Entropy on Hubbard trees

Tan Lei 谭蕾, Université d'Angers

(most figures are produced by W. Thurston and G. Tiozzo)

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Organization of the mini course

– Definition of the topological entropy

– For $f : z \mapsto z^2 + c$, $c \in [-2, \frac{1}{4}]$, study various aspects of the entropy on $[-\beta_c, \beta_c]$

1. Misiurewitcz-Szlenk's lap numbers
2. Milnor-Thurston's kneading theory
3. Douady's external angle model
4. Thurston's angle-pair model
5. Tiozzo's parameter section model $\cdots$

– Case of complex polynomials : entropy on the Hubbard tree and other variants

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- The topological entropy $h(X, f) := \sup_{\mathcal{U},\text{ open cover of } X} h(X, \mathcal{U}, f)$.

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This sounds difficult to compute. But it is equal to the growth of many interesting dynamical quantities, some of them are (sometimes) computable. The variation $f \mapsto h(f)$ is also interesting.

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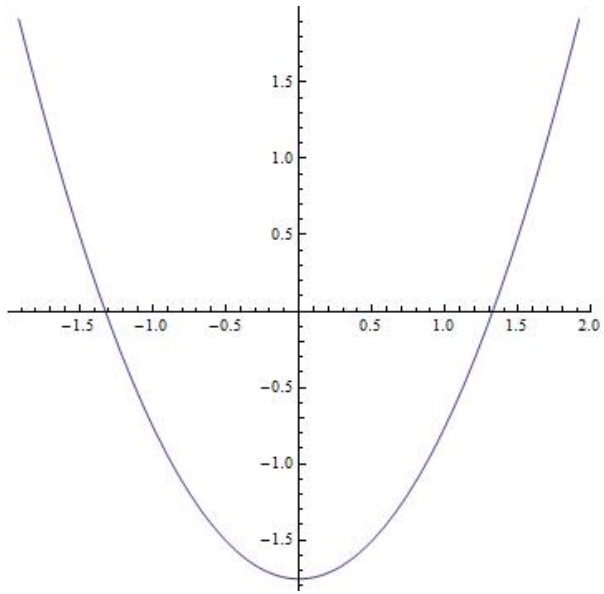

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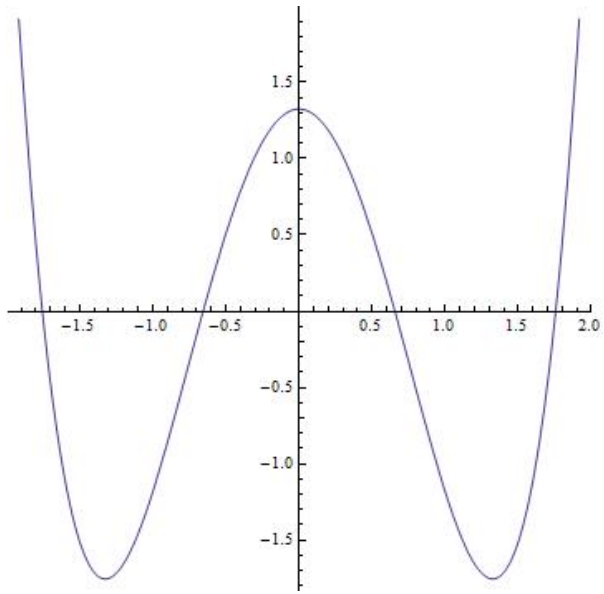

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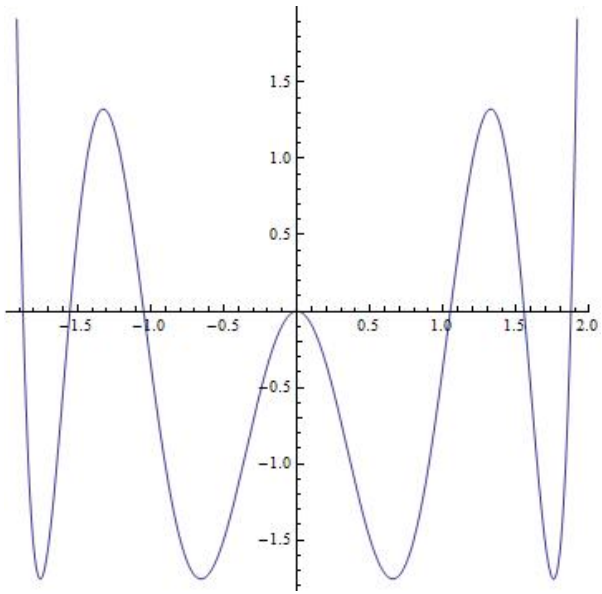

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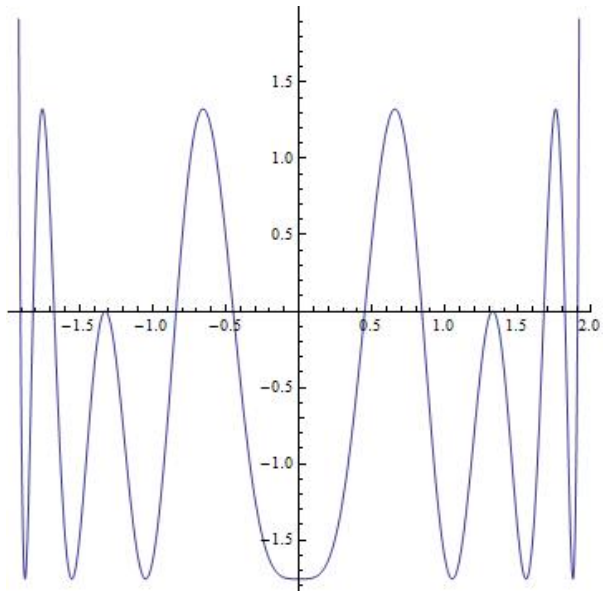

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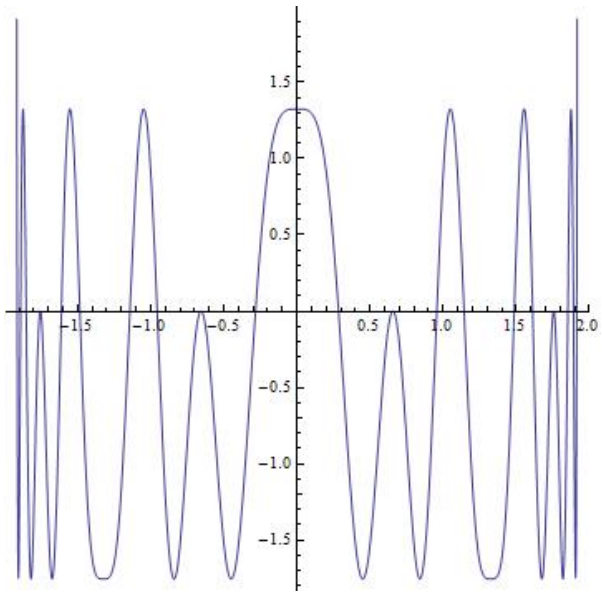

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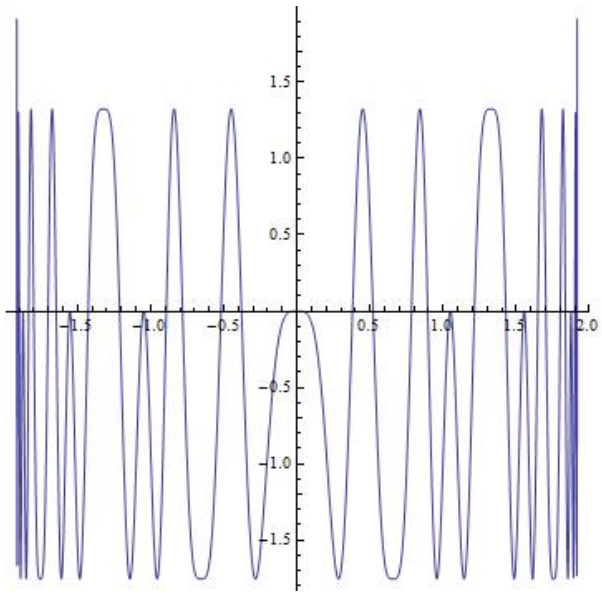

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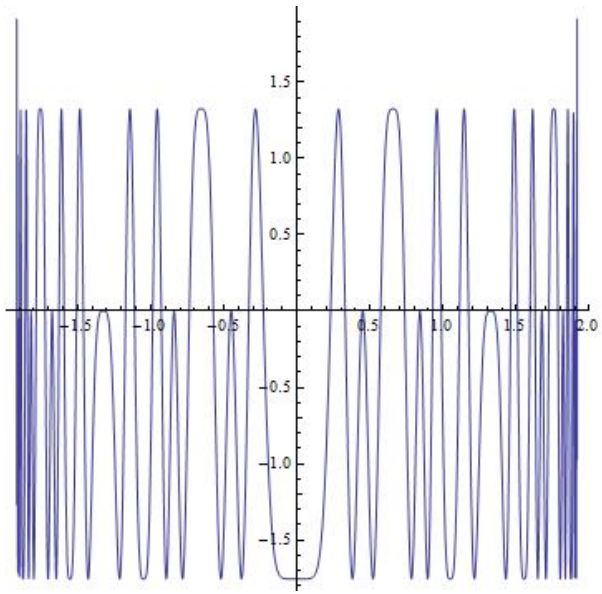

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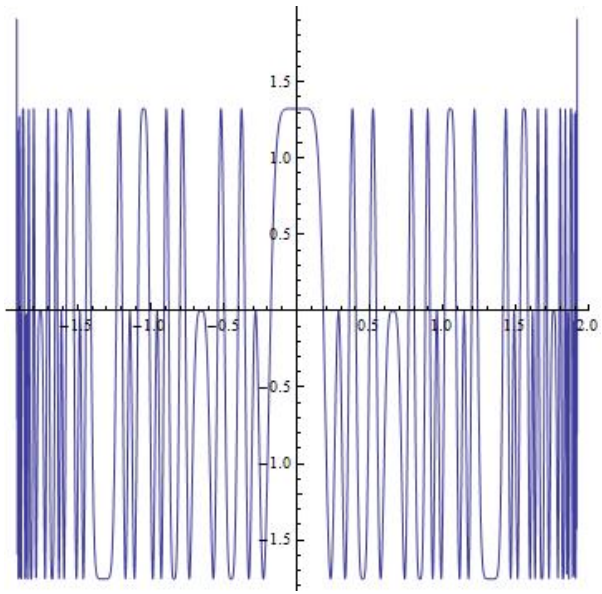

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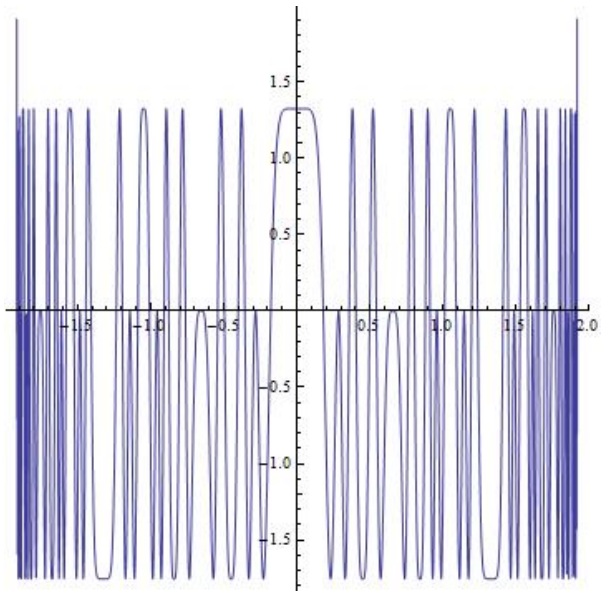

Proof. For $\gamma(t) = \sum_{i \ge 0} \gamma_i t^i$,, $\ell(t) = \sum_{i \ge 0} \ell_{i+1} t^i$: $\ell(t)(1-t) = 1 + \gamma(t)$.

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It follows that

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$$\text{log-growth} \#\bigcup_{j=0}^{n-1} f^{-j}S \geq \text{log-growth} N(\bigvee^n \mathcal{U}) = h(I,\mathcal{U},f).$$

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Question. Is the growth of $\#f^{-n}(\alpha_c)$ equal to **s**? ($\alpha_c < 0$ is the other fixed point...) Can we replace max by min?

§3. Milnor-Thurston's kneading theory : $\mathbf{s} := \exp(h(I, f))$

Theorem 2. If $\mathbf{s} > 1$, the the kneading series $D(t) = \sum_{i \geq 0} \pm t^i$
$\left[+ \text{ if } (f^{i+1})'|_0 \nearrow, \text{ and } - \text{ if } \searrow \right]$
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Cases of z^2 : $D(t) = \sum t^i = \dfrac{1}{1-t}$,

of $z^2 - 2$: $D(t) = 1 - \sum_{i \geq 1} t^i = 1 - \dfrac{t}{1-t} = \dfrac{1-2t}{1-t}$, $\mathbf{s} = 2$;

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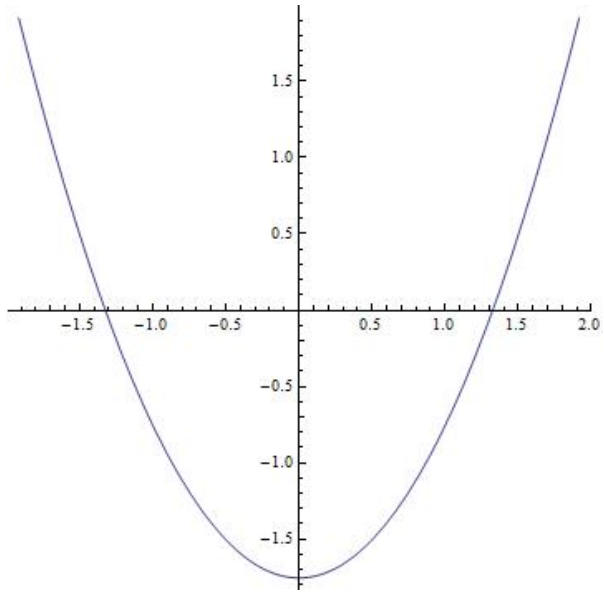

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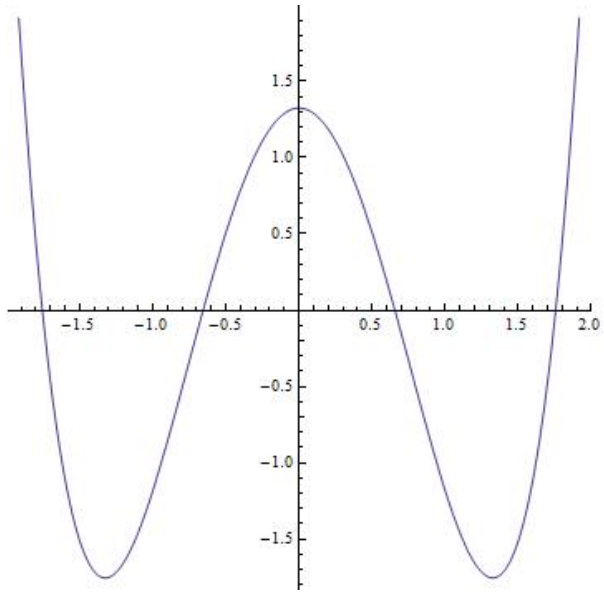

