Rays landing at Misiurewitz and parabolic parameters

CUI Guizhen and TAN Lei

Academy of Mathematics and Systems Science, CAS
Universite d'Angers

Holbaek, September 2014

Accessing Misiurewicz polynomials

Douady-Hubbard (Kiwi for higher degree)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

We give a new (intrinsic) proof that works for more general settings.

Set $f_r : z \mapsto z^2 + c(r)$. Construct via surgery $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ uniformly on } n \\ & & r & \longrightarrow & 0 \end{array}$$

Accessing Misiurewicz polynomials

Douady-Hubbard (Kiwi for higher degree)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

We give a new (intrinsic) proof that works for more general settings.

Set $f_r : z \mapsto z^2 + c(r)$. Construct via surgery $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ uniformly on } n \\ & & r & \longrightarrow & 0 \end{array}$$

Accessing Misiurewicz polynomials

Douady-Hubbard (Kiwi for higher degree)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

We give a new (intrinsic) proof that works for more general settings.

Set $f_r : z \mapsto z^2 + c(r)$. Construct via surgery $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & f_r & \xrightarrow{\text{so}} & g & \\ \uparrow & \uparrow & & \| & \\ n & f_{r,n} & \longrightarrow & g & \text{uniformly on } n \\ & r & \longrightarrow & 0 & \end{array}$$

Accessing Misiurewicz polynomials

Douady-Hubbard (Kiwi for higher degree)

For $g(z) = z^2 + c$ such that 0 is strictly preperiodic, and θ such that the external ray $R_c(\theta)$ lands at c, there is a path $c(r)_{r>0}$ such that $c(r) \in R_{c(r)}(\theta)$ with potential e^r and $c(r) \to c$ as $r \to 0$.

We give a new (intrinsic) proof that works for more general settings.

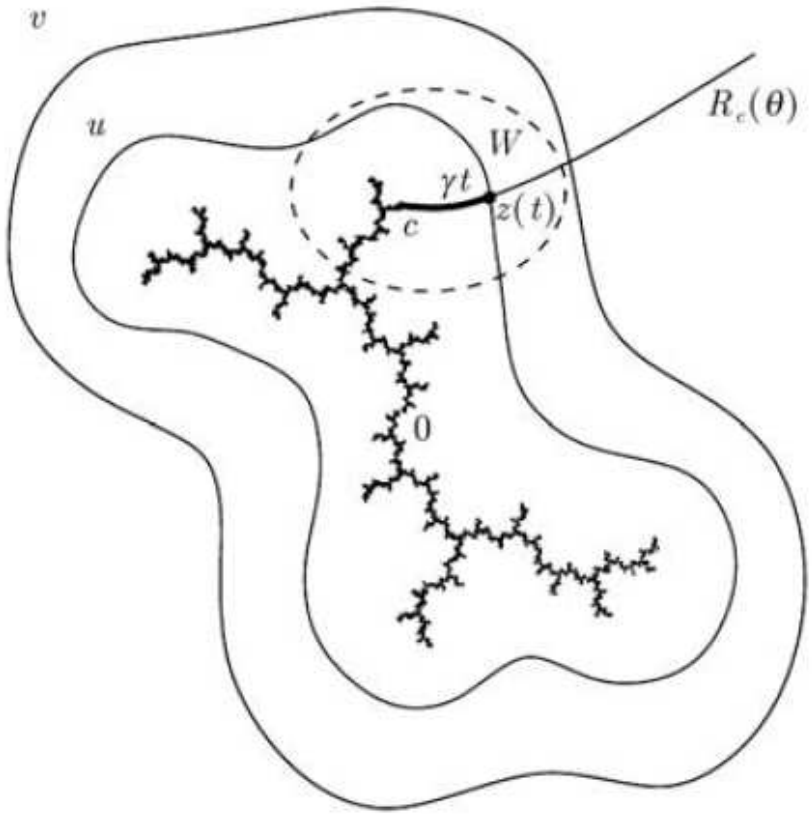
Set $f_r : z \mapsto z^2 + c(r)$. Construct via surgery $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & f_r & \xrightarrow{\text{so}} & g & \\ \uparrow & \uparrow & & \| & \\ n & f_{r,n} & \longrightarrow & g & \text{uniformly on } n \\ & r & \longrightarrow & 0 & \end{array}$$

The vertical convergence is due to Thurston's algorithm (no need of Thurston's theorem).

