What is a core and its entropy for a polynomial ?

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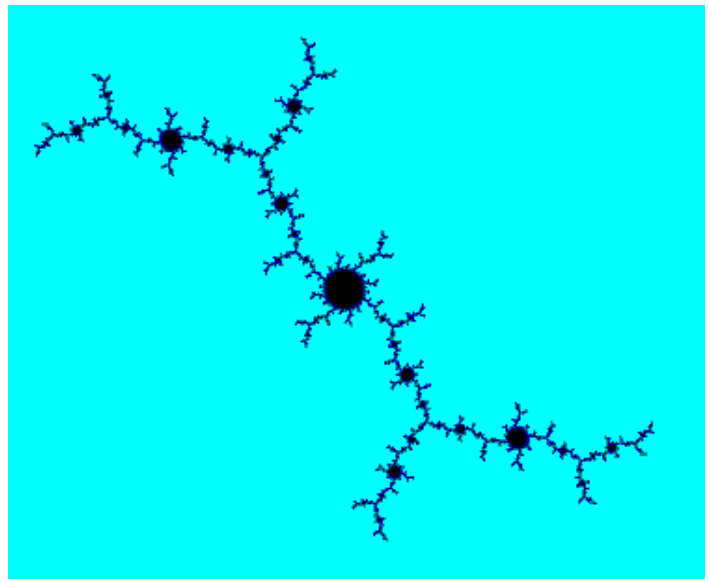
Core(a polynomial f**)= ?**



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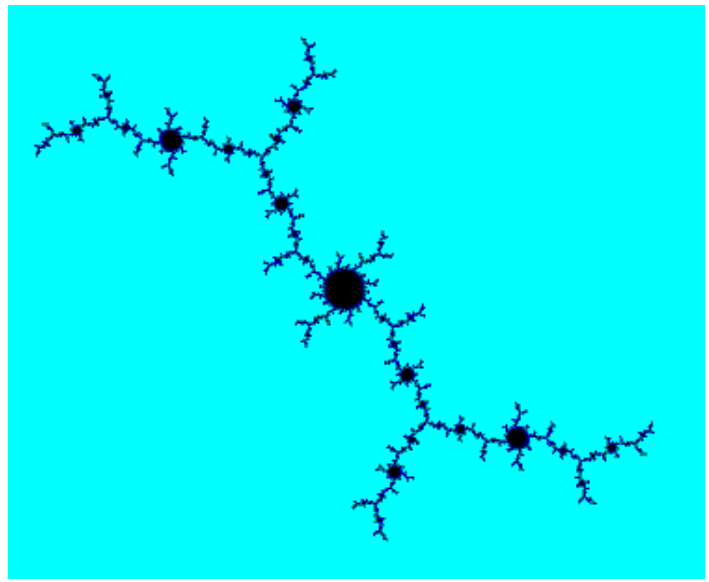

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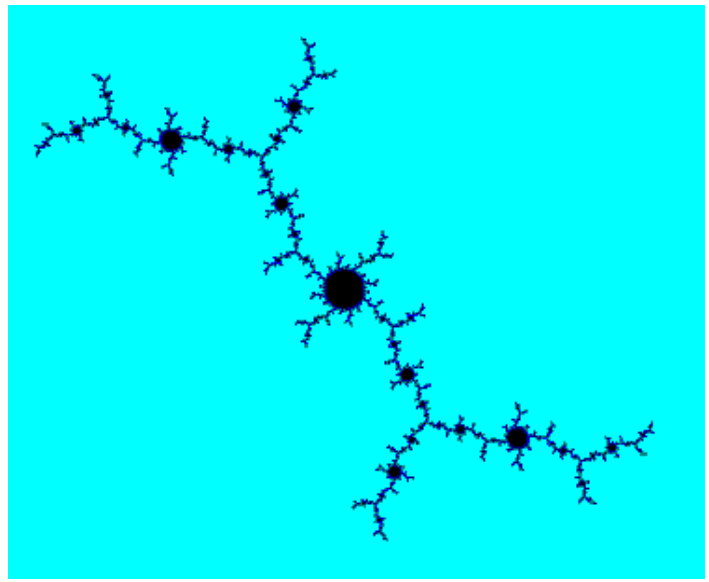