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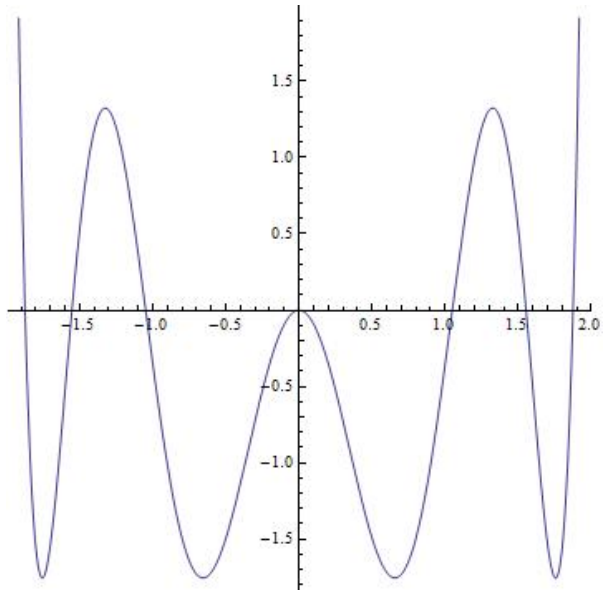

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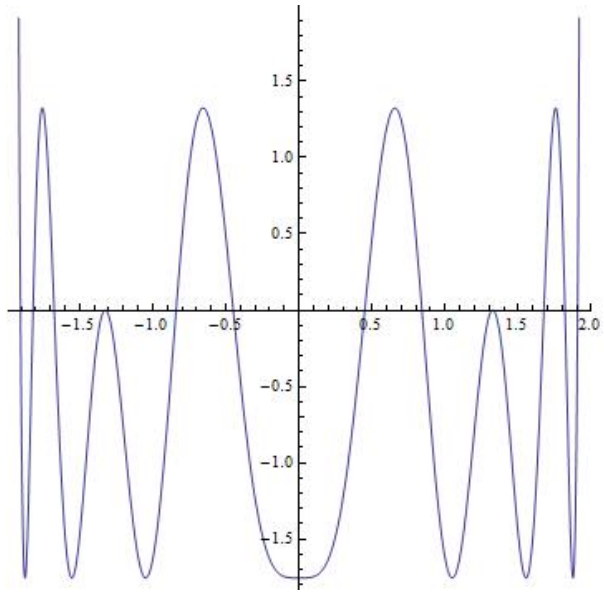

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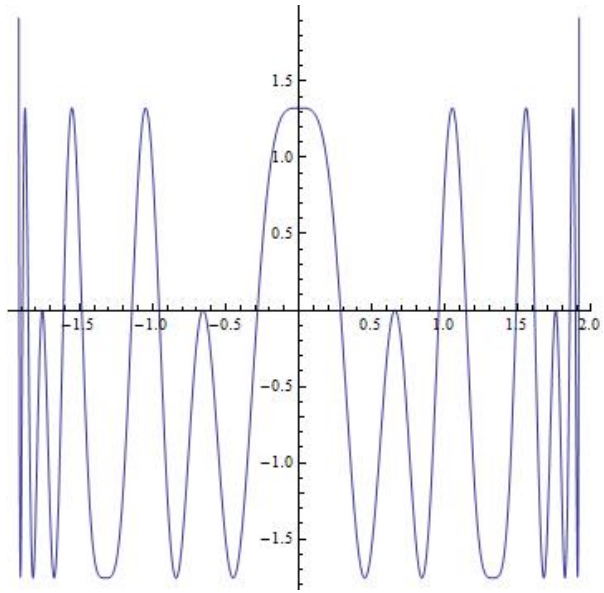

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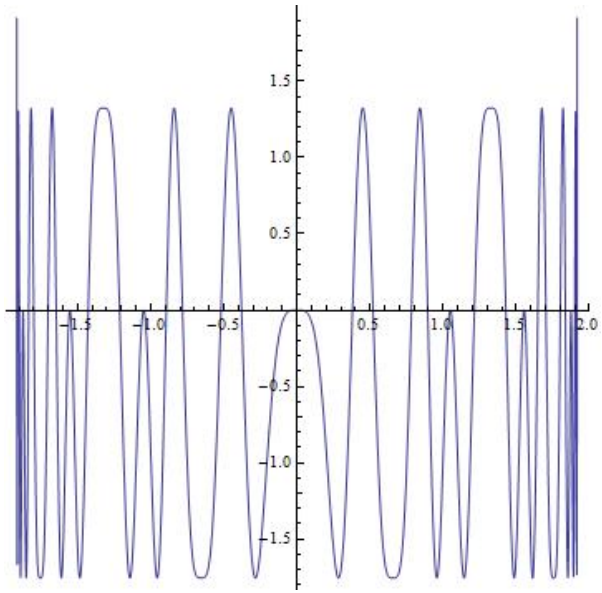

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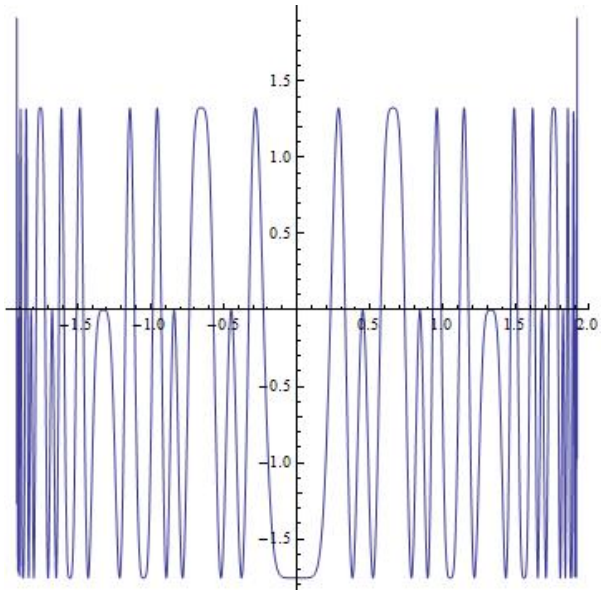

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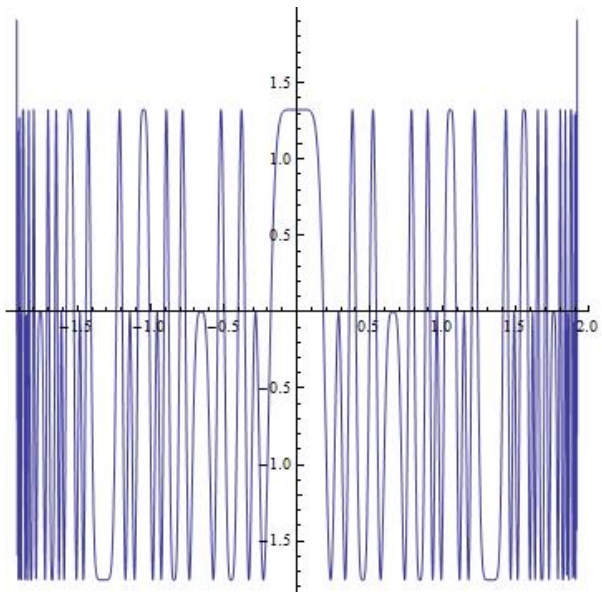

Itinerary and kneading

For $n \geq 0$, define the step function $\theta_n(x) = \begin{cases} +1 & \text{if } f^{n+1} \nearrow \\ -1 & \text{if } f^{n+1} \searrow \end{cases}$.

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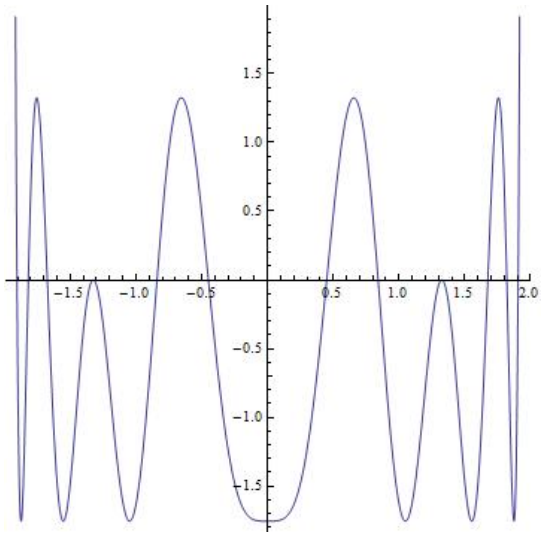

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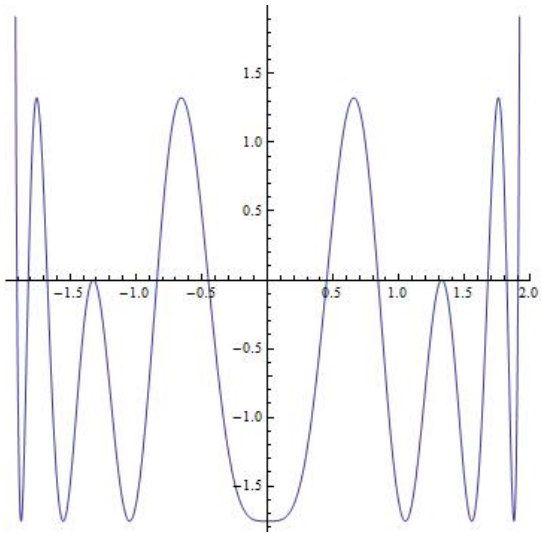

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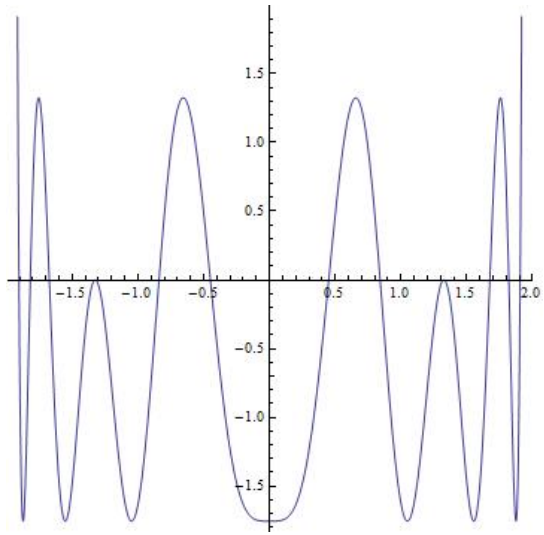

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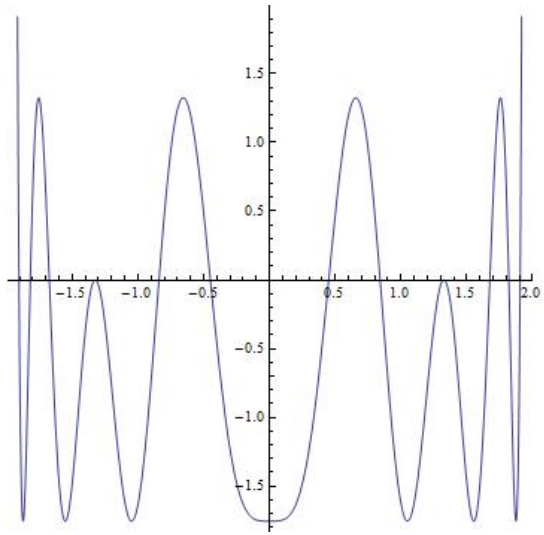

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$itin(0^+)$	$+1$	ε_1	ε_2	$\cdots$	ε_n	$\cdots\cdots$	$= +1 itin(f(0)^+)$
$itin(0^-)$	-1	ε_1	ε_2	$\cdots$	ε_n	$\cdots\cdots$	$= -1 itin(f(0)^+)$

$\Theta(0^+)$	$+1$	ε_1	$\varepsilon_1\varepsilon_2$	$\cdots$	$\varepsilon_1\cdots\varepsilon_n$	$\cdots\cdots$
$itin(0^+)$	$+1$	ε_1	ε_2	$\cdots$	ε_n	$\cdots\cdots$

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$itin(0^+)$	$+1$	ε_1	ε_2	$\cdots$	ε_n	$\cdots\cdots$
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$\Theta(0^+)$	$+1$	ε_1	$\varepsilon_1\varepsilon_2$	$\cdots$	$\varepsilon_1\cdots\varepsilon_n$	$\cdots\cdots$
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$\Theta(0^-)$	-1	$-\varepsilon_1$	$-\varepsilon_1\varepsilon_2$	$\cdots$	$-\varepsilon_1\cdots\varepsilon_n$	$\cdots\cdots$

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$\Theta(0^+)$	$+1$	ε_1	$\varepsilon_1\varepsilon_2$	$\cdots$	$\varepsilon_1\cdots\varepsilon_n$	$\cdots\cdots$
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So $\Theta(0^+) = (-1) \times \Theta(0^-)$ (change of signs, not concatenation).

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If x is not a precritical point, then $\Theta(x^+) = \Theta(x^-)$.

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If x is not a precritical point, then $\Theta(x^+) = \Theta(x^-)$.

What happens if $f^i(x) = 0$ (with i minimal) ?

$\Theta(0^+)$	$+1$	ε_1	$\varepsilon_1\varepsilon_2$	$\cdots$	$\varepsilon_1\cdots\varepsilon_n$	$\cdots\cdots$
$itin(0^+)$	$+1$	ε_1	ε_2	$\cdots$	ε_n	$\cdots\cdots$
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So in both cases $\Theta(x^\pm) = \theta_0 \cdots \theta_{i-1}\Theta(0^\pm)$.

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If x is not precritical, $\theta(x^+, t) = \theta(x^-, t)$. Otherwise one uses $D(x, t) := \dfrac{\theta(x^+, t) - \theta(x^-, t)}{2} = \sum_{i \geq 0} \dfrac{\theta_i(x^+) - \theta_i(x^-)}{2} t^i$ to measure the discontinuity at x.

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Theorem 2 says that $D(\tau)$ is null (and hence $D(x, \tau)$ is null $\forall x$) for $\tau = \dfrac{1}{\mathbf{s}}$.

Theorem 2. If $\mathbf{s} > 1$, the kneading series $D(t) = \sum_{i \geq 0} \theta_i(0^+) t^i$ admits $1/\mathbf{s}$ as a zero, and $D(z) \neq 0$ on $\{|z| < 1/\mathbf{s}\}$.

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The **kneading sequence** is $itin(f(0)^+) = \varepsilon_1 \cdots \varepsilon_n \cdots$. And $D(t) = 1 + \sum_{n\geq 1} \varepsilon_1 \cdots \varepsilon_n \cdot t^n$.

Theorem 3. If $\mathbf{s} > 1$, the map $x \mapsto \frac{1}{\mathbf{s}}\theta(x^-, \frac{1}{\mathbf{s}})$ is continuous and semi-conjugates f to the symmetric tent map of slope $\mathbf{s}$ with min=-1.