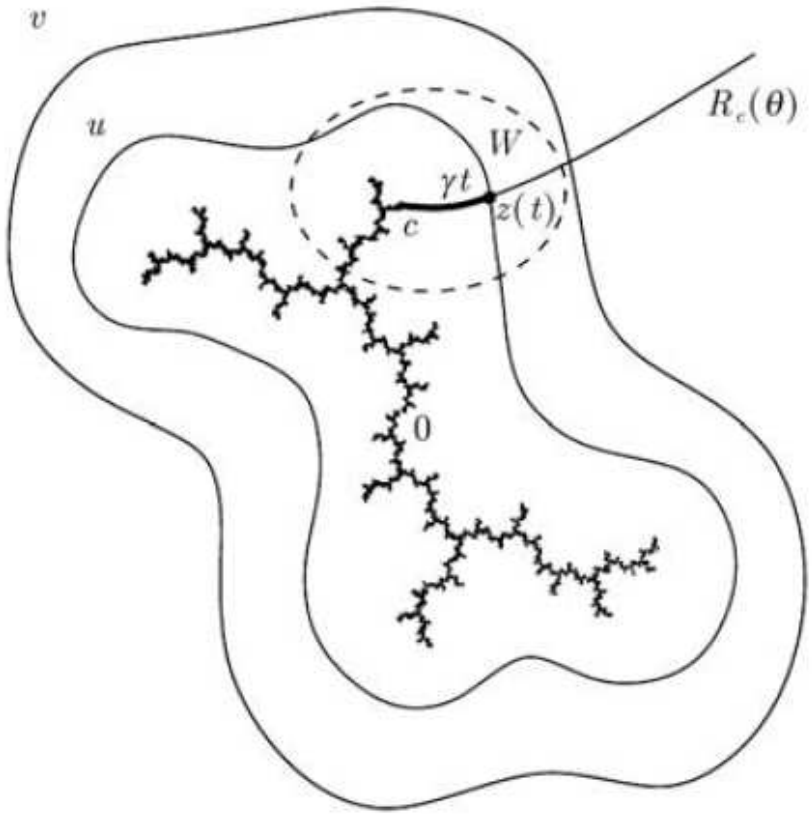
The horizontal one is due to a distortion control on pseudo-conjugacies.

Surgery on g



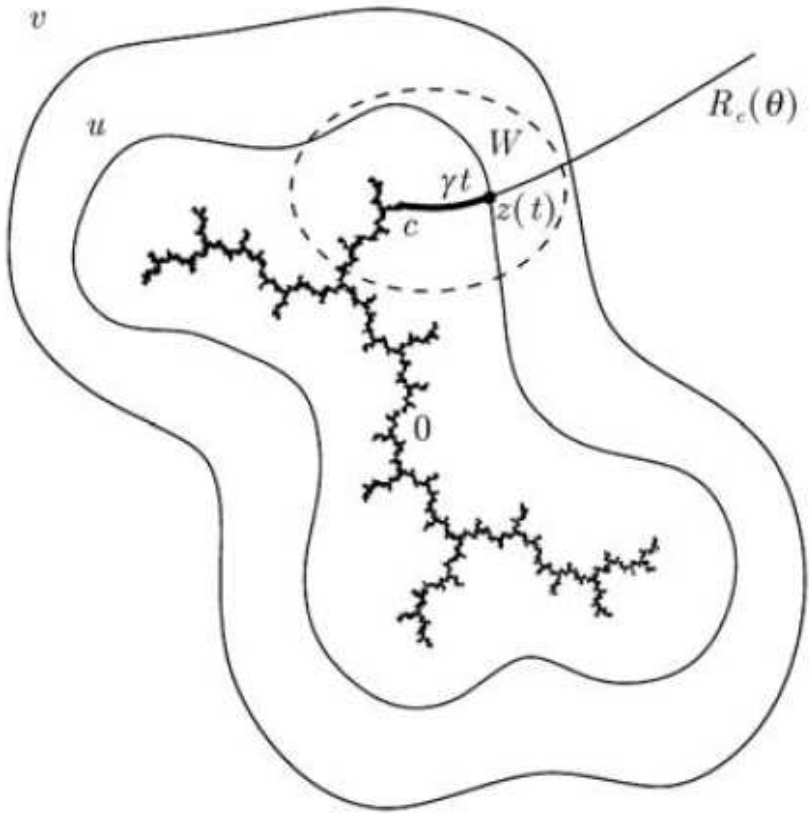
1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. Pullback the standard complex structure infinitely many times to get an F_r-invariant structure and integrate using Measurable Riemann Mapping Theorem.

Surgery on g



1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. Pullback the standard complex structure infinitely many times to get an F_r-invariant structure and integrate using Measurable Riemann Mapping Theorem.

Surgery on g



1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. Pullback the standard complex structure infinitely many times to get an F_r-invariant structure and integrate using Measurable Riemann Mapping Theorem (this does not work!).

1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ of 0 with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. pullback step by step the standard complex structure and integrate, to get

$$\begin{array}{ccc} & \xrightarrow{p_{r,n}} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow[p_{r,n-1}]{} & \end{array}$$

with $p_{r,n}$ univalent outside

$$\bigcup_{j=0\dots n} F_r^{-j} U_0(r) = \bigcup_{j=0\dots n} g^{-j} U_0(r).$$

3. So for $V =$ exterior of a large equipotential,

$$g = F_r \Big| \begin{array}{ccc} V & \xrightarrow{p_{r,n},\text{univalent}} & \\ & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1},\text{univalent}]{} & \end{array}$$

4. Then

$$\begin{array}{ccc} \infty & f_r & \\ \uparrow & \uparrow & \\ n & f_{r,n} & \longrightarrow \quad g \text{ unif. on } n, \\ & r & \longrightarrow \quad 0 \end{array}$$

with $f_r : z \mapsto z^2 + c(r)$ and $c(r) \in R_{c(r)}(\theta)$ (of potential $e^{t(r)}$).

