Séminaire systèmes dynamiques et géométrie

Angers, 9 oct. 2012

Modèle de Bill Thurston de l'espace des polynômes dynamiques

This presentation is jointly prepared by

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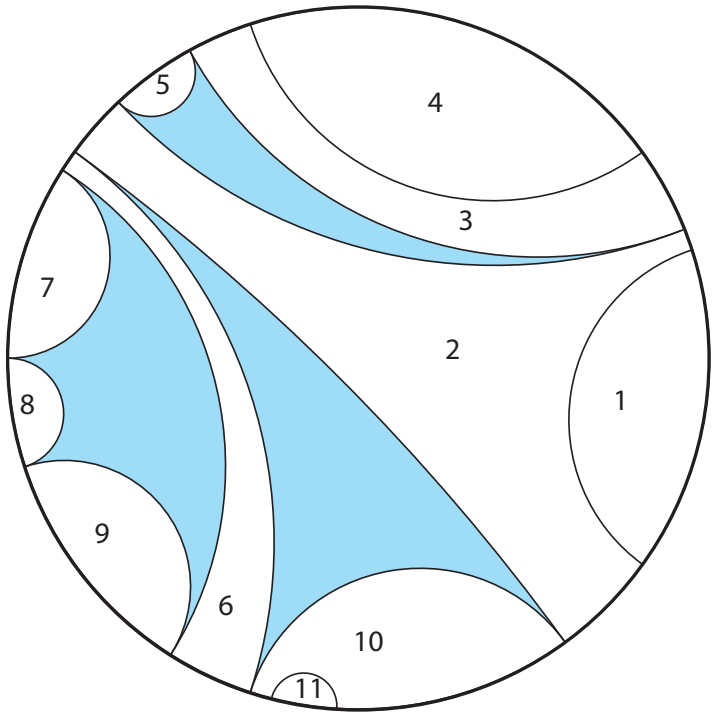
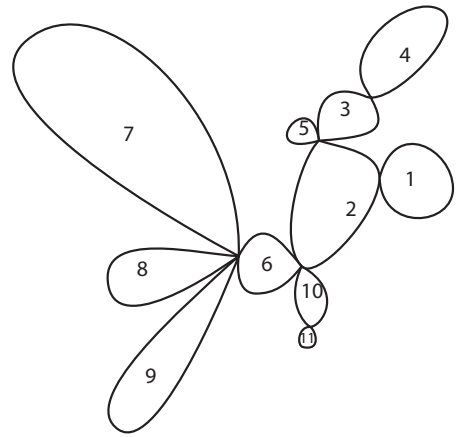
0. Lamination and quotient

A *treelike* lamination on the circle is a closed equivalence relation such that for any two distinct equivalence classes, their convex hulls are disjoint.

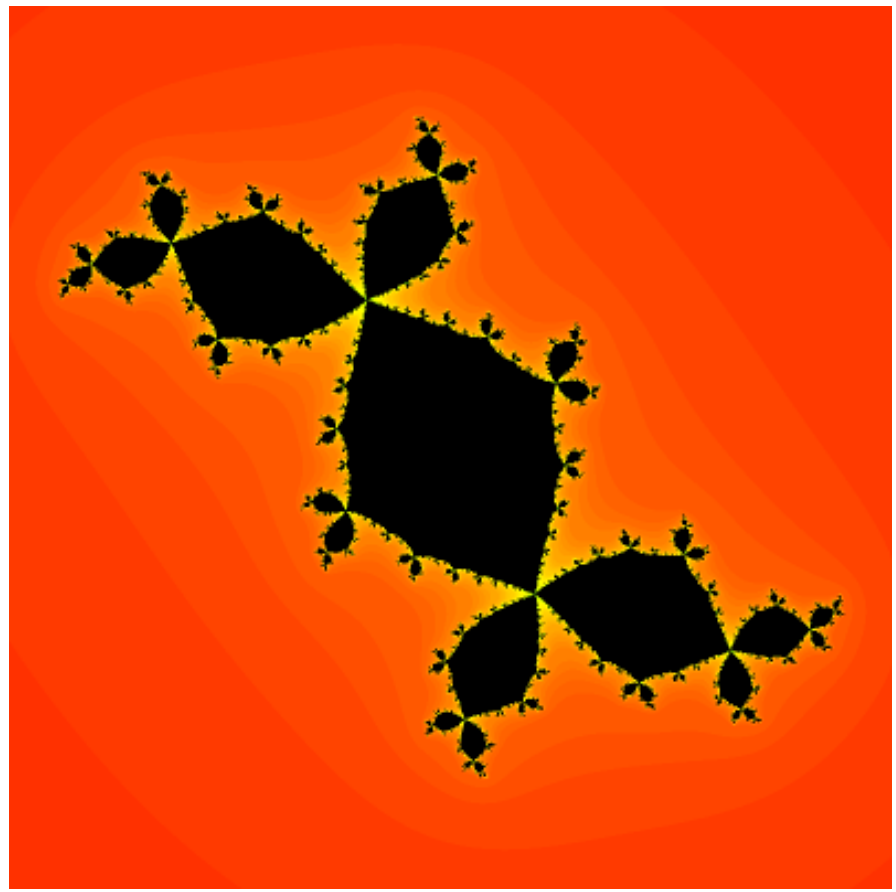




The leaves and shaded areas can be shrunk to points by a pseudo-isotopy of the plane to obtain (topologically) the figure on the right.

Iteration of a polynomial P

A degree d monic polynomial $P : \mathbb{C} \to \mathbb{C}$ behaves like z^d near ∞ (in fact can be conjugate to z^d), so ∞ attracts points out of a compact set. This defines K_P (the filled Julia set) as the set of points NOT attracted by ∞.



