b_K$ for any $J \neq K$, then $KerJac(H : \mathbb{C}^{\mathcal{J}} \to L_{\mathcal{J}}) = Span[1, 1, \cdots, 1]$.

Note that $L_{\mathcal{J}}$ is a hyperplan in $\mathbb{C}^{\mathcal{J}}$ transversal to the vector $(1, 1, \cdots, 1)$.

Proof. On one hand,

$$\sum_K \frac{\partial H_J}{\partial b_K} = \underset{M}{\text{average}} \int_{b_M}^{b_J} \sum_K \omega_K\, \mathrm{d}y$$

$$= \underset{M}{\text{average}} \int_{b_M}^{b_J} -\Phi_b'(y)\, \mathrm{d}y$$

$$= \underset{M}{\text{average}}(\Phi_b(b_M) - \Phi_b(b_J)) = 0$$

On the other hand, let $v : \mathcal{J} \to \mathbb{C}$ be a non-zero vector in the kernel.

Claim. If $b_K \neq b_J$ for $J \neq K$, the ω_K's are linearly independent.

Proof. Notice that ω_K has a zero of order $\beta_K - 1$ at b_K and a zero of order β_J for any $J \neq K$. A linear combination $\sum u_K \omega_K$ has its β_K'th derivative evaluated at b_K equal to u_K times a non-zero complex number (recall that $b_J \neq b_K$ if $J \neq K$). So $\sum u_K \omega_K \equiv 0$ iff $u_K = 0$ for every K.

Now set $\omega = \sum_K v_K \omega_K$ and $g(z) = \operatorname*{average}_M \int_{b_M}^{z} \omega(y)\, \mathrm{d}y$. Then g is a non-zero polynomial, and g is the unique primitive of ω whose values at b_J, $J \in \mathcal{J}$ have a zero as weighted average. Then, for any $J \in \mathcal{J}$,

$$0 = \sum_K v_K \frac{\partial H_J}{\partial b_K} = \operatorname*{average}_M \int_{b_M}^{b_J} \sum_K v_K \omega_K \, \mathrm{d}y$$

$$= \operatorname*{average}_M \int_{b_M}^{b_J} \omega(y)\, \mathrm{d}y = g(b_J)$$

For each pair K, J, the polynomial $\omega_K(y)$ has a factor $(y - b_J)^s$ with $s \geq \beta_J - 1$. So $\omega(y)$ has a factor $(y - b_J)^s$ with $s \geq \beta_J - 1$. As $g'(z) = \omega(z)$ and $g(b_J) = 0$, we know that b_J is a zero of g with multiplicity $t_J \geq \beta_J$. Now each ω_K is a polynomial of degree $m - 1$. So $\deg(\omega) \leq m - 1$, and $\deg(g) \leq m$. Recall that $\sum_J \beta_J = m$. As $b_J \neq b_K$ for $J \neq K$, we know that $\deg(g) \geq \sum_J t_J \geq \sum_J \beta_J = m$. So all the inequalities in the present paragraph are in fact equalities and $g = a\Phi_b$. So

$$0 \equiv g(z) - a\Phi_b(z) = \operatorname*{average}_M \int_{b_M}^{z} \sum_K (v_K - a)\omega_K(y)\, \mathrm{d}y$$

and $0 \equiv g'(z) - a\Phi_b'(z)$. By linear independency, $v_K \equiv a$.

Conclusion. So H restricted to the hyperplane of $\mathbb{C}^{\mathcal{J}}$ $L_{\mathcal{J}} = \{b : \mathcal{J} \to \mathbb{C}, \sum_J \beta_J b_J = 0\}$ minus the small diagonals $\Delta_{\mathcal{J}}$ is locally invertible. Moreover

$$\Delta_{\mathcal{J}} = \bigcup_{\substack{J_1, J_2 \in \mathcal{J} \\ J_1 \neq J_2}} L_{\mathcal{K}(J_1, J_2)}$$

and $L_{\mathcal{K}(J_1,J_2)}$ is a critical component of $H : L_{\mathcal{J}} \to L_{\mathcal{J}}$ with multiplicity $|J_1| + |J_2| + 1$.