1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ of 0 with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. pullback step by step the standard complex structure and integrate, to get

$$\begin{array}{ccc} & \xrightarrow{p_{r,n}} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow[p_{r,n-1}]{} & \end{array} \quad \text{with } p_{r,n} \text{ univalent outside} \quad \bigcup_{j=0\dots n} F_r^{-j} U_0(r) = \bigcup_{j=0\dots n} g^{-j} U_0(r).$$

3. So for $V =$ exterior of a large equipotential,

$$g = F_r \Big\downarrow \begin{array}{ccc} V & \xrightarrow{p_{r,n},\text{univalent}} & \\ & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1},\text{univalent}]{} & \end{array}$$

4. Then

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ unif. on } n, \\ & & r & \longrightarrow & 0 \end{array}$$

with $f_r : z \mapsto z^2 + c(r)$ and $c(r) \in R_{c(r)}(\theta)$ (of potential $e^{t(r)}$).

1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ of 0 with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. pullback step by step the standard complex structure and integrate, to get

$$\begin{array}{ccc} & \xrightarrow{p_{r,n}} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow[p_{r,n-1}]{} & \end{array}$$

with $p_{r,n}$ univalent outside

$$\bigcup_{j=0\dots n} F_r^{-j} U_0(r) = \bigcup_{j=0\dots n} g^{-j} U_0(r).$$

3. So for $V =$ exterior of a large equipotential,

$$\begin{array}{ccc} V & \xrightarrow{p_{r,n},\text{univalent}} & \\ g = F_r \downarrow & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1},\text{univalent}]{} & \end{array}$$

4. Then

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ unif. on } n, \\ & & r & \longrightarrow & 0 \end{array}$$

with $f_r : z \mapsto z^2 + c(r)$ and $c(r) \in R_{c(r)}(\theta)$ (of potential $e^{t(r)}$).

1. Replace g by a quasi-regular $F_r = g$ except on a small $U_0(r)$ of 0 with $F_r(0) =$ the point on $R_c(\theta)$ landing at c, of potential $e^{t(r)}$.
2. pullback step by step the standard complex structure and integrate, to get

$$\begin{array}{ccc} & \xrightarrow{p_{r,n}} & \\ F_r \downarrow & & \downarrow f_{r,n} \\ & \xrightarrow[p_{r,n-1}]{} & \end{array} \quad \text{with } p_{r,n} \text{ univalent outside} \quad \bigcup_{j=0\dots n} F_r^{-j} U_0(r) = \bigcup_{j=0\dots n} g^{-j} U_0(r).$$

3. So for $V =$ exterior of a large equipotential,

$$g = F_r \Big| \begin{array}{ccc} V & \xrightarrow{p_{r,n},\text{univalent}} & \\ & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1},\text{univalent}]{} & \end{array}$$

4. Then

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ unif. on } n, \\ & & r & \longrightarrow & 0 \end{array}$$

with $f_r : z \mapsto z^2 + c(r)$ and $c(r) \in R_{c(r)}(\theta)$ (of potential $e^{t(r)}$).

Horizontal convergence: use $p_{r,n}$

Let: g a rational map post-cr. finite on $\mathcal{J}ulia$;

X_0 a finite collection of cycles in $\mathcal{J}$; $\{U_x$ disc-neighborhood$\}_{x\in X_0}$;

$X =$ grand orbit of X_0; (by pullback) $\{U_y\}_{y\in X\smallsetminus X_0}$;

For $0<r<1$, $U_x(r)$ is of radius r (in the uniformized coordinate), $U_y(r)$...

Theorem. Let $V \subset\subset \mathcal{F}atou$. $\forall\ n\geq 1$

a) Let $p_{r,n}$ normalized univalent outside $\bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}$. Then $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

b) Assume $\begin{cases} V\text{: } g(V)\subset V \text{ and} \\ f_{r,n} \text{ same degree} \\ \text{and } p_{r,n} \text{ s.t.} \end{cases}$ $\begin{array}{ccc} V & \xrightarrow{p_{r,n}} & \\ g\downarrow & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1}]{} & \end{array}$

Then $f_{r,n} \underset{r\to 0}{\longrightarrow} g$ uniformly on n.

Horizontal convergence: use $p_{r,n}$

Let: g a rational map post-cr. finite on $\mathcal{J}ulia$;

X_0 a finite collection of cycles in $\mathcal{J}$; $\{U_x$ disc-neighborhood$\}_{x\in X_0}$;

$X =$ grand orbit of X_0; (by pullback) $\{U_y\}_{y\in X\smallsetminus X_0}$;

For $0<r<1$, $U_x(r)$ is of radius r (in the uniformized coordinate), $U_y(r)$...