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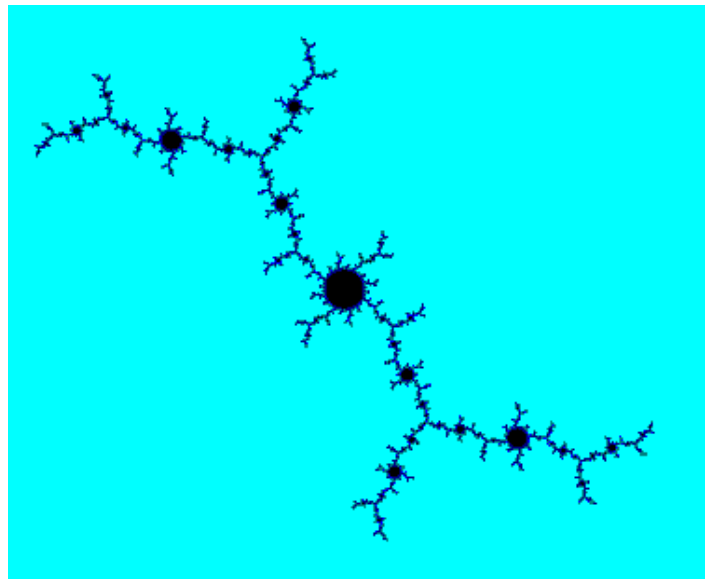

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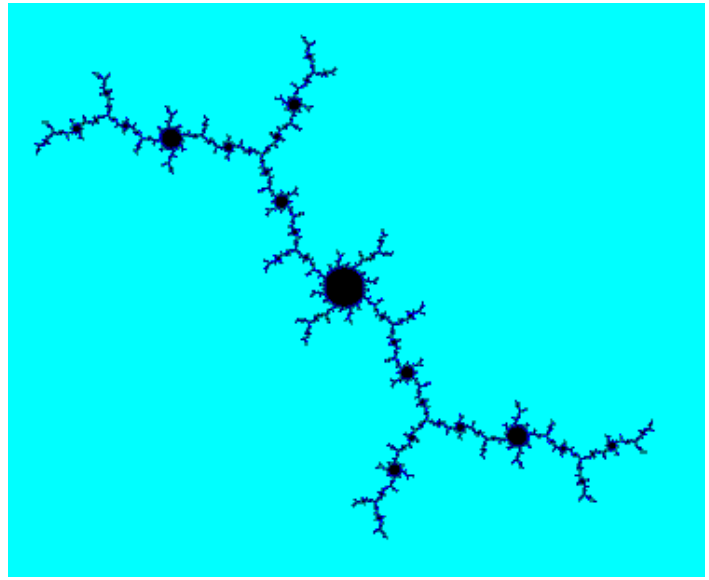

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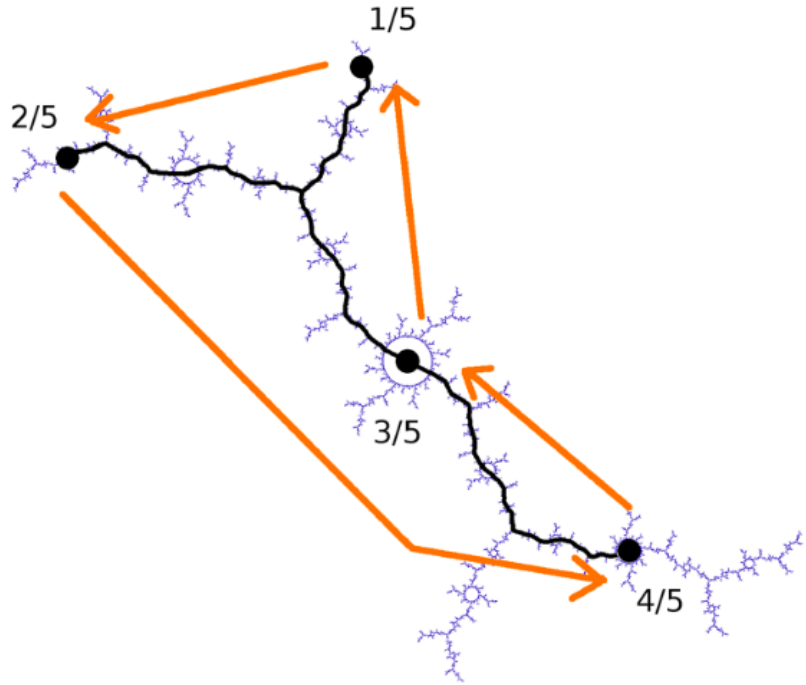
The Hubbard tree H_c of $f_c : z \mapsto z^2 + c$, if exists, is the smallest topologically finite tree in K_c containing the orbit of the critical point 0. It is automatically forward invariant.