If K_P is connected (and locally connected), the extended conjugacy induces a treelike lamination L_P whose quotient is K_P. This (infinite) lamination L_P is 'determined' (generated under pullback by z^d) by a finite lamination m_P called 'primitive' majors.

Thus studying the parameter space $\{P, K_P \text{ connected}\}$ amounts to study $\{L_P\}$ or $\{m_P\}$.

Main question. Understand the space of primitive majors of degree d.

I. A model for the parameter space

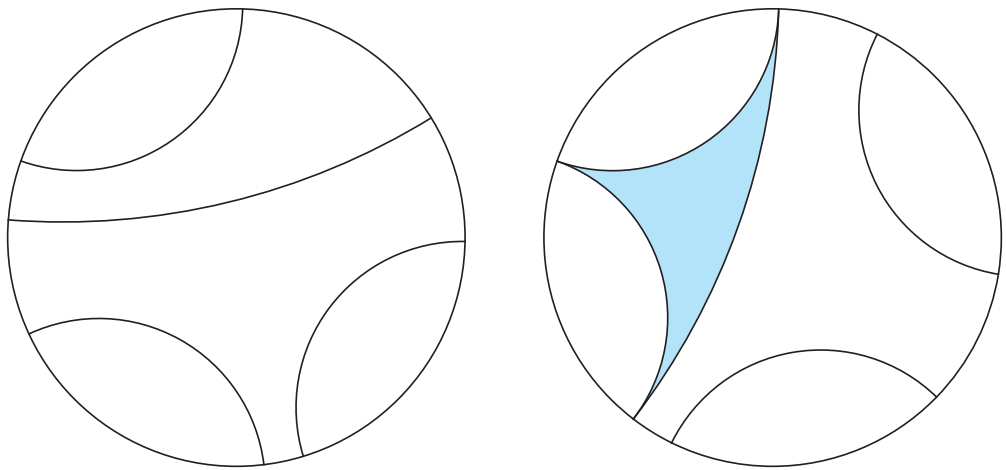
From Bill's talk at Topology Festival (May 4-7, 2012)

It turns out that the set of all "critical" degree d polynomials can be approximately described by collections of arcs of the disk whose endpoints have angle between them of the from k/d. It took me a while to realize that the set of all such arcs, along with the limiting cases where some endpoints coincide and there are additional implicit arcs, describe a spine for sets of d disjoint points in $\mathbb{C}$, that is, its fundamental group is the braid group and higher homotopy groups are trivial.

The space of primitive majors of degree d

A primitive degree-d major is a collection of disjoint leaves and polygons in $\mathbb{D}$ each of whose vertices are identified under $z \mapsto z^d$, with the total criticality $d-1$ (a k-gon has *criticality* $k-1$.)

Two examples of degree 5 primitive majors (critical portraits).



Their space is denoted by $PM(d)$. Each $m \in PM(d)$ induces a pseudo-metric $met(m)$ on S^1. One may equip $PM(d)$ a metric :

$$d(m, m') = \sup_{x,y \in S^1} |met(m)(x,y) - met(m')(x,y)|.$$

Degrees 2 and 3 (from Bill's draft)

A primitive quadratic major is just a diameter of the unit circle, so PM(2) is itself a circle. This corresponds to the fact that the 2-strand braid group is $\mathbb{Z}$.
A primitive cubic minor (major) is either an equilateral triangle inscribed in the circle, or a pair of chords that each cut off a segment of angle-length $2\pi/3$. There are a circle's worth of equilateral triangles. To each primitive cubic that consists of a pair of leaves, there is associated a unique diameter that bisects the central region. Thus, the space of two-leaf primitive cubic majors fibers over S^1, with fiber an interval. When the diameter is turned around by an angle of π, the interval maps to itself by reversing the orientation, so this is a Moebius band. The boundary of the Moebius band is wrapped attached to the circle of equilateral triangle configurations, wrapping around 3 times, since, given an equilateral triangle, you can remove any of its three edges to get a limit of two-leaf majors.

$$\text{the Moebius band} \quad [0,\tfrac{1}{2}]\times]0,\tfrac{1}{6}[/\{0,\epsilon\}\sim\{\tfrac{1}{2},\tfrac{1}{6}-\epsilon\}.$$

