Partial self-coverings on disconnected Riemann surfaces

February 10, 2008

Joint work with Cui Guizhen
Tan Lei, Université de Cergy-Pontoise

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Objective here: Extend the theory to other settings.

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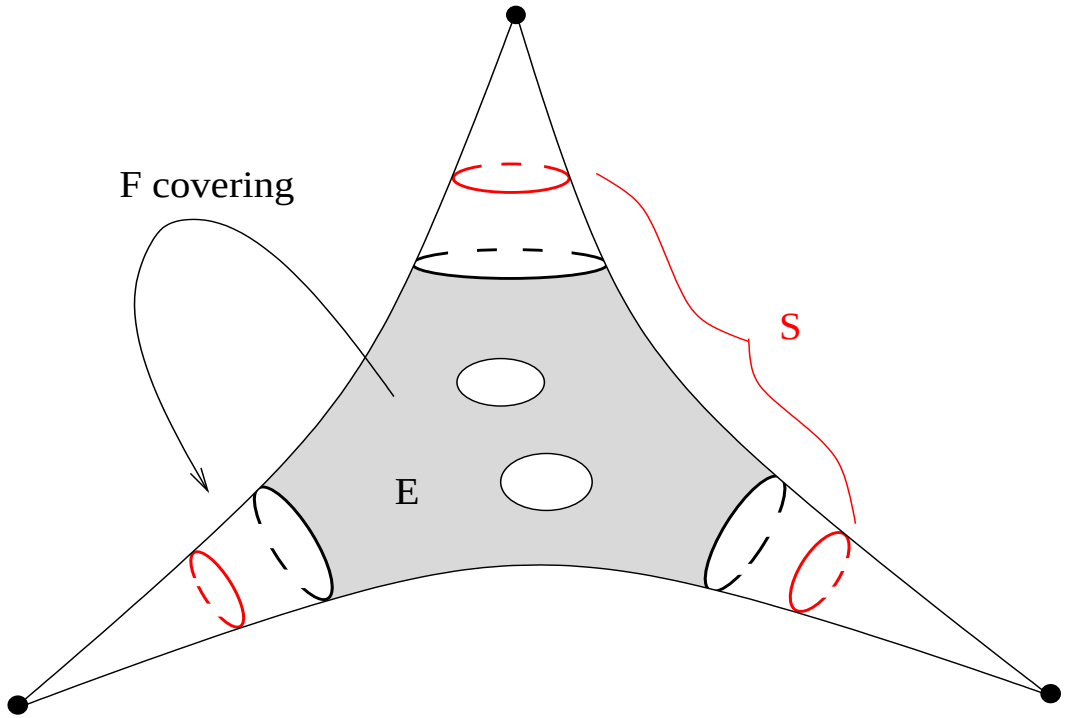
(One may also allow F to be a postcritically finite branched covering, or/and S^2 to be S_g.)

Example 1: Thurston-type

$$\begin{cases} \mathcal{S} \text{ connected}, \\ \mathcal{S} \supset\supset \mathcal{E} \text{ connected}, \\ i_* : \pi_1(\mathcal{E}) \to \pi_1(\mathcal{S}) \text{ surjective} \end{cases} \overset{Def.}{\Longrightarrow} \begin{cases} \mathcal{E} \| \mathcal{S} \\ F \text{ is of } \textbf{Thurston-type}. \end{cases}$$



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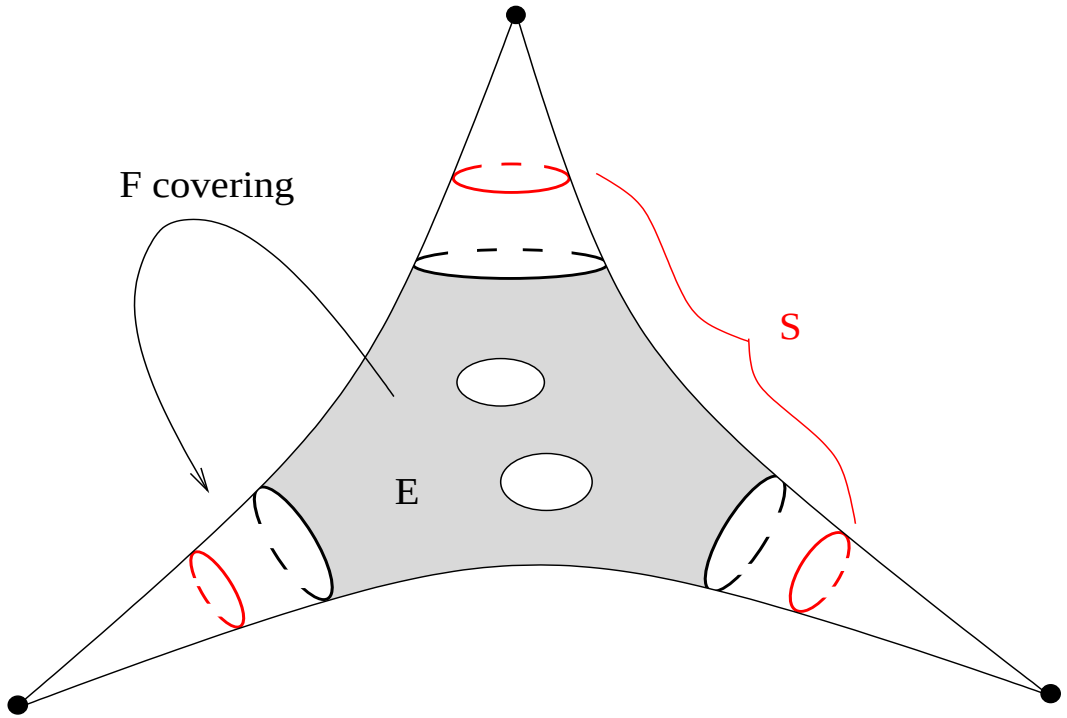
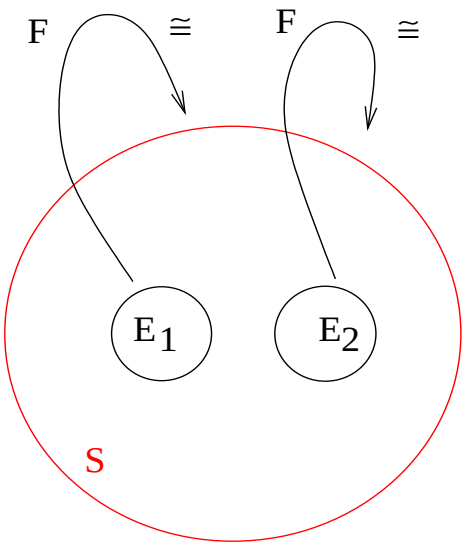
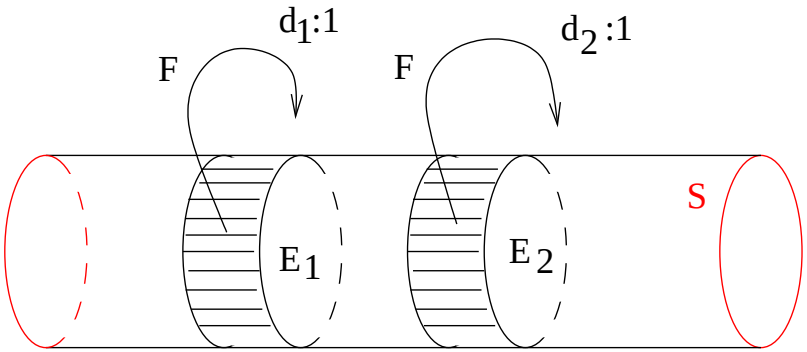
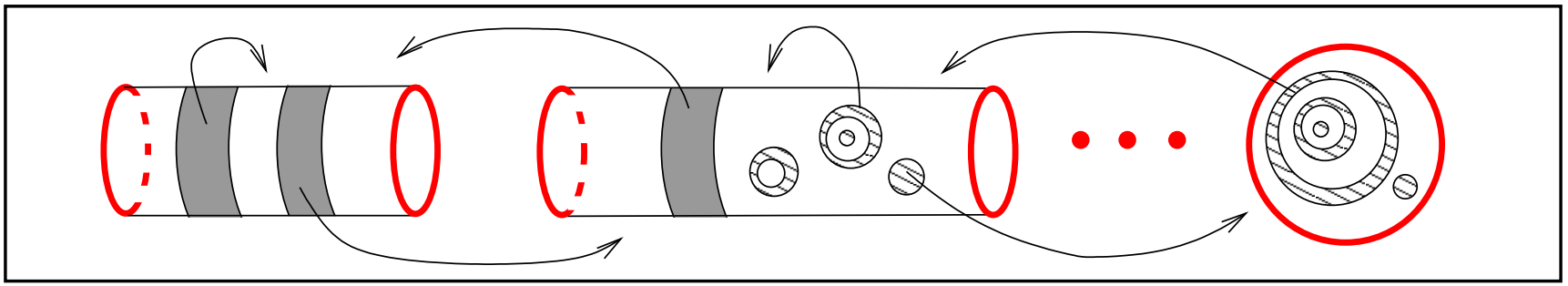


Such F extends to a postcritically finite branched covering $\widehat{F}$ of S^2 (Thurston map).

Example 2: disc-annular type: $\mathcal{S}$ has only disclike or annular components







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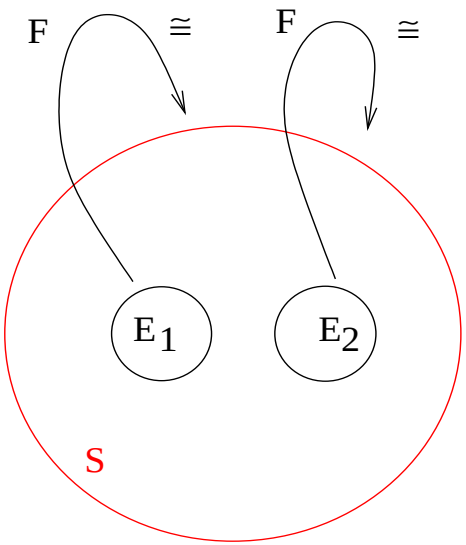

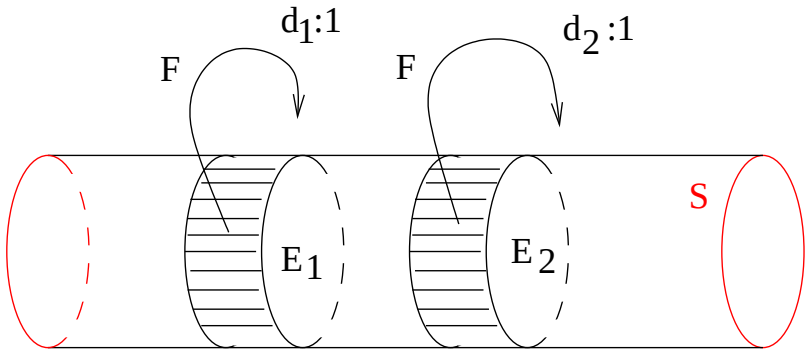

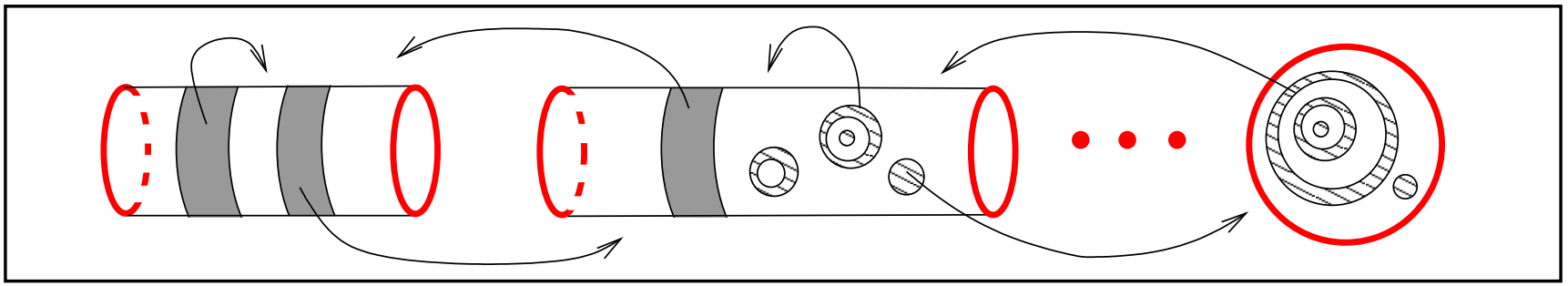

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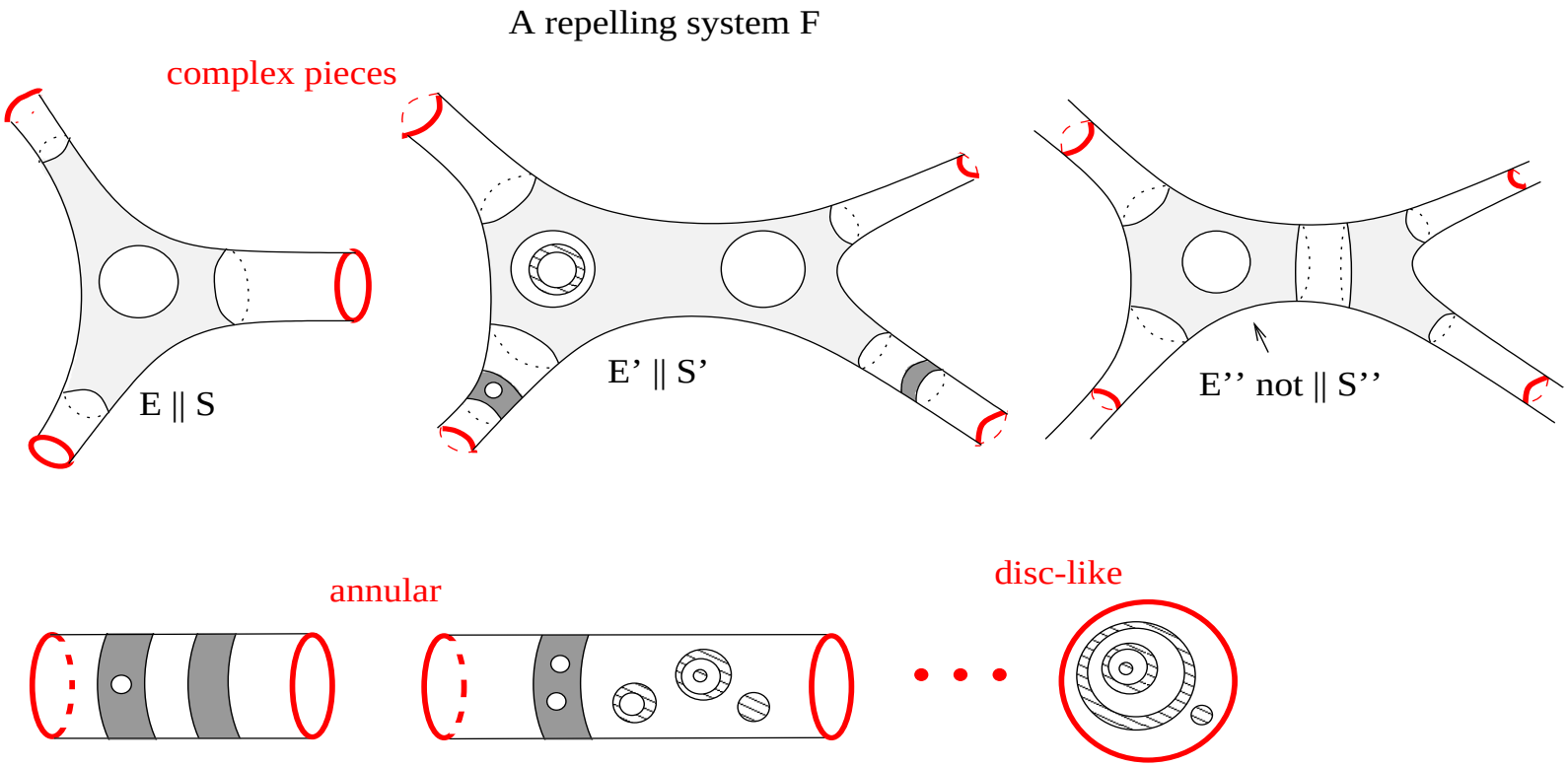
A more general example