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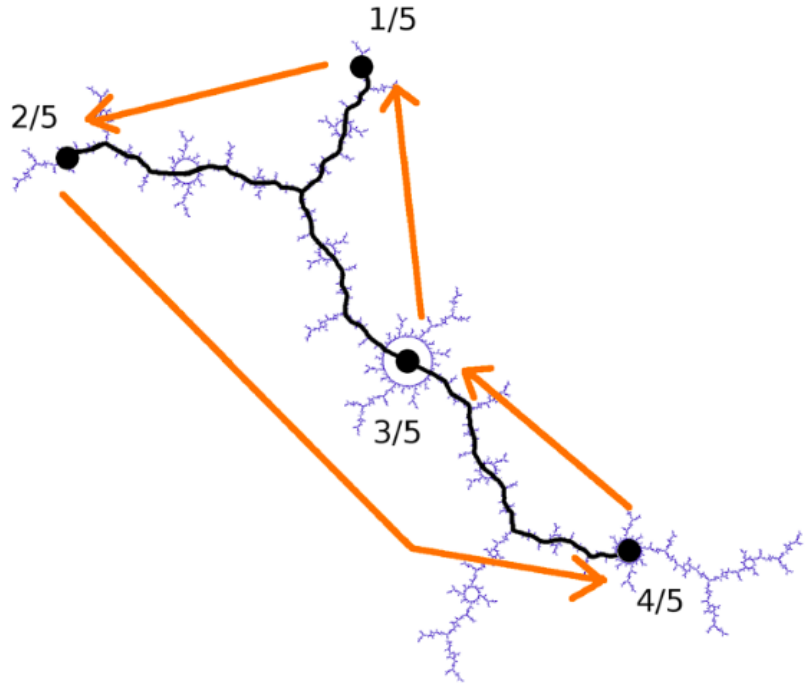


H_c exists for many "good" values of c.
If postcritically finite, $\exists$ Markov partition and matrix M_c and

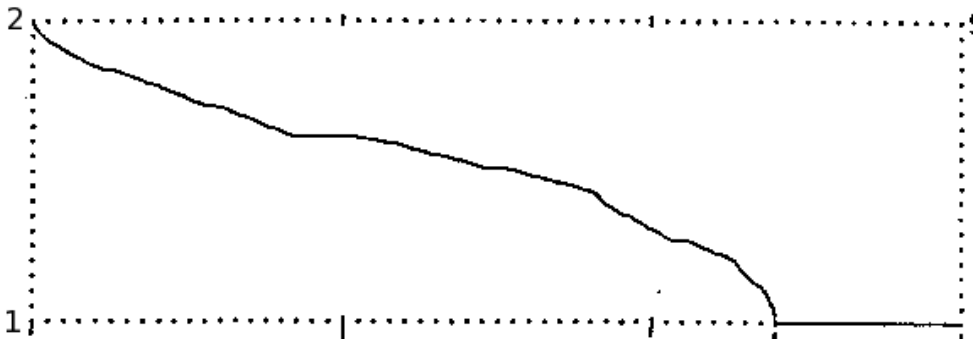
$$e^{h_{top}(f_c, H_c)} = \lambda_{\text{leading}}(M_c).$$

Lemma : When H exists, $e^{h_{top}(f,H)} = d^{\dim(\text{Bi-acc}(f))}$

The idea came from Milnor-Thurston's study on the real quadratic family

$$f_c(z) := z^2 + c \quad c \in [-2, \frac{1}{4}], \; h(c) := h_{top}(f_c, [c, c^2 + c])$$

Theorem. $c \mapsto \mathbf{s}(c) = exp(h(c))$ is continuous (Milnor-Thurston) and weakly decreasing (Douady), $2 \searrow 1$.



For the monotonicity, complex methods using external rays are more powerful...

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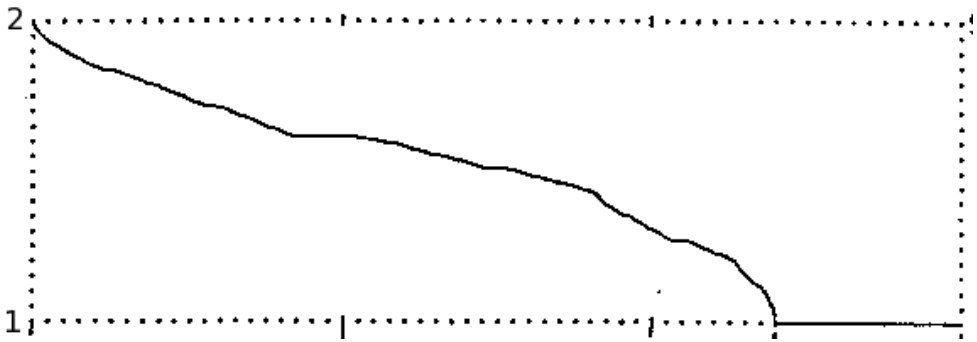