Theorem. Let $V \subset\subset \mathcal{F}atou$. $\forall\ n\geq 1$

a) Let $p_{r,n}$ normalized univalent outside $\bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}$. Then $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

b) Assume $\begin{cases} V\text{: } g(V)\subset V \text{ and} \\ f_{r,n} \text{ same degree} \\ \text{and } p_{r,n} \text{ s.t.} \end{cases}$ $\begin{array}{ccc} V & \xrightarrow{p_{r,n}} & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_{r,n}} \\ V & \xrightarrow[p_{r,n-1}]{} & \end{array}$

Then $f_{r,n} \underset{r\to 0}{\longrightarrow} g$ uniformly on n.

Horizontal convergence: use $p_{r,n}$

Let: g a rational map post-cr. finite on $\mathcal{J}ulia$;

X_0 a finite collection of cycles in $\mathcal{J}$; $\{U_x$ disc-neighborhood$\}_{x\in X_0}$;

$X =$ grand orbit of X_0; (by pullback) $\{U_y\}_{y\in X\smallsetminus X_0}$;

For $0<r<1$, $U_x(r)$ is of radius r (in the uniformized coordinate), $U_y(r)$...

Theorem. Let $V \subset\subset \mathcal{F}atou$. $\forall\ n\geq 1$

a) Let $p_{r,n}$ normalized univalent outside $\bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)}$. Then $p_{r,n}|_V \underset{r\to 0}{\longrightarrow} id$ uniformly on n.

b) Assume $\begin{cases} V\text{: } g(V)\subset V \text{ and} \\ f_{r,n} \text{ same degree} \\ \text{and } p_{r,n} \text{ s.t.} \end{cases}$ $\begin{array}{c|c} V & \xrightarrow{p_{r,n}} \\ g\downarrow & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1}]{} \end{array}$

Then $f_{r,n} \underset{r\to 0}{\longrightarrow} g$ uniformly on n.

Theorem. Let $V \subset\subset \mathcal{F}atou$. $\forall\, n \geq 1$

a) Let $p_{r,n}$ normalized univalent outside $\bigcup_{y \in g^{-n}(X_0)} \overline{U_y(r)}$. Then $p_{r,n}|_V \underset{r \to 0}{\longrightarrow} id$ uniformly on n.

b) Assume $\begin{cases} V\text{: } g(V) \subset V \text{ and} \\ f_{r,n} \text{ same degree} \\ \text{and } p_{r,n} \text{ s.t.} \end{cases}$ $\begin{array}{ccc} V & \xrightarrow{p_{r,n}} & \\ g \downarrow & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1}]{} & \end{array}$

Then $f_{r,n} \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

Proof of a)$\Longrightarrow$b). $p_{r,n}|_V \underset{r \to 0}{\longrightarrow} id$ uniformly on n implies $f_{r,n}|_V \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

By control of degree, $f_{r,n}|_{\widehat{\mathbb{C}}} \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

In order to prove a), we control first the distortion of $p_{r,n}$ from being globally conformal (i.e. Möbius transformation). We then use the normalization to control the spherical distance.

Theorem. Let $V \subset\subset \mathcal{F}atou$. $\forall\, n \geq 1$

a) Let $p_{r,n}$ normalized univalent outside $\bigcup_{y \in g^{-n}(X_0)} \overline{U_y(r)}$. Then $p_{r,n}|_V \underset{r \to 0}{\longrightarrow} id$ uniformly on n.

b) Assume $\begin{cases} V\text{: } g(V) \subset V \text{ and} \\ f_{r,n} \text{ same degree} \\ \text{and } p_{r,n} \text{ s.t.} \end{cases}$ $\begin{array}{ccc} V & \xrightarrow{p_{r,n}} & \\ g \downarrow & & \downarrow f_{r,n} \\ V & \xrightarrow[p_{r,n-1}]{} & \end{array}$