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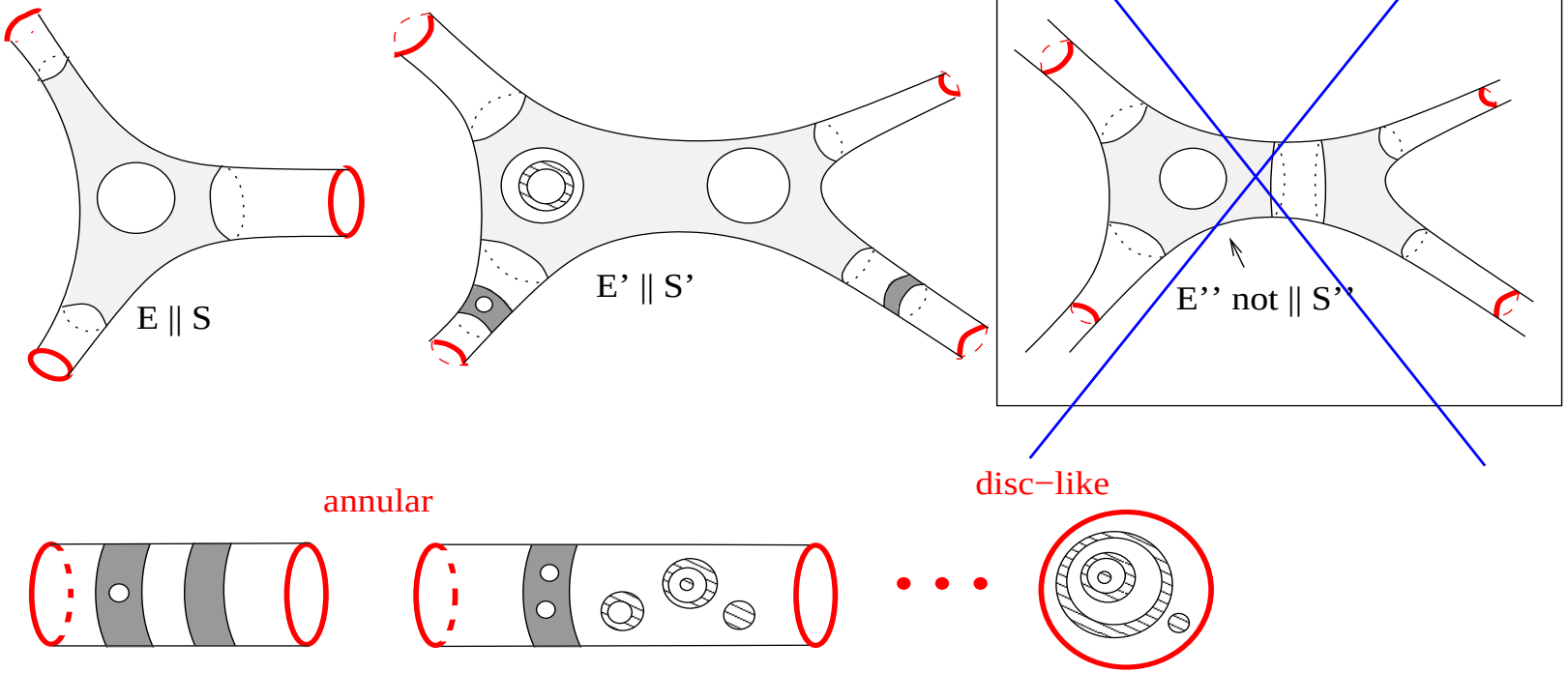
Comb. Lemma: After pullback and fill−in, F can be reduced to

A repelling system of constant complexity



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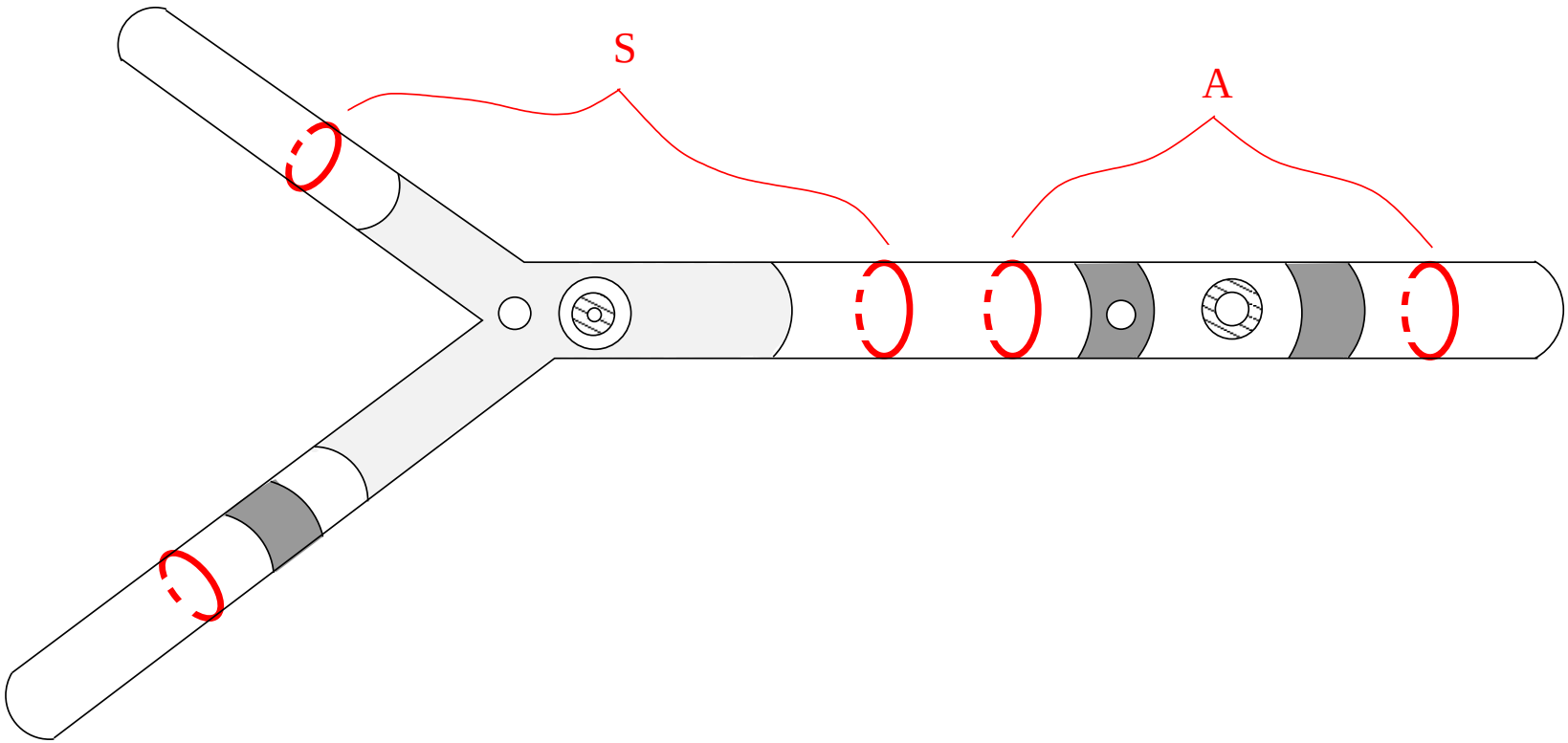

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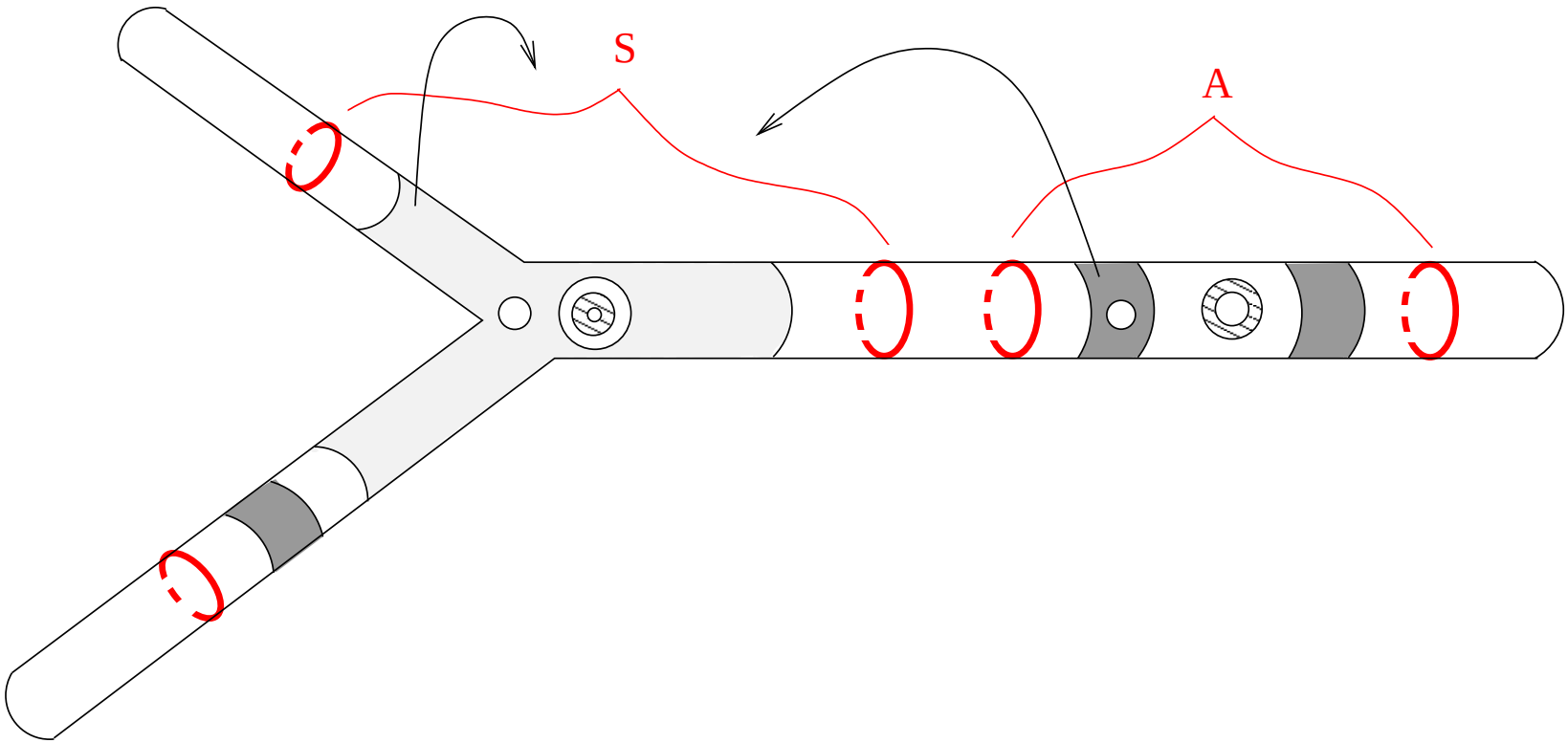

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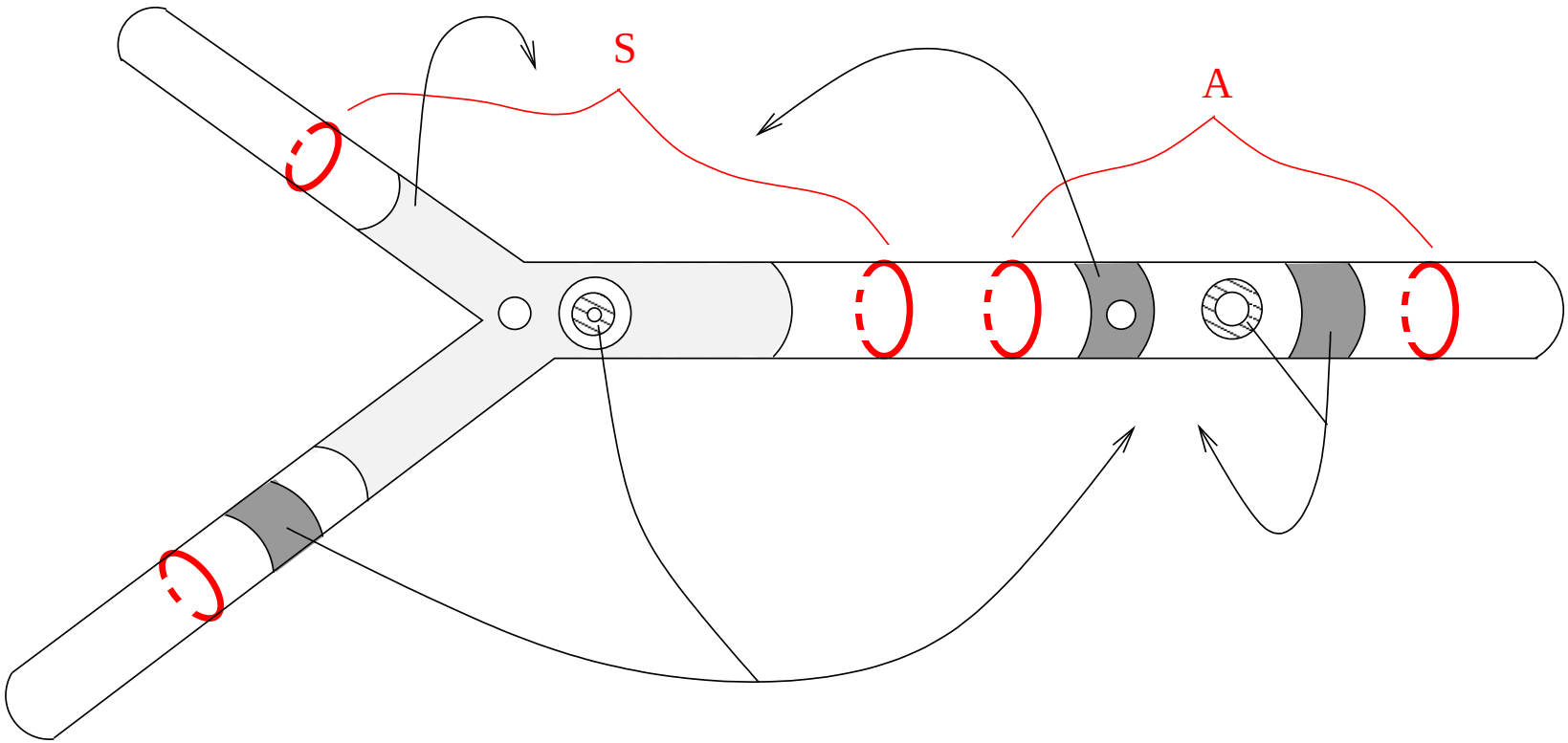

Combinatorial equivalence

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For repelling systems: $(F, \mathcal{E}, \mathcal{S}), (F', \mathcal{E}', \mathcal{S}')$,

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(If F is a Thurston map replace $\mathcal{S}$ by $S^2 \setminus P_F$).

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$\overset{6.}{\underset{\text{M.R.M.T.}}{\Longrightarrow}}$ get rid of constant complexity.

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$$\boxed{\text{no obstructions}} \overset{\text{Theorem 1}}{\Longrightarrow} \sim_{comb} \text{a holomorphic model.}$$

Theorem 2 A repelling system $(F, \mathcal{E}, \mathcal{S})$ (of constant complexity) has no Thurston obstructions

$$\Longleftrightarrow \begin{cases} \partial\mathcal{S} \text{ is not an obstruction, and} \\ \text{each } \widehat{F}_i \text{ is unobstructed (in the sense of Thurston).} \end{cases}$$

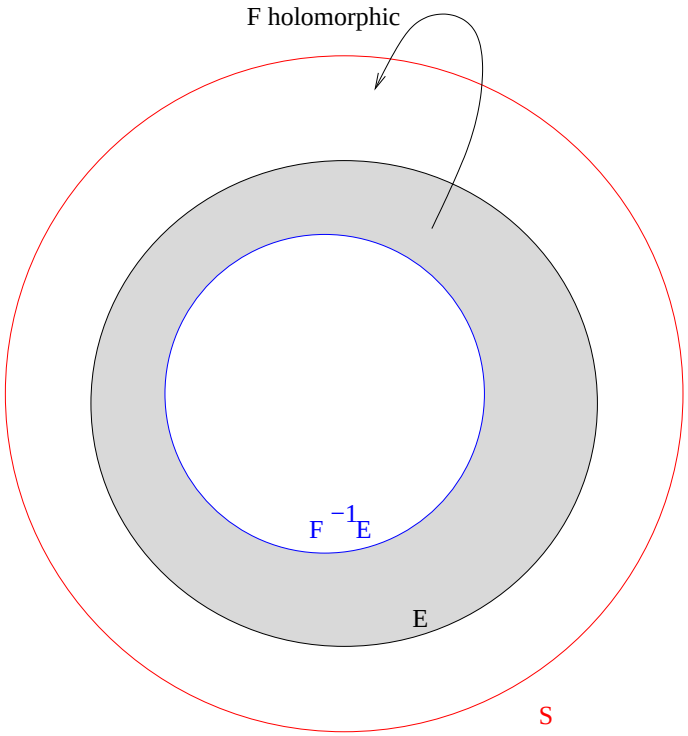
$$\Longleftrightarrow$$

$$\begin{cases} \partial\mathcal{S} \text{ is not an obstruction, and} \\ \text{each } \widehat{F}_i \text{ is } \sim_{comb} \text{a rational map (in the sense of Thurston).} \end{cases}$$

Rigidity Any two hol. models of F are hybrid-equivalent in the sense of Douady-Hubbard (they have the same inner class $= R_i$ for each $\widehat{F}_i$ + gluing data).

rigidify the outer class = add analytic structures

Assume $\mathcal{S} \subset \prod_j \overline{\mathbb{C}}_j \setminus$ holes, F is *holomorphic* on $\mathcal{E} \setminus F^{-1}(\mathcal{E})$.