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$$T_w(x) = \begin{cases} -wx - 1, & x \in [-\frac{1}{w-1}, 0] \\ wx - 1, & x \in [0, \frac{1}{w-1}] \end{cases} \qquad \begin{cases} \phi_-(y) = \frac{-1-y}{w} \\ \phi_+(y) = \frac{1+y}{w} \end{cases}$$

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For an IFS of affine contractions $\phi_1, \cdots, \phi_m$, there is a unique attractor K such that $K = \phi_1(K) \cup \cdots \phi_m(K)$. Moreover, there is a continuous surjective encoding $\pi : \{1, \cdots, m\}^{\mathbb{N}} \to K$ such that $\pi(\varepsilon \mathbf{u}) = \phi_\varepsilon(\pi(\mathbf{u}))$.

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Remark of Tiozzo. The growth $\mathbf{s} > 1$ is also the greatest positive value w so that the inverse IFS of T_w encodes the minimal value -1 of T_w by the itinerary of $f(0)^+$.

And for this $\mathbf{s}$, every itinerary of f is realized by an itinerary of $T_{\mathbf{s}}$ (detailed are not provided). □

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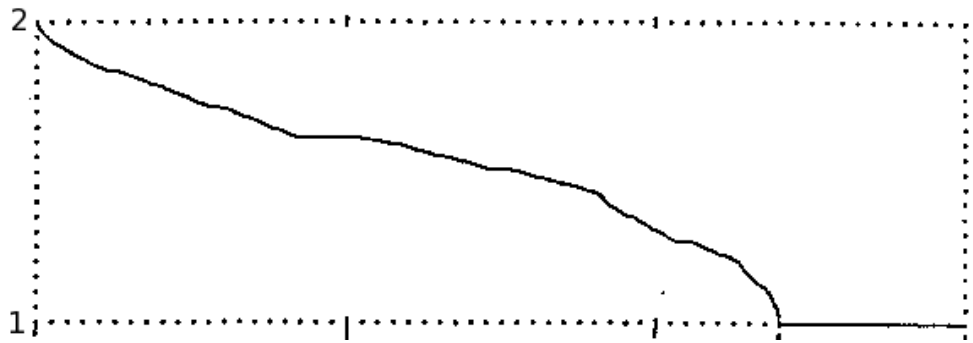
Theorem 5. $c \mapsto \mathbf{s}(c) = exp(h(l_c, f_c))$ is continuous and weakly decreasing (Douady), $2 \searrow 1$.



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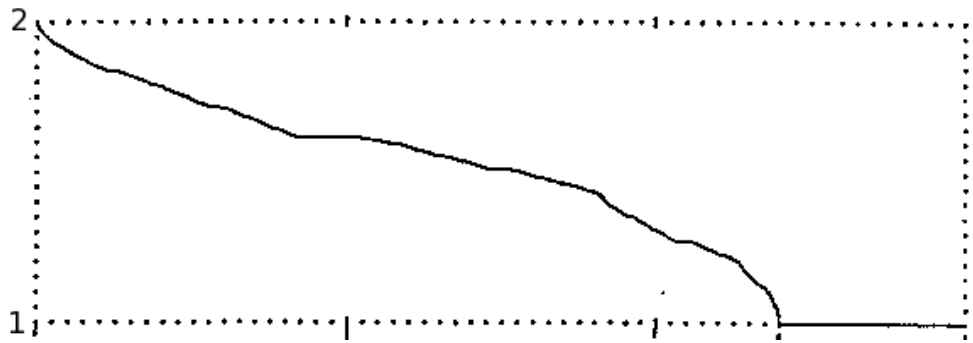

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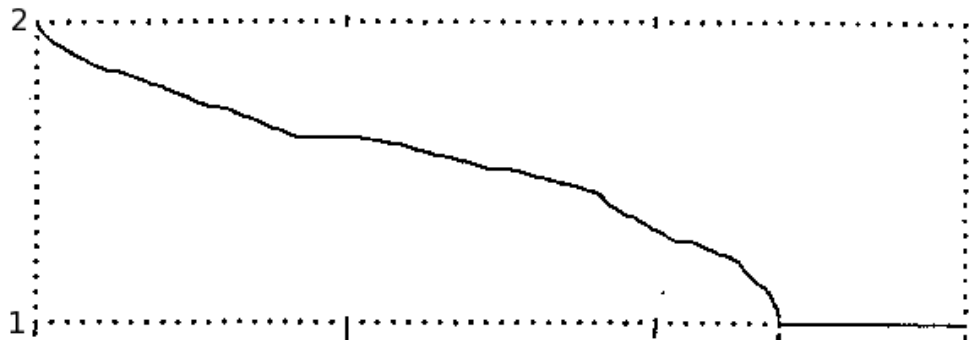

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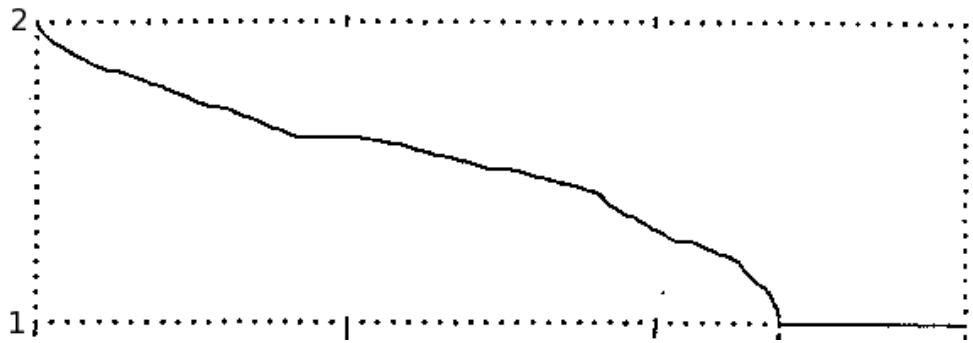

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For the monotonicity, complex methods using external rays are much more powerful... □

§4. Complex view of real $z^2 + c$, Douady's circle model

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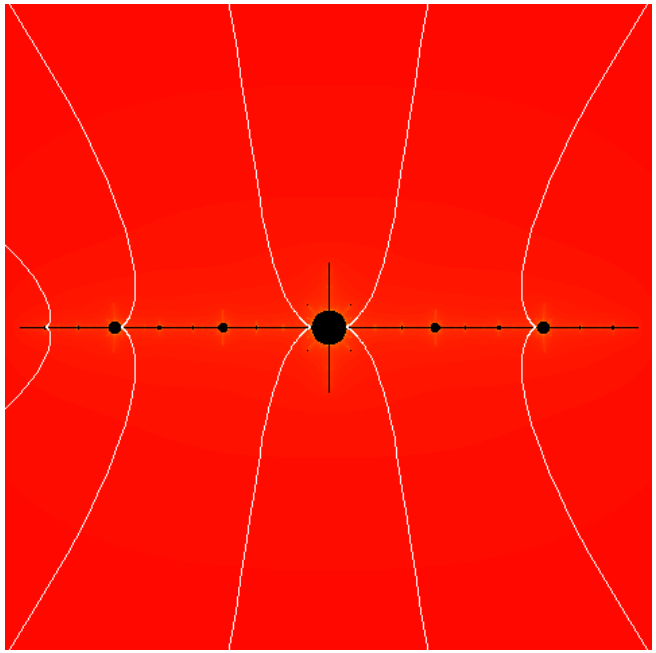
For $c \in [-2, \frac{1}{4}]$, the set $\hat{\mathbb{C}} \setminus K_c$ is simply-connected, there is a conformal map $\Phi_c : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus K_c$ semi-conjugating z^2 to f_c. The external rays $\{R_c(t), t \in S^1\}$ coming from radial rays outside $\overline{\mathbb{D}}$, foliate the outside of K_c, and are invariant by f : $f_c(R_c(t)) = R_c(2t)$. $K_c \cap \mathbb{R} = I_c$. This translates the entropy on I_c to an entropy of the **symbolic dynamics** :



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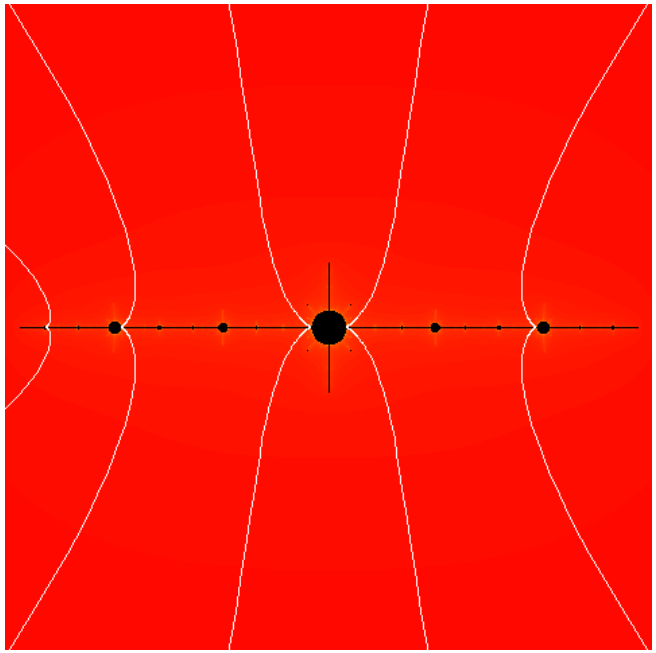

Douady's formula ('93) : $Y_c := \{t \in S^1 \ : \ R_c(t)$ lands on $\mathbb{R}\}$ is a closed invariant subset of the angle doubling map

$$q : t \mapsto 2t, \ S^1 \to S^1, \quad \text{and} \quad h(Y_c, q) = h(I_c, f_c).$$

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Proposition 1. If $X = X_1 \cup X_2$, with X_1 and X_2 compact, $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, then $h(X, f) = \sup\big(h(X_1, f), h(X_2, f)\big)$.

Proposition 2. Let Z be a closed subset of X such that $f(Z) \subset Z$. Suppose that for any $x \in X$, the distance of $f^n(x)$ to Z tends to 0, uniformly on any compact set in $X - Z$. Then $h(X, f) = h(Z, f)$.

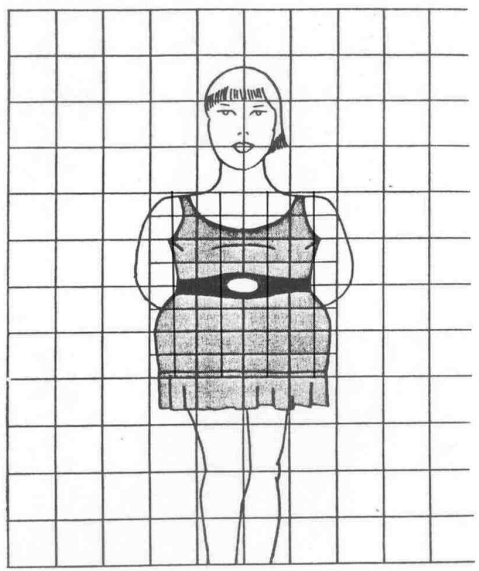
Proposition 3. For surjective semi-conjugacies

$$\begin{array}{ccc} Y & \xrightarrow{q} & Y \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{f} & X \end{array}, \qquad \begin{array}{ccc} Y_c & \xrightarrow{q} & Y_c \\ \pi\downarrow & & \downarrow\pi \\ J_c\cap\mathbb{R} & \xrightarrow{f_c} & J_c\cap\mathbb{R} \end{array}$$

$h(X, f) \leq h(Y, q)$ and $\sup_x \#\pi^{-1}(x) < \infty \Longrightarrow h(X, f) = h(Y, q)$.

In Douady's case $h(I_c, f_c) = h(K_c \cap \mathbb{R}, f_c) \overset{Prop.2}{=}$
$h(J_c \cap \mathbb{R} \sqcup Attractors, f_c) \overset{Prop.1}{=} h(J_c \cap \mathbb{R}, f_c) \overset{Prop.3}{=} h(Y_c, q)$.

§5. Entropy and dimension on the torus



Fix a degree $d \geq 2$. For every n, partition the unit square into d^{2n} small squares of size $\frac{1}{d^n}$. For a closed subset A, denote by $\nu(A, n)$ the number of standard level-n squares intersecting A and, whenever the limit exist,

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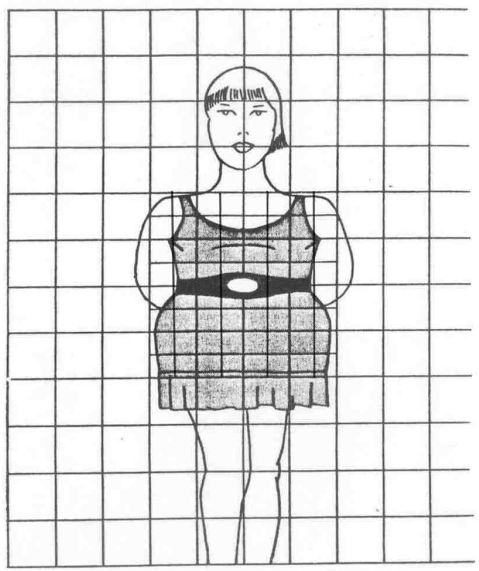

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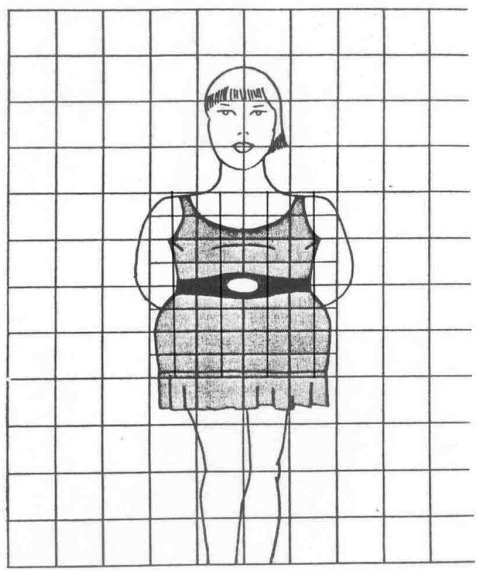

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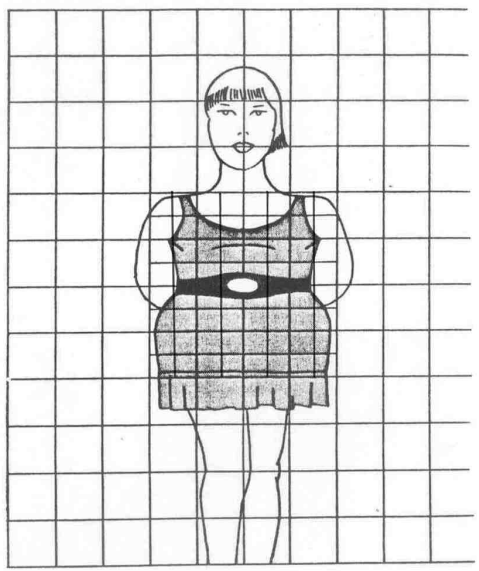

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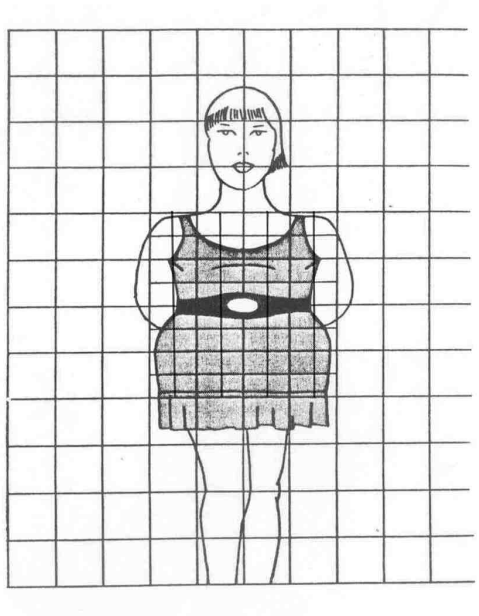

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Theorem. For $F : \mathbb{T}^2 \to \mathbb{T}^2, (x, y) \mapsto (dx, dy)$, $F(A) \subset A$, A closed,

- both limits exist, $d(A) = \text{H.dim}(A)$, $d(A) = \frac{h(A)}{\log d}$ (adapting Furstenberg's similar result on S^1)
- $h(A) = h(A, F)$ (adapting Misiurewicz-Szlenk)

Existence of the limit

$$\{S_{n+m}\} \hookrightarrow \{(S_n, S_m)\}$$

$$\begin{array}{ccc} & & \mathbb{T}^2 \\ & \nearrow_{F^{\circ n}} & \cup \\ S_n & & S_m \\ \cup & \nearrow_{F^{\circ n}} & \\ S_{n+m} & & \end{array}$$

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So $\nu(A, n+m) \leq \nu(A, n) \cdot \nu(A, m)$.