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(Complex methods) Douady's formula

$Y_c := \{t \in S^1,\ R_c(t) \text{ lands on } [c, c^2+c]\}$

is a closed invariant subset of
$q : t \mapsto 2t,\ S^1 \to S^1$, and

airplane1.png

$$h_{top}(q, Y_c) = h(c).$$

When $-2 \leftarrow c$, $Y_c \nearrow$, so $h(c) \nearrow$.
A single function, various subsets...

W. Thurston's torus model, 2011

airplane1.png

1. A single map :
$F : \begin{pmatrix} s \\ t \end{pmatrix} \mapsto 2 \begin{pmatrix} s \\ t \end{pmatrix}$, $\mathbb{T}^2 \to \mathbb{T}^2$.
$A_c := \{(t, 1-t) \in \mathbb{T}^2, t \in Y_c\}$,
$=$ bi-angles landing at a common point on $\mathbb{R}$, is closed forward invariant set, with the same entropy.

2. Connect entropy to dimension :

$$2^{\dim A_c} \overset{\text{Thurston}}{=} e^{\text{Core-entropy}(f_c)} \overset{\text{Tiozzo}}{=} 2^{\dim Y_c}.$$

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Combinatorial approach of W. Thurston

Given $d \geq 2$, consider the torus expanding covering

$$F : \mathbb{T}^2 \to \mathbb{T}^2, \ \begin{pmatrix} s \\ t \end{pmatrix} \bmod 1 \mapsto \begin{pmatrix} d \cdot s \\ d \cdot t \end{pmatrix} \bmod 1$$

W. Thurston : $\{B(m)$ non-closed, invariant subset $\mid m\}$
$\longrightarrow$ core-entropy(polynomials).

To define
- $m, B(m)$ then
- Growth$(m) :=$ $d^{\dim(B(m))}$
- A core $T \subset B(m)$ (if exists) is a 'closed' invariant attractor.

Results :
- $\exists$ core T $\Longrightarrow$ Growth$(m) = e^{h_{top}(F,T)}$
- P pcf $\Longrightarrow \exists m_P$, Growth$(m_P) = e^{h_{top}(P,\text{Hubbard-tree})}$
- m rational $\Longrightarrow \exists$ core
- m rational $\Longrightarrow$ implement a matrix Γ_m s.t. $\lambda(\Gamma_m) = Growth(m)$
- Parameter space
- For any m, Growth$(m) = \exp h_{\mathbb{T}^2}(F, B(m))$ in Bowen's sense