Then $f_{r,n} \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

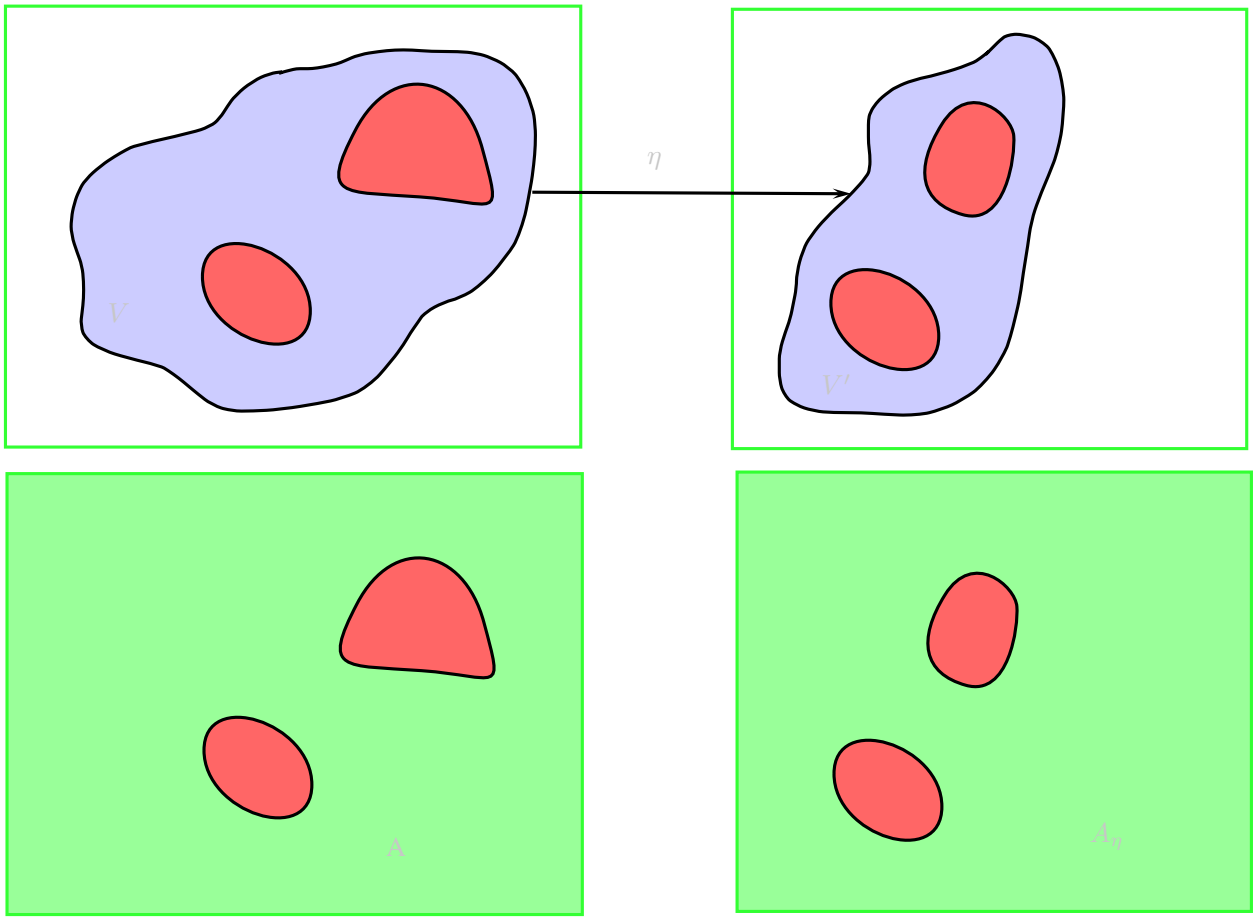
Proof of a)$\Longrightarrow$b). $p_{r,n}|_V \underset{r \to 0}{\longrightarrow} id$ uniformly on n implies $f_{r,n}|_V \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

By control of degree, $f_{r,n}|_{\widehat{\mathbb{C}}} \underset{r \to 0}{\longrightarrow} g$ uniformly on n.

In order to prove a), we control first the distortion of $p_{r,n}$ from being globally conformal (i.e. Möbius transformation). We then use the normalization to control the spherical distance.

To prove a): Distortion of univalent maps

Given $V \subset \widehat{\mathbb{C}}$ open and $\eta : V \xrightarrow{\text{univalent}} \widehat{\mathbb{C}}$,
how far is η being globally conformal?