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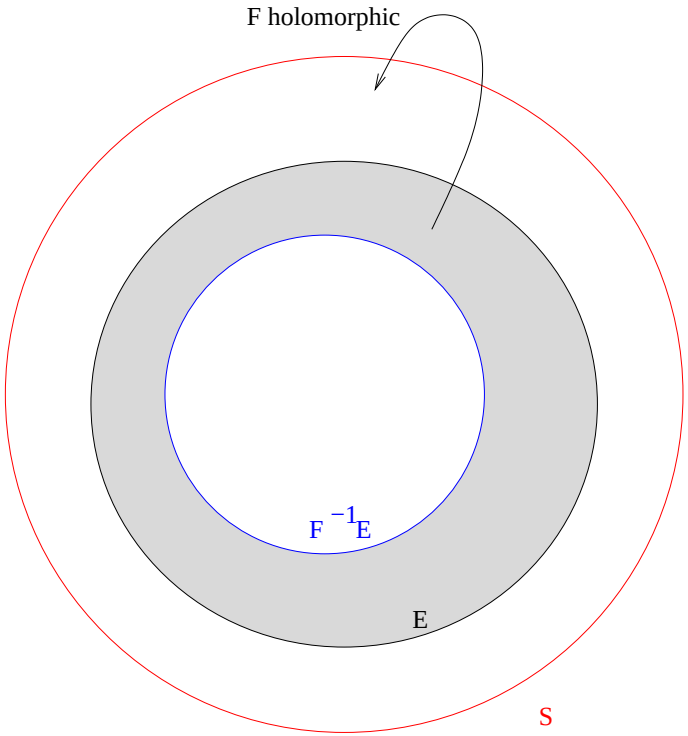
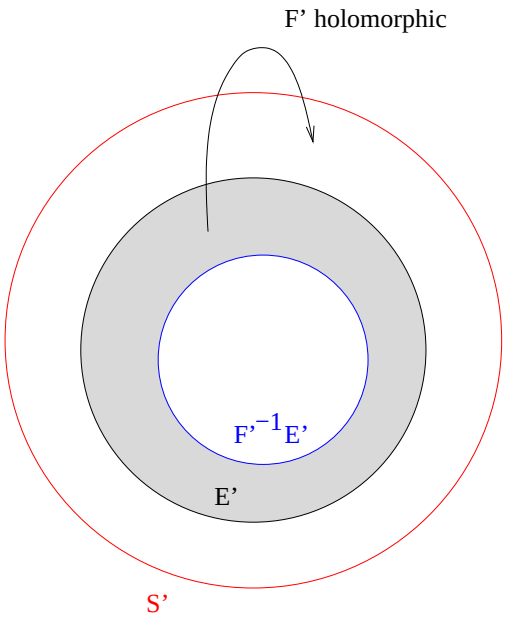
$\Phi \circ F = F' \circ \Psi$, and,





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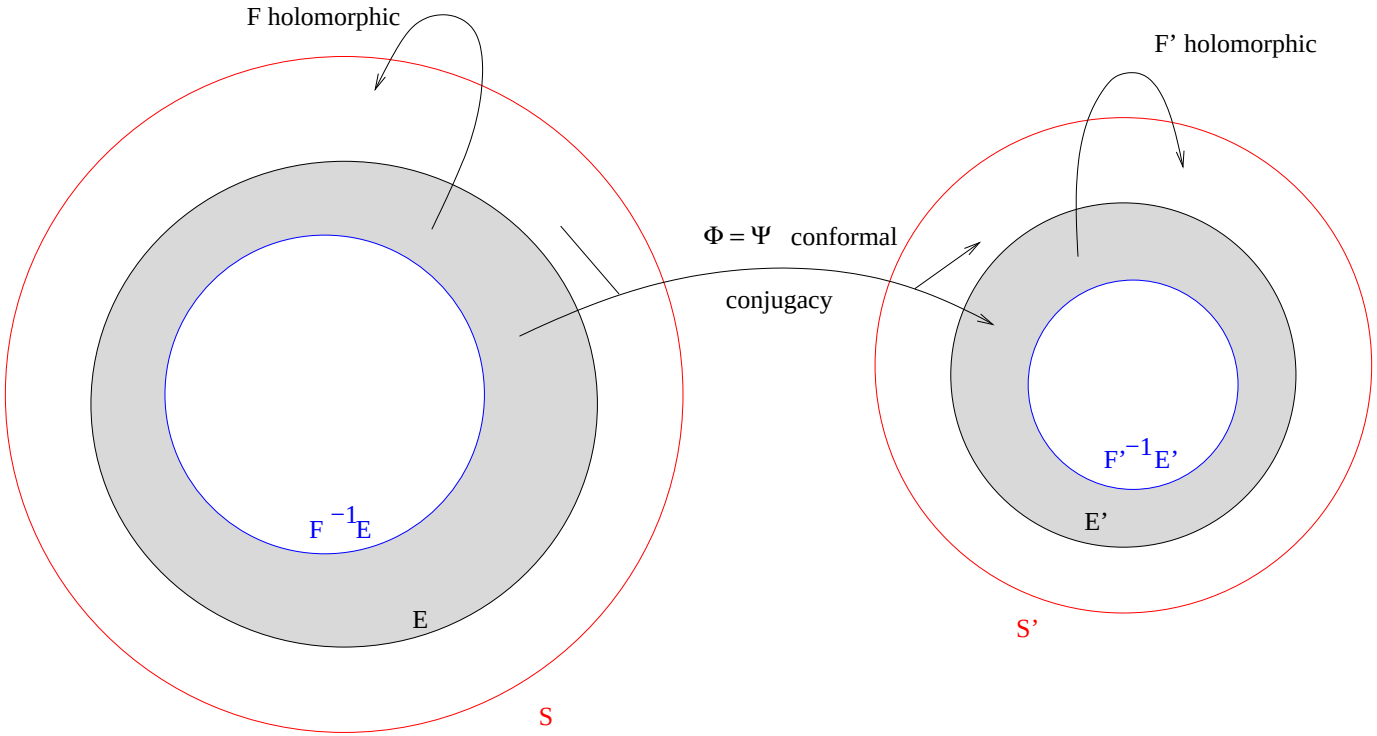

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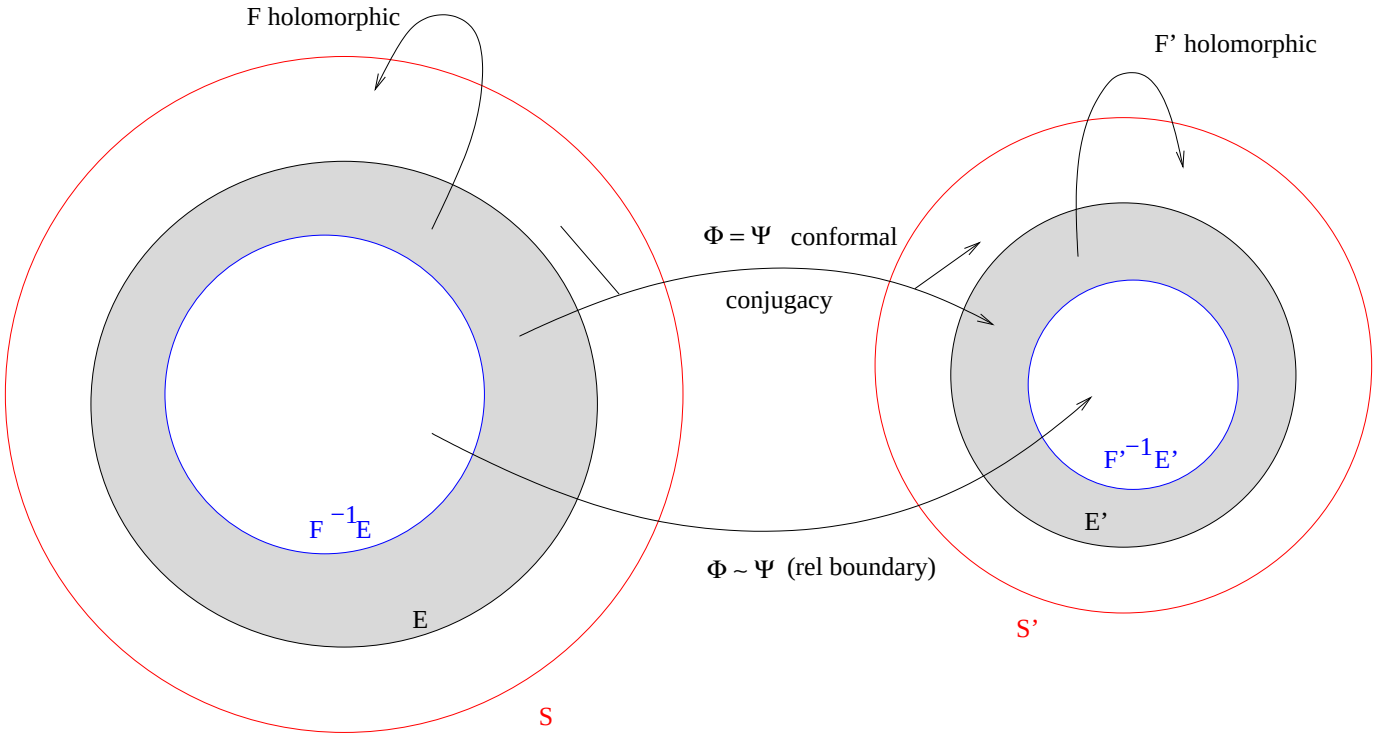

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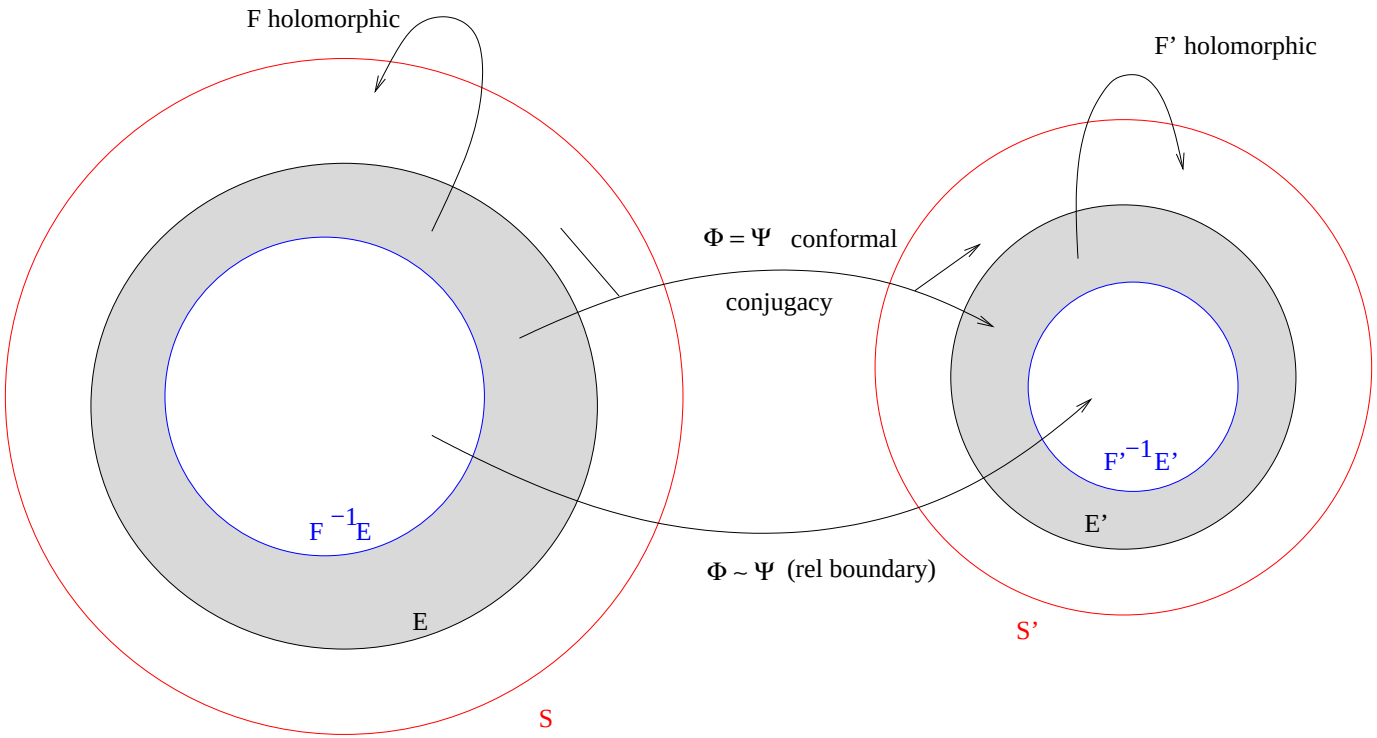

A hol. model of F is **unique** (up to conformal conjugacy).

Corollary: For hyperbolic maps. Let $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic type branched coverings, holomorphic on $U(acc(\mathcal{P}_G))$.

$\boxed{\text{no obstructions}} \Longrightarrow \exists !$ rational map $R \sim_{ana-comb} G$.

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$+$ distortion controls, plumbing and pinching surgeries:

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Pinching paths converge to cusp boundary points.

Is $F \sim_{comb}$ hol.-model?

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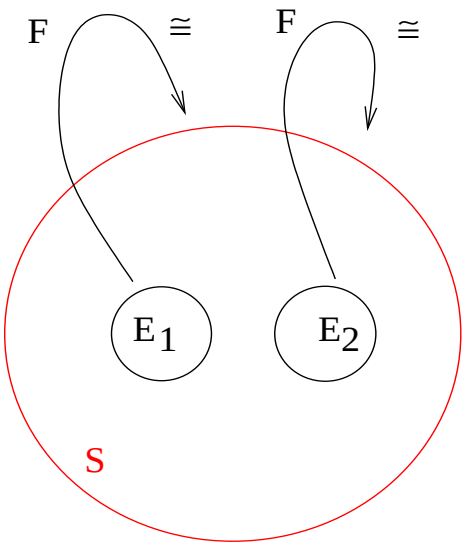

Usually $F \not\sim_{conj}$ a hol.-model

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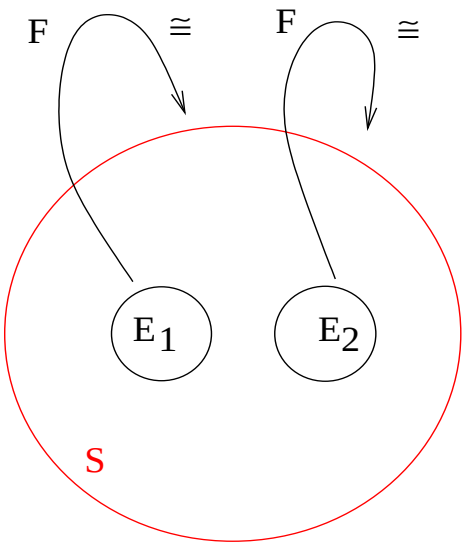

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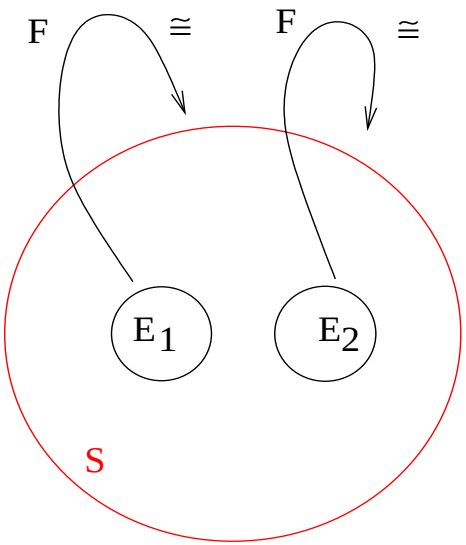

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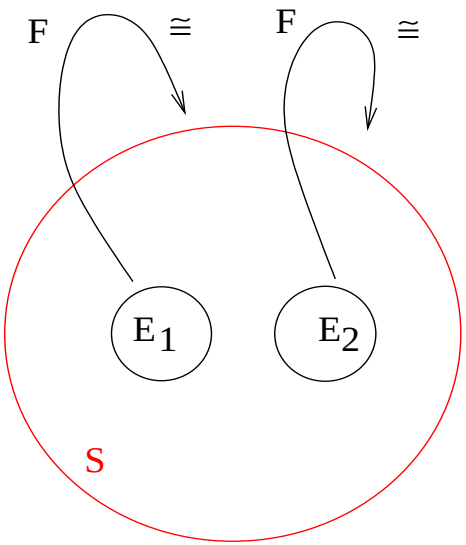

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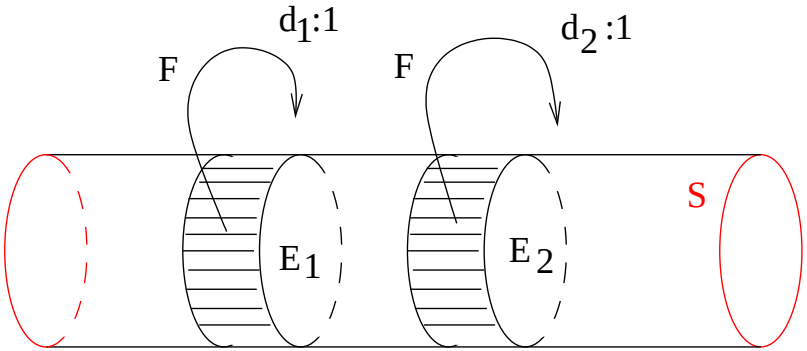