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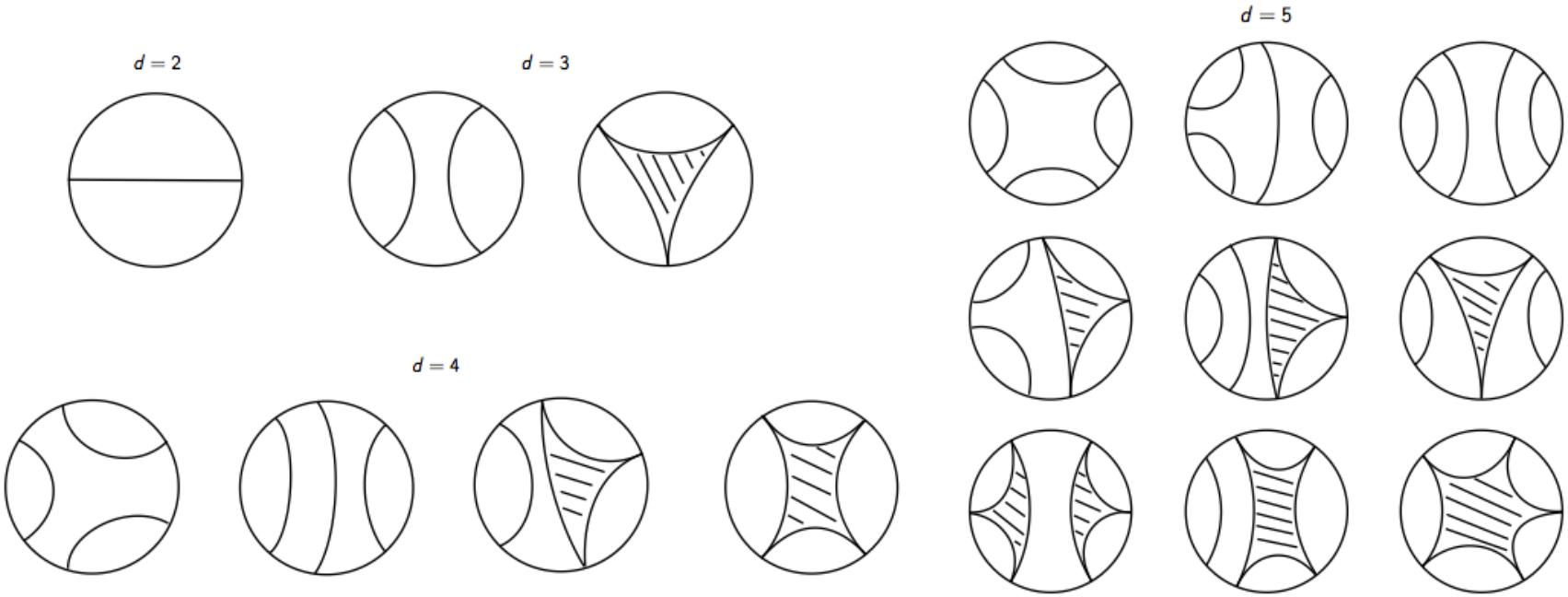
A primitive major (critical portrait) m of degree d = {disjoint hyperbolic leaves and ideal polygons in $\mathbb{D}$} s.t. $\mathbb{D} \setminus m$ has d regions, each touches S^1 in a union of closed (non-point) intervals of total length $1/d$.



Tomasini : a closed form for number $\tilde{p}_d$ of combinatorial classes.

d	2	3	4	5	6	7	8	9	10	11	12	
$\tilde{p}_d$	1	2	4	9	27	94	364	1529	6689	30230	140114	$\cdots$

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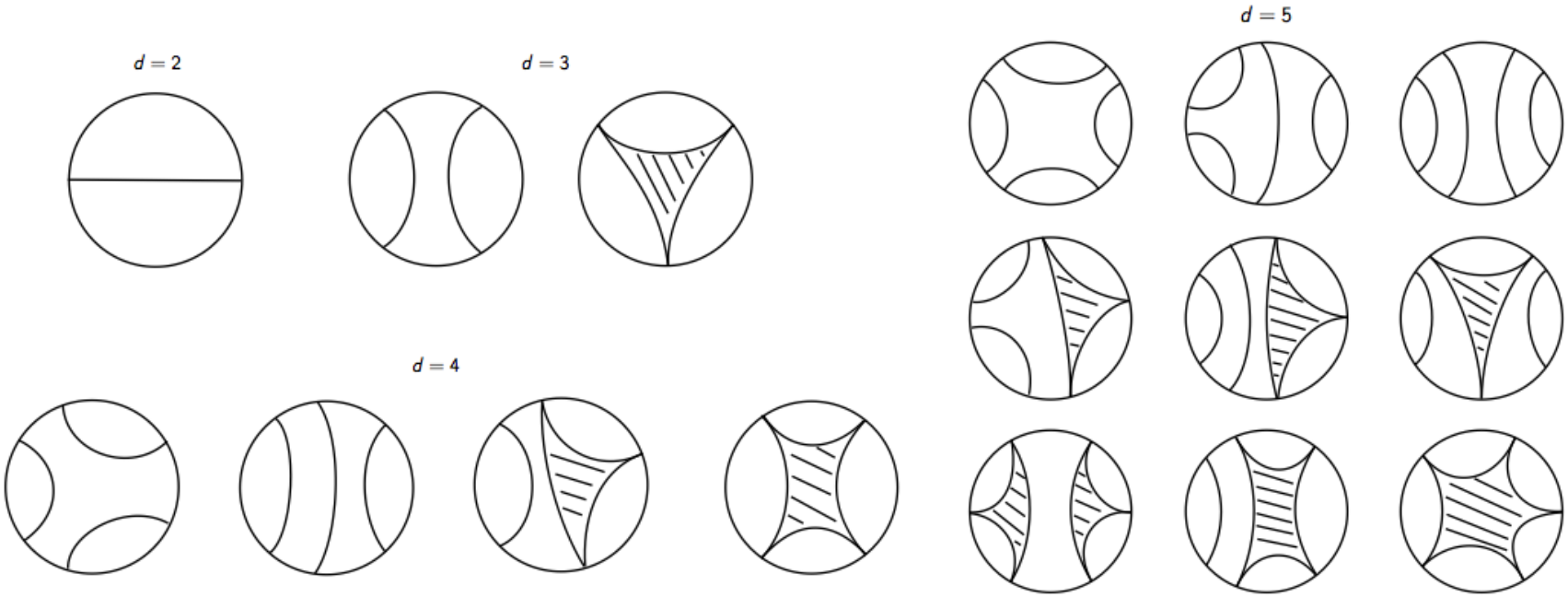