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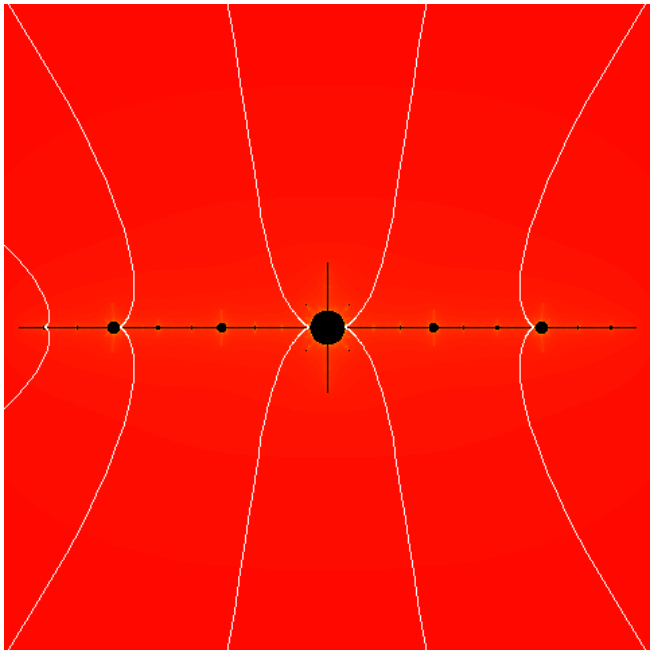
We will see that (F, A), $(F, B(A))$ for various A parametrizes combinatorially all degree d polynomials, with $h(A)$ as the core entropy.

§6. Thurston's bi-circle (torus) model, '2001



The set $A_c := \{(t, 1-t) \in \mathbb{T}^2, t \in Y_c\}$, consisting of bi-angles landing at a common point on $\mathbb{R}$, is a closed forward invariant set of $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto 2\begin{pmatrix} s \\ t \end{pmatrix}$, $\mathbb{T}^2 \to \mathbb{T}^2$ with the same entropy.

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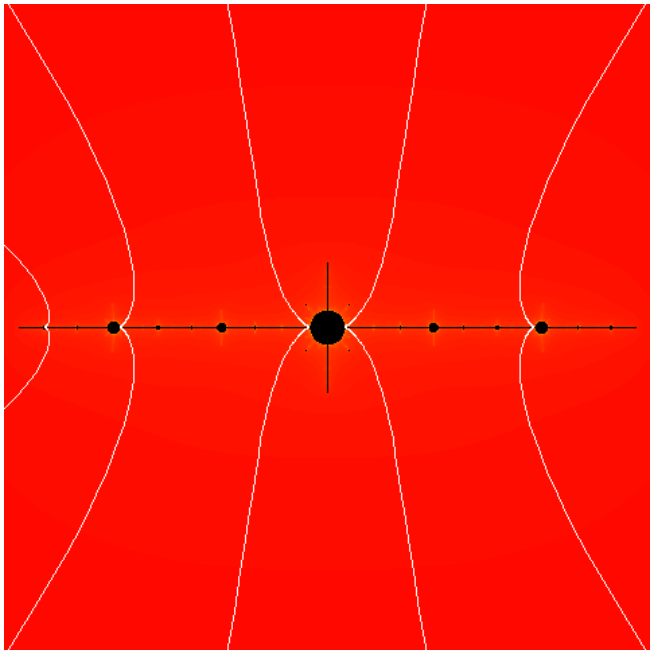

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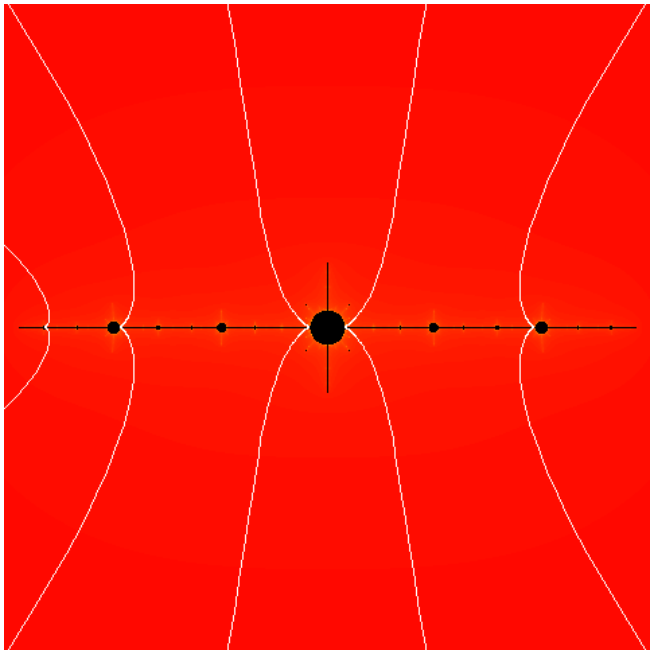

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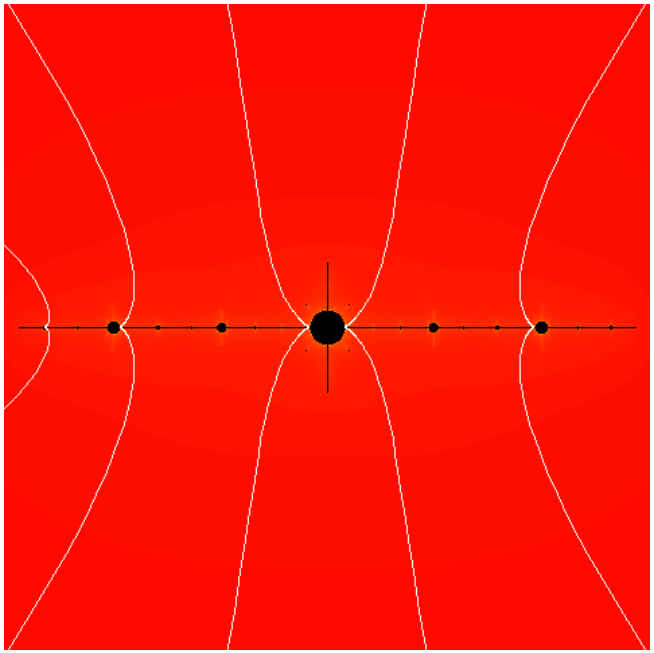

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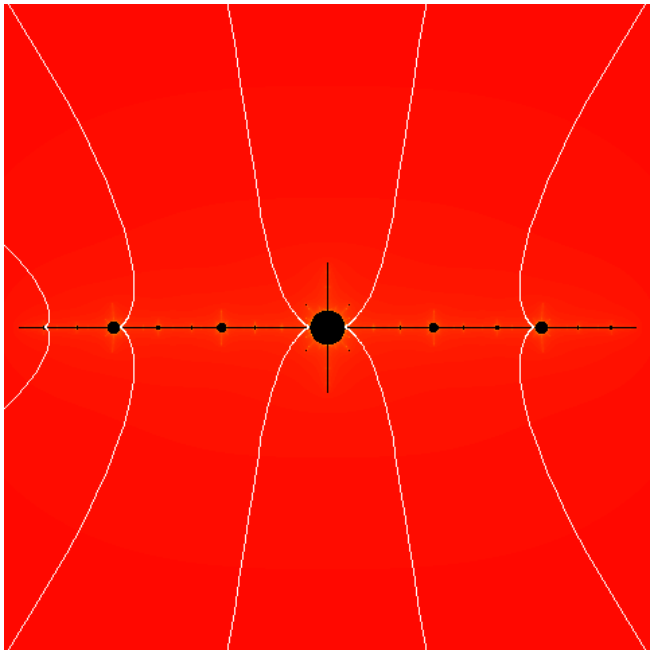

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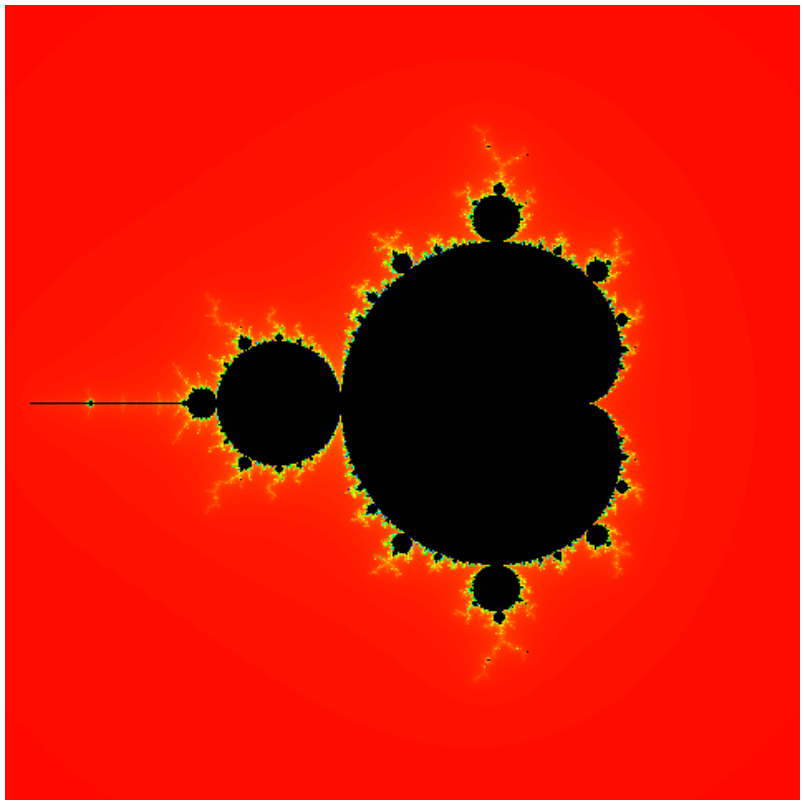
Consequently, the set $B(A_c) := \bigcup_n F^{-n}(A_c)$, consisting of bi-angles landing at a common point on the Julia set, has the same dimension as A_c.

§7. Parameter view of real $z^2 + c$

The Mandelbrot set $\mathcal{M}$ is defined by

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^{\circ n}(0) \nrightarrow \infty\}$$

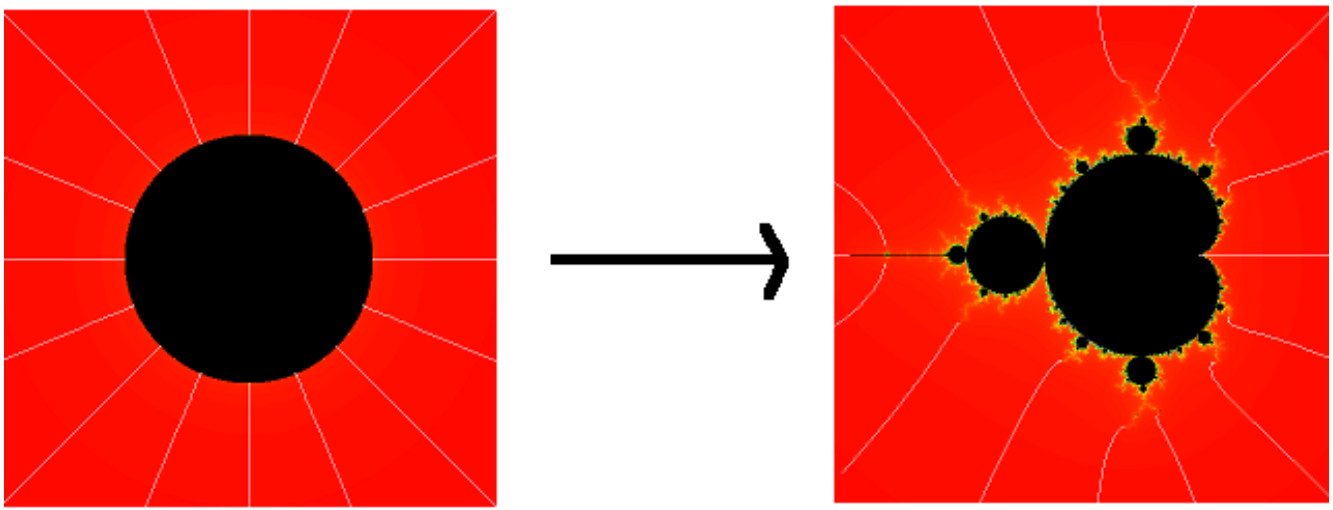


External rays

Since $\hat{\mathbb{C}} \setminus \mathcal{M}$ is simply-connected (by Douady-Hubbard and Sibony), there is a conformal map :

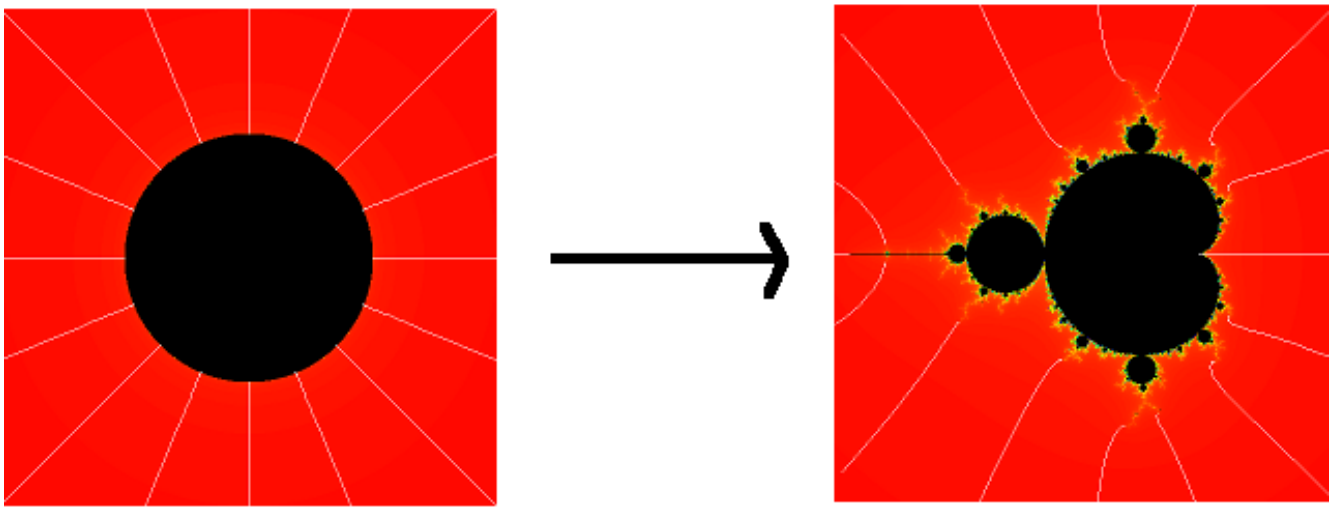
$$\Phi_{\mathcal{M}} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \hat{\mathbb{C}} \setminus \mathcal{M}$$

parametrizing the outside and the boundary of $\mathcal{M}$ by external angles.