A **passport** is a pair of partitions $\mathcal{V} \preceq \mathcal{C}$ of I such that each part of $\mathcal{V}$ has at most $m+1$ preimages (each part C of $\mathcal{C}$ counts as $|C|+1$ preimages). Set $\mathcal{J}^* := L_{\mathcal{J}} - \Delta_{\mathcal{J}}$ **the exact stratum**. We want to count the number of polynomials in a given passport, computed by

$$\mu(H_{\mathcal{V}\preceq\mathcal{C}}) := \#H^{-1}(v) \cap \mathcal{C}^* = \#H_{\mathcal{C}}^{-1}(v) \cap \mathcal{C}^*, \quad v \in \mathcal{V}^*$$

and its unordered version $\mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}})$. We have

$$\mu(H_{\mathcal{V}\preceq\mathcal{C}}) \cdot \mu(\tau) \overset{b}{=} \mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}}) \cdot \mu(\alpha)$$

with $\mu(\tau) = |aut(\mathcal{V})|$, $\mu(\alpha) = |aut(\mathcal{V} \preceq \mathcal{C})| \cdot \prod_{V\in\mathcal{V}} |aut(\mathcal{C} \cap \{V\})|$.

- $|aut(\mathcal{V})|$ counts the symmetry in the shapes of the Young diagrams after turning each diagram into L-shape.
- $|aut(\mathcal{V} \preceq \mathcal{C})|$ counts the symmetry of the pairing $\mathcal{V} \preceq \mathcal{C}$ (i.e. of the shapes of the Young diagrams).
- $\prod_{V\in\mathcal{V}} |aut(\mathcal{C} \cap \{V\})|$ counts the symmetry of critical points inside the Young diagram for each V.

We need also the notion of the degree of the variety $H_{\mathcal{C}}^{-1}(L_{\mathcal{V}})$.

Lemma. $\deg H_{\mathcal{C}}^{-1}(L_{\mathcal{V}}) \cdot d^{|\mathcal{V}|-1} \stackrel{a}{=} \mu(H_{\mathcal{V}\preceq\mathcal{C}}) \stackrel{b}{=} \dfrac{\mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}}) \cdot \mu(\alpha)}{\mu(\tau)}$

Proof. Equality b is obvious. Equality a follows from $\deg H_{\mathcal{C}}^{-1}(L_{\mathcal{V}}) \cdot d^{\dim H_{\mathcal{C}}^{-1}(L_{\mathcal{V}})} = \deg \mathcal{V} \cdot \mu(H_{\mathcal{V}\preceq\mathcal{C}})$ with $\deg \mathcal{V} = 1$ and $\dim H_{\mathcal{C}}^{-1}(L_{\mathcal{V}}) = \dim L_{\mathcal{V}} = |\mathcal{V}| - 1$. □

Now $H^{-1}(L_{\mathcal{V}}) \cap L_{\mathcal{C}} = \bigcap_i H^{-1}(L_{\mathcal{V}_i}) \cap L_{\mathcal{C}_i}$ for $(\mathcal{V}_i \preceq \mathcal{C}_i)_i$ the collection of the primitive partitions in $\mathcal{V}$ (only one non-trivial partition in $\mathcal{V}_i$). One gets then $\mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}})$ by

$$\begin{array}{ccccc} \deg H^{-1}(L_{\mathcal{V}_i}) \cap L_{\mathcal{C}_i} & \stackrel{a}{\longleftarrow} & \mu(H_{\mathcal{V}_i\preceq\mathcal{C}_i}) & \stackrel{b}{\longleftarrow} & \mu(\mathcal{LL}_{\mathcal{V}_i\preceq\mathcal{C}_i}) \\ \prod_i \downarrow & & & & \\ \deg H^{-1}(L_{\mathcal{V}}) \cap L_{\mathcal{C}} & \stackrel{a}{\longrightarrow} & \mu(H_{\mathcal{V}\preceq\mathcal{C}}) & \stackrel{b}{\longrightarrow} & \mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}}). \end{array}$$

It remains to compute directly $\mu(H_{\mathcal{V}\preceq\mathcal{C}})$ or $\mu(\mathcal{LL}_{\mathcal{V}\preceq\mathcal{C}})$ for every primitive passport $\mathcal{V} \preceq \mathcal{C}$. See [ZL]. Here we will compute them for L-shape passports and Maxwell passports.

Case 1. $\mathcal{V} = \mathcal{C}$ (i.e. each crv has a unique critical preimage, the Young's diagram is a collection of L-shapes). In this case
$|aut(\mathcal{V})| = |aut(\mathcal{V} \preceq \mathcal{C})|$ and
$\mu(\mathcal{LL}_{\mathcal{C}}) = \mu(H_{\mathcal{C}=\mathcal{C}}) = \text{multiplicity}(H_{\mathcal{C}}) = d^{|\mathcal{C}|-1} = d^{|\mathcal{V}|-1}$.
Notice that only the number of critical values is involved.
In the largest stratum,
$\mu(\mathcal{LL}_{\mathcal{I}}) = \mu(H_{\mathcal{I}=\mathcal{I}}) = \text{multiplicity}(H_{\mathcal{I}}) = d^{m-1} = d^{d-2}$.