message of 26 march 2011

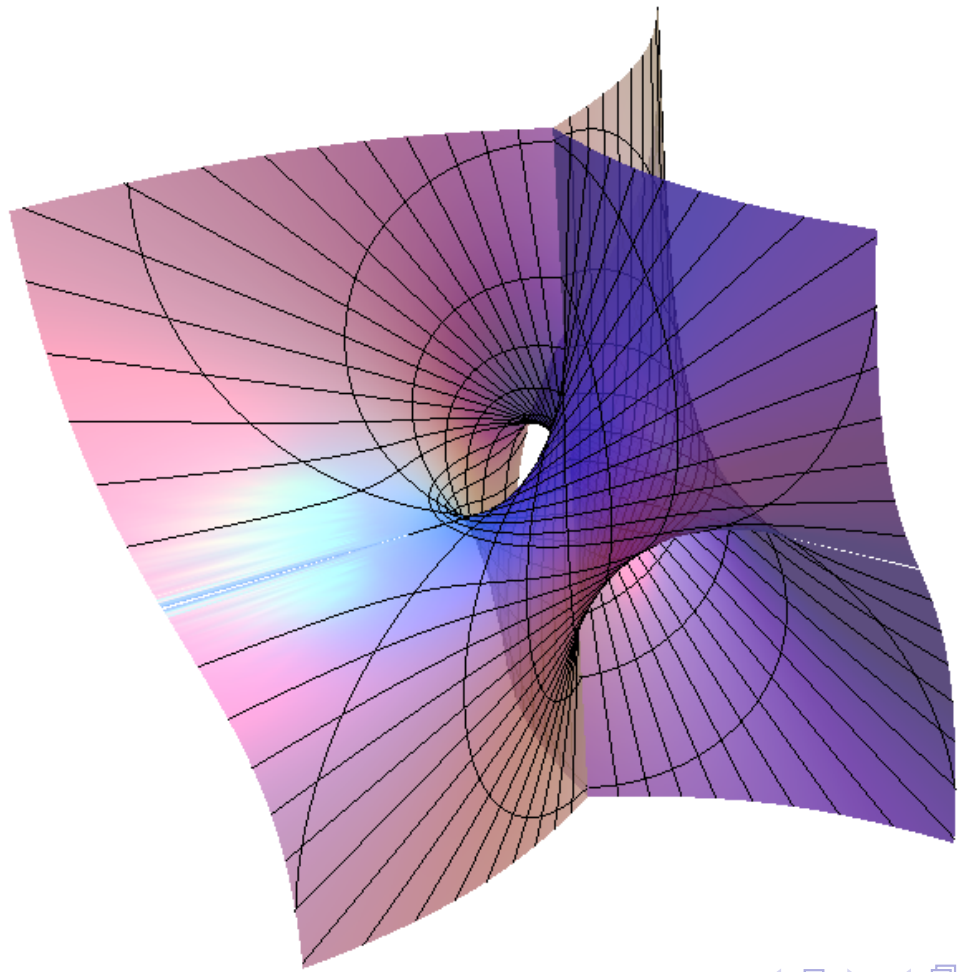
This figure can be embedded in S^3, which should somehow connect to the parameter space picture, as follows : You can make a Moebius band by twisting a long thin strip 3 half-turns, so that its boundary is a trefoil knot. In S^3, if you make the Moebius band wider and wider, eventually you can make the boundary collide with itself in a circle, where it wraps around the circle 3 times. Another way to say this : take two great circles in S^3 that are polar to each other ($\pi/2$ apart). Draw a perpendicular arc between them. Now rotate one circle at a speed of 2 while rotating the other great circle at a speed of 3. The arc sweeps out a 2-complex topologically equivalent to the set of cubic major sets. This is also a spine for the complement of the discriminant locus for cubic polynomials, but I'm not sure how that description fits in.

messages of 27 march 2011

You can think of S^3 as $\mathbb{R}^3$ together with a point at infinity, with the correspondence given by stereographic projection.... So a convenient choice of great circles $\pi/2$ apart is the z-axis together with the circle in the xy plane. It would be easy to make a movie of the trajectory of the unit line segment [0,1] on the x axis as the z-axis is rotated twice (I think this is just $(0, 0, \tan(2 \arctan \theta))$ while the x-axis is rotated 3 times $(cos(3\theta), sin(3\theta), 0)$.

Here's a picture with the role of the unit circle and the z-axis interchanged. Maybe this is easier to visualize. A neighborhood of the unit circle in the xy plane is a 3/2 twisted Moebius band. The surface at first appears to separate S^3 into 3 regions, but there are 3 tunnels connecting the regions, so the complement is actually connected. If you go in a loop through all three tunnels, you will make a trefoil knot : this structure can be thought of as a spine for the complement of the trefoil knot. The trefoil knot also happens to be the locus in the unit sphere in $\mathbb{C}^2$ where the discriminant of a cubic polynomial, $z^3 + az + b$ is 0.

I think you can describe this surface implicitly as the (a,b) coefficients of polynomials as above such that the minimum distance between roots is achieved for at least two pairs of roots, where $|a|^2 + |b|^2 = 1$. I don't know exactly how this relates either to the cubic major sets or to the parameter space, but it's suggestive.



braid group presentation

The spine gives a graphic description for one presentation of the 3-strand braid group,

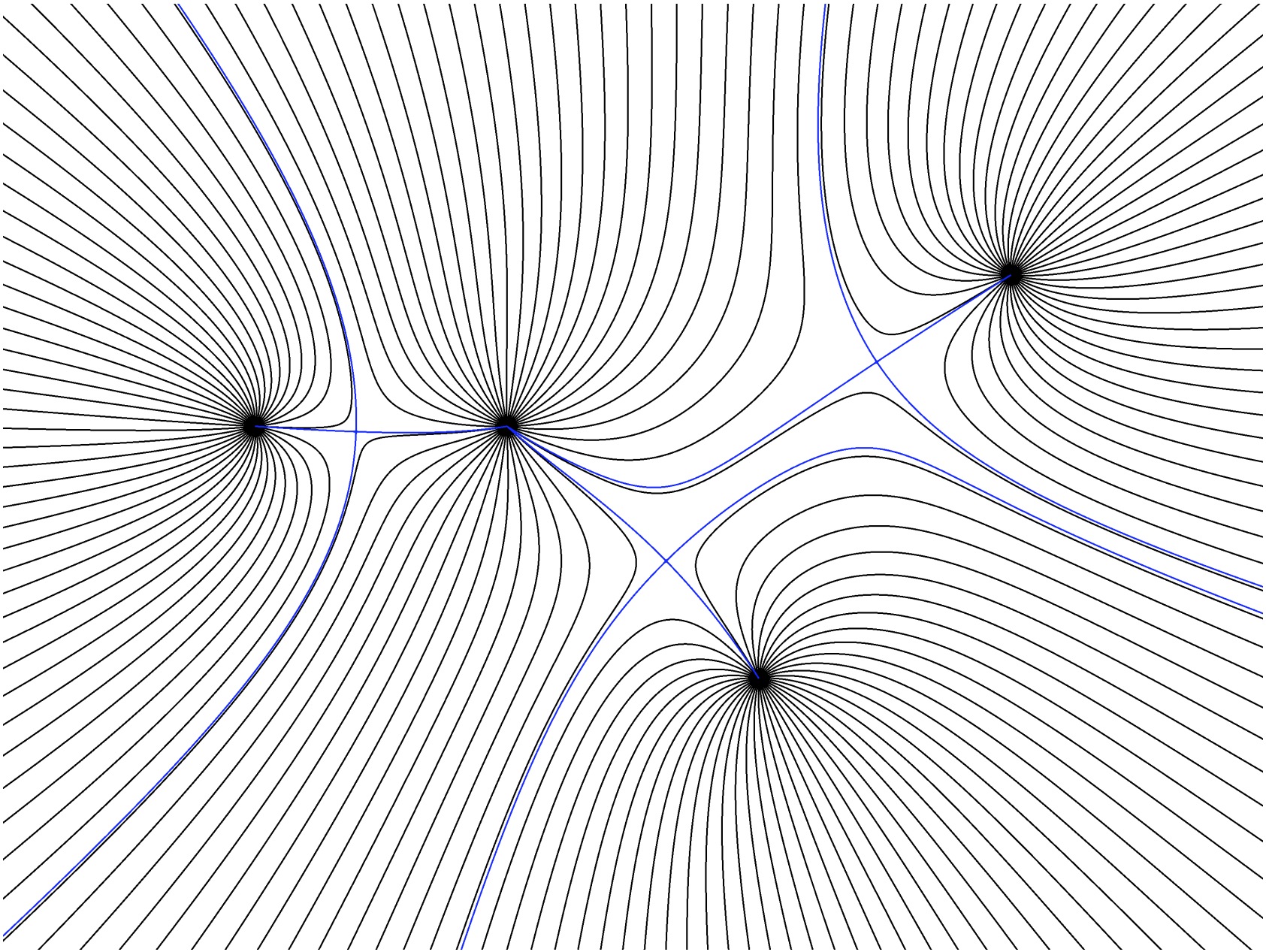
$$B_3 = \langle a, b | a^2 = b^3 \rangle$$

