# Grothendieck's dessins d'enfants
# Children's drawings of Grothendieck

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# Two key results

Definition. A complex polynomial $P \in \mathbb{C}[X]$ is called Belyi if it has only two critical values, at 0 and 1 (can be chosen to be at other points).

Two such polynomials $f, g$ are covering equivalent if there is an affine map $M(z) = az + b$, $a \neq 0$, such that

$$\begin{array}{ccc} \overline{\mathbb{C}} & \stackrel{M}{\rightarrow} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array} .$$

**Theorem** (Grothendieck). In each equivalence class there is a representative polynomial $P$ with algebraic numbers as coefficients.

**Theorem** (Belyi) (1) Given any finite set $A$ of algebraic numbers there is a polynomial $P$ with rational coefficients such that $P(A) \cup \{\text{critical values of } P\} \subset \{0, 1\}$.

(2) Given any rational map $f$ with algebraic numbers as critical values (for example $f \in \overline{\mathbb{Q}}[X]$), there is a polynomial $P$ with rational coefficients such that cr. val$(P \circ f) \subset \{0, 1, \infty\}$.

# Basic facts

If $cv(f) \subset A$, then $cv(P \circ f) \subset P(A) \cup cv(P)$.

Proof. Pullback by $P$ then by $f$ a small disc outside $P(A) \cup cv(P)$.

$\mathbb{Q} = \{\text{rational numbers}\}$,

$\overline{\mathbb{Q}} = \{\text{algebraic numbers}\} = \{a \in \mathbb{C} | P(a) = 0 \text{ for some } P \in \mathbb{Q}[X]\}$.

$\overline{\mathbb{Q}}$ is countable, dense in $\mathbb{C}$, a field, algebraically closed. Moreover, the roots of a polynomial system in $\overline{\mathbb{Q}}[X_1, \cdots, X_n]$ form either an unbounded set, or a finite set with algebraic coordinates.

# Proof of Grothendieck's thm : $\exists P \circ M$ monic and fixing 0

Any $P$ monic, $P(0) = 0$ and $cv(P) \subset \{0, 1\}$ must have algebraic coefficients.

Proof. Let $P$ be monic. Let $\{a_i, i \in I\} = P^{-1}(0)$, with multiplicity $s_i$ for $a_i$, and $\{b_j, j \in J\} = P^{-1}(1)$ with multiplicity $t_j$ for $b_j$, $d = \sum s_i = \sum t_j$. Then

$$\prod_I (z - a_i)^{s_i} = P(z) = \prod_J (z - b_j)^{t_j} + 1 \ .$$

Equalizing the coefficients on both sides above we get that $\mathbf{a} = \{a_i\}$ and $\mathbf{b} = \{b_j\}$ must satisfy a system of polynomial equations $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, with $\mathbf{F} = \mathbf{F}_{s_i, t_j} \in (\mathbb{Q}[\mathbf{a}, \mathbf{b}])^d$. It is known

$$\{\text{zeros of } \mathbf{F}\} = \begin{cases} \text{either an unbounded algebraic variety} \\ \text{or a finite set with algebraic coordinates} \end{cases} .$$

Choose now the $s_i$ and $t_j$'s so that the critical values of $P$ are in $\{0, 1\}$, in other words, $\sum(s_i - 1) + \sum(t_j - 1) = d - 1$ ($s_i$ and $t_j$'s are called the **passport of** $P$ ).