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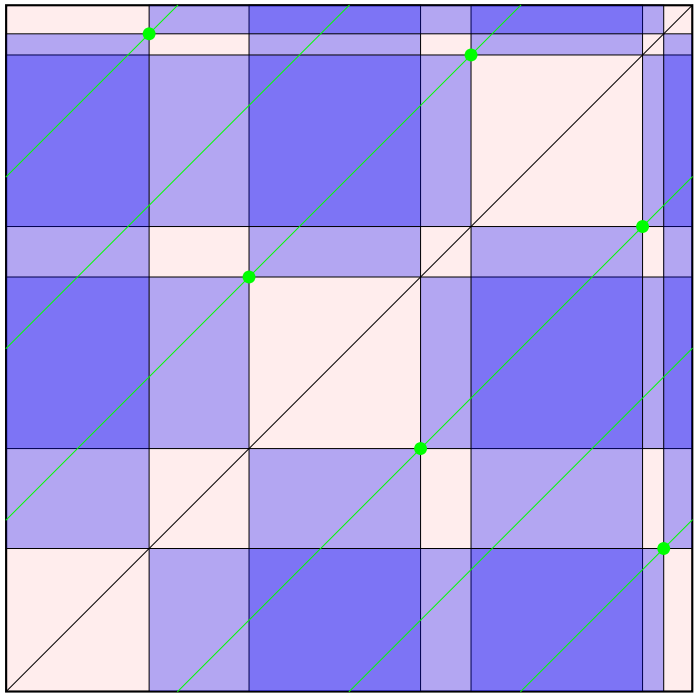
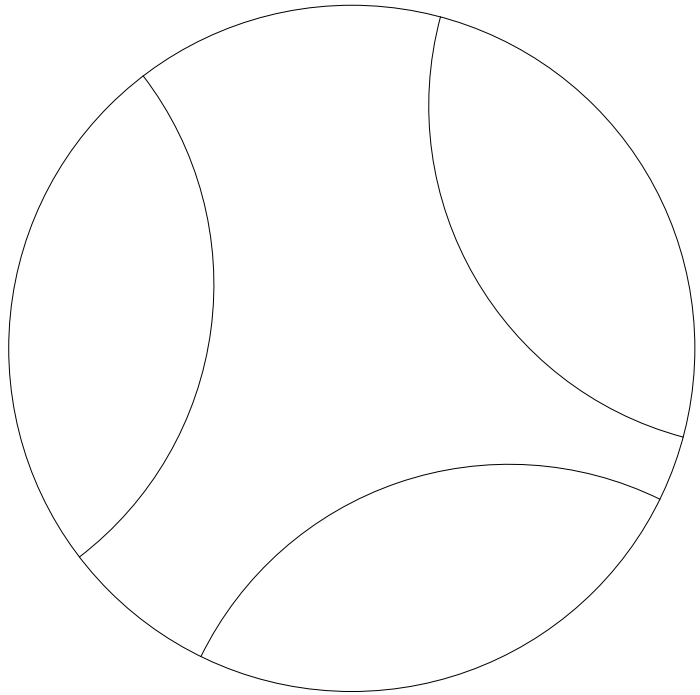
monic polynomial $P \overset{?}{\longleftrightarrow}$ primitive major m

Case 1. All critical points of P escape and escape with the same rate. Then $P \longleftrightarrow$ unique m by pulling back the external rays landing at the critical values and record only the rays landing at the critical points.

Case 2. P is pcf. $P \longrightarrow$ finitely many m's. m is necessarily rational.