where a is represented by the core circle of the Moebius band, and b is the circle of equilateral triangles, the two loops being connected via a short arc to a common base point. The presentation is an amalgamated free product of two copies of $\mathbb{Z}$ over subgroups of index 2 and 3. As braids, a is a 180 degree flip of three strands in a line, while b is a 120 degree rotation of 3 strands forming a triangle ; the square of a and the cube of b is a 360 degree rotation of all strands, which generates the center of the 3-strand braid group.

Theorem (around 1 april 2011). The space $PM(d)$ *embeds in the space of monic centered degree d (non-dynamical) polynomials as a spine for the set of polynomials with distinct roots, that is, the complement of the discriminant locus. The spine consists of polynomials whose critical values are all on the unit circle.* Thus $\pi_1(PM(d))$ is the d-strand braid group and all higher homotopy groups are trivial (i.e. $PM(d)$ is a $K(B_d, 1)$ space).

Proof. Take any degree d polynomial P with no multiple zeros, and look at $\log(P)$, thought of as a map from $\mathbb{C} \setminus$ *roots* to an infinite cylinder. For each critical point, draw the two separatrices going upward (i.e., this is the curve through each critical point of P that maps by P to a vertical half-line on the cylinder $\leftrightarrow$ a ray in $\mathbb{C}$ pointed opposite the direction to the origin).
Then make the finite lamination in a disk that joins the ending angles of these separatrices. This is a degree-d major set. Conversely, for each major set, there is a contractible family of polynomials whose separatrices end that at the corresponding pairs of angles.
To pick a canonical representative of each of these families : look at polynomials whose critical points (values) are on the unit circle. This forms a spine for the complement of the discriminant locus for degree d.

Proof of $\Longleftarrow$: Pullback the critical value radial spider.



They are also : separatrices of the gradient flow for $|P|$, $|P|^2$, $\log|P|$, or of the Newton flow $-\frac{P}{P'}$ (Newton tree)...

Proof of $\Longrightarrow$:

a primitive major $\longrightarrow$ a critical value radial spider + Hurwicz data $\longrightarrow$ reconstruction of P through cut-paste and Riemann Mapping Theorem.

A generic critical value radial spider $\longrightarrow$ d^{d-2} Hurwicz data. The counting is more complicated in the degenerate case.

P has only simple roots $\longleftrightarrow$ cr. values $\neq 0$. In this case one can slide the cr. values to the unit circle generating the same primitive major. q.e.d.

Show H.H. Rugh's scilab code...

II. Binding entropy and the torus presentation

The **Binding Entropy** h for a postcritically finite polynomial map f of degree d is the topological entropy of the action of $(\theta, \eta) \mapsto (d\theta, d\eta)$ on pairs of external angles that end at the same point in the Hubbard tree H_f of f.

It is also the topological entropy of $f|_{H_f} : H_f \to H_f$, and $\lambda = e^h$ is the leading eigenvalue of the transition matrix of f on the edges of H_f.

For $\lambda = e^h$, the sequence λ^n approximates the growth of

the number of edges in $f^{-n}H_f \cap H_f$ (the lap number);
the maximal edge decomposition of an edge of H_f under f^{-n};
the maximal number of nth-preimages of a point in H_f;
the number of n-periodic ends in H_f;
the number of n-periodic angles landing on H_f;
the number of depth-n puzzle pieces cutting $H_f(??)$; $\cdots$

$\frac{1}{\lambda}$ = radius of conv. of the generating functions.

Torus presentations of laminations (around 21 April 2011)

Consider $Q : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, $(\theta, \eta) \mapsto (2\theta, 2\eta)$ the angle doubling map.