# Geometric proof of the boundedness of $\mathbf{F(a,b)=0}$

Let $(\mathbf{a},\mathbf{b})=\{a_i,b_j\}$ be a solution of $\mathbf{F}(\mathbf{a},\mathbf{b})=\mathbf{0}$.
Let $P(z)=\prod_I (z-a_i)^{s_i}=\prod_J (z-b_j)^{t_j}+1$ . Thus the critical values of $P$ are in $\{0,1\}\subset[0,1]$.

$$\begin{array}{ccc} \mathbb{C}\smallsetminus\overline{\mathbb{D}} & & \mathbb{C}\smallsetminus P^{-1}([0,1]) \\ z^d\downarrow & & \downarrow P \\ \mathbb{C}\smallsetminus\overline{\mathbb{D}} & \overset{\text{Riemann}}{\longrightarrow} \ \frac{1}{4}(z+\frac{1}{z})+\frac{1}{2} & \mathbb{C}\smallsetminus[0,1] \end{array}$$

Now using $P(0)=0$ and the Koebe 1/4-theorem, we have
$\{a_i,b_j \mid i\in I, j\in J\}=P^{-1}(\{0,1\})\subset P^{-1}([0,1])\subset \mathbb{D}(0,\frac{4}{4^{\frac{1}{d}}})$. □

(Bezout $\Longrightarrow$ the number of solutions = the number of normalized Belyi polynomials with a given passport $=d^{d-2}$ counted with multiplicity.)

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$$\begin{array}{ccc} \mathbb{C}\setminus\overline{\mathbb{D}} & \overset{\exists \frac{1}{4^{1/d}}z+O(1)}{\underset{\text{biholomorphic}}{\longrightarrow}} & \mathbb{C}\setminus P^{-1}([0,1]) \\ z^d\downarrow & & \downarrow P \\ \mathbb{C}\setminus\overline{\mathbb{D}} & \overset{\text{Riemann}}{\longrightarrow} & \mathbb{C}\setminus[0,1] \\ & \frac{1}{4}(z+\frac{1}{z})+\frac{1}{2} & \end{array}$$

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# Action of the absolute Galois group $\Gamma = Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$

Known fact : $Fix(\Gamma) = \mathbb{Q}$.

Let $P(X) = \sum a_i X^i$ with $a_i \in \overline{\mathbb{Q}}$. Let $\sigma \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

$$\begin{array}{ccccccccc} a_i & P & \overset{derivative}{\longrightarrow} & P' & \overset{take\ zeros}{\longrightarrow} & cpts & \overset{P}{\longrightarrow} & cvs \\ \sigma\downarrow & \sigma\downarrow & & \sigma\downarrow & & \downarrow\sigma & & \downarrow\sigma \\ \sigma(a_i) & Q & \overset{derivative}{\longrightarrow} & Q' & \overset{take\ zeros}{\longrightarrow} & cpts & \overset{Q}{\longrightarrow} & cvs \end{array}$$

alg. conjugacy

$$\begin{array}{ccc} \overline{\mathbb{Q}} \ni z & \overset{P}{\longrightarrow} & P(z) \\ \sigma\downarrow & & \downarrow\sigma \\ \sigma(z) & \overset{Q}{\longrightarrow} & Q(z) \end{array}$$

$\sigma(P \circ L) = \sigma(P) \circ \sigma(L)$.

# Proof of Belyi's theorem

Given any finite set $A$ of algebraic numbers there is a polynomial $P \in \mathbb{Q}[X]$ such that $P(A) \cup \{\text{critical values of } P\} \subset \{0, 1\}$.

**Step 1**. Reducing the number of points outside $\mathbb{Q}$ :

$$B_0 = A \smallsetminus \mathbb{Q} \quad \xrightarrow[\text{say } d^o = d]{P_0 \in \mathbb{Q}[X]} \quad 0$$

$$\downarrow$$

$$B_1 = \{0\} \cup cv(P_0) \smallsetminus \mathbb{Q} \quad \xrightarrow[d^o \leq d-1]{P_1 \in \mathbb{Q}[X]} \quad 0$$

$$\downarrow$$

$$\{0\} \cup cv(P_1) \smallsetminus \mathbb{Q} \quad \cdots$$

with $P_1(z) = \prod_{\delta \in B_1} (z - \delta)$. Any $\sigma \in \Gamma$ must fix $P_0$ thus permutes $B_1$ and fixes the coefficients of $P_1$. So these coefficients are all in $\mathbb{Q}$.