Primitive major $m \longrightarrow$ the binding set $B(m) \subset \mathbb{T}^2$





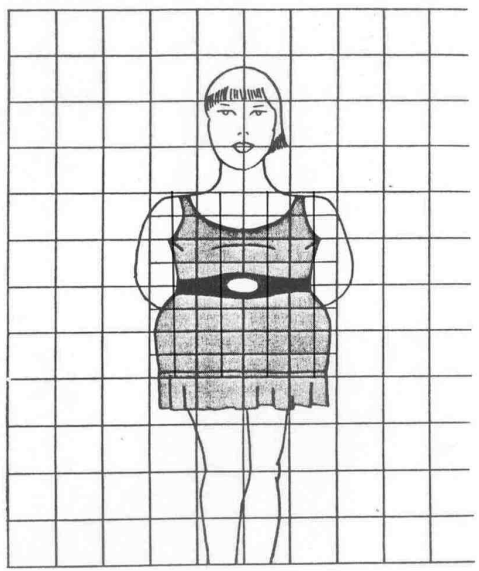
$\mathbb{D} \setminus m$ has d regions R_i, with $\overline{R_i} \cap S^1 =$ a union of closed intervals $J_1 \cup \cdots \cup J_k$ of total length $1/d$. Set $\mathbb{T}^2 \supset G(m) := \cup_{R_i}(J_1 \cup \cdots \cup J_k) \times (J_1 \cup \cdots \cup J_k)$ (the light pink rectangles). The map $F|_{G(m)} : G(m) \to \mathbb{T}^2$ is a degree d 'conformal repellor'. Let $K(m)$ be its non-escaping locus. Set $B(m) := K(m) \setminus$ diagonal.

$m \longrightarrow G(m)$ and then $B(m)$ whose dimension we want to measure

In the quadratic and cubic cases :

See deg2-deg3-TorusWithLamination-Growth

When a core T (closed inv. attractor subset of $B(m)$) exists



Given $d \geq 2$ and $T \subset \mathbb{C}/\mathbb{Z}^2$ compact. Set

- $\nu(T, n) := \#$ closed $\frac{1}{d^n}\mathbb{Z}^2$-gridded tiles intersecting T ;
- if exists, $D(T) := \lim_{n\to\infty} \frac{\log \nu(T,n)}{\log d^n}$.

Theorem (Furstenberg or earlier ?)
For $F : (x, y) \mapsto (dx, dy)$, T closed, $F(T) \subset T$,

- $D(T)$ exists
- $D(T) \cdot \log d = h_{top}(F, T)$
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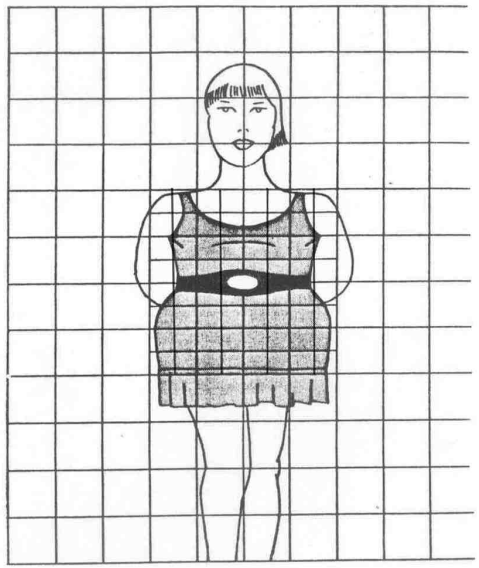

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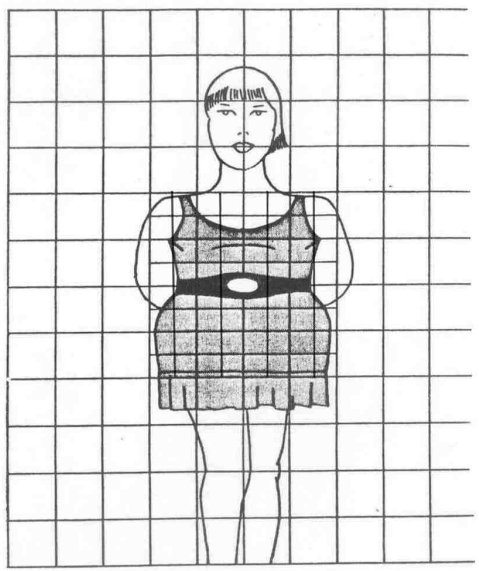

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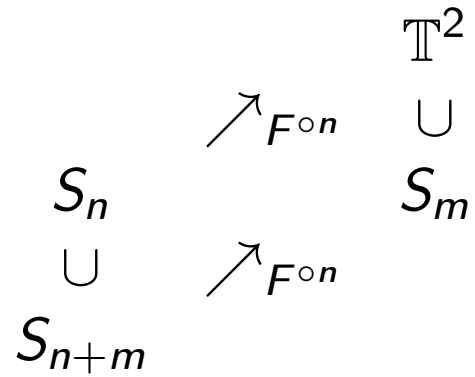
Existence of growth rate

For tiles intersecting T :

$$\{(n+m)\text{-tiles}\} \overset{\text{injects}}{\hookrightarrow} \{(n\text{-tile}, m\text{-tile})\}$$



So $\nu(T, n+m) \leq \nu(T, n) \cdot \nu(T, m)$ and their growth rate exists.