Case 2. $\mathcal{V} = \mathcal{K}(C_1, C_2)$, so $|\mathcal{V}| = |\mathcal{C}| - 1$ (one critical value has two critical preimages, the other critical value has only one critical preimage). Then
$\mu(H_{\mathcal{V}\preceq\mathcal{C}}) + (\#H^{-1}(v) \cap \mathcal{V}^*)(|C_1| + |C_2| + 1) = \text{multipl}(H_{\mathcal{C}}) = d^{|\mathcal{C}|-1}$
So $\mu(H_{\mathcal{K}(C_1,C_2)\preceq\mathcal{C}})$
$= -d^{|\mathcal{V}|-1}(|C_1| + |C_2| + 1) + d^{|\mathcal{C}|-1} = d^{|\mathcal{V}|-1}(d - (|C_1| + |C_2| + 1))$

• $|aut(\mathcal{V})|$ counts the symmetry in the shapes of the Young diagrams after turning each diagram into L-shape. The symmetry involving $K(C_1, C_2)$ is equal to k if $|C_1| + |C_2|$ is equal to the cardinality of $k-1$ other parts of $\mathcal{C}$. So $\mu(\tau) = k \cdot |aut(\mathcal{C} - \{C_1, C_2\})|$.

• $|aut(\mathcal{V} \preceq \mathcal{C})|$ counts the symmetry of the shapes of the Young diagrams. Clearly the part $K(C_1, C_2)$ can not be permuted with the other parts. So $|aut(\mathcal{V} \preceq \mathcal{C})| = |aut(\mathcal{C} - \{C_1, C_2\})|$.

• $\prod_{V \in \mathcal{V}} |aut(\mathcal{C} \cap \{V\})|$ counts the symmetry of critical points inside the Young diagram for each V. It returns a 1 if $V \neq K(C_1, C_2)$, and gives 1 if $V = K(C_1, C_2)$ but $|C_1| \neq |C_2|$, or 2 ! if $V = K(C_1, C_2)$ but $|C_1| = |C_2|$. So

$$\mu(\mathcal{LL}) = \begin{cases} d^{|\mathcal{V}|-1}\Big(d - (|C_1| + |C_2| + 1)\Big) \cdot \dfrac{k}{2} & \text{if } |C_1| = |C_2| \\ d^{|\mathcal{V}|-1}\Big(d - (|C_1| + |C_2| + 1)\Big) \cdot k & \text{if } |C_1| \neq |C_2| \end{cases},$$

where $k-1$ is the number of other parts of $\mathcal{C}$ with cardinality $|C_1| + |C_2|$.

In case of a simple Maxwell passport, i.e 1 critical value has two simple critical points, all other crv are simple, we have $k = 1$, $|\mathcal{V}| = d - 2$, so $\mu(\mathcal{LL}) = \dfrac{d^{d-3}(d-3)}{2}$.

In case of a simple primitive passport such that all but one crv are simple, and the remaining critical value has two distinct critical points, $k = 1$ and

$$\mu(\mathcal{LL}) = \begin{cases} \dfrac{d^{|\mathcal{V}|-1}(d-(|C_1|+|C_2|+1))}{2} & \text{if } |C_1| = |C_2| \\ d^{|\mathcal{V}|-1}(d-(|C_1|+|C_2|+1)) & \text{if } |C_1| \neq |C_2| \end{cases}.$$

Case 3. $\mathcal{V}$ is a contraction of 1,2,3, denoted by (123). It's case 1 stratum to (12)3, 1(23) and (13)2, in which we know the numbers $H^{-1}(v)$ by case 1. Then
$\mu(H_{\mathcal{V}\prec\mathcal{C}}) = \deg(H_{\mathcal{C}}) - (deg(H_{\mathcal{V}}) \cdot multi + \mu(H_{\mathcal{V}\prec(12)3}) \cdot multi + ...)$
Case s. $\mathcal{V}$ is a contraction of $1, 2, 3, \cdots s$. Use the results at previous levels, calculate the multiplicities...
This is a recurrent way to compute everything. But a closed form as in [ZL] is better.

To do :

Decide which stratum is at the boundary of which one (the cell decomposition structure)

Compute the monodromy in each stratum.

IMG

Generalize H to other dynamical cases : periodic with permutations, preperiodic cases. Compute the number of fixed points as in BEK.

Is each strict stratum connected (hard question) ?

Rational map case ?