So $\exists P \in \mathbb{Q}[X]$ such that $Y := P(A) \cup \{\text{critical values of } P\} \subset \mathbb{Q}$.

# Step 2. Reducing the number of points inside $\mathbb{Q}$

Given any finite set $Y \subset \mathbb{Q}$, we will show that there is $L \in \mathbb{Q}[X]$ such that $L(Y) \cup \{\text{critical values of } L\} \subset \{0, 1\}$.

For any increasing triple $a, b, c \in \mathbb{Q}$,

$$a < b < c \xrightarrow[\alpha,\beta\in\mathbb{Q}]{\alpha z+\beta} 0 < \frac{p}{p+q} < 1 \xrightarrow[\text{magic map}]{z^p(1-z)^q} 0 < s,\ s \in \mathbb{Q}.$$

□

# Dessin d'enfants

For any meromorphic map $F : S \to \overline{\mathbb{C}}$ with $S$ a compact Riemann surface such that $cv(F) \subset \{0, 1, \infty\}$. The set $D_F := F^{-1}([0, 1])$ is called the dessin of $F$.

In the case of a Belyi polynomial $P$, $D_P$ is an embedded tree in $\mathbb{C}$ whose vertices are bi-colored.

The set of covering equivalence classes $Belyi(\overline{\mathbb{C}})/\sim = Belyi(\overline{\mathbb{Q}})/\sim$ is in bijection with the set of embedded bi-colored trees (up to $homeo^+$ preserving the coloring).

Any bounded arc linking 0 and 1 will lift to a homeomorphic dessin.

# Action of the absolute Galois group $\Gamma = Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$

Let $P(X) = \sum a_i X^i$ with $a_i \in \overline{\mathbb{Q}}$. Let $\sigma \in Aut_{\mathbb{Q}}(\overline{\mathbb{Q}})$.

$$\begin{array}{ccccccc} a_i & P & \stackrel{derivative}{\longrightarrow} & P' & \stackrel{take\ zeros}{\longrightarrow} & cpts & \stackrel{P}{\longrightarrow} & cvs \\ \sigma\downarrow & \sigma\downarrow & & \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma \\ \sigma(a_i) & Q & \stackrel{derivative}{\longrightarrow} & Q' & \stackrel{take\ zeros}{\longrightarrow} & cpts & \stackrel{Q}{\longrightarrow} & cvs \end{array}$$

alg. conjugacy

$$\begin{array}{ccc} \overline{\mathbb{Q}} \ni z & \stackrel{P}{\longrightarrow} & P(z) \\ \sigma\downarrow & & \downarrow\sigma \\ \sigma(z) & \stackrel{Q}{\longrightarrow} & Q(z) \end{array} \qquad \sigma(P \circ L) = \sigma(P) \circ \sigma(L).$$

**Theorem**. The group $\Gamma$ acts faithfully : 1. on $Belyi(\overline{\mathbb{Q}})$ and on $Belyi(\overline{\mathbb{Q}})/\sim$ (Lenstra-Schneps) ; 2. on the set of dynamical Belyi polynomials and commutes with iterations (Pilgrim). The pull-back dessins Hausdorff-converge to the Julia set, which contains a dense set of algebraic numbers (Pilgrim).

**Open problem**. See the action on trees.

# Proof of faithfulness

Let $id \neq \gamma \in \Gamma$. Then $\exists \alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ such that $\gamma(\alpha) \neq \alpha$. Set $f$ such that $f'(z) = z(z-1)^2(z-\alpha)^3$. By Belyi $\exists g \in \mathbb{Q}[X]$ such that $g \circ f \in Bel(\overline{\mathbb{Q}})$. Then $\gamma.(g \circ f)$ can not be equivalent to $g \circ f$.