Almost all (and conjecturally all) rays land at some point on the boundary of $\mathcal{M}$.

Douady's monotonicity argument ('93)



The following chain of associations are trivially monotone :

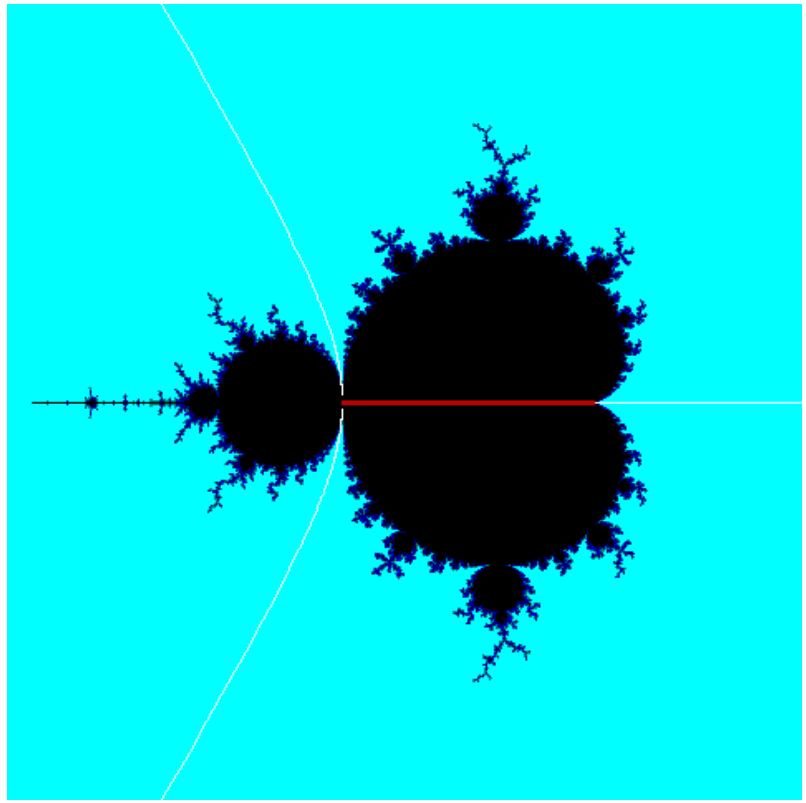
$$[-2, \frac{1}{4}] \ni c \quad \overset{\text{external angle}}{\longrightarrow} \quad \theta \in [0, \frac{1}{2}] \to \text{forbidden region }]\theta, 1-\theta[$$

$$\longrightarrow \quad X_\theta := \{t \in S^1 \mid 2^n t \notin\ ]\theta, 1-\theta[\,\}$$

$$\longrightarrow \quad h(X_\theta, q) \underset{\text{less trivial}}{\overset{q(Y_c)=X_\theta}{=}} h(I_c, f_c).$$

Tiozzo's section theorem ('2012)

Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\,\}$.



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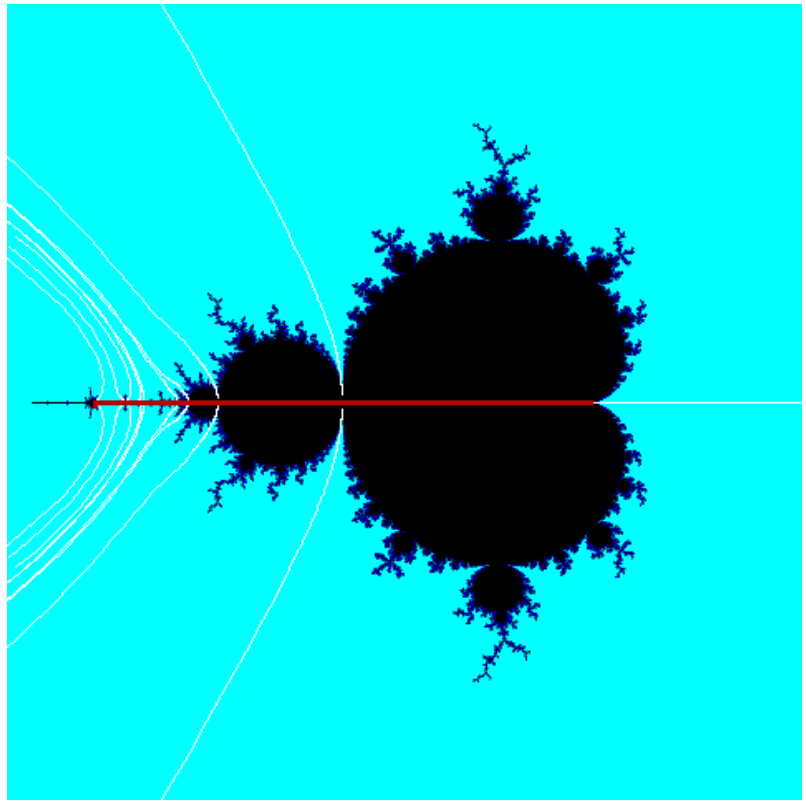

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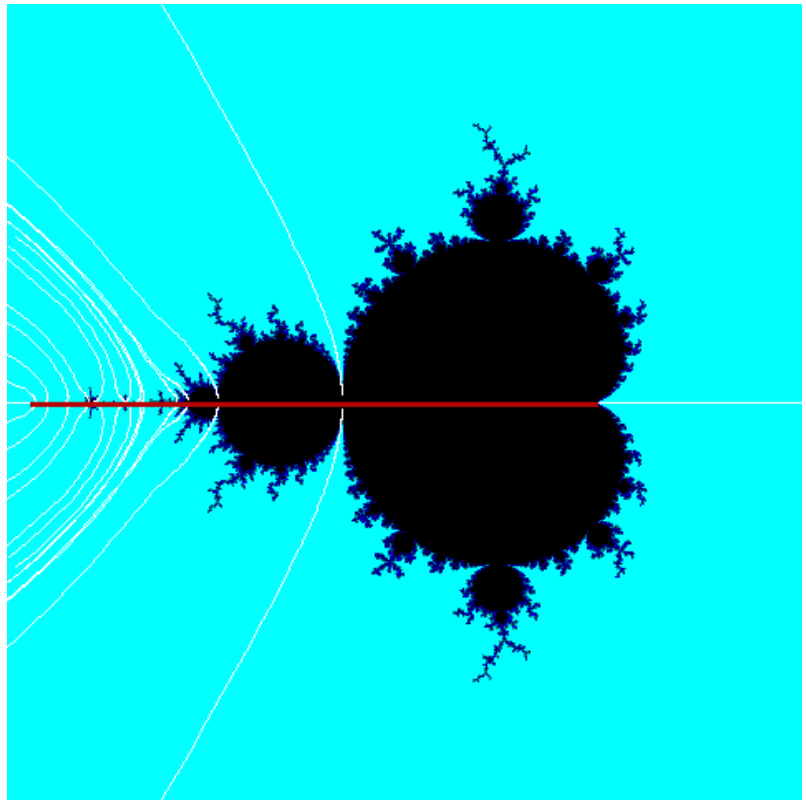

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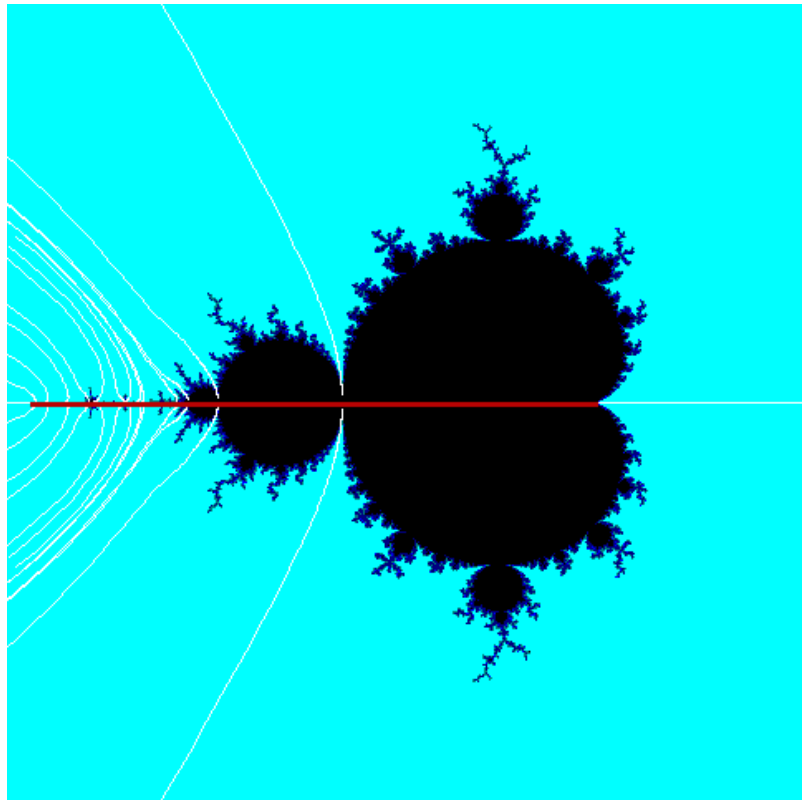

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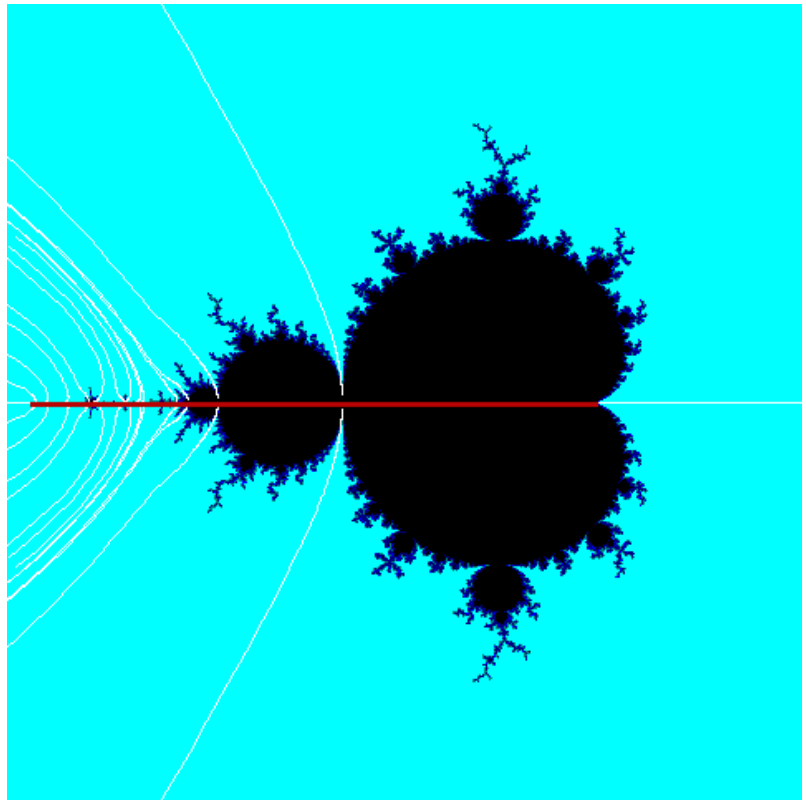

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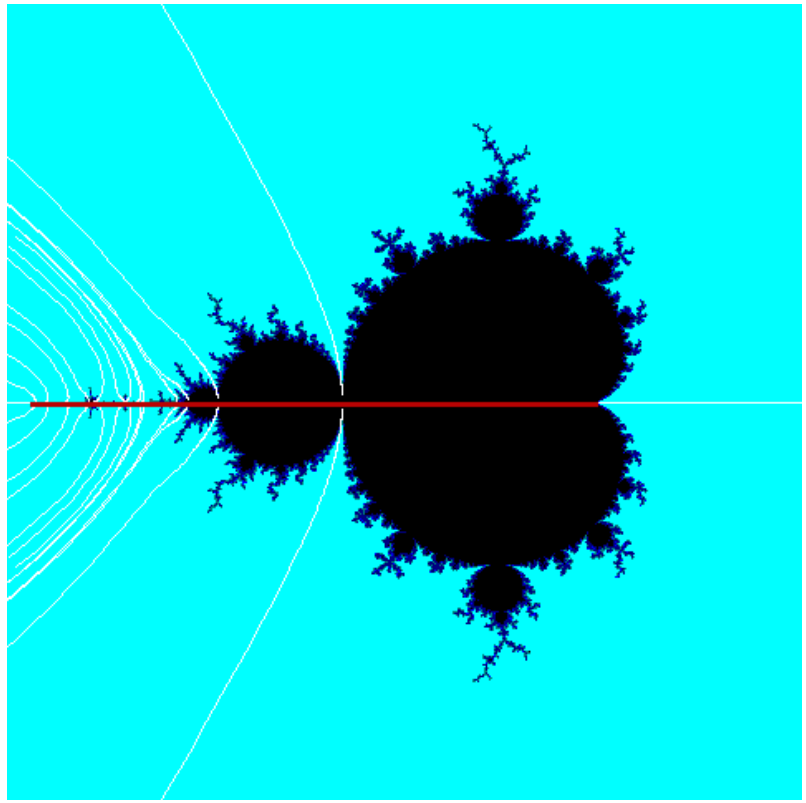

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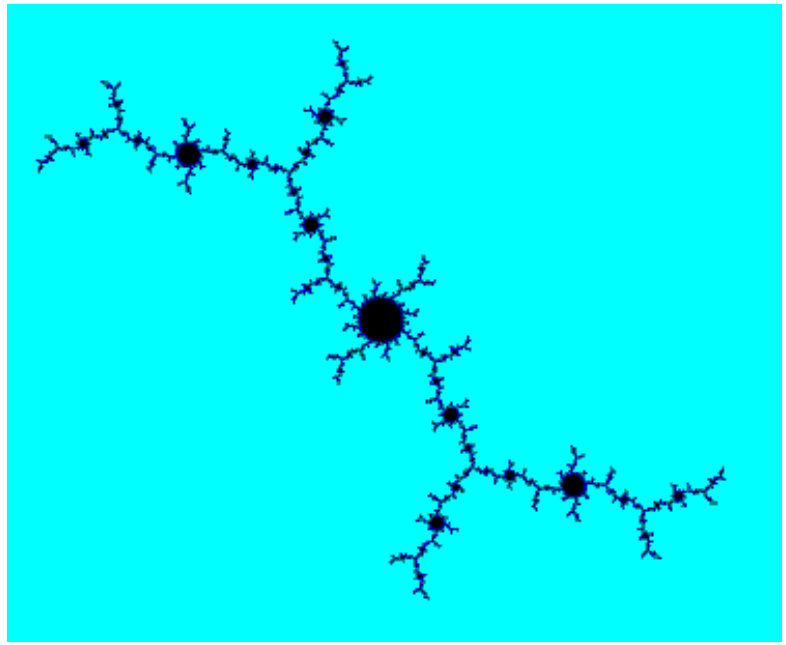
The proof uses techniques coming from α-continued fractions.