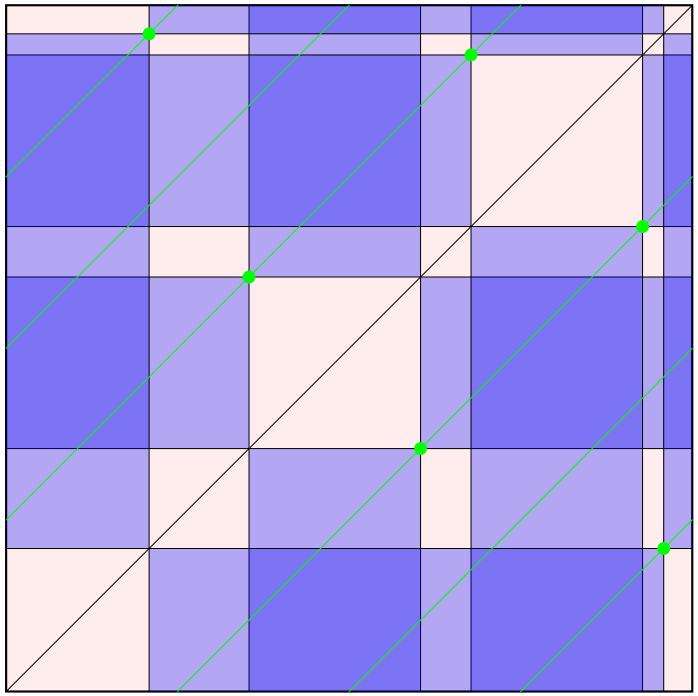
For any θ, let L_θ be the union of the two consecutive closed squares of size $\frac{1}{2}$ along the diagonal of $\mathbb{R}^2$, starting from $(\frac{\theta}{2}, \frac{\theta}{2})$.

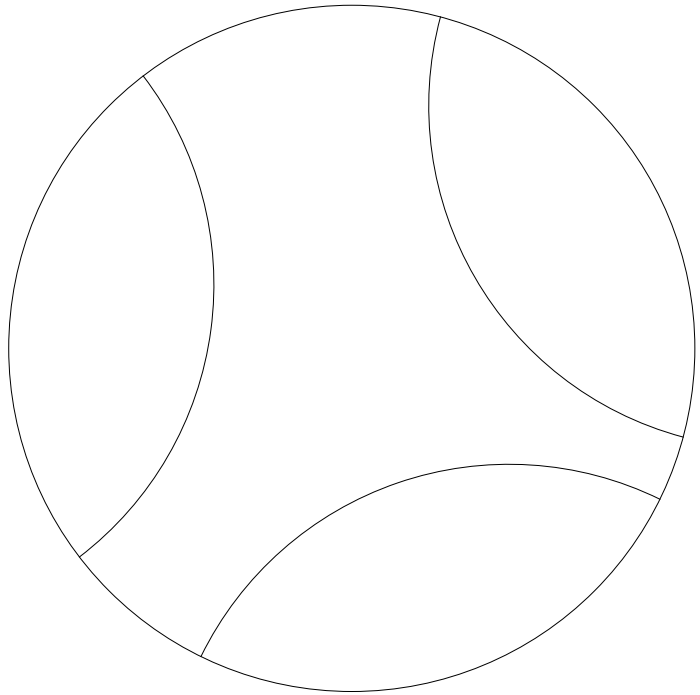
Let K_θ be the non escaping set of the map $Q|_{L_\theta} : L_\theta/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, and Δ be the diagonal.

Then : $K_\theta \setminus \Delta$ describes the binding set, or the set of pairs of angles having the same itinerary relative to the two closed half circles.

If θ is rational, $2^{\dim_H(K_\theta \setminus \Delta)} = \lambda(z \mapsto z^2 + c_\theta)$, *since* $K_\theta \setminus \Delta$ *has a measure that is locally multiplied by* λ *under* $\eta \mapsto 2\eta$ *which expands the metric of the circle by 2, so the measure is a conformal density of dimension* $\log \lambda / \log 2$.

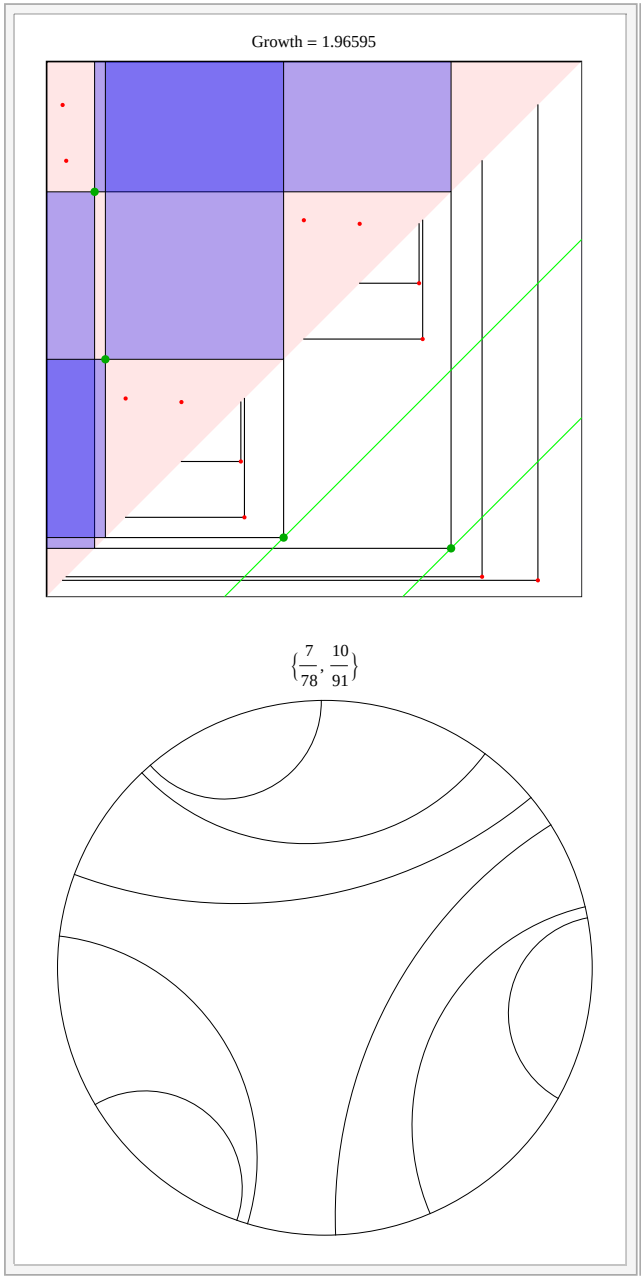
The set L for higher degree polynomials.