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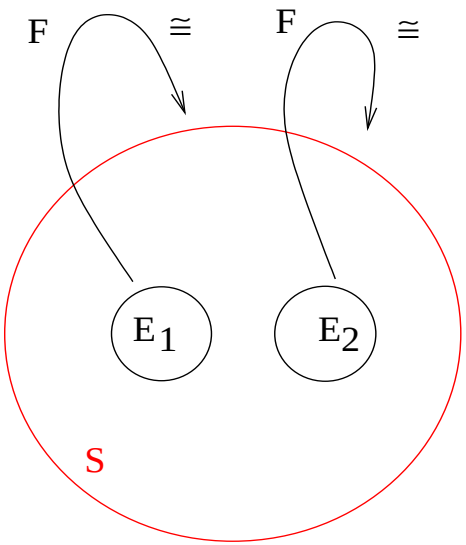

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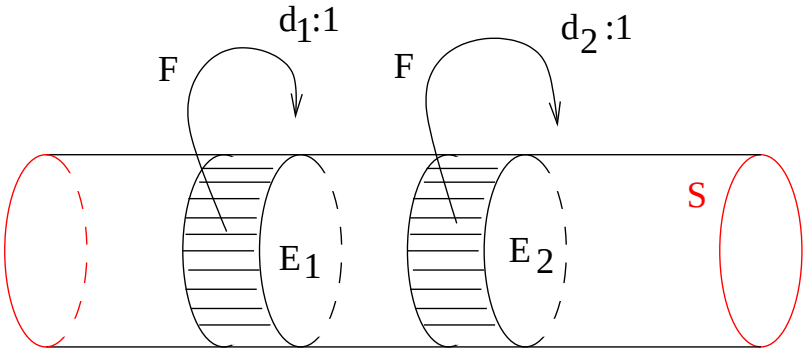
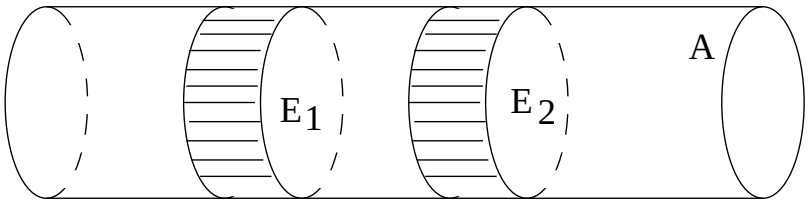
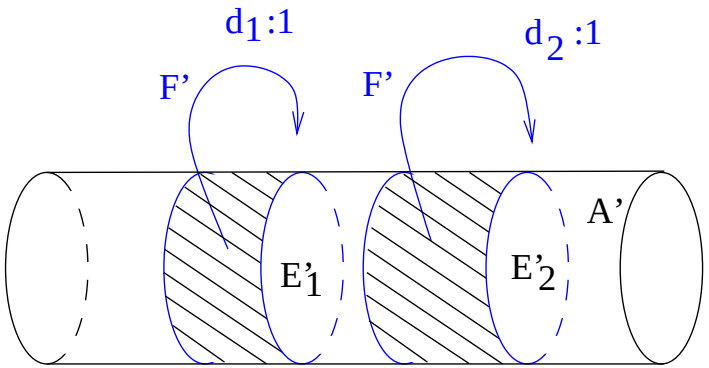


$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

$\sum_i \frac{1}{d_i} < 1$ (**moduli** prob.)

To find F' hol., Φ, θ s.t. $F' \circ \Phi \circ \theta|_{\mathcal{E}} = \Phi \circ F$, $\theta \sim_{\partial A} id$





1. Construct at first a model hol. rep. sys. $F' : E_i' \to A'$
   This is possible precisely because $\sum \frac{1}{d_i} < 1$.

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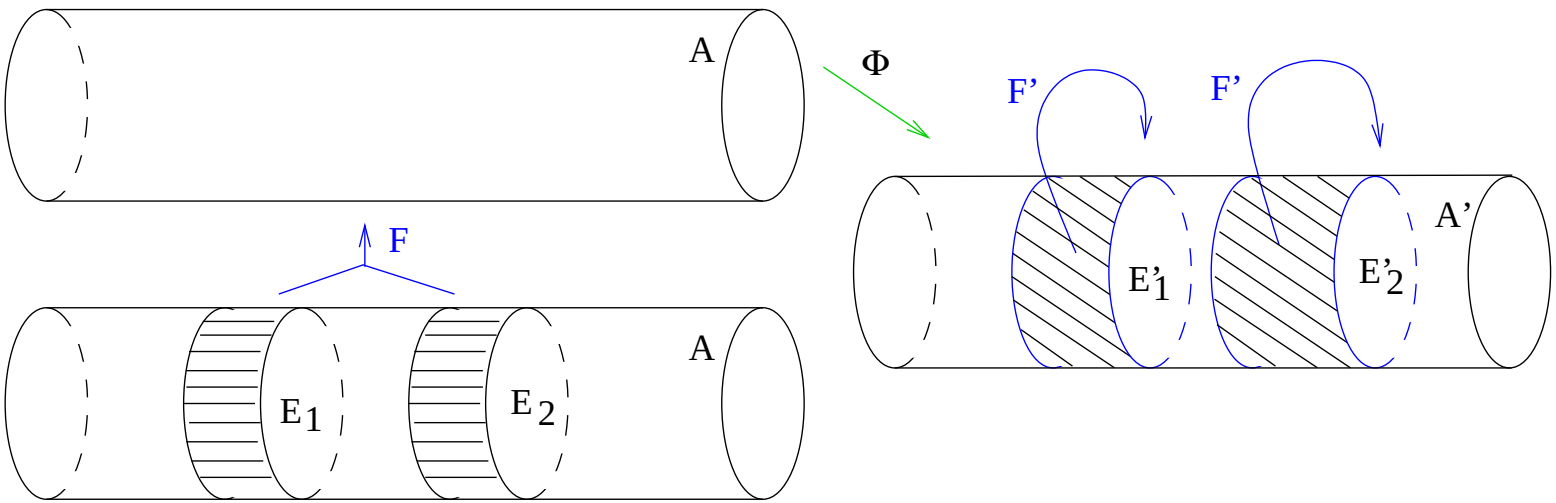

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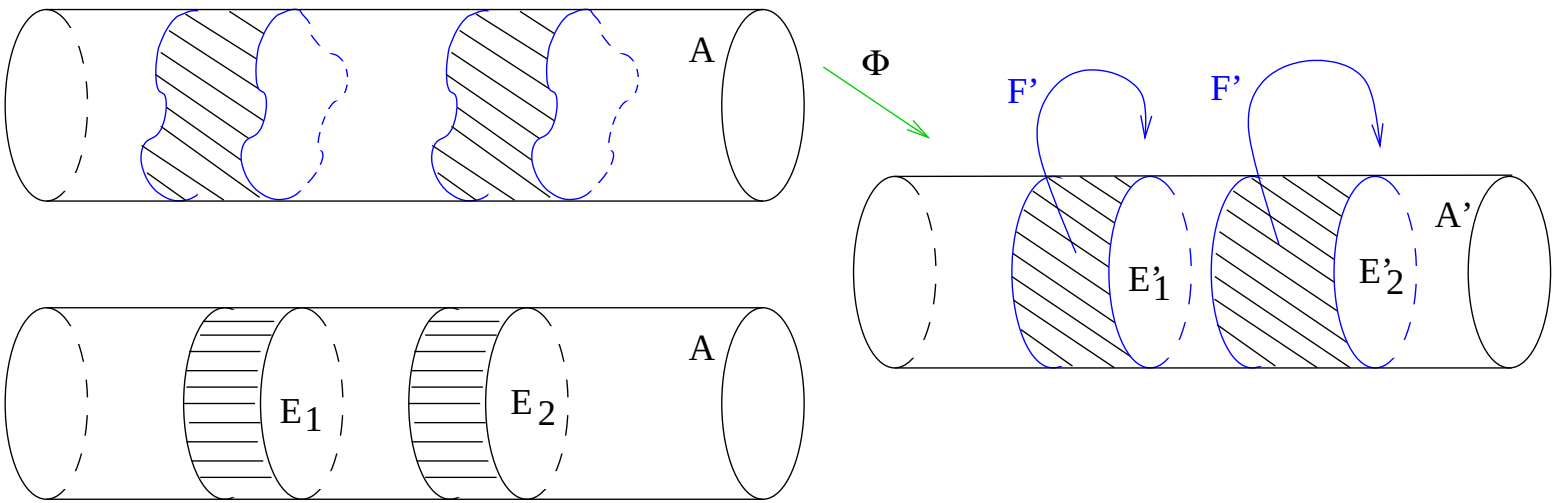

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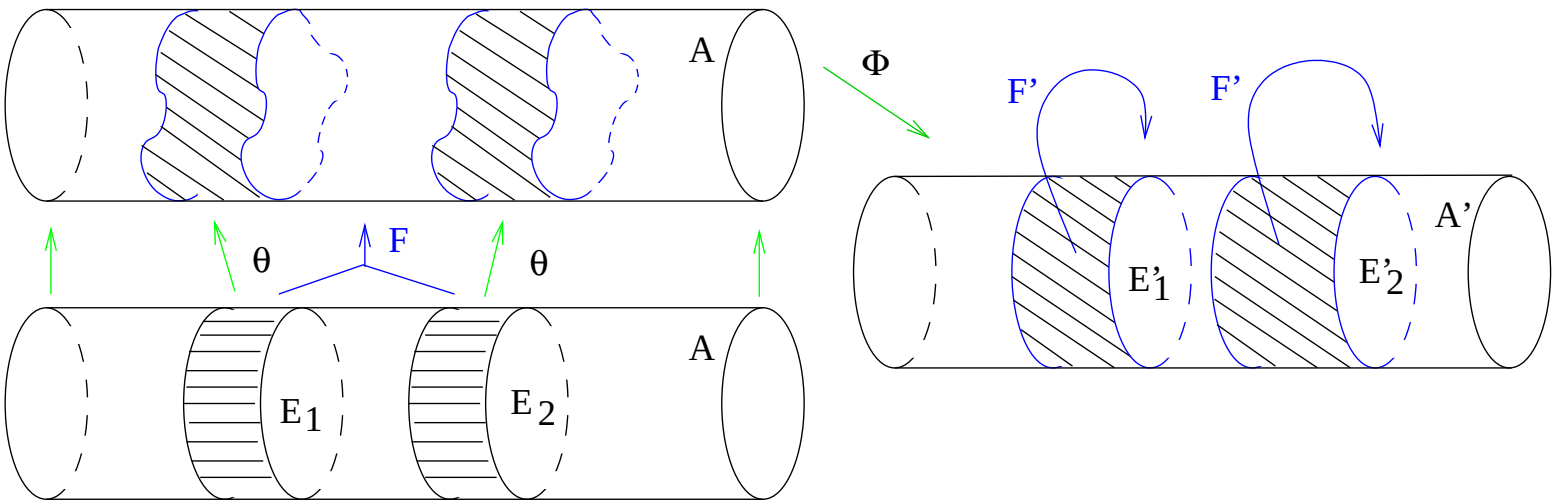

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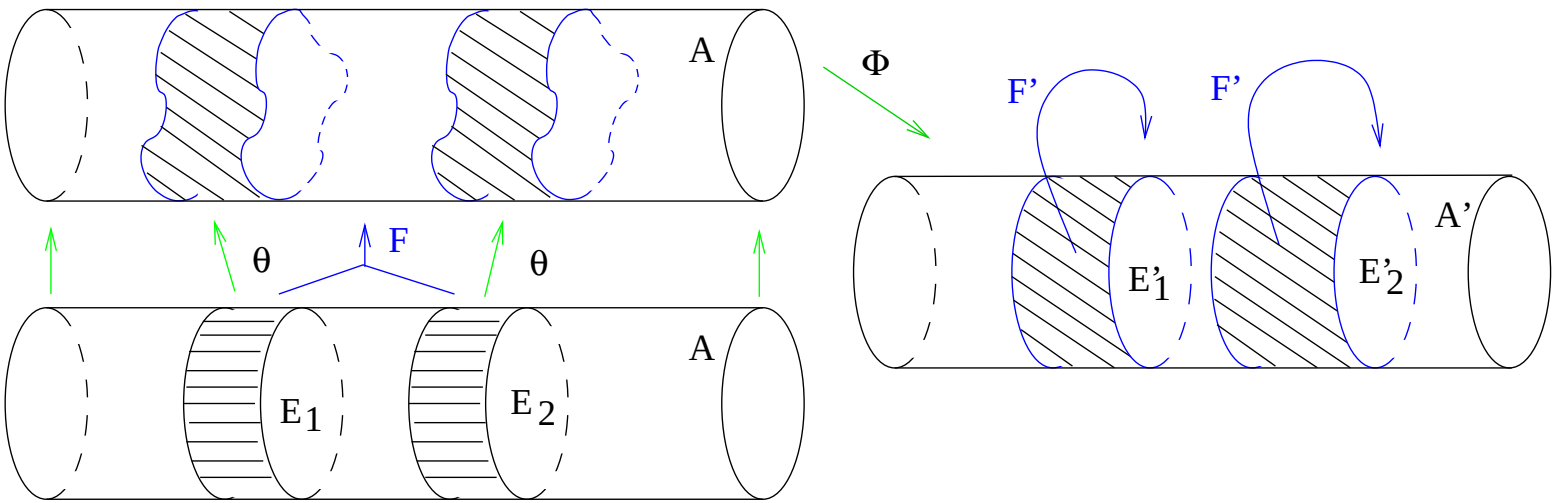

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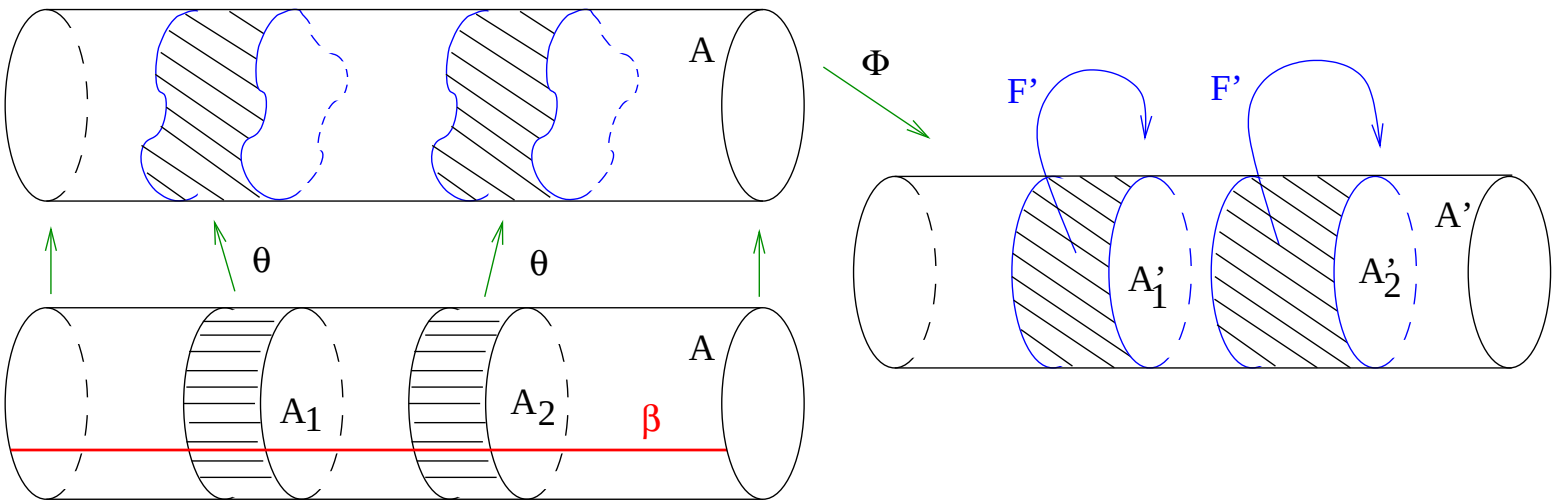