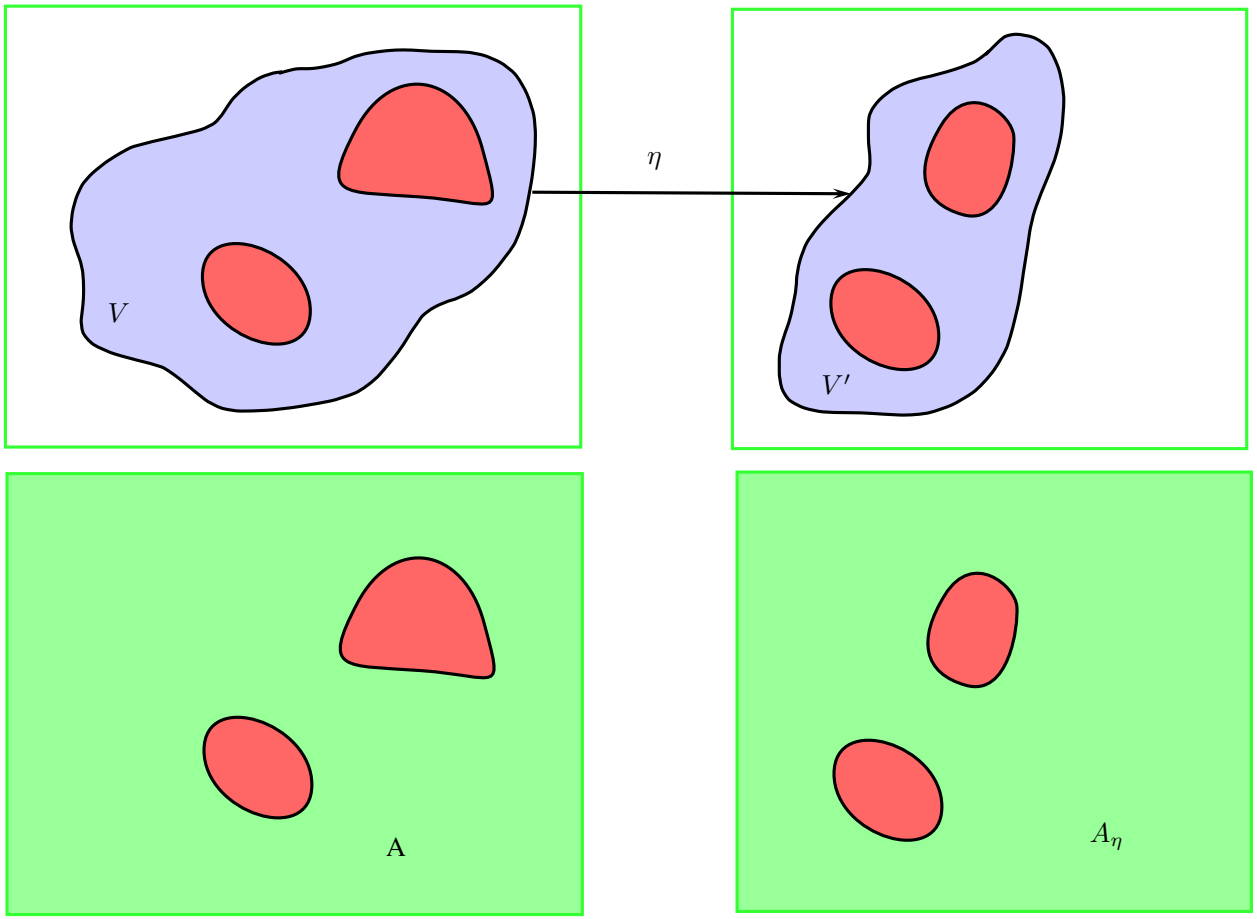


Define $\mathcal{D}(\eta, V) = \sup \left\{ |\mathrm{mod} A - \mathrm{mod} A_\eta| \;\middle|\; A \supset \mathbb{C} \setminus V \right\}$.

Example: $\mathcal{D}(z^2, \mathbb{H}) = \infty$.

To prove a): Distortion of univalent maps

Given $V \subset \widehat{\mathbb{C}}$ open and $\eta : V \xrightarrow{\text{univalent}} \widehat{\mathbb{C}}$,
how far is η being globally conformal?



Define $\mathcal{D}(\eta, V) = \sup\left\{ |\mathrm{mod}A - \mathrm{mod}A_\eta| \;\middle|\; A \supset \mathbb{C} \setminus V \right\}$.

Example: $\mathcal{D}(z^2, \mathbb{H}) = \infty$.

Conformal distortion of $p_{r,n}$

Let: g a rational map post-cr. finite on $\mathcal{J}$;

X_0 a finite collection of cycles in $\mathcal{J}$; $\{U_x$ disc-neighborhood$\}_{x\in X_0}$;

$X =$ grand orbit of X_0; (by pullback) $\{U_y\}_{y\in X\smallsetminus X_0}$;

For $0<r<1$, $U_x(r)$ is $D(0,r)$ (after uniformization), $U_y(r)$...

Theorem (conformal distortion of $p_{r,n}$)

Let $V \subset\subset \mathcal{F}_g$. Then $\exists\, C(r) \leq \infty$ with $C(r) \xrightarrow[r\searrow 0]{} 0$ such that

$$\forall\, n \geq 0,\ \forall\, p_{r,n} \text{ univalent outside } \bigcup_{y\in g^{-n}(X_0)} \overline{U_y(r)},\quad \mathcal{D}(p_{r,n}, V) \leq C(r).$$

Proof in 3 steps.

1. An extremal length estimate: A annulus.

$$\rho_0 = \textbf{extremal metric} \text{ if } \quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$$

$$\rho \text{ is } \textbf{length increasing} \text{ if } \quad \ell_{\text{loop}}(\rho) \quad \ell_{\text{arc}}(\rho)$$

Key lemma

Let $Z \subset A$ compact and φ univalent on $A \setminus Z$. Let A_φ bounded by $\varphi(\partial A)$. Then

$$|\text{mod}A_\varphi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0} \quad \forall\, \rho \text{ on } A \setminus Z \text{ and length increasing.}$$

Proof. φ pushes ρ to a metric on A_φ preserving area and lengths. So

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} \leq \text{mod}A_\varphi \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$$

$$\leq \text{mod}A_\varphi - \text{mod}A \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} - \text{mod}A \overset{\text{length-inc.}}{\leq} \text{area}_\rho - \text{area}_{\rho_0} \ .$$

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} - \text{mod}A \overset{\text{length-inc.}}{\geq} \frac{\ell_{\text{arc}}(\rho_0)^2}{\text{area}_\rho} - \text{mod}A = \frac{\text{area}_{\rho_0}^2}{\text{area}_\rho} - \text{area}_{\rho_0} \geq \text{area}_{\rho_0} - \text{area}_\rho.$$

1. An extremal length estimate: A annulus.

$\rho_0 =$ **extremal metric** if $\quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$

ρ is **length increasing** if $\quad \ell_{\text{loop}}(\rho) \quad \ell_{\text{arc}}(\rho)$

Key lemma

Let $Z \subset A$ compact and φ univalent on $A \setminus Z$. Let A_φ bounded by $\varphi(\partial A)$. Then

$$|\text{mod}A_\varphi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0} \quad \forall\, \rho \text{ on } A \setminus Z \text{ and length increasing.}$$