§8. How to define core-entropy(P) for a non-real poly. P ?



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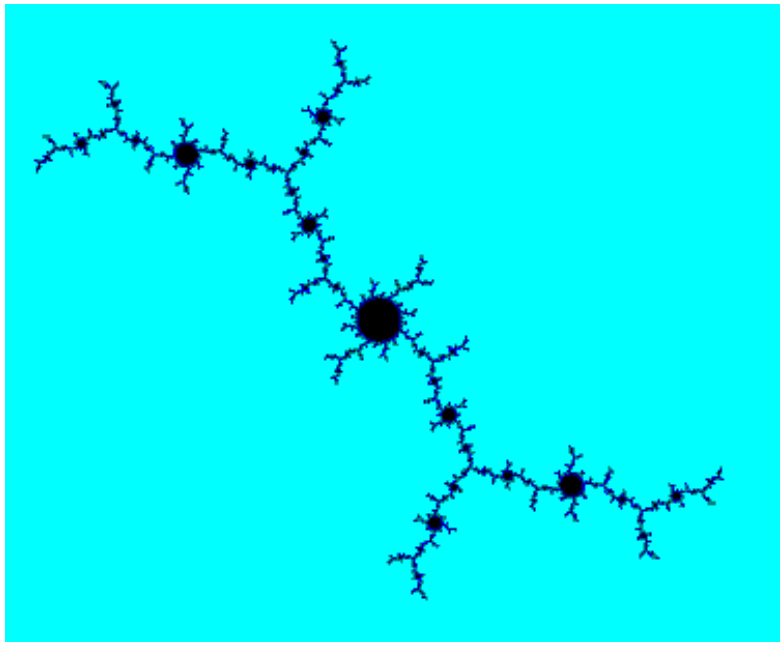

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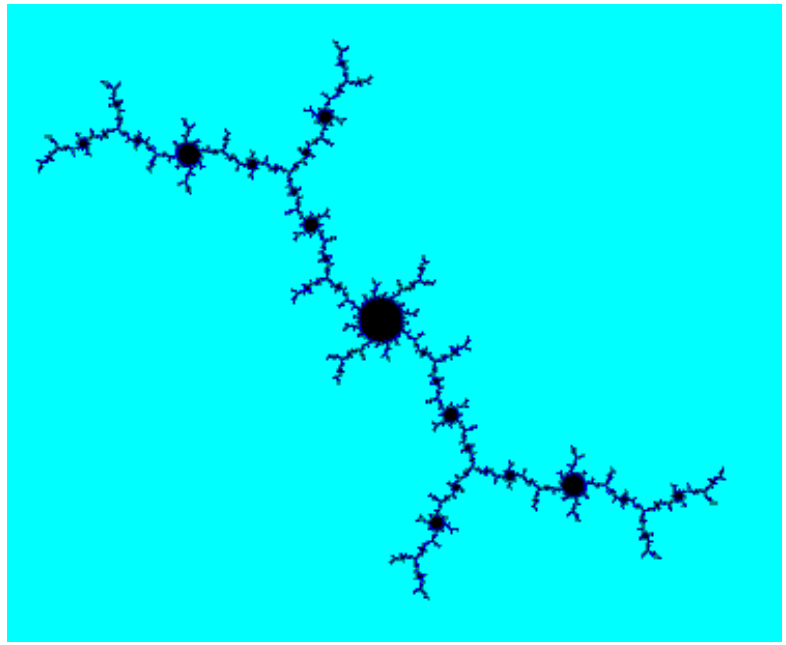

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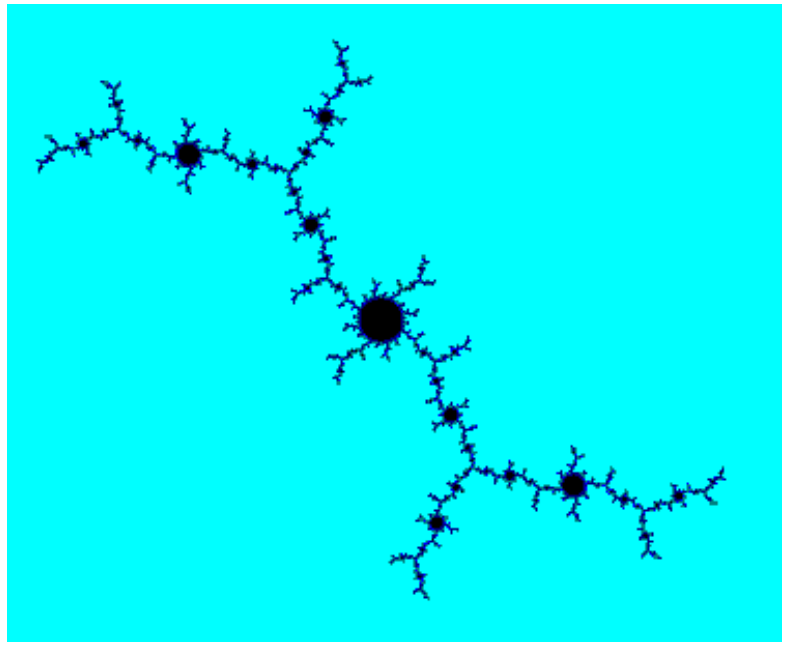
$\{P, Post.cr(P) < \infty\}$

§8. How to define core-entropy(P) for a non-real poly. P ?



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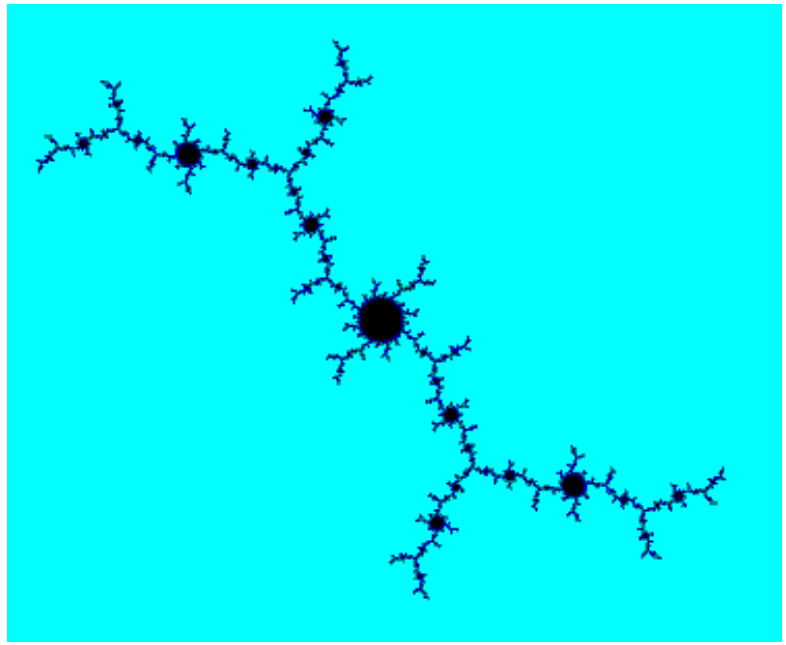

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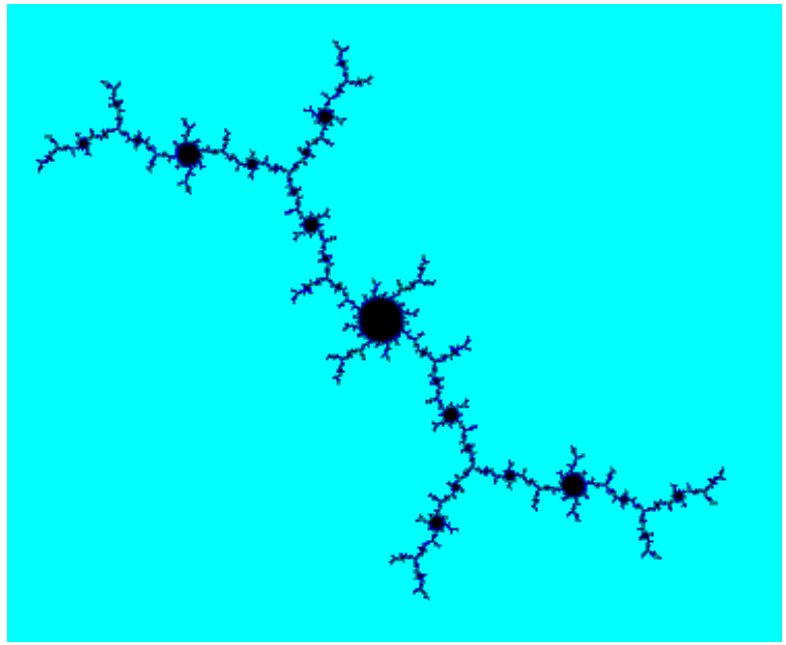

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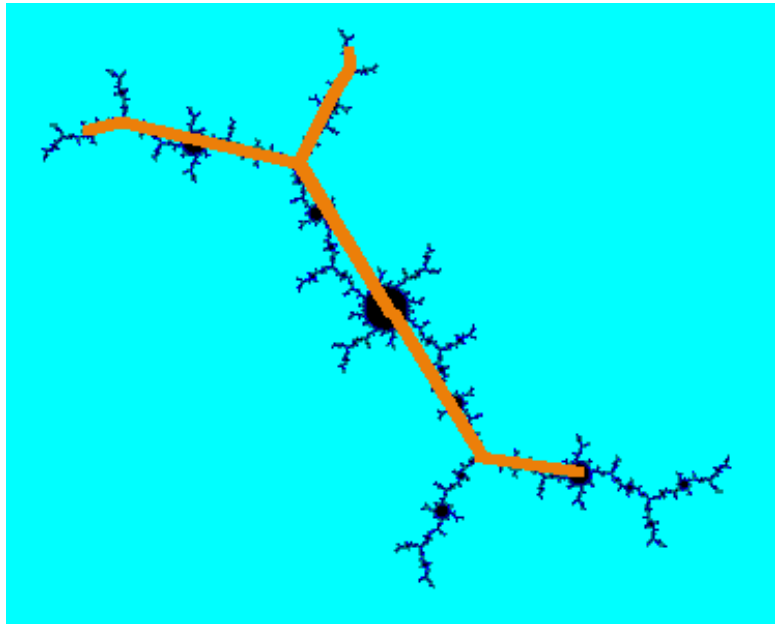
$\{P, Post.cr(P) < \infty\} \subset \{P, \exists \text{tree}\} \subset \{P, K_P \text{ connected}\} \subset$ {all polynomials}.

The Hubbard tree H_c of $f_c : z \mapsto z^2 + c$, if exists, is the smallest topologically finite tree in K_c containing the orbit of the critical point 0. It is automatically forward invariant.



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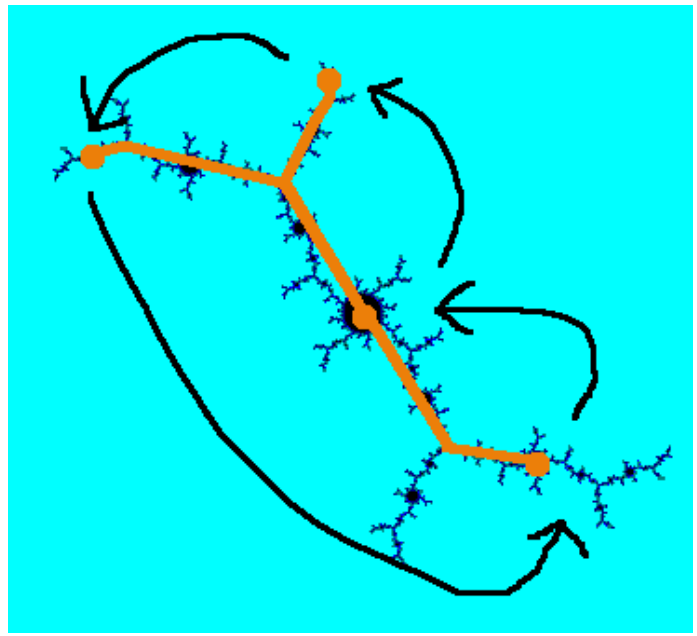

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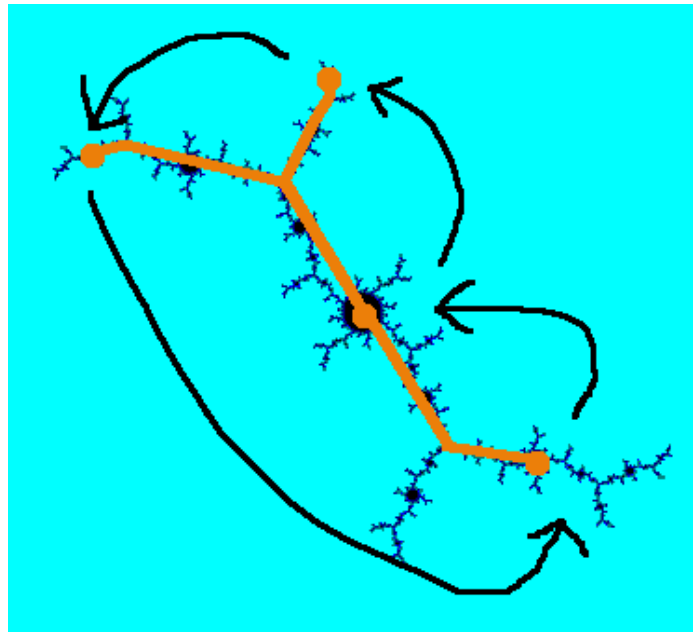

For many "good" values of c, the Hubbard tree, and many larger forward invariant trees, exist, and all give the same entropy (see Tiozzo's preprint). For even better values of c (such that 0 has a finite orbit), the tree admits a Markov partition and matrix and

$$\text{Core-entropy}(f_c) = \log \lambda_{\text{leading}}(\text{Markov matrix}).$$

§9. The postcritically finite case

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With the Hubbard tree H_f playing the role of the interval, $exp(h(H_f, f))$ is the growth of

the number of edges in $f^{-n}H_f \cap H_f$ (the lap number);
the maximal edge decomposition of an edge of H_f under f^{-n};
the maximal number of nth-preimages of a point in H_f;
the number of n-periodic ends in H_f;
the number of n-periodic angles landing on H_f; $\cdots$

If all critical points of a polynomial f are strictly preperiodic f is said 'Misiurewicz'.

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- For the 'antenna' c_n of the $\dfrac{1}{n}$-th limb of M, the growth number is the largest positive real root of $(-1)^n x(x^{n+1} - x^n - 2)$.

§10. Other models

There are other models to encode the core (binding) entropy, such as Bartholdi-Dudko-Nekrashevych's automata model. Relating them all and studying parameter variations is part of the current research. Many results/questions are also valid for higher degree polynomials.

§11. Thurston's parametrization by primitive majors

Thurston used primitive majors (critical portrait) to parametrize degree d polynomials and to encode their core entropy (partially proved by Gao and T.)