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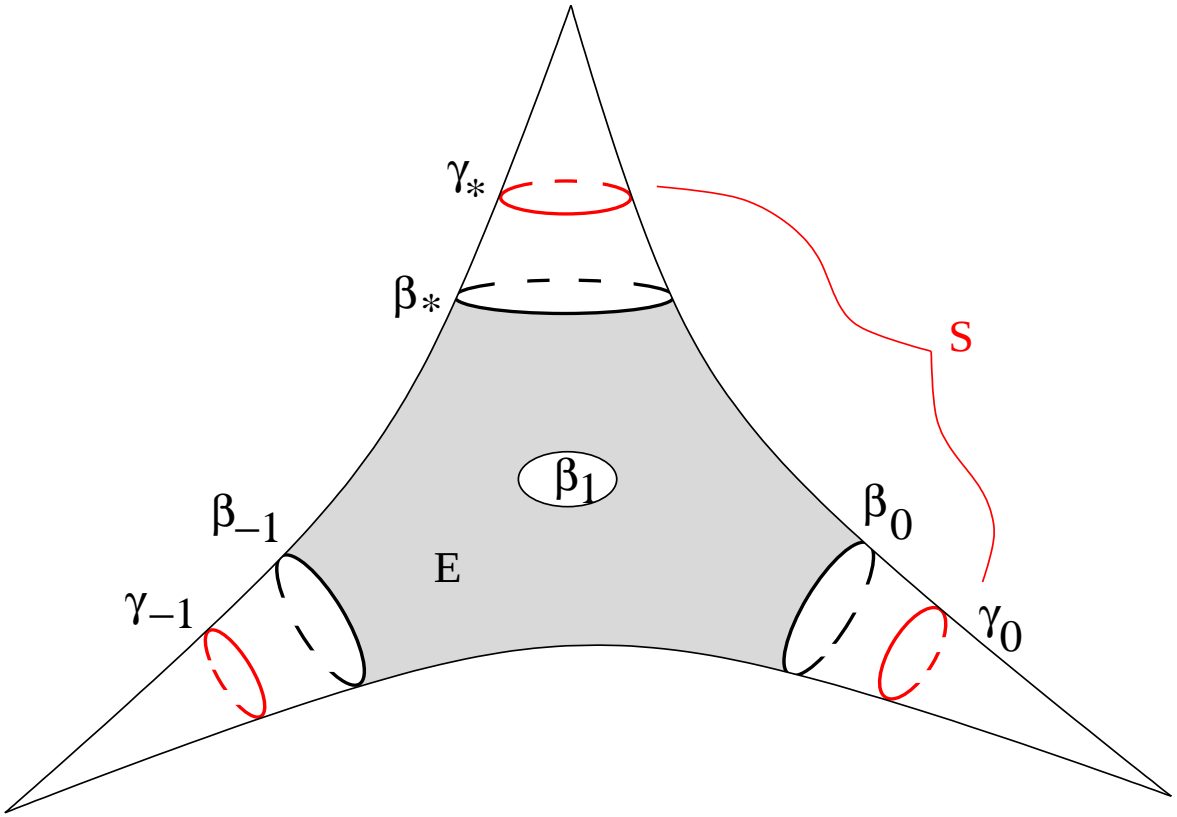
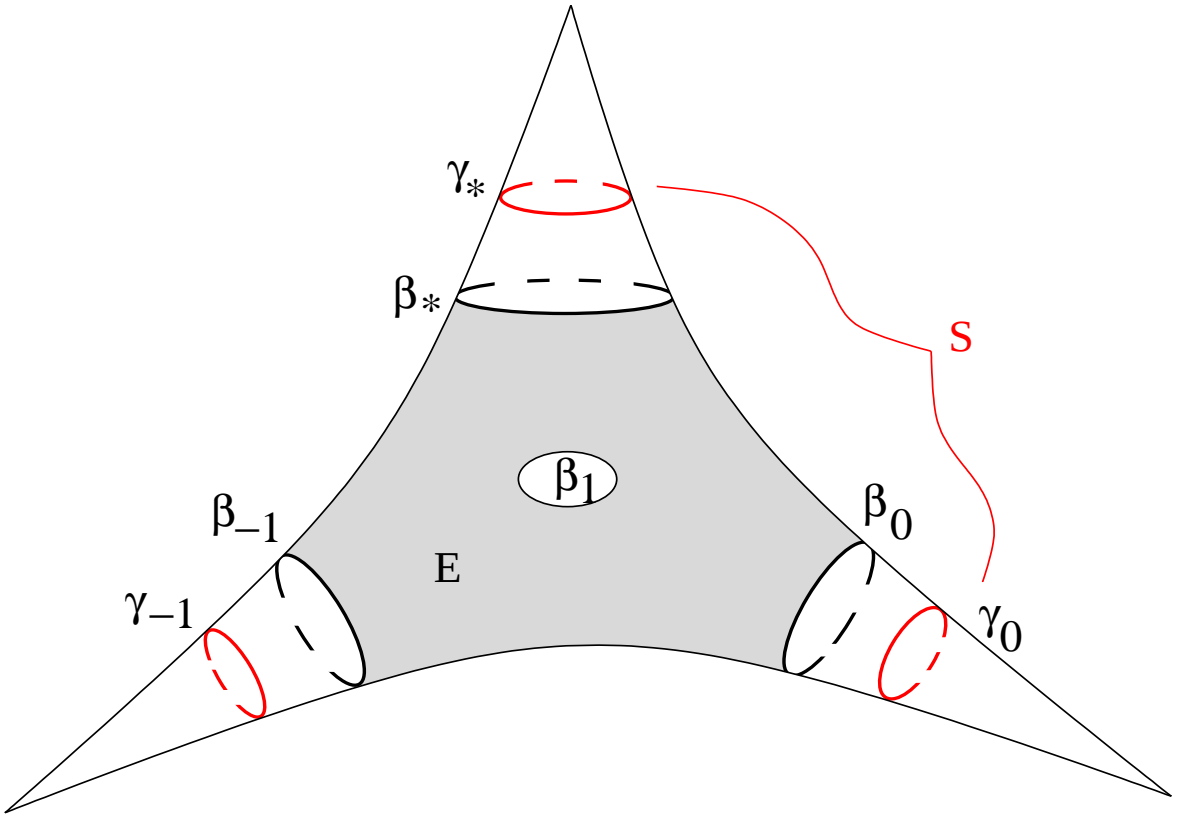
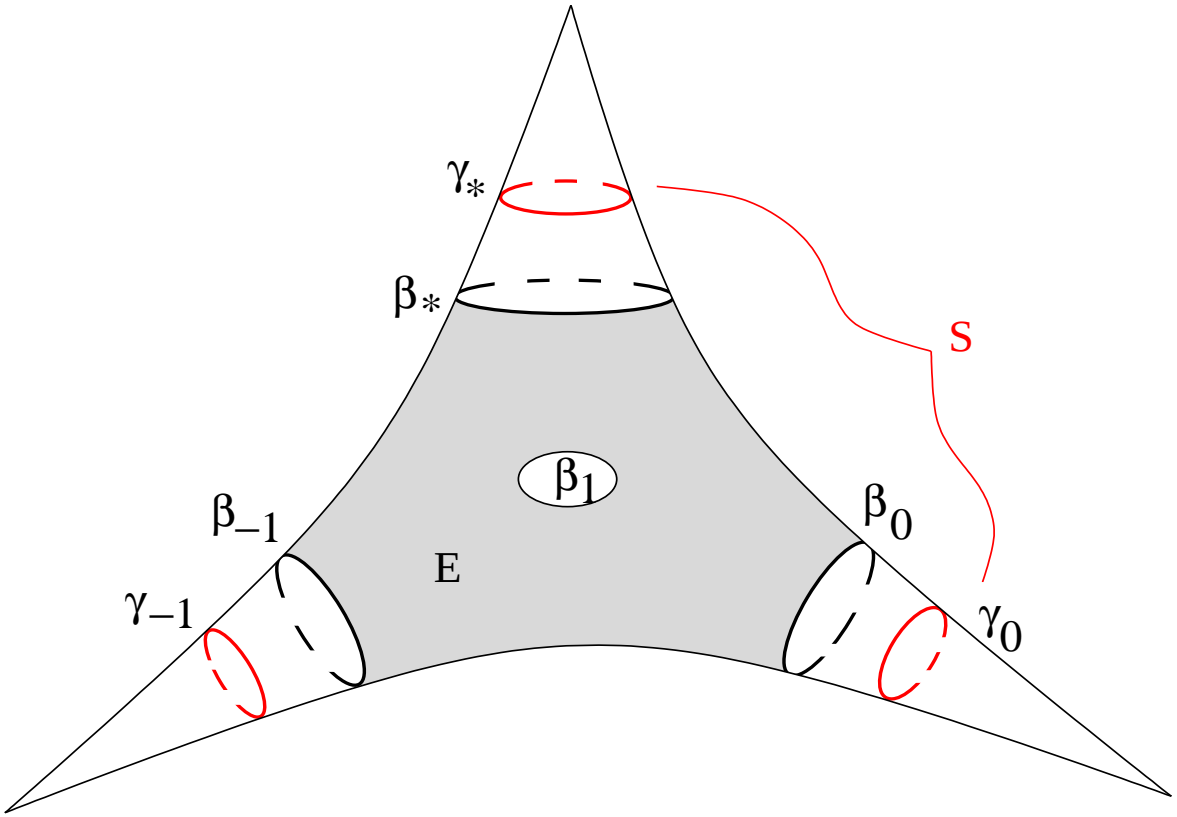
5. Extend θ as a homeo. $A \to A$. Then $F' \circ \Phi \circ \theta|_{E_i} = \Phi \circ F$.
6. Via Dehn twist to modify the extension so that $\theta(\beta) \sim_{\partial A} \beta$.
   This guarantees: $\theta \overset{\text{isotop.}}{\sim} id$ rel. ∂A.


γ_*
β_*
S
β_1
β_{-1}
β_0
E
γ_{-1}
γ_0



Let $F : E \to S$ be a covering of degree 2. This induces a covering $F : \{\beta_0, \beta_{-1}, \beta_*, \beta_1\} \to \{\gamma_0, \gamma_{-1}, \gamma_*\}$ of degree 2. Then
$F \sim_{comb}$ a hol. model $\iff$



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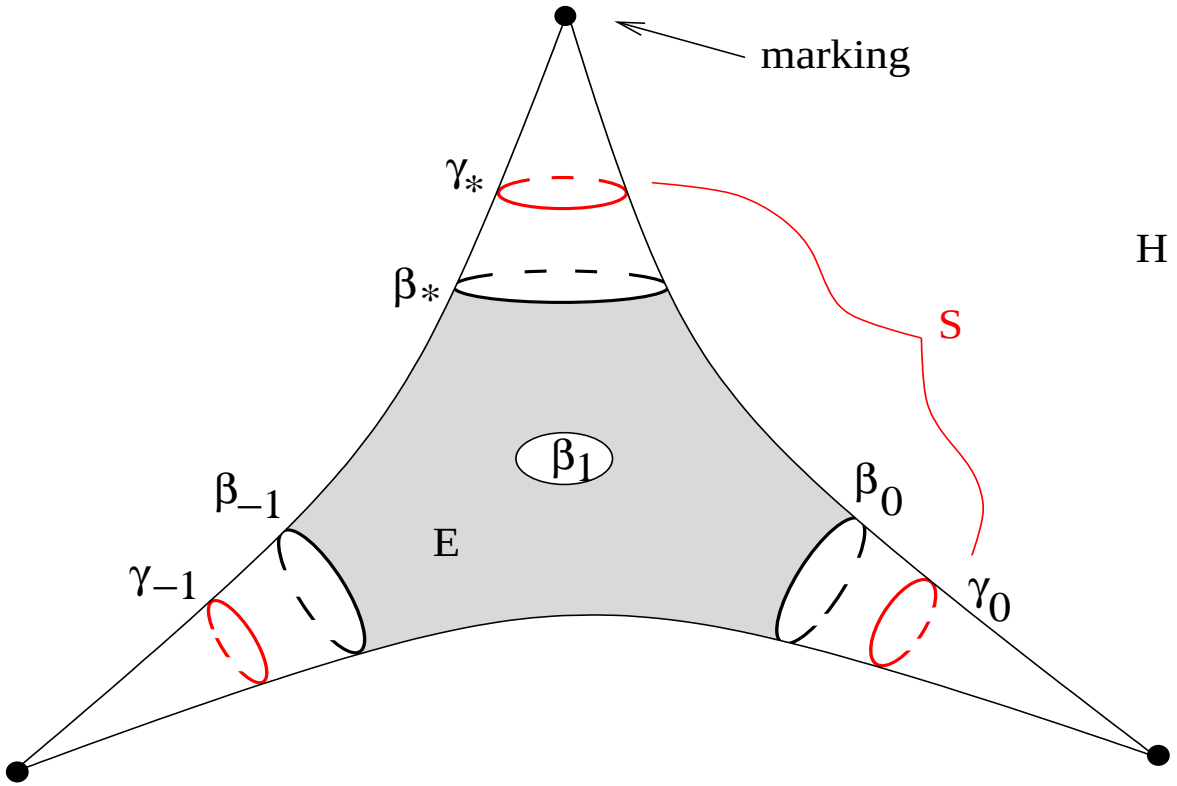
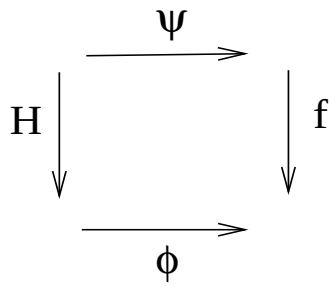
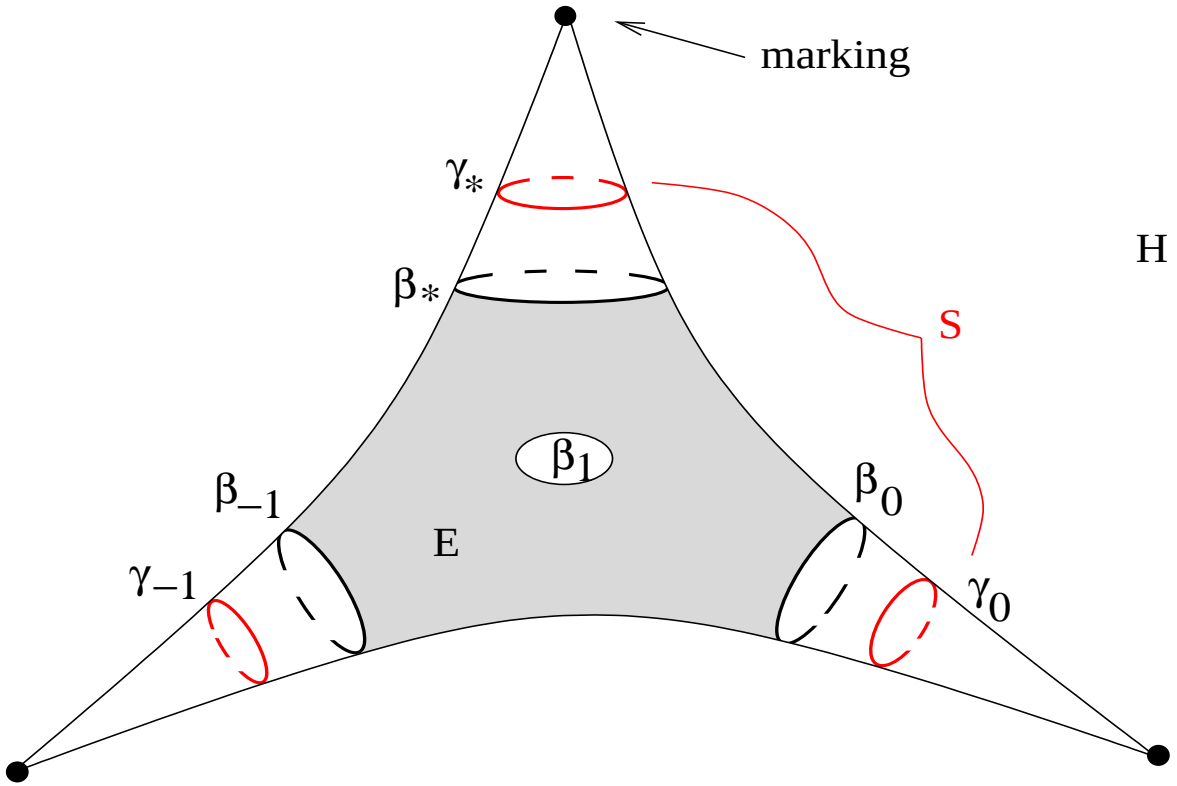
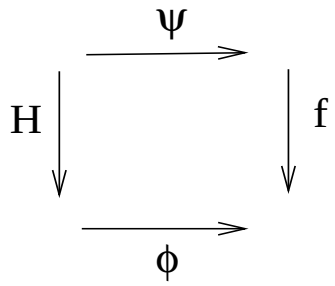




Proof. Mark one point on each corner.