Proof. φ pushes ρ to a metric on A_φ preserving area and lengths. So

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} \leq \text{mod}A_\varphi \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$$

$$\leq \text{mod}A_\varphi - \text{mod}A \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} - \text{mod}A \overset{\text{length-inc.}}{\leq} \text{area}_\rho - \text{area}_{\rho_0}\ .$$

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} - \text{mod}A \overset{\text{length-inc.}}{\geq} \frac{\ell_{\text{arc}}(\rho_0)^2}{\text{area}_\rho} - \text{mod}A = \frac{\text{area}_{\rho_0}^2}{\text{area}_\rho} - \text{area}_{\rho_0} \geq \text{area}_{\rho_0} - \text{area}_\rho.$$

1. An extremal length estimate: A annulus.

$\rho_0 =$ **extremal metric** if $\quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$

ρ is **length increasing** if $\quad \ell_{\text{loop}}(\rho) \quad \ell_{\text{arc}}(\rho)$

Key lemma

Let $Z \subset A$ compact and φ univalent on $A \setminus Z$. Let A_φ bounded by $\varphi(\partial A)$. Then

$$|\text{mod}A_\varphi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0} \quad \forall\, \rho \text{ on } A \setminus Z \text{ and length increasing.}$$

Proof. φ pushes ρ to a metric on A_φ preserving area and lengths. So

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} \leq \text{mod}A_\varphi \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$$

$$\leq \text{mod}A_\varphi - \text{mod}A \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} - \text{mod}A \overset{\text{length-inc.}}{\leq} \text{area}_\rho - \text{area}_{\rho_0}\ .$$

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} - \text{mod}A \overset{\text{length-inc.}}{\geq} \frac{\ell_{\text{arc}}(\rho_0)^2}{\text{area}_\rho} - \text{mod}A = \frac{\text{area}_{\rho_0}^2}{\text{area}_\rho} - \text{area}_{\rho_0} \geq \text{area}_{\rho_0} - \text{area}_\rho.$$

1. An extremal length estimate: A annulus.

$\rho_0 =$ **extremal metric** if $\quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$

ρ is **length increasing** if $\quad \ell_{\text{loop}}(\rho) \quad \ell_{\text{arc}}(\rho)$

Key lemma

Let $Z \subset A$ compact and φ univalent on $A \setminus Z$. Let A_φ bounded by $\varphi(\partial A)$. Then

$$|\text{mod}A_\varphi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0} \quad \forall\, \rho \text{ on } A \setminus Z \text{ and length increasing.}$$

Proof. φ pushes ρ to a metric on A_φ preserving area and lengths. So

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} \leq \text{mod}A_\varphi \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$$

$$\leq \text{mod}A_\varphi - \text{mod}A \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} - \text{mod}A \overset{\text{length-inc.}}{\leq} \text{area}_\rho - \text{area}_{\rho_0} \ .$$

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} - \text{mod}A \overset{\text{length-inc.}}{\geq} \frac{\ell_{\text{arc}}(\rho_0)^2}{\text{area}_\rho} - \text{mod}A = \frac{\text{area}_{\rho_0}^2}{\text{area}_\rho} - \text{area}_{\rho_0} \geq \text{area}_{\rho_0} - \text{area}_\rho.$$

1. An extremal length estimate: A annulus.

$\rho_0 =$ **extremal metric** if $\quad \ell_{\text{loop}}(\rho_0) = 1 \quad \ell_{\text{arc}}(\rho_0) = \text{area}_{\rho_0} = \text{mod}A$

ρ is **length increasing** if $\quad \ell_{\text{loop}}(\rho) \quad \ell_{\text{arc}}(\rho)$

Key lemma

Let $Z \subset A$ compact and φ univalent on $A \setminus Z$. Let A_φ bounded by $\varphi(\partial A)$. Then

$$|\text{mod}A_\varphi - \text{mod}A| < \text{area}_\rho - \text{area}_{\rho_0} \quad \forall\, \rho \text{ on } A \setminus Z \text{ and length increasing.}$$

Proof. φ pushes ρ to a metric on A_φ preserving area and lengths. So

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} \leq \text{mod}A_\varphi \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2}.$$

$$\leq \text{mod}A_\varphi - \text{mod}A \leq \frac{\text{area}_\rho}{\ell_{\text{loop}}(\rho)^2} - \text{mod}A \overset{\text{length-inc.}}{\leq} \text{area}_\rho - \text{area}_{\rho_0}\ .$$

$$\frac{\ell_{\text{arc}}(\rho)^2}{\text{area}_\rho} - \text{mod}A \overset{\text{length-inc.}}{\geq} \frac{\ell_{\text{arc}}(\rho_0)^2}{\text{area}_\rho} - \text{mod}A = \frac{\text{area}_{\rho_0}^2}{\text{area}_\rho} - \text{area}_{\rho_0} \geq \text{area}_{\rho_0} - \text{area}_\rho.$$

2. Univalent maps outside nested or disjoint disks

Let $0 < \lambda < 1$ a factor, and $1 \geq \mathbf{s}(r) > 0$ a function on [0,1[with $\mathbf{s}(r) \underset{r \searrow 0}{\longrightarrow} 0$.