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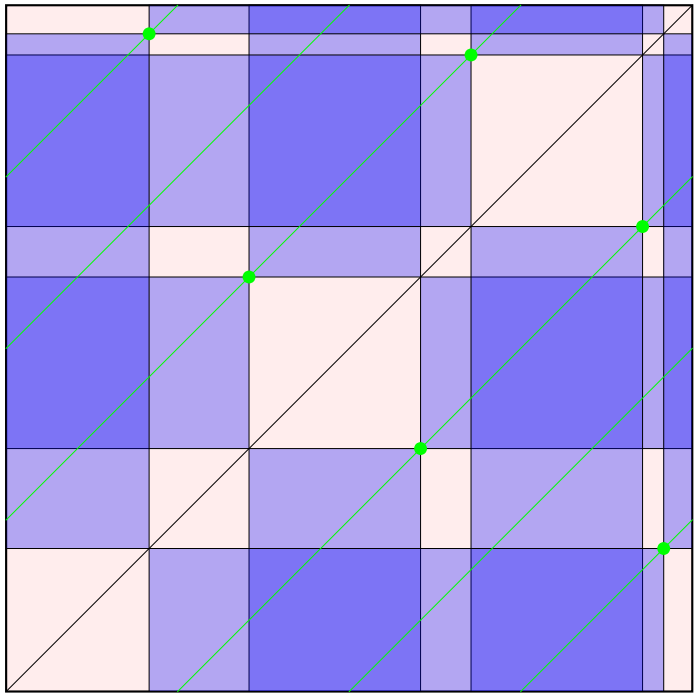
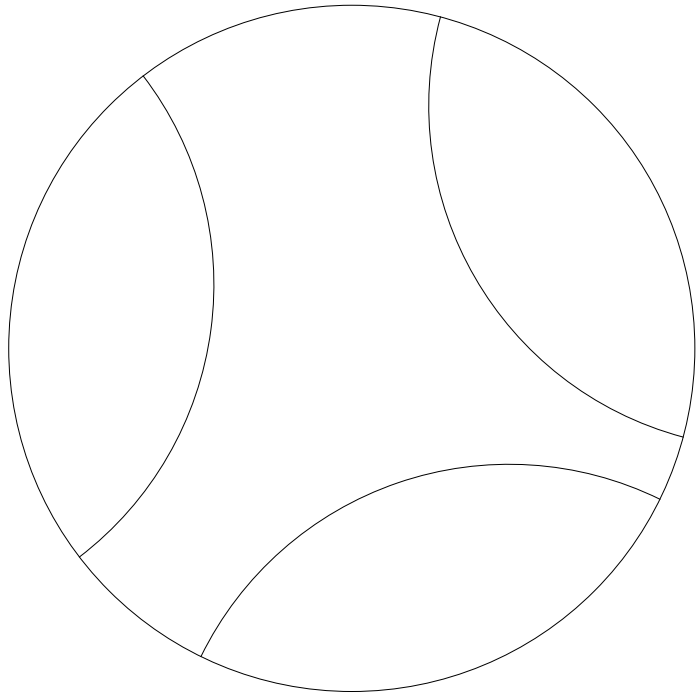
From a primitive major m to its binding set $B(m)$

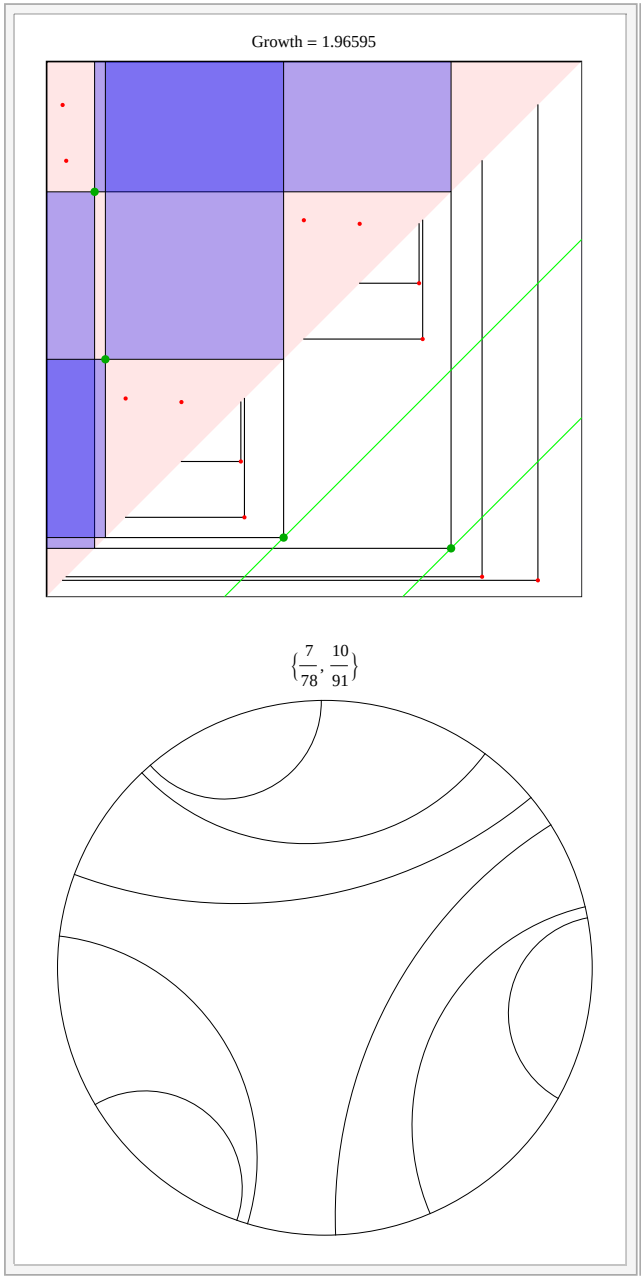


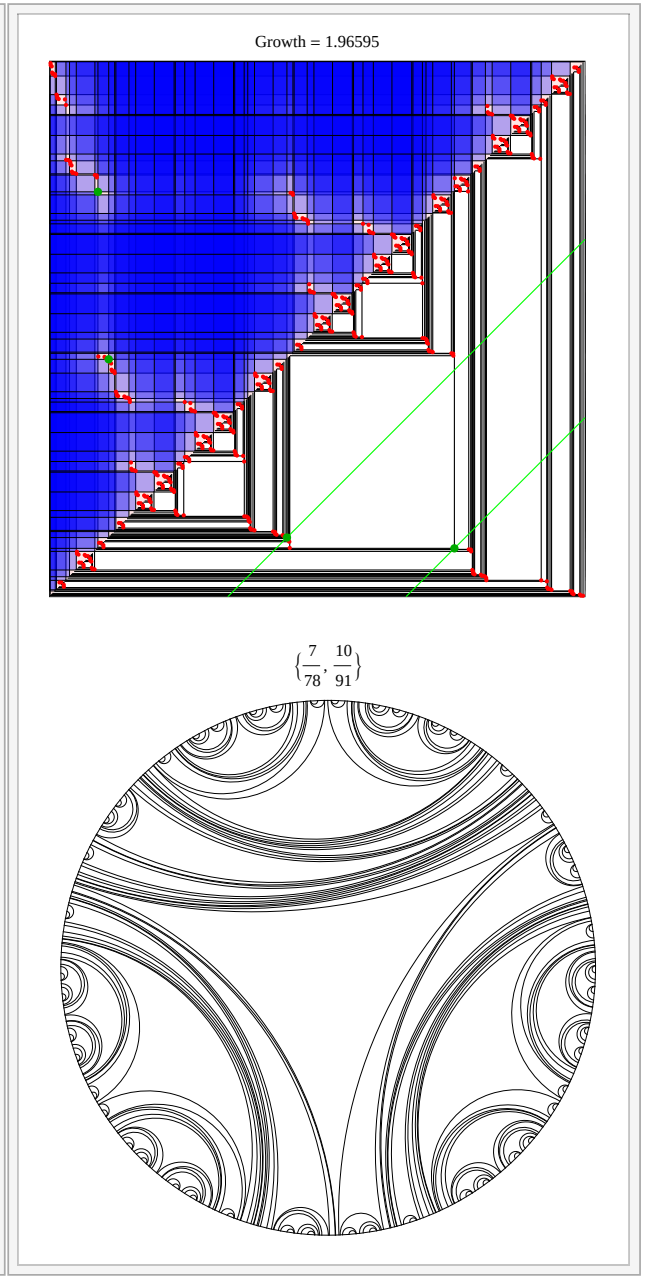


A primitive major m of degree d is a finite lamination in $\mathbb{D}$ so that each region of $\mathbb{D} \setminus m$ touches S^1 in a union of one or more closed intervals $J_1 \cup \cdots \cup J_k$ of total length $1/d$. The product $(J_1 \cup \cdots \cup J_k) \times (J_1 \cup \cdots \cup J_k)$ is a union of light pink rectangles.

The set $K(m)$ is the non-escaping locus under $F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto d \begin{pmatrix} s \\ t \end{pmatrix}$.

The binding set, whose dimension is to be measured





§12. Thurston's entropy algorithm bypassing trees and dimensions (march 2011)

Start from any rational angle $\theta \neq 0$ and separate the circle into two halves by $\theta/2, \theta + 1/2$, each half is a closed segment in the circle (so the cutting points belong to both halves).

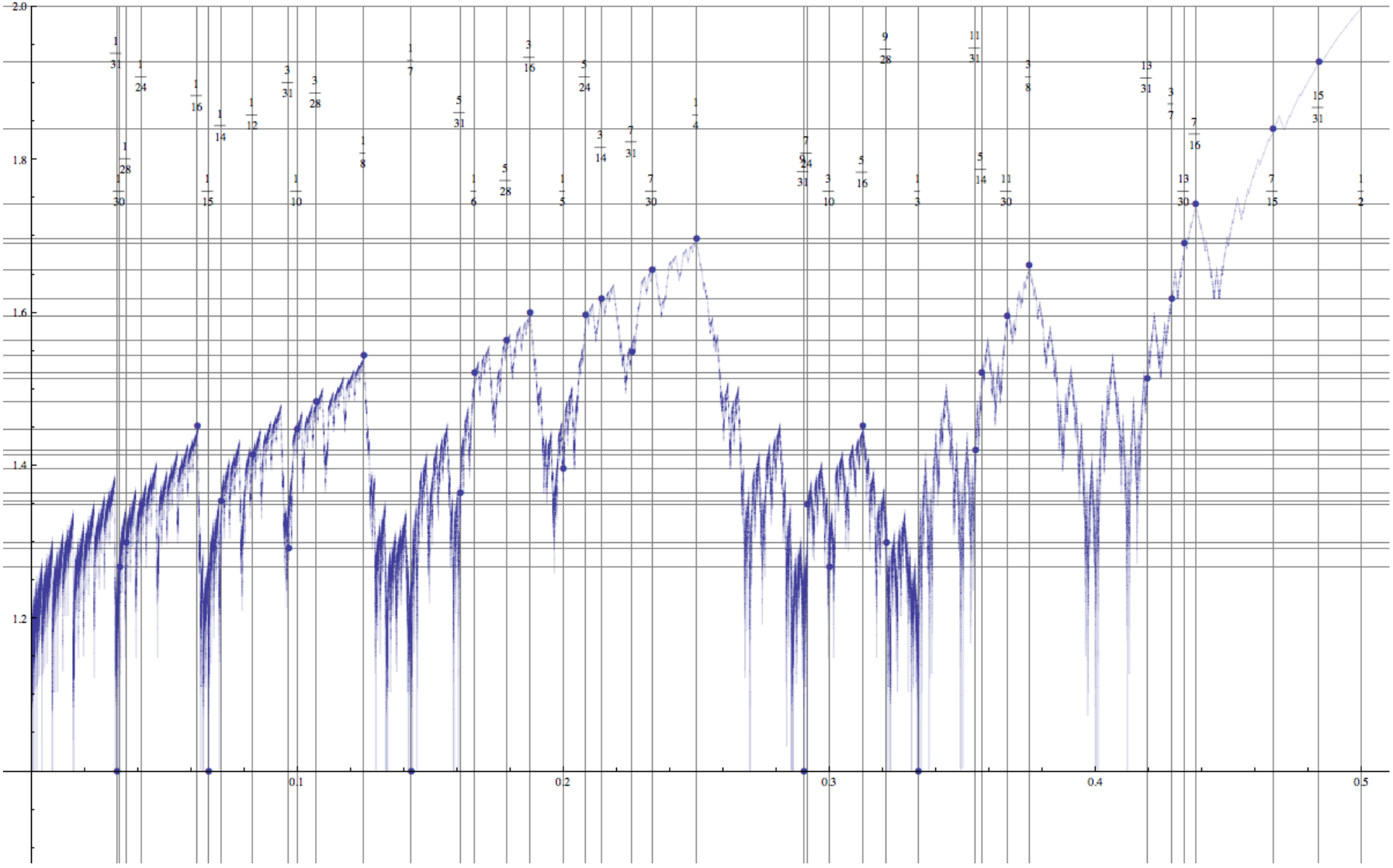
In the following all the $\theta/2$ should be replaced by $\theta + 1/2$ if the latter is periodic.