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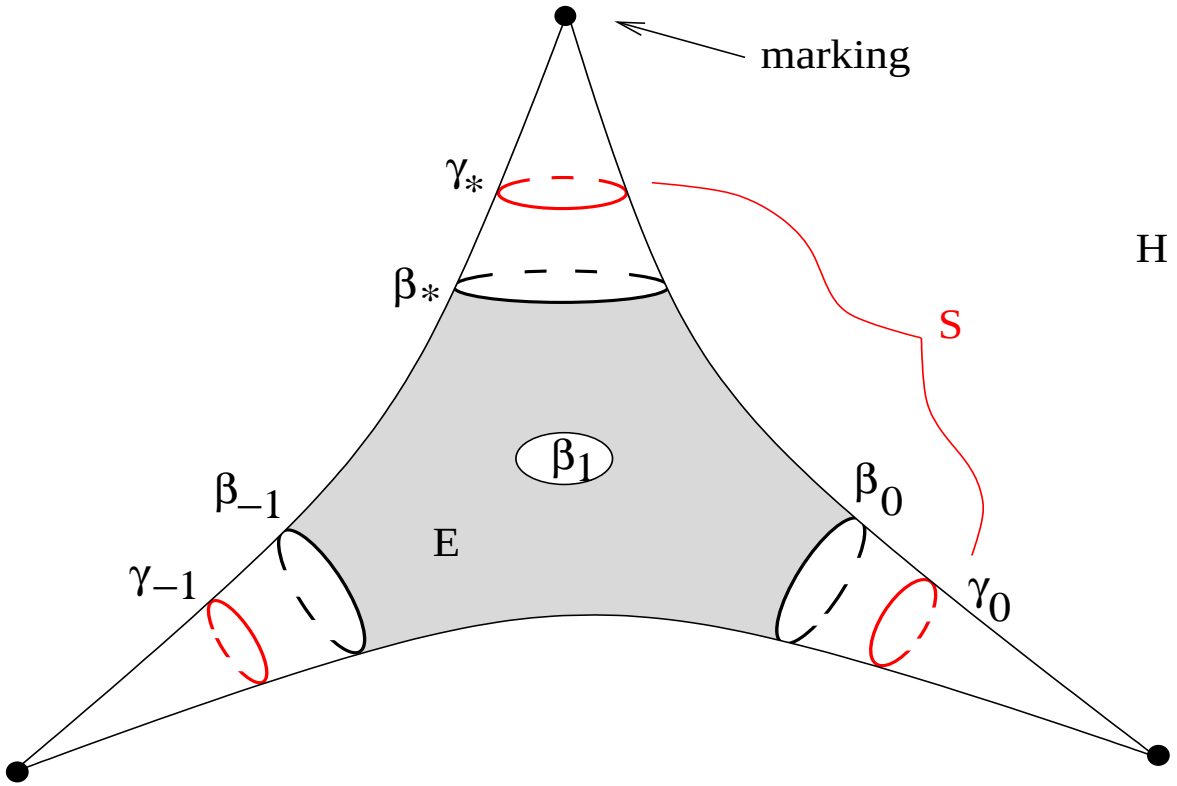
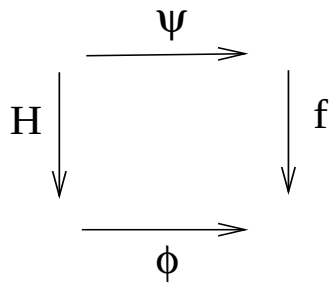
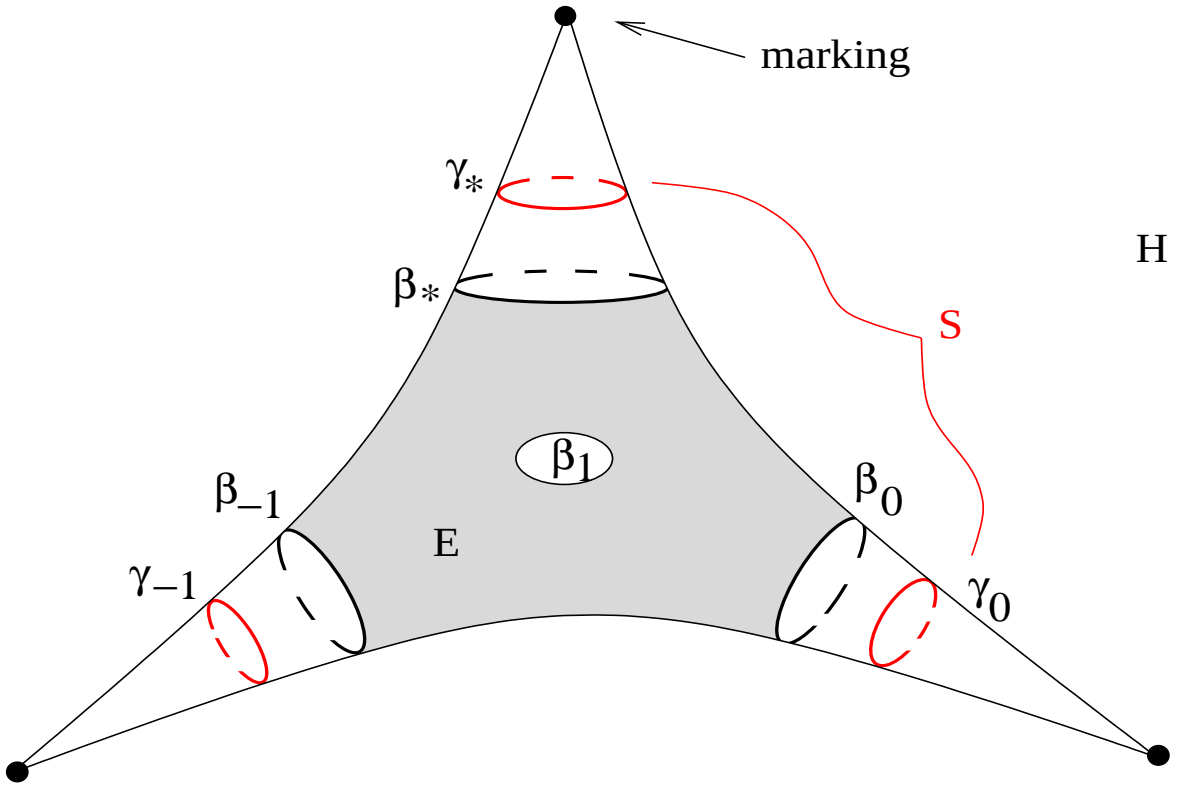
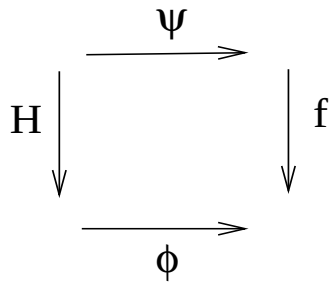




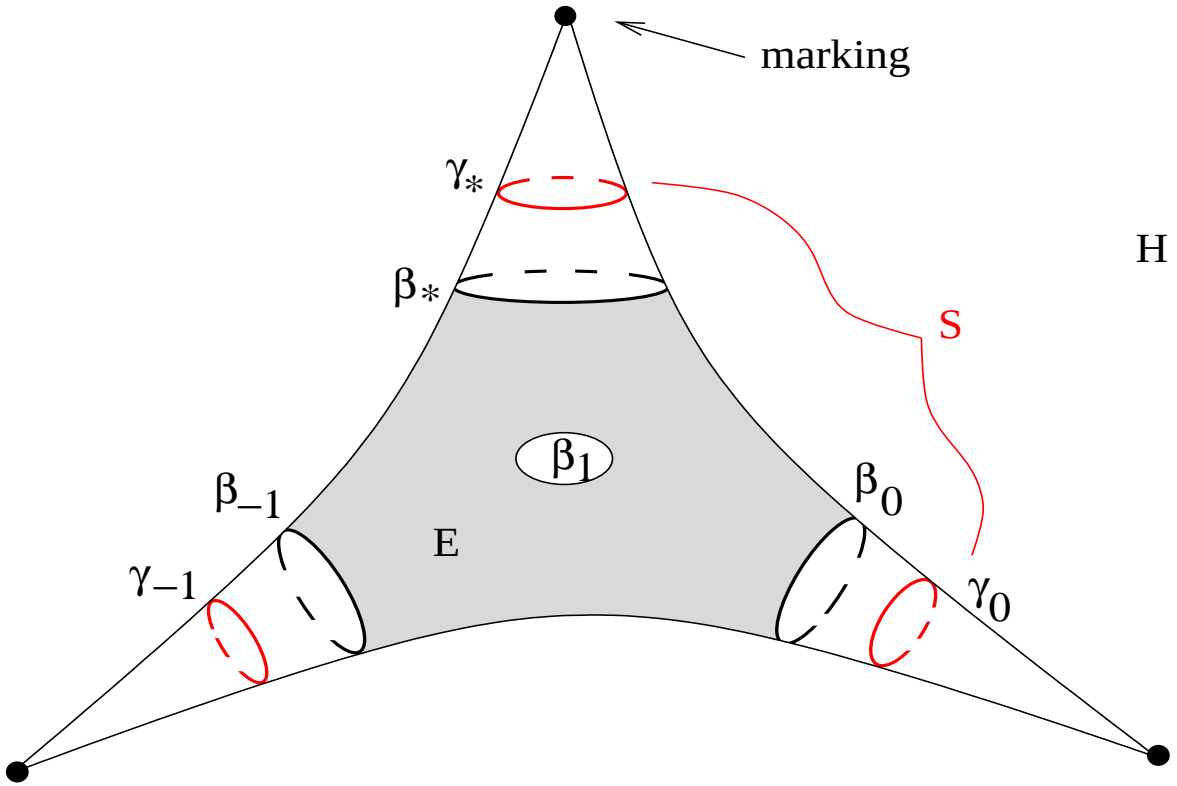
Proof. Mark one point on each corner. Extend F as a branched cover H with $P_H \subset \mathcal{P}$, and H

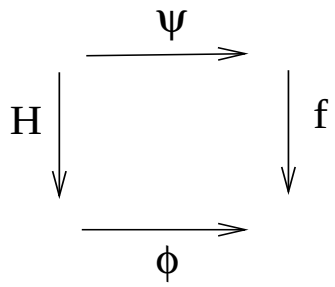




Proof. Mark one point on each corner. Extend F as a branched cover H with $P_H \subset \mathcal{P}$, and H holomorphic outside S.

$\#\mathcal{P} = 3 \overset{\text{Thurston}}{\Longrightarrow}$ there are $\phi = id$, ψ homeomorphism, and f a rational map, such that $f \circ \psi = \phi \circ H$, and ψ isotopic to ϕ rel $\mathcal{P}$.





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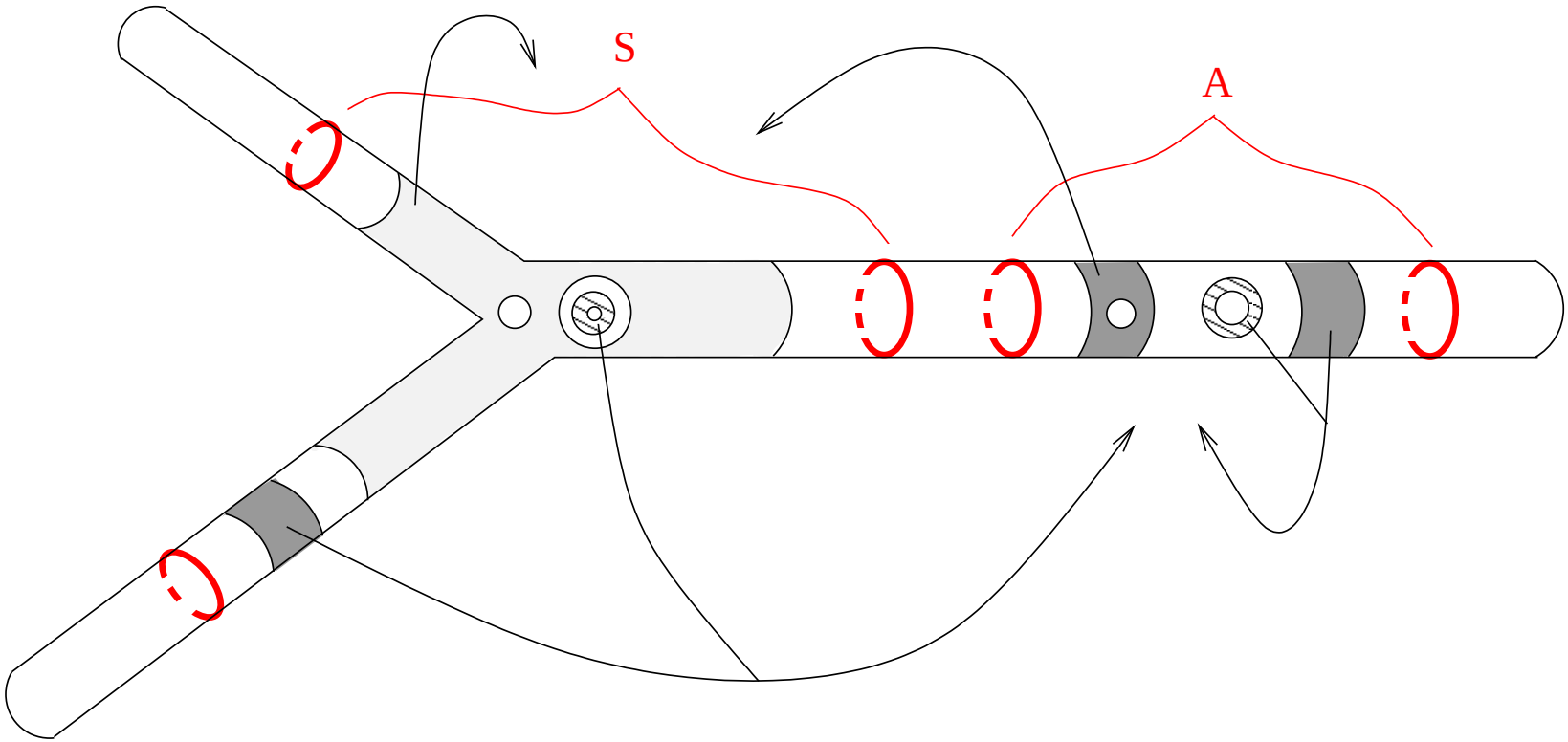
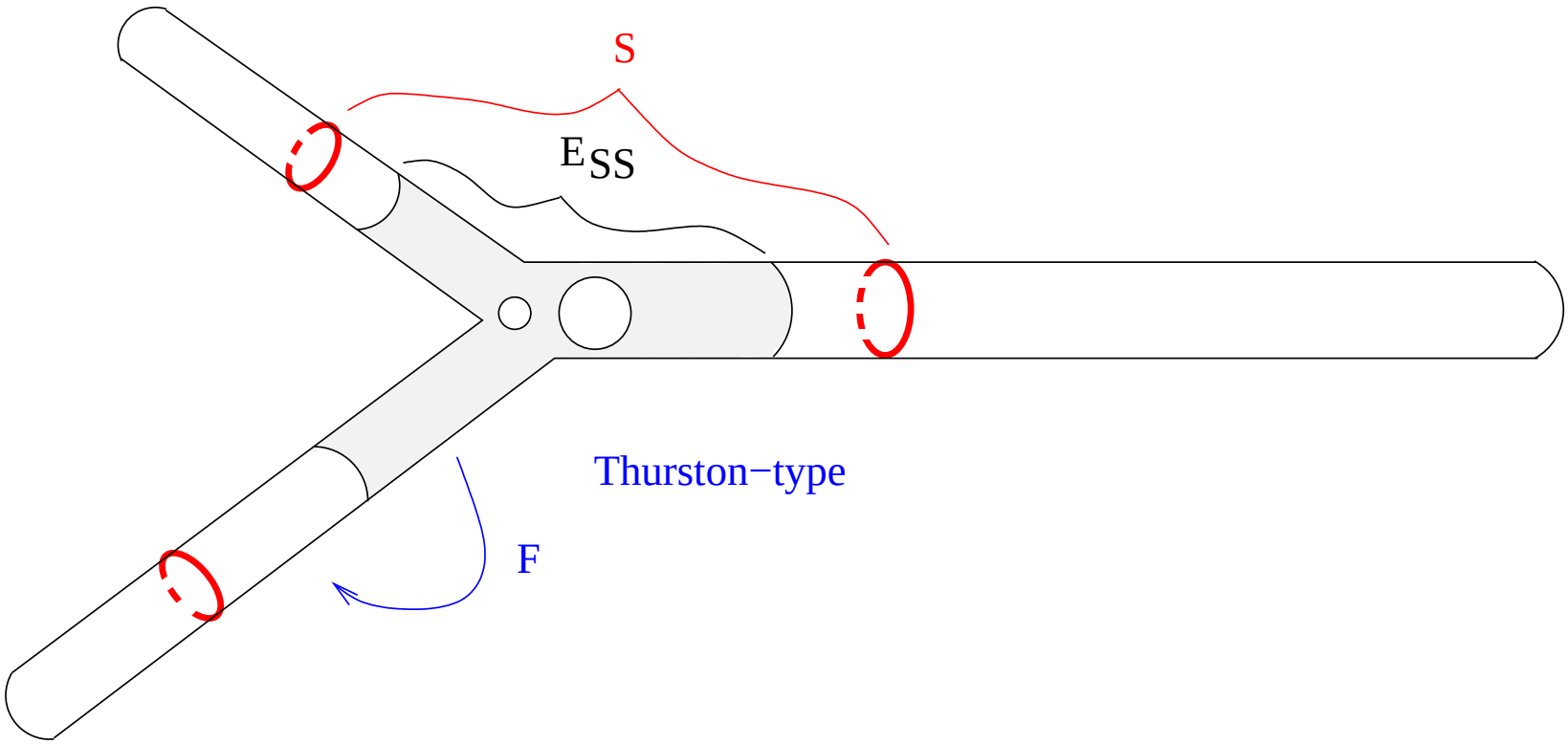
$\#\mathcal{P} = 3 \overset{\text{Thurston}}{\Longrightarrow}$ there are $\phi = id$, ψ homeomorphism, and f a rational map, such that $f \circ \psi = \phi \circ H$, and ψ isotopic to ϕ rel $\mathcal{P}$.

$\lambda < 1 \overset{\text{Shishikura}}{\Longrightarrow}$ ϕ, ψ can be modified so that $\phi = \psi$ outside S, and ψ isotopic to ϕ rel $\overline{\mathbb{C}} \setminus S$. Therefore $f \circ \psi|_E = \phi|_S \circ F$.