Let $\{D_y\}_{y \in Y}$ finitely many marked discs. They are $(\lambda, \mathbf{s})$-**nested** if :
(1) Any two are either disjoint or nested (but $\neq$).
(2) If $D_y \subset D_x$ and D_y touches $D_x(r)$, then $D_y \subset D_x(\mathbf{s}(r))$ (near the center they are not too long)
(3) $\text{area}\Big(\bigcup_{D_y \subsetneq D_x} D_y \Big) \leq \lambda \, \text{area} D_x$.

Theorem (a priori bound)

$\exists \infty \geq C(r) \underset{r \searrow 0}{\longrightarrow} 0$ depending only on $(\lambda, \mathbf{s})$, such that for any set $\{D_y\}_{y \in Y}$ of $(\lambda, \mathbf{s})$-nested discs and any disjoint $\overline{V}$,

$$\forall \text{ univalent map } \phi \text{ outside } \bigcup \overline{D_y(r)}, \quad \mathcal{D}(\phi, V) \leq C(r) \cdot M(\cup D_y, V),$$

where $M(D, V)$ is a constant depending only on D and V.

Proof using Key lemma

Fix $r << 1$ (so that $\mathbf{s}(r) < 1$) and $A \supset \cup D_y$ with $\partial A \subset V$.

For $k \geq 1$, among the discs D_x of nesting depth k, those touching some $D_y(r)$ are contained in $D_y(\mathbf{s}(r))$, the others form a union J_k. Define $J_k(\mathbf{s}(r))$ by reducing the discs' sizes.

Define $\rho_k(z)$ inductively on k by

$$\rho_k(z) = \begin{cases} \rho_{k-1}(z) & \text{on } A - J_k, \\ \rho_0(z)(1 - \sqrt{\mathbf{s}(r)})^{-k} & \text{on } J_k - J_k(\mathbf{s}(r)), \\ 0 & \text{on } J_k(\mathbf{s}(r)). \end{cases}$$

When reaching the deepest nest, the metric stabilizes to ρ, which increases length, $= 0$ on $\cup D_x(r)$ and

$$0 \leq \text{area}_\rho(A) - \text{area}_{\rho_0}(A) \leq C(r) \cdot M(D, V)$$

Our ϕ is univalent map outside $\cup D_x(r)$.

$$|\text{mod}A - \text{mod}A_\phi)| \overset{\text{Key lemma}}{\leq} \text{area}_\rho A - \text{area}_{\rho_0} A \leq C(r) \cdot M(D, V).$$

So $\mathcal{D}(\phi, V) \leq C(r) \cdot M(D, V)$.

3. Turn overlapped discs to nested discs

Let: g a rational map post-cr. finite on $\mathcal{J}$;

X_0 a finite collection of cycles in $\mathcal{J}$; $\{U_x$ disc-neighborhood$\}_{x\in X_0}$;

$X =$ grand orbit of X_0; (by pullback) $\{U_y\}_{y\in X\smallsetminus X_0}$;

For $0 < r < 1$, $U_x(r)$ is $D(0,r)$ (after uniformization), $U_y(r)$...

This infinitely many discs are overlapped. But for any given n, the finitely many discs of $g^{-n}(X_0)$ can be turned nested with a common $(\lambda, \mathbf{s})$ therefore $C(r)$...

Vertical convergence

$$\begin{array}{cc} \infty & f_r \\ \uparrow & \uparrow \\ n & f_{r,n} \end{array}$$: (forgetting r) F is Thurston equivalent to f through a pair ϕ_0, ϕ_1 (they can be constructed by hand in this case, no need to apply Thurston's theorem).

Use Thurston's pullback map, get f_n. The sequence converges to $f = f_r$.

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ & \xleftarrow{\phi_3} & & \xrightarrow{p_3} & \\ f\downarrow & & F\downarrow & & \downarrow f_3 \\ & \xleftarrow{\phi_2} & & \xrightarrow{p_2} & \\ f\downarrow & & F\downarrow & & \downarrow f_2 \\ & \xleftarrow{\phi_1} & & \xrightarrow{p_1} & \\ f\downarrow & & F\downarrow & & \downarrow f_1 \\ & \xleftarrow{\phi_0} & & \xrightarrow{p_0} & \end{array}$$

$p_n \circ \phi_n^{-1}$ are more and more conformal and remain K-qc., any limit map must be conformal thus must be the identity. Thus f_n converges to $f = f_r$.

Accessing parabolic maps from stable directions

Same procedure: Construct via surgery and Thurston's algorithm $f_{r,n}$ such that:

$$\begin{array}{ccccc} \infty & & f_r & & \\ \uparrow & & \uparrow & & \\ n & & f_{r,n} & \longrightarrow & g \text{ uniformly on } n \\ & & r & \longrightarrow & 0 \end{array}$$

Extra difficulties:

Surgery on the space and the map

The vertical convergence (DO need Thurston's theorem).

(The horizontal one is identical.)

Thank you !