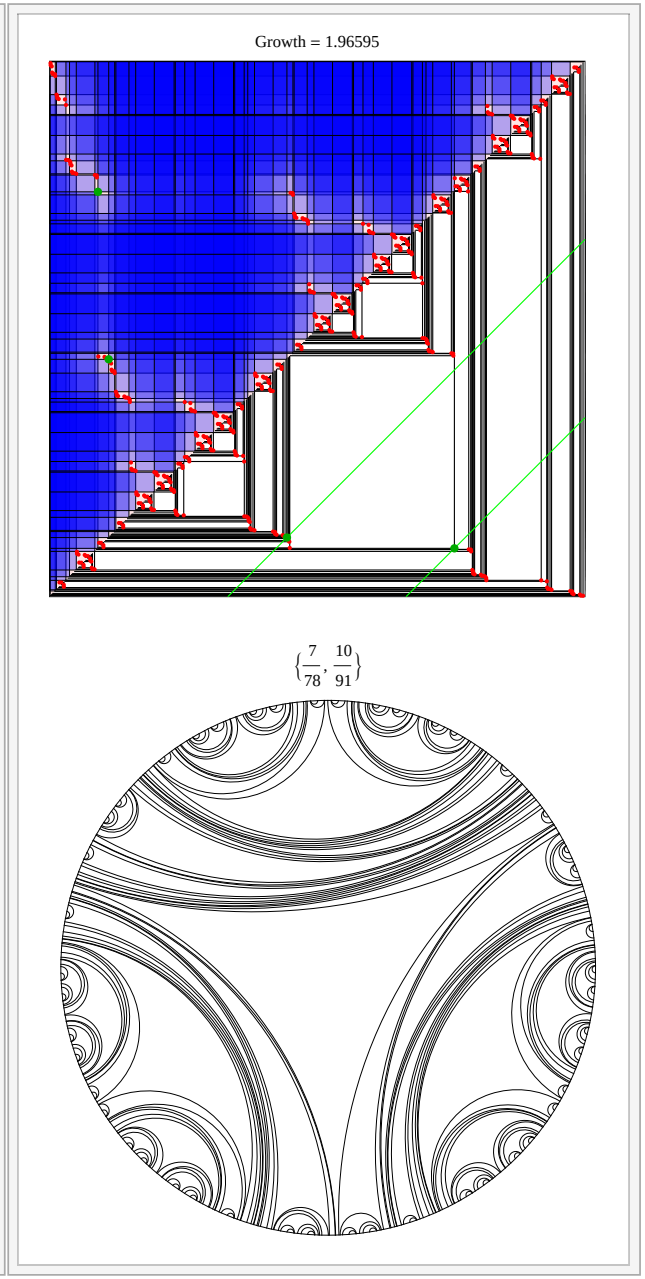




Given a primitive major $m \in PM(d)$, each region of $\mathbb{D} \setminus m$ touches S^1 in a union of one or more closed intervals $J_1 \cup \cdots \cup J_k$ of total length $1/d$. The product $(J_1 \cup \cdots \cup J_k) \times (J_1 \cup \cdots \cup J_k)$ is a union of k^2 rectangles whose total area is $1/d^2$ that maps under the degree d^2 covering map $(x, y) \mapsto (d \cdot x, d \cdot y)$ to the entire torus.

The binding set, whose dimension is to be measured





III. The entropy algorithm bypassing trees and dimensions (march 2011)

Start from any rational angle $\theta \neq 0$ and separate the circle into two halves by $\theta/2, \theta + 1/2$, each half is a closed segment in the circle (so the cutting points belong to both halves).