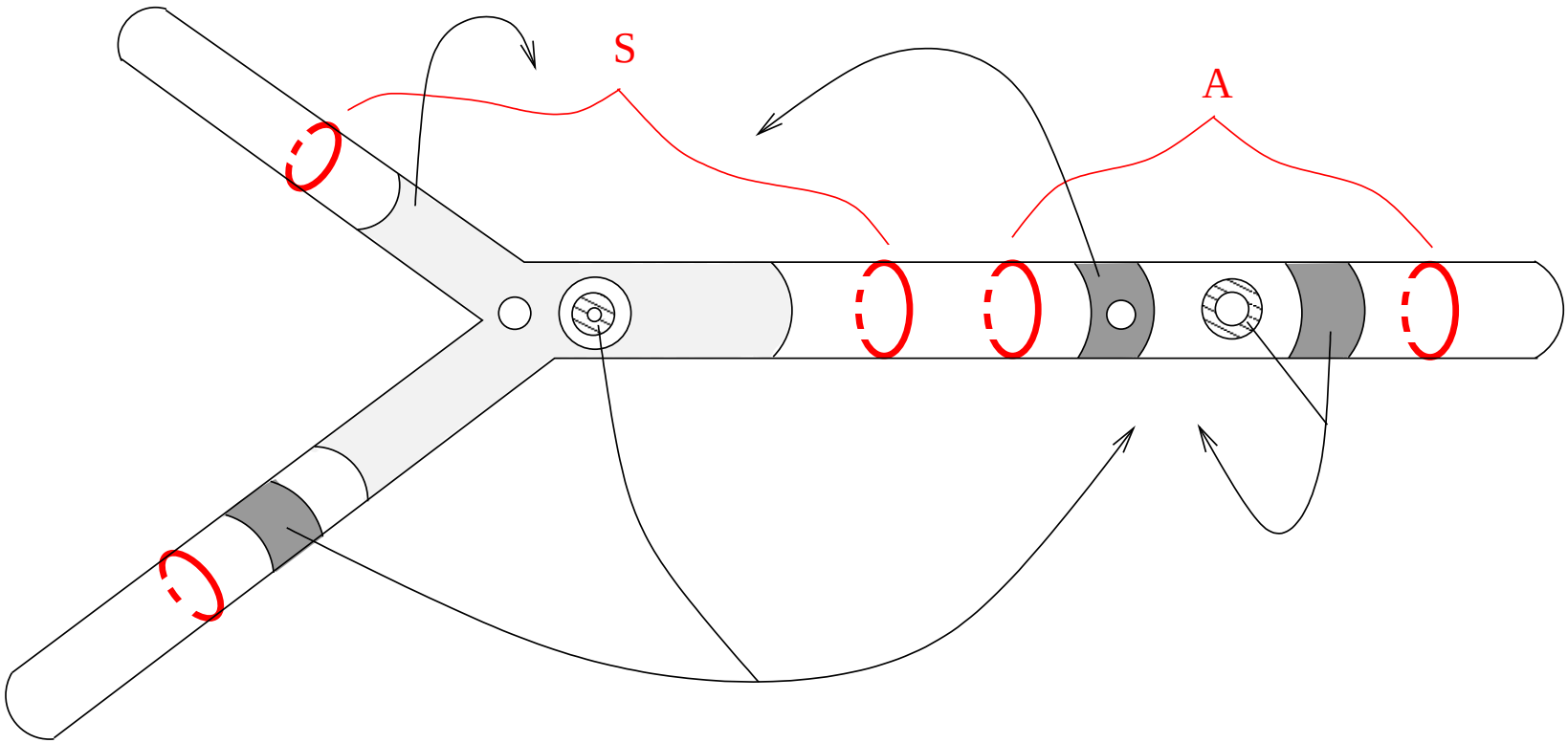


$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$



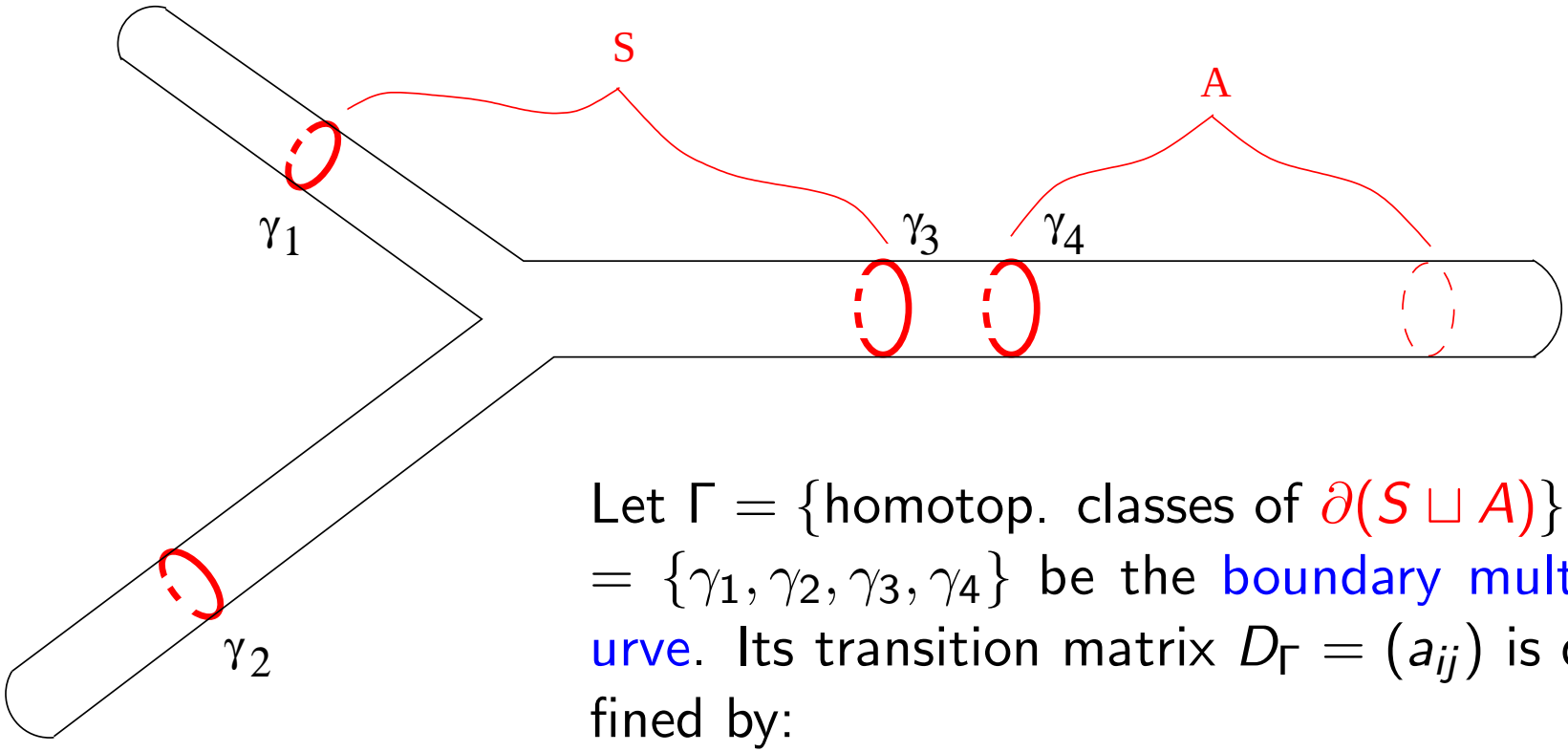
$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

- (F, E_{SS}, S) has no Thurston obstructions
  ((F, E_{SS}, S) is called a renormalization of F).
-

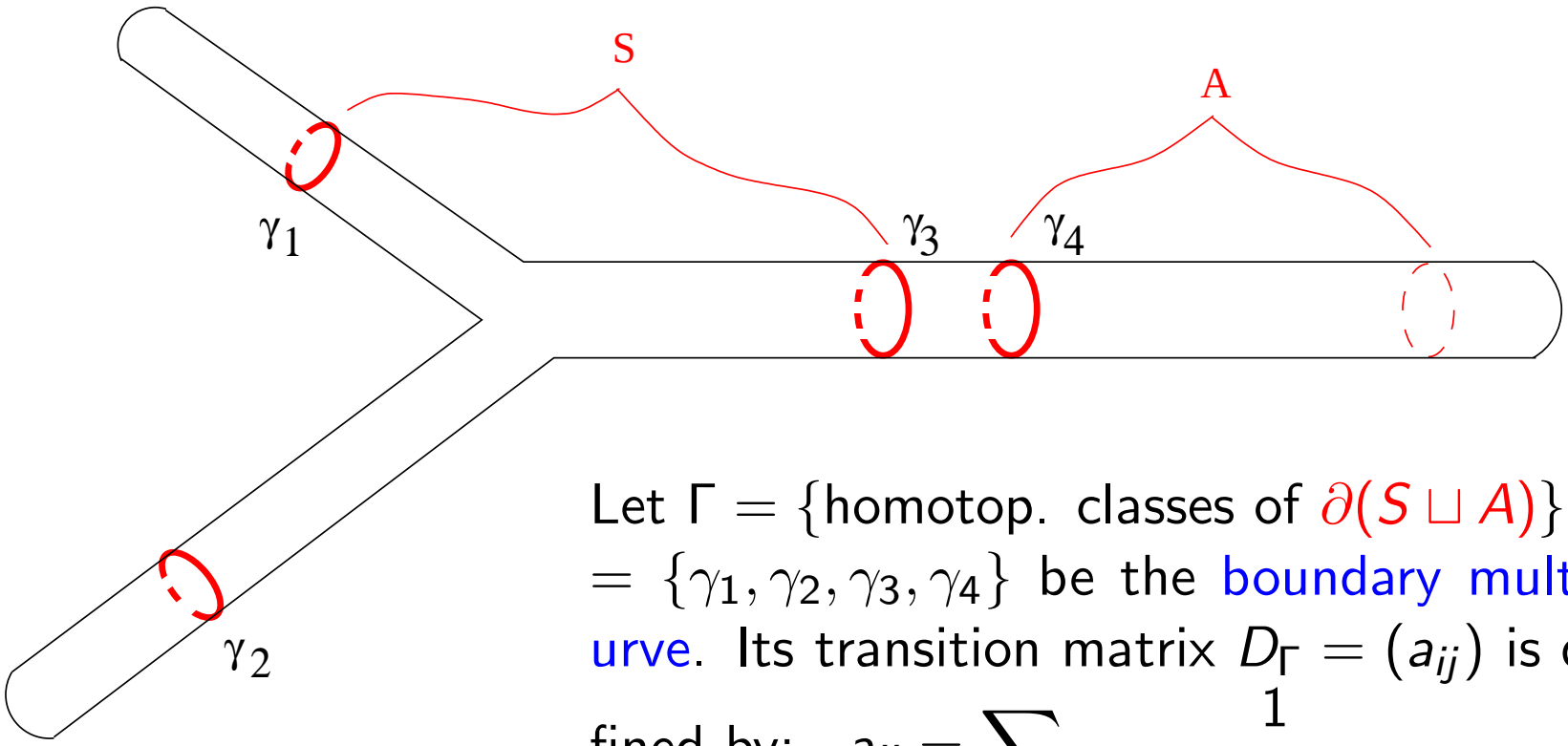


$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

$\begin{cases} \bullet\ (F, E_{SS}, S) \text{ has no Thurston obstructions} \\ \quad ((F, E_{SS}, S) \text{ is called a renormalization of } F). \\ \bullet\ \partial(S \sqcup A) \text{ is not an obstruction} \end{cases}$



Let $\Gamma = \{\text{homotop. classes of } \partial(S \sqcup A)\}$ $= \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the boundary multicurve. Its transition matrix $D_\Gamma = (a_{ij})$ is defined by:



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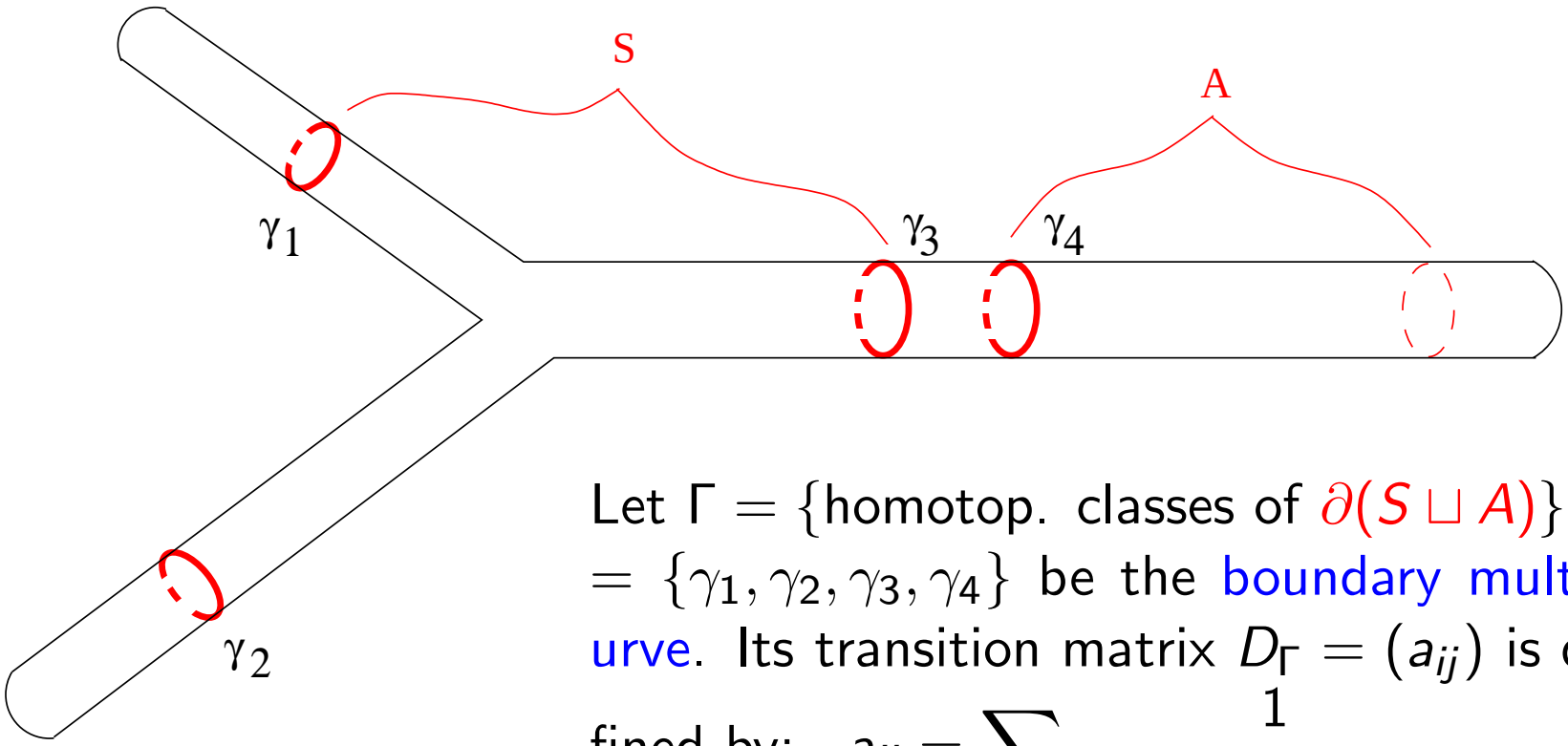

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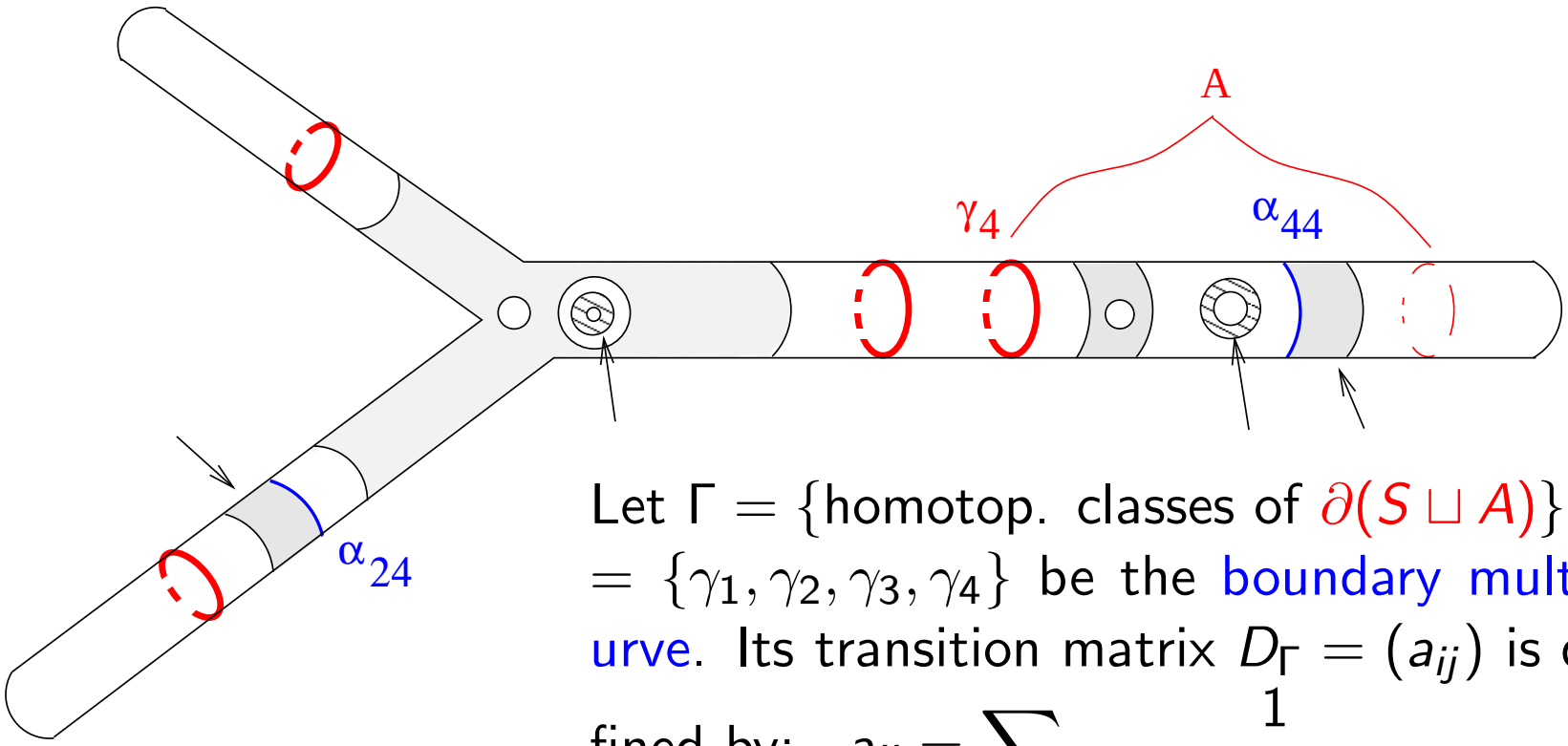