Assume the contrary, $\gamma.(g \circ f) = g \circ f \circ M$, with $M$ affine. So $g \circ \gamma.f = \gamma.(g \circ f) = g \circ f \circ M$. By the decomposition theorem, there is affine $B$ such that $\gamma.f = B \circ f \circ M$. This implies that $M$ transports the critical points of $f$ to that of $\gamma.f$. But due to the pairwise distinct powers of the critical points, $M$ must fixe 0 and 1 and so much be identity. But $M(\alpha) = \gamma.\alpha \neq \alpha$. A contradiction.

Decomposition theorem. If $g_1 \circ f_1 = g_2 \circ f_2$ with the same degree decomposition, then $f_2 = affine(f_1)$.

Proof. Look at the preimage of a line segment linking the critical values.

# Dessin as trajectories of polynomial vector fields

A Belyi passport $(s_i, t_j)$ is called clean if $t_j \equiv 2$. Any Belyi polynomial $Q$ can be turned into a clean $B$ by post composing $z \mapsto 4z(1-z)$. In terms of the dessin, one just needs to add a vertex in each edge to be the new colored 1 vertices and color the old vertices by 0. So $\boxed{d \text{ old vertices}} \Longrightarrow \boxed{d-1 \text{ new vertices}}$.

**Theorem (Pilgrim)**. Any clean Belyi polynomial is isomorphic to a $B(z)$ of the form $-\prod_{I,|I|=d}(z-a_i)^{s_i}$, with $a_i$ pairwise distinct, $\sum_I a_i = 0$ and $\sum_I s_i = deg(B) = \sum_{j=1}^{d-1} t_j = 2(d-1)$. Any $B$ induces a monic centered polynomial $P_B(z) = \prod_I (z-a_i)$ and

$$\dot{z} = P_B(z) \underset{pullsback}{\overset{B^*}{\longleftarrow}} \dot{z}^2 = 4(z^2 - z^3)$$

with separatrices $B^{-1}([-\infty, 0])$. The edges of the dessin $B^{-1}([0,1])$ are trajectories. Each $a_i$ receives $s_i$ separatrices. The vector field $\dot{z} = P_B(z)$ is generic, monic centered with constant analytic invariant $(= \dfrac{\pi i}{d-1})$.

# Bijection : from vector fields to Belyi polynomials

Let $P(z) = \prod_{I,|I|=d}(z - a_i)$ be a monic centered polynomial such that the vector field $P(z)\frac{\partial}{\partial z}$ is generic : all roots are simple and no homoclinic connections of the $2(d-1)$ separatrix germs at $\infty$ (in particular each $a_i$ is either a source or a sink)

Each $a_i$ receives $s_i > 0$ separatrices (as $a_i$ is not a center), and $\sum_I s_i = 2(d-1)$ (every separatrix germ lands at some $a_i$).

So $(s_1, \cdots, s_d)$ is an integer partition of $2(d-1)$.

Within the same landing pattern there is a unique choice of $\mathbf{a} = (a_i)_I$ such that the analytic invariant of $P$ is $\{\frac{\pi i}{d-1}, \cdots, \frac{\pi i}{d-1}\}$. For this $P$, set $B_P(z) = -\prod_I (z - a_i)^{s_i}$. Then (by Pilgrim) $B_P$ is a clean Belyi polynomial. This is obtained by uniformizing and then squaring the zones of $\dot{z} = P(z)$ to get a global holomorphic map $B$ which turns out to be $B_P$.

**Counting result** The number of normalized clean Belyi polynomial of given degree is the catalan number.

(Tomasini) There is a closed form for the number of equivalence classes of clean Belyi polynomials as well as the number of combinatorial invariants of generic polynomial vector fields.

**Question** Given $(s_1, s_2, \cdots, s_d)$ a partition of $2(d-1)$, how many landing patterns are there ? A landing pattern is a non-crossing partition of $\{0, \cdots, 2d-3\}$ into $d$ sets of cardinality $s_1, \cdots, s_d$ so that each set contains only numbers of the same parity (either all are odd or all are even).