1. Form the set of all possible unordered pairs $\{2^n\theta, 2^l\theta\}$ so that both l, n are ≥ -1 and the two angles are unequal. Consider them as a basis of an abstract linear space.
2. Compute the leading eigenvalue of the linear map :

$$\{2^n\theta, 2^l\theta\} \mapsto \begin{cases} \{2^{n+1}\theta, 2^{l+1}\theta\} & \text{if } 2^n\theta, 2^l\theta \text{ are in the same half-circle} \\ \{2^{n+1}\theta, \theta\} + \{\theta, 2^{l+1}\theta\} & \text{otherwise} \end{cases}$$

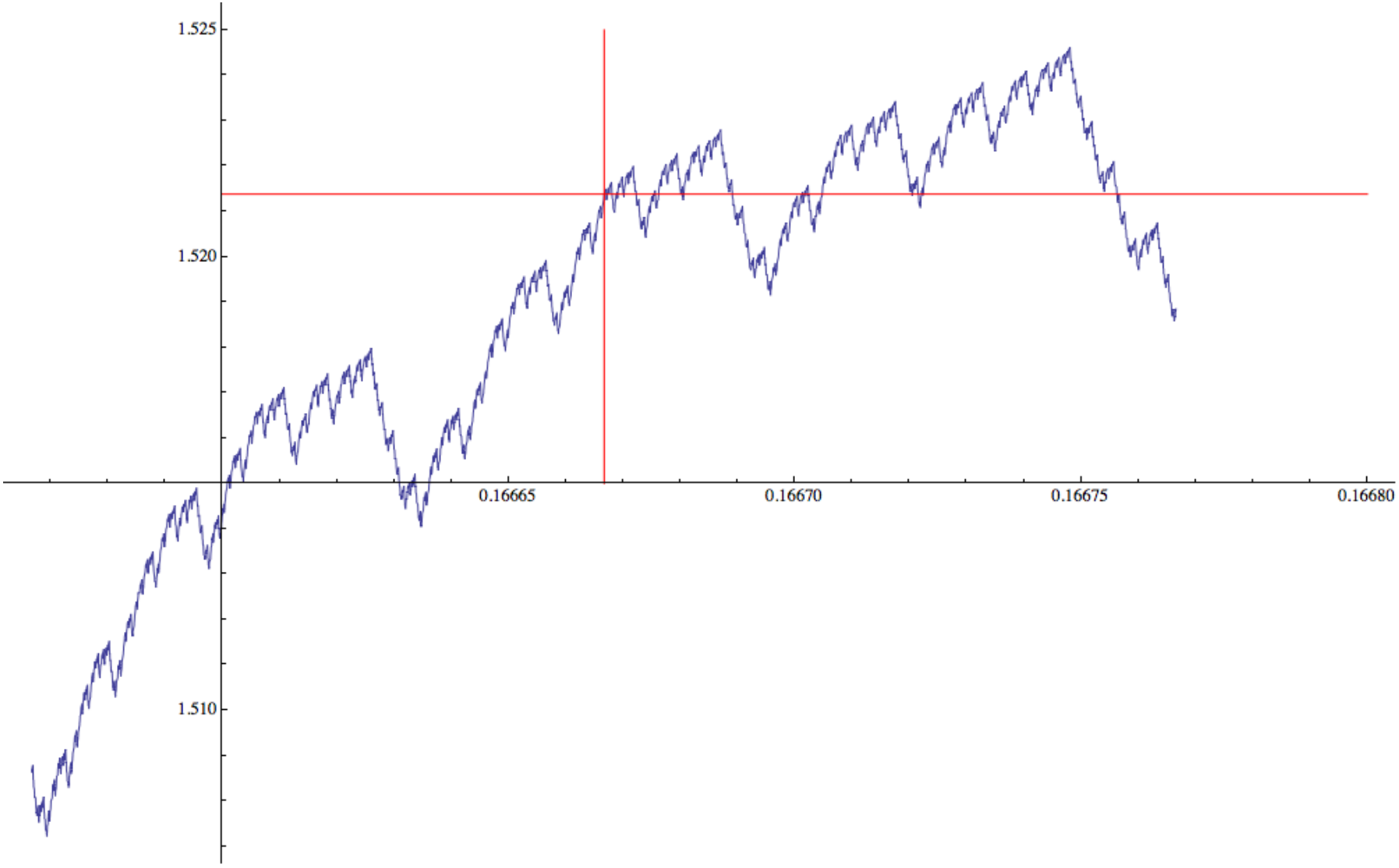
3. You get e^h. A similar algorithm works for any degree.

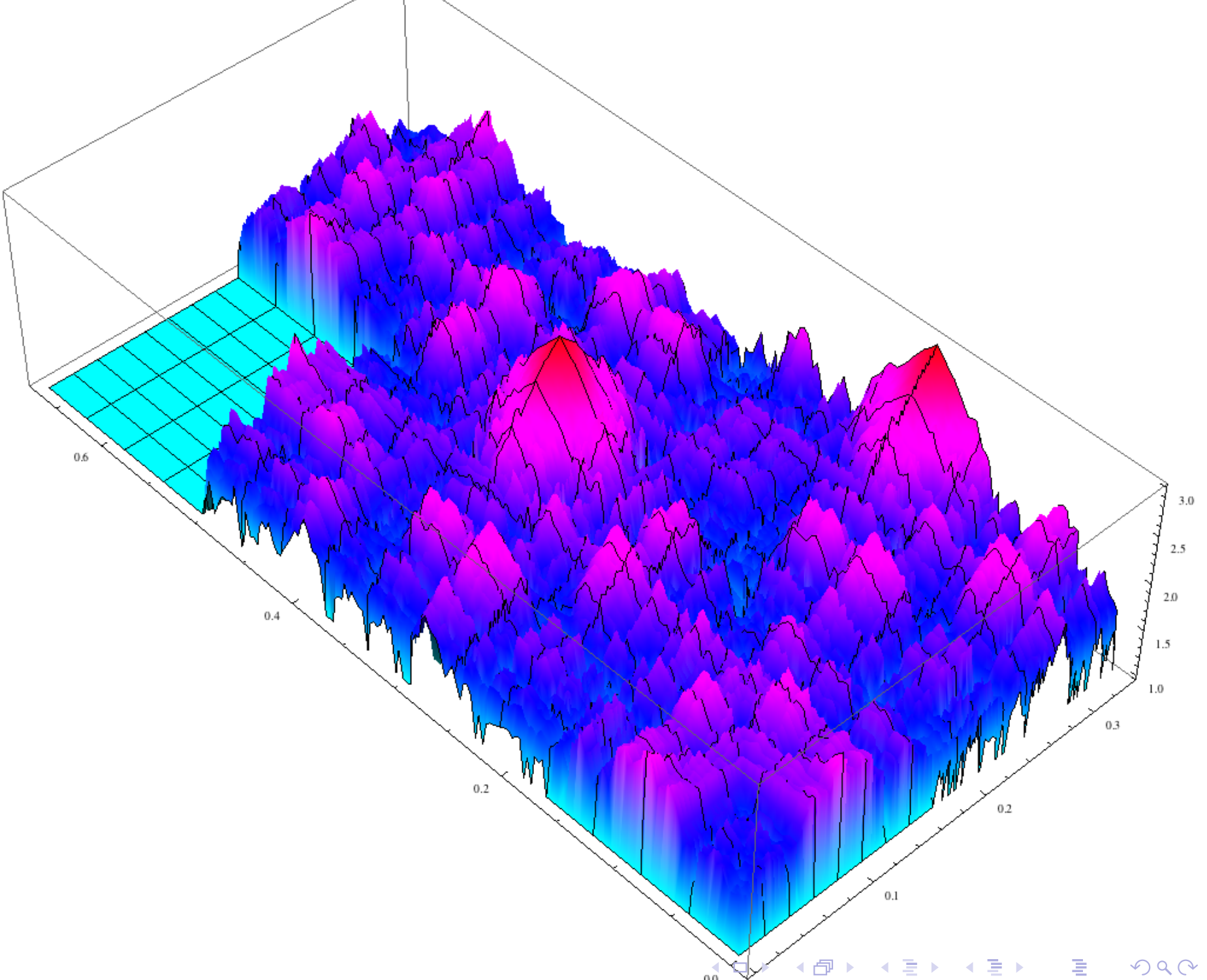
Thurston's plot $\mathbb{Q} \ni \theta \mapsto$ core entropy of c_θ



Continuous ? Local-max = pseudo-center ? See works of Alsedà-Fagella, Bruin-Schleicher, Jung, Tiozzo...

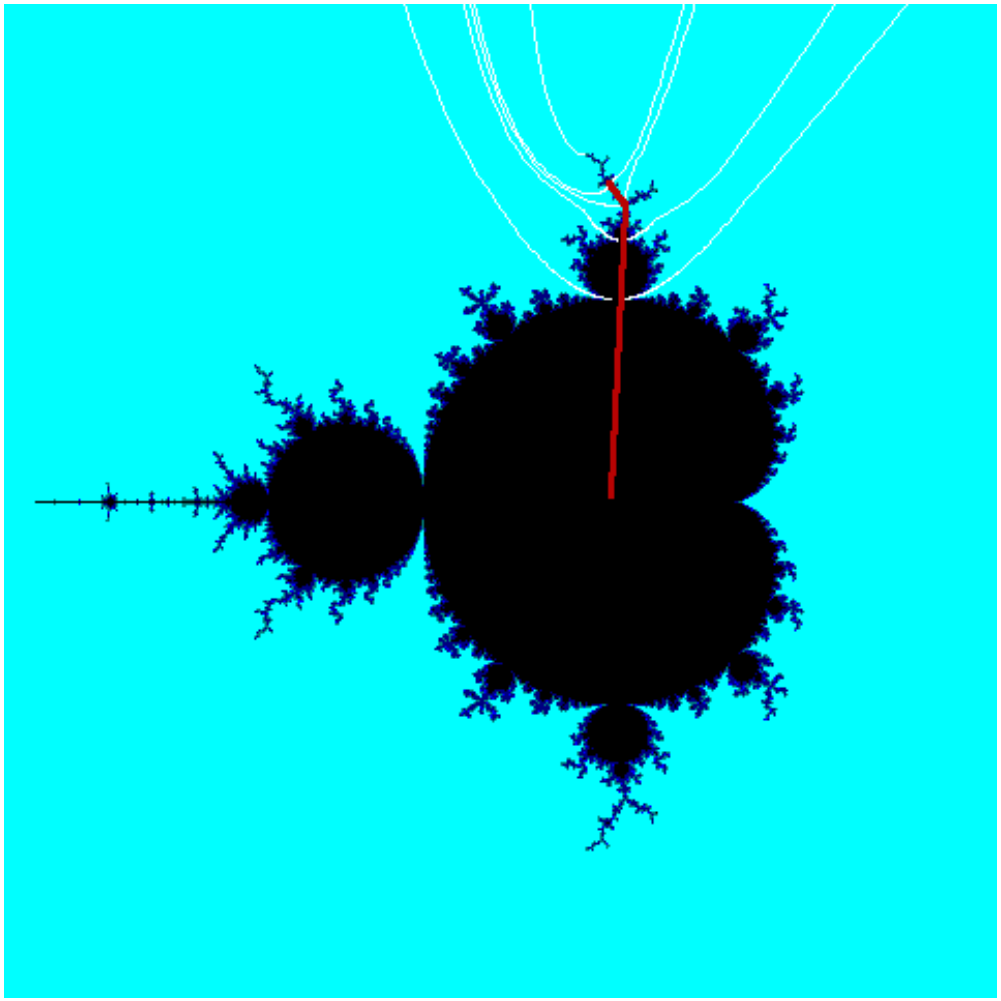
and a zoom at 1/6




0.6
0.4
0.2
0.0
0.1
0.2
0.3
1.0
1.5
2.0
2.5
3.0

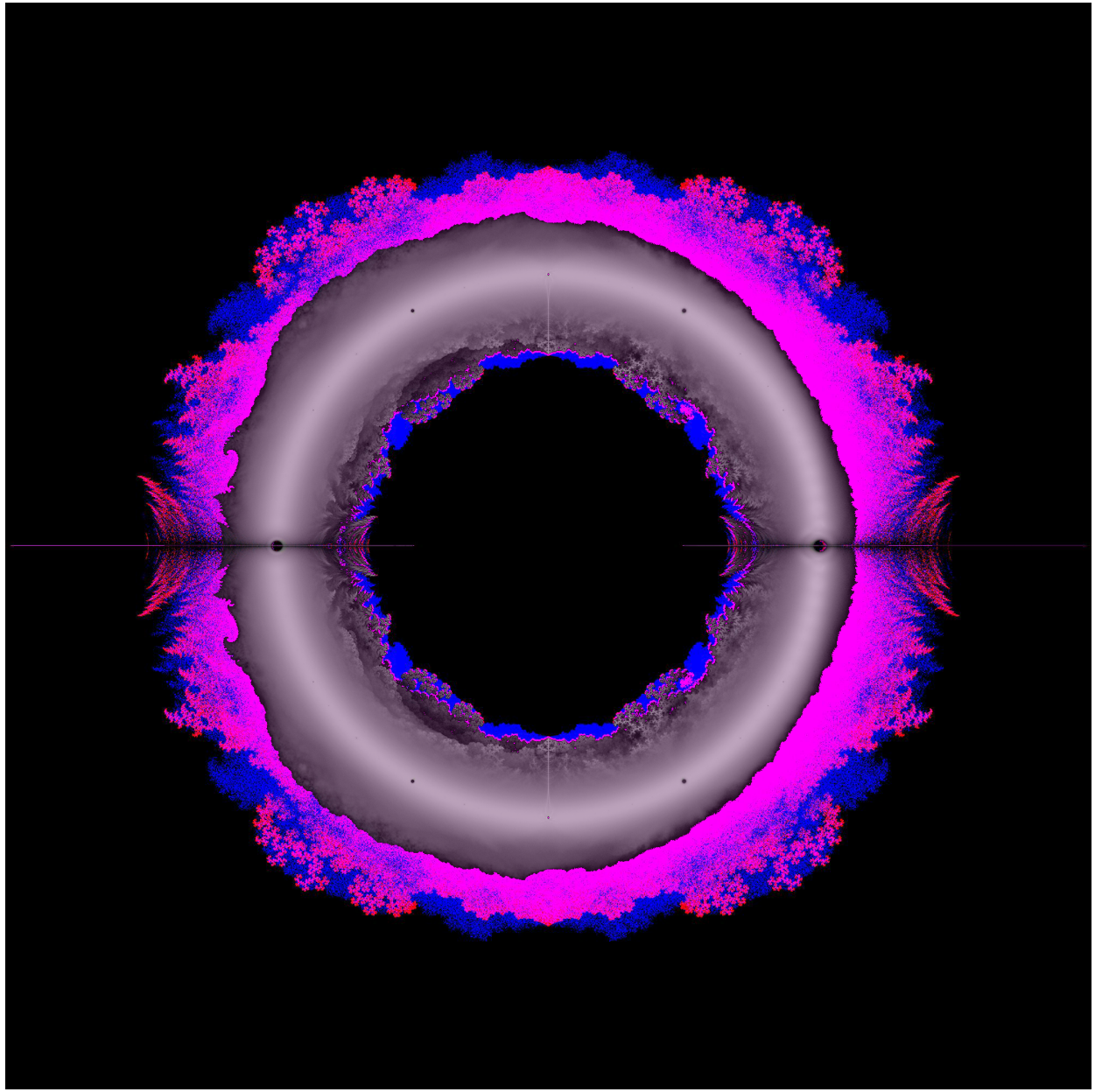
§13. Monotonicity and continuity along the veins

The Mandelbrot set has a tree-like structure of veins, with the real segment as a particular vein.



See works of Jung, Li and Tiozzo ...

Appendix. Realization of Perron numbers by W. Thurston



A Perron number is $s > 1$ as a root of a minimal monic polynomial $p(z)$ with coefficients in $\mathbb{Z}$ and with $s > |other\ root\ of\ p|$.

Remark. All core-entropies of (real) postcritically finite polynomials are (weak) Perron numbers.

Theorem (Thurston 2011). The converse is true : every Perron number λ is realized by a real postcritically finite polynomial Q, and $degree(p) \leq degree(Q)$.

Question. Which λ are realized by quadratic Q's ? or degree d Q's ?

Remark. All $\dfrac{1}{\text{core-entropy}}$ of real quadratic postcritically periodic polynomials are roots of $D(t) = 1 + \sum_{i \geq 1} \pm t^i$ with periodic coefficients. So

$$D(t) = \frac{1 \pm t \pm t^2 \pm \cdots \pm t^n}{1 \pm t^m} \overset{w=\frac{1}{t}}{=} \frac{t^n(w^n \pm w^{n-1} \pm \cdots \pm 1)}{1 \pm t^m} = \frac{t^n P_c(w)}{1 \pm t^m}$$

with the core-entropy as the positive largest root of $P_c(w)$.

$$D(\frac{1}{w}) = \frac{t^n P_c(w)}{1 \pm t^m}$$

with the core-entropy as the positive largest root of $P_c(w)$.

gray set :=$\{$roots of such $P_c(w)$ with degree $\leq N\} \subset$
red set :=$\{$roots of monic $P(w)$ with ± 1 as coefficients and with degree $\leq N\} \subset$
blue set :=$\{$roots of monic $P(w)$ with $\pm 1, 0$ as coefficients, with $P(0) \neq 0$ and with degree $\leq N\}$.

Thank you !

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Gràcies to the organizers !