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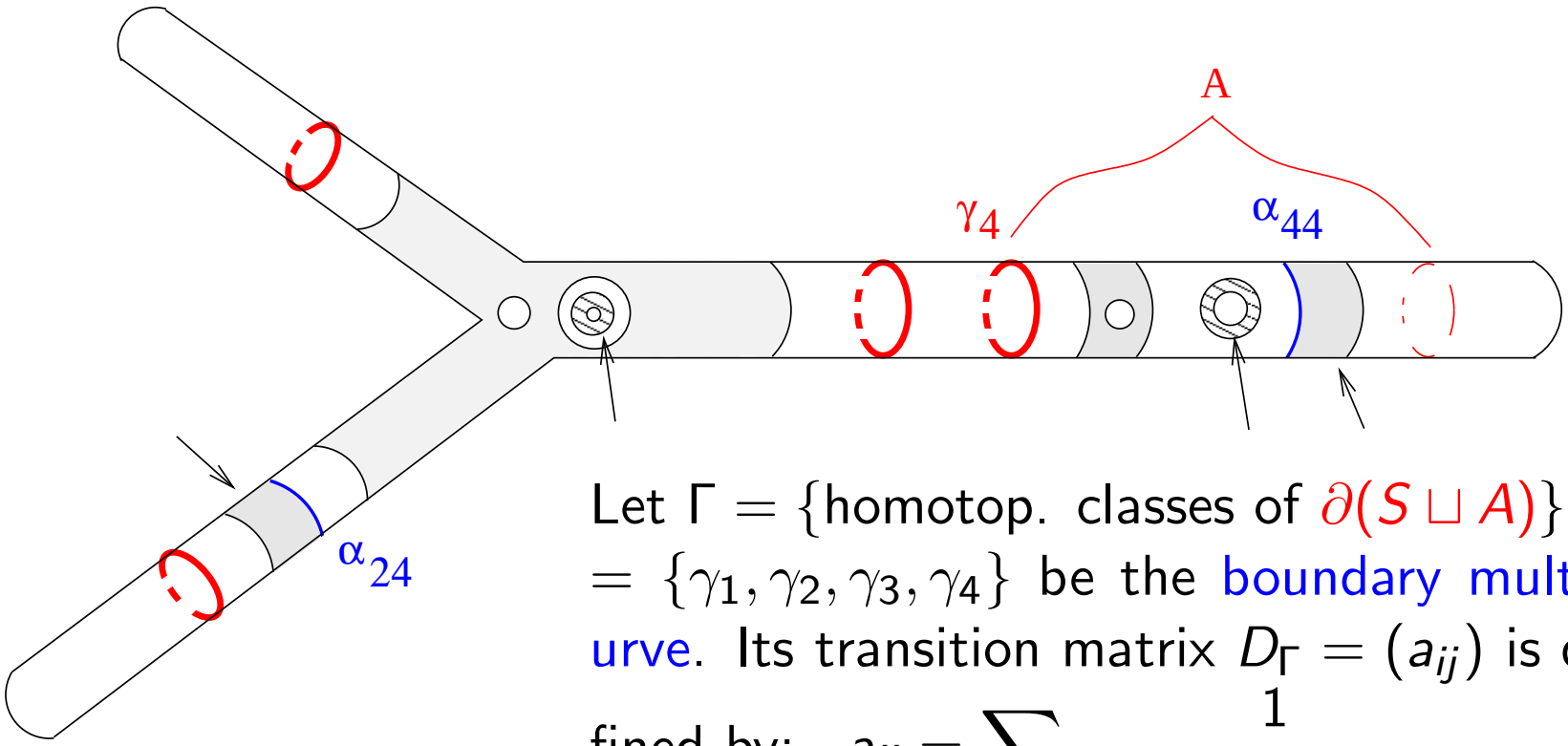
For example: $a_{44} = \frac{1}{deg(F : \alpha_{44} \to \gamma_4)}$, $a_{24} = \frac{1}{deg(F : \alpha_{24} \to \gamma_4)}$.



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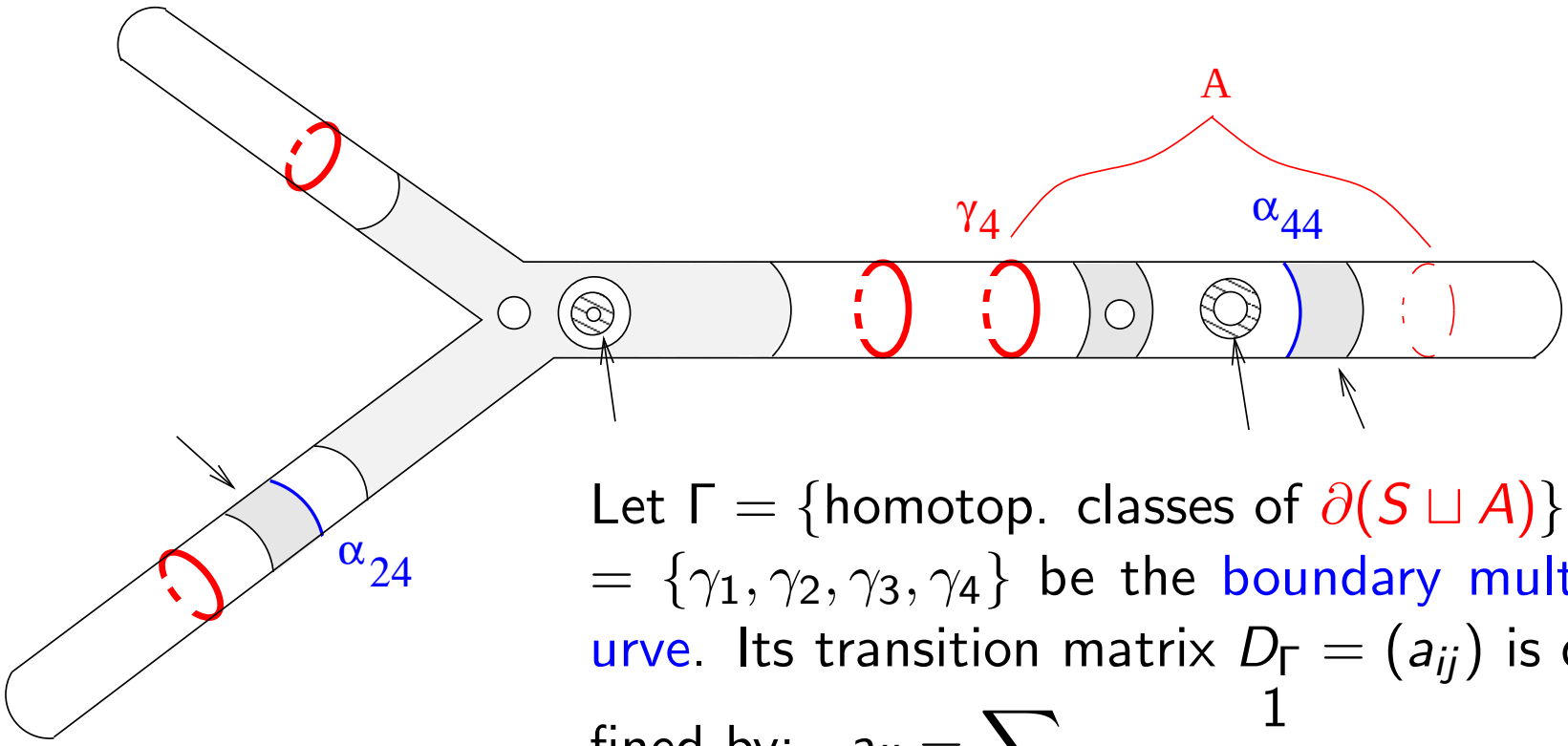
$\boxed{\lambda(D_\Gamma) < 1}$ corresponds to $\sum \frac{1}{d_i} < 1$, deals with moduli problems.



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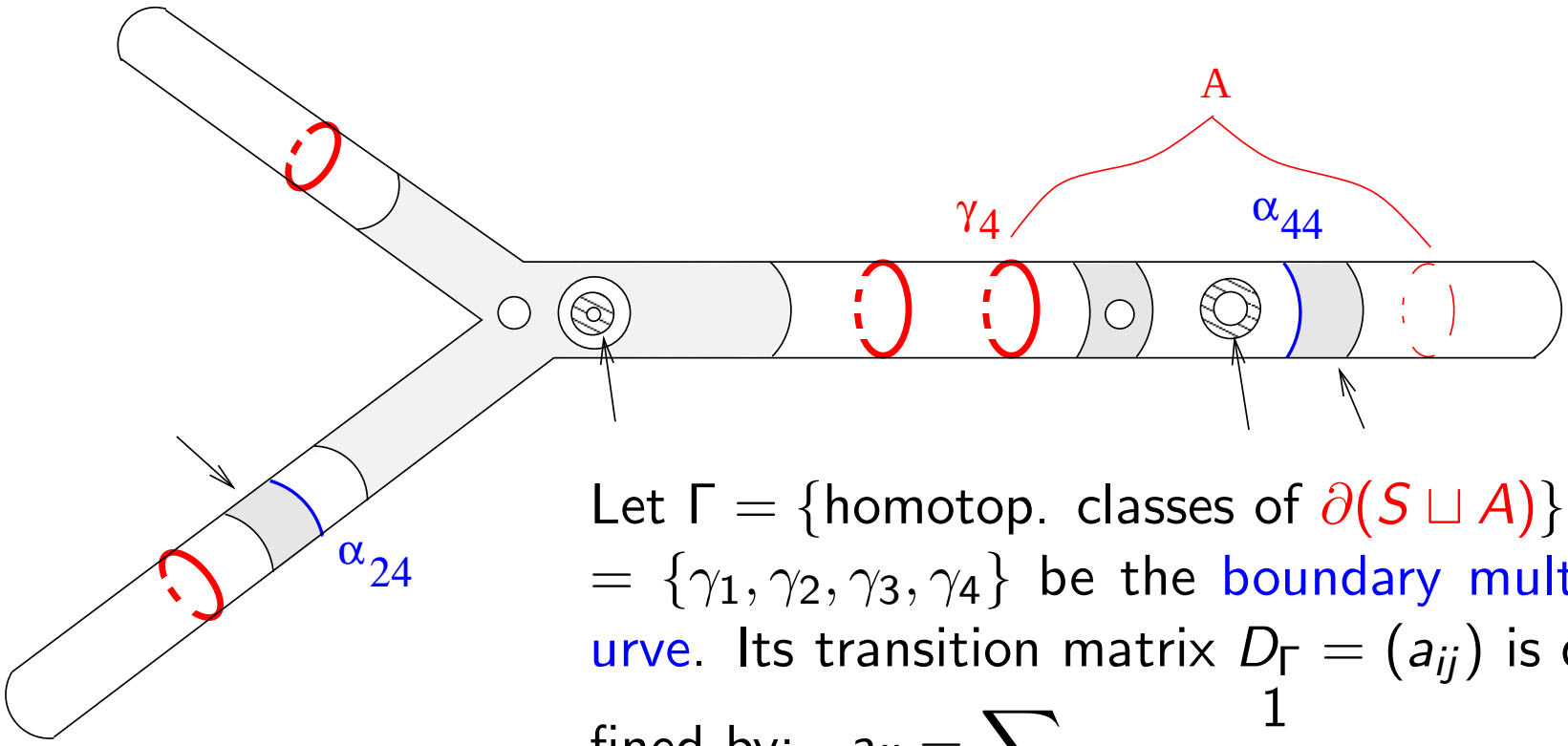
F has a boundary obstruction if $\lambda(D_\Gamma) \geq 1$.



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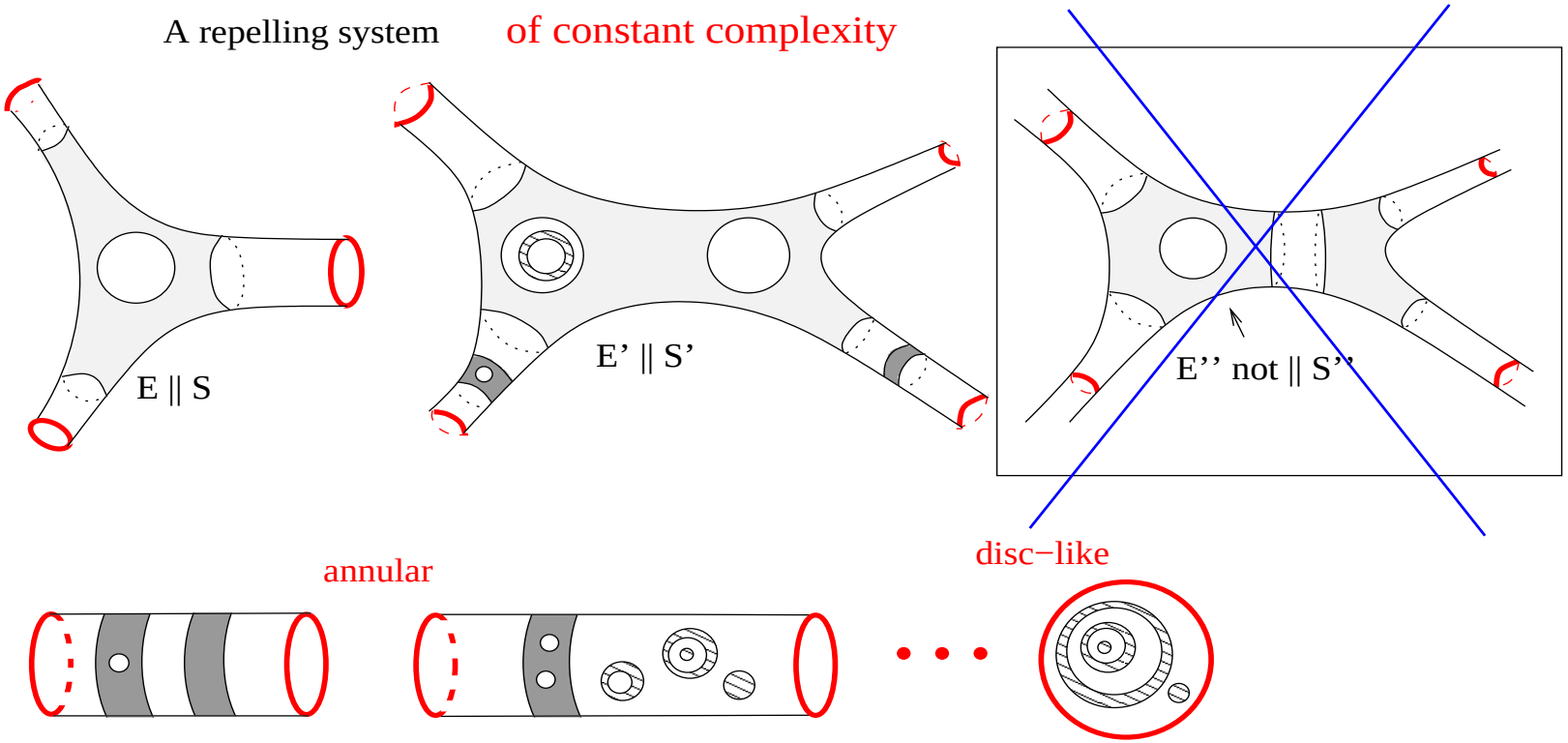
$F \sim_{comb}$ a hol.-model $\Longleftrightarrow$

$\begin{cases} \bullet\ (F, E_{SS}, S) \text{ has no Thurston obstructions,} \\ \bullet\ \lambda(D_\Gamma) < 1. \end{cases}$

Constant complexity ≈ disc-annular type

Comb. Lemma: After pullback and fill−in, F can be reduced to

A repelling system of constant complexity