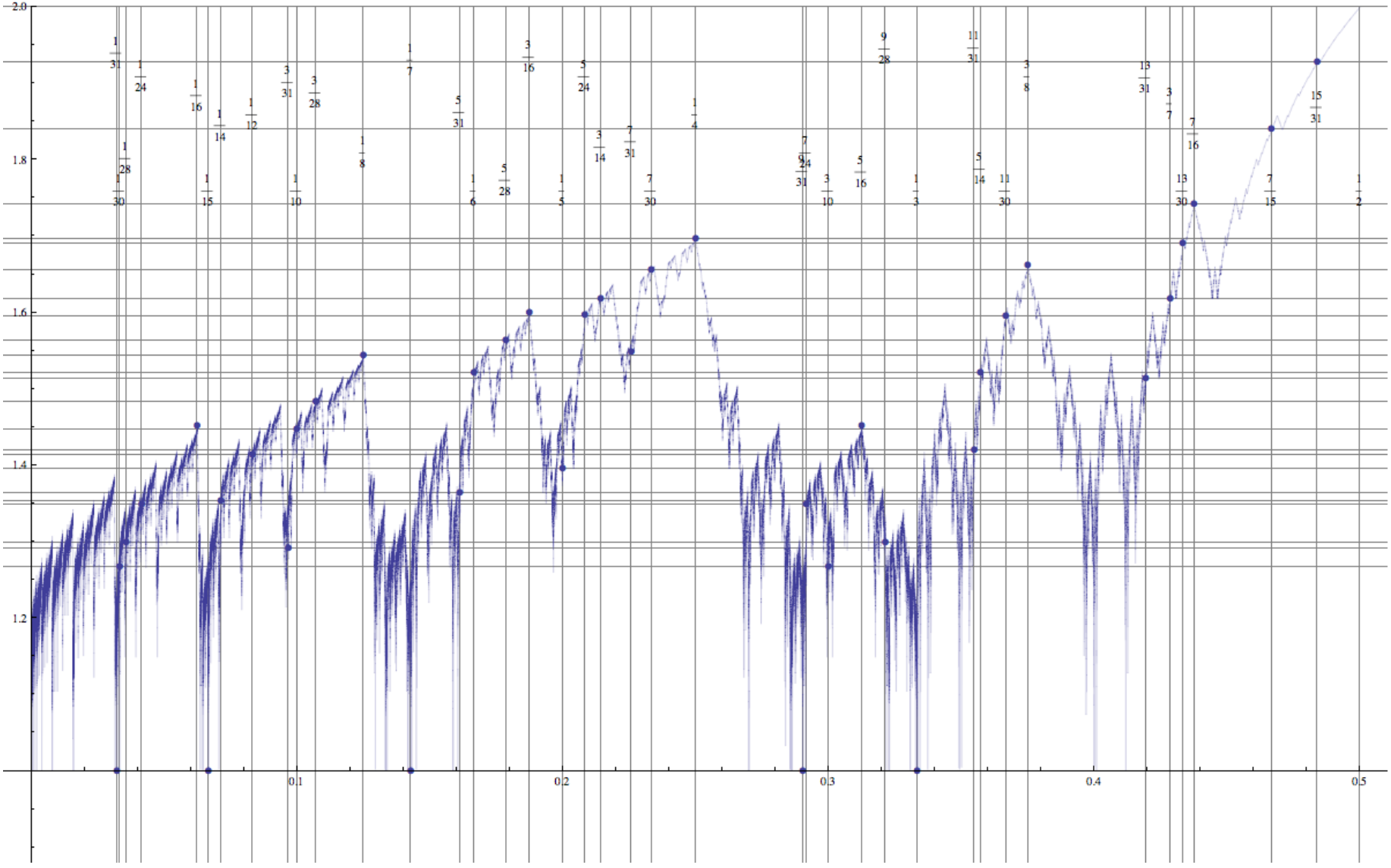
In the following all the $\theta/2$ should be replaced by $\theta + 1/2$ if the latter is periodic.

1. Form the set of all possible unordered pairs $\{2^n\theta, 2^l\theta\}$ so that both l, n are ≥ -1 and the two angles are unequal. Consider them as a basis of an abstract linear space.
2. Compute the leading eigenvalue of the linear map :

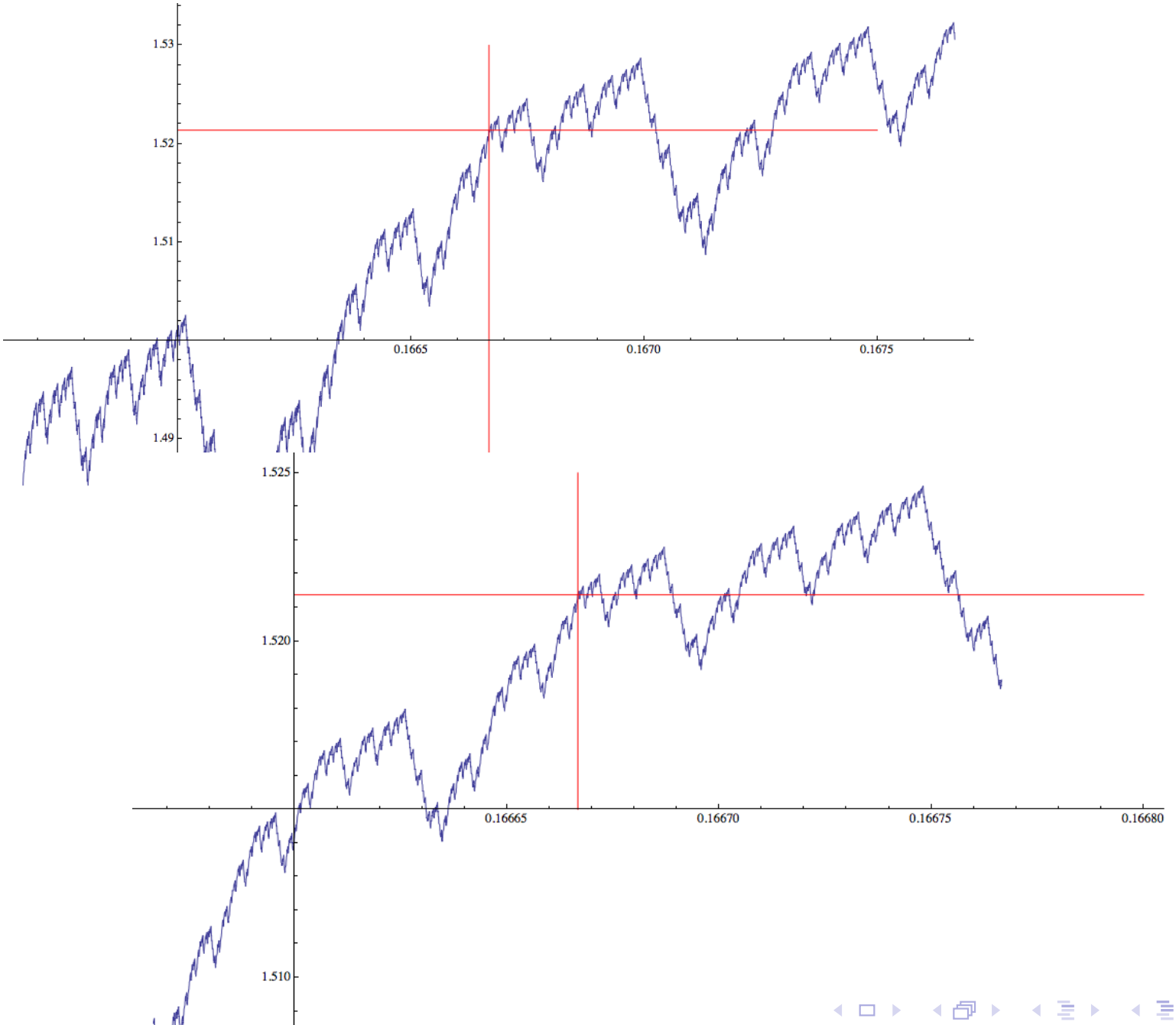
$$\{2^n\theta, 2^l\theta\} \mapsto \begin{cases} \{2^{n+1}\theta, 2^{l+1}\theta\} & \text{if } 2^n\theta, 2^l\theta \text{ are in the same half-circle} \\ \{2^{n+1}\theta, \theta\} + \{\theta, 2^{l+1}\theta\} & \text{otherwise} \end{cases}$$

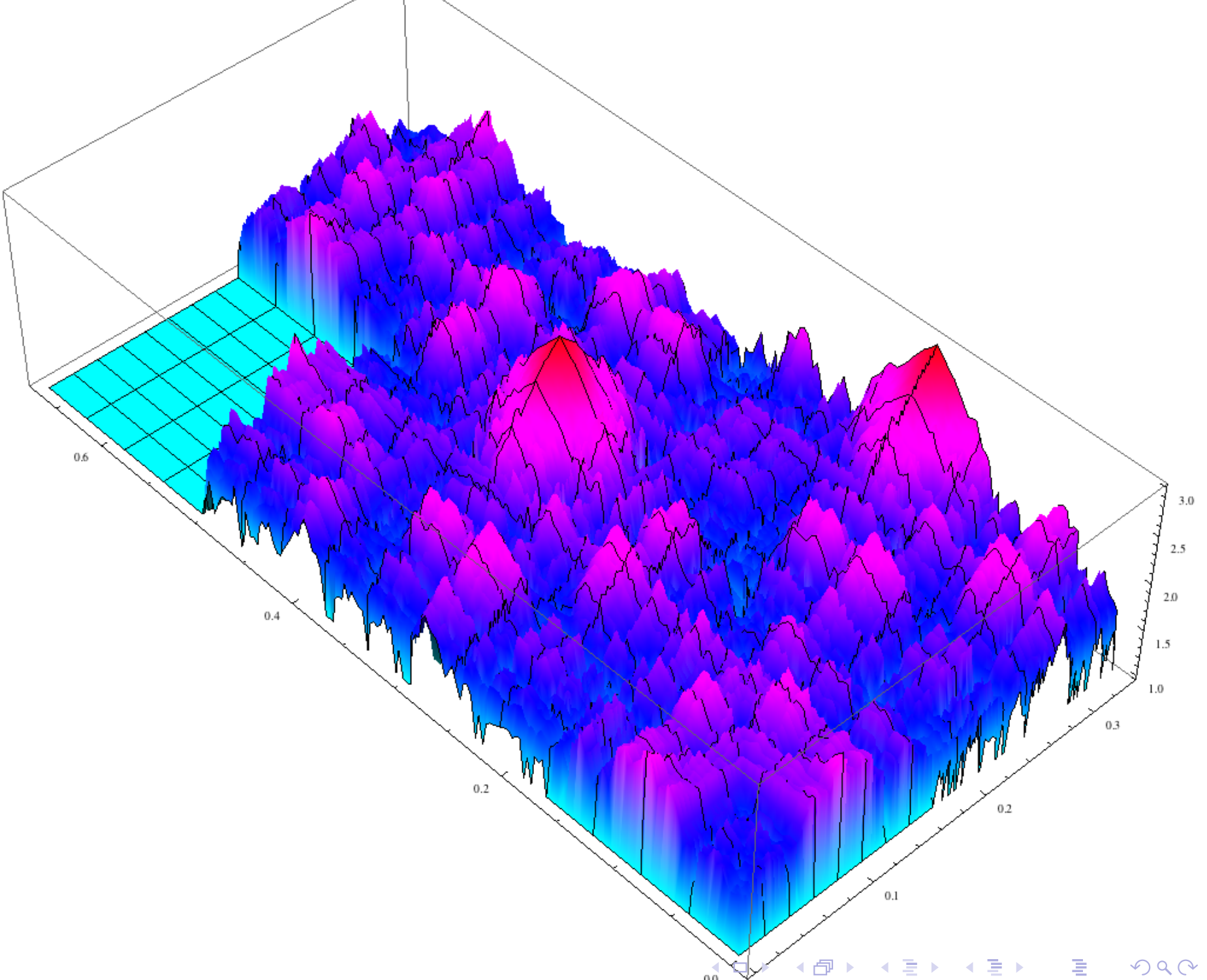
3. You get e^h. A similar algorithm works for any degree.

The plots (on March 13, 2011)



and two zooms at 1/6 (the same day)




0.6
0.4
0.2
0.0
0.1
0.2
0.3
1.0
1.5
2.0
2.5
3.0

IV. Continuity (? ?)

From a message of Bill on March 19 2012 : *Hubbard now thinks he can prove the continuity of entropy, as does Dierk (...). I think I know how to give a complete proof as well, but I'll wait and see what Hamal and Dierk come up with.*

The first issue is to define the binding entropy for any angle...

See work of Chris Penrose, Branner-Hubbard, Kiwi, Poirier, Tao Li, Dujardin-Favre, Bruin-Schleicher, Bardoldi-Dudko-Nekrashevych, Tiozzo, among others...