The relation in the general setting, following W. Thurston

$$\begin{array}{lll}
 & \text{Hubbard tree } H_P & \longrightarrow e^{h_{top}(P,H_P)} = \lambda(M(H_P)) \\
\text{post-cr-finite } P & S^1 \supset \{\text{angles landing on } H_P\} & \to d^{\dim(\text{angles})} \\
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$T(m) = \text{closure}\{\text{leaves in } B(m) \text{ separating the post-}m \text{ angles}\}$; Γ_m will be defined below.

Theorem (proofs are being completed by Gao Y., see also W. Jung) $T(m)$ is a core of $B(m)$.
All three lower right quantities are equal, and
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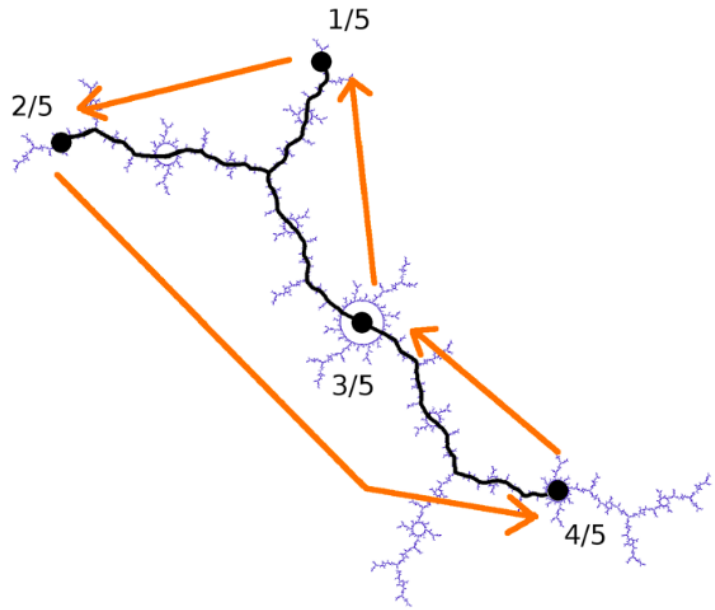
\updownarrow & - - - - - - - - - - - - - - - - - - & \\
 & \mathbb{T}^2 \supset B(m) & \longrightarrow \quad d^{\dim B(m)} \\
\text{rational } m \longrightarrow & \text{comb. tree } T(m) & \longrightarrow \quad e^{h_{top}(F,T(m))} \\
 & \text{computable matrix } \Gamma_m & \longrightarrow \quad \lambda(\Gamma_m)
\end{array}
$$

$T(m) = \text{closure}\{\text{leaves in } B(m) \text{ separating the post-}m \text{ angles}\}$; Γ_m will be defined below.

Theorem (proofs are being completed by Gao Y., see also W. Jung) $T(m)$ is a core of $B(m)$.
All three lower right quantities are equal, and
= top right three quantities.

W. Thurston's entropy matrix $\Gamma_{\{1/10,3/5\}}$ bypassing trees and dimensions (march 2011)



Label the post-critical points by external angles.

basis={pairs}$=\left\{\left\{\frac{1}{5},\frac{2}{5}\right\},\left\{\frac{1}{5},\frac{3}{5}\right\},\left\{\frac{1}{5},\frac{4}{5}\right\},\left\{\frac{2}{5},\frac{3}{5}\right\},\left\{\frac{2}{5},\frac{4}{5}\right\},\left\{\frac{3}{5},\frac{4}{5}\right\}\right\}$

Linear map Γ :

$\left\{\frac{1}{5},\frac{2}{5}\right\}\to\left\{\frac{2}{5},\frac{4}{5}\right\}\quad\left\{\frac{1}{5},\frac{3}{5}\right\}\to\left\{\frac{2}{5},\frac{1}{5}\right\}\quad\left\{\frac{1}{5},\frac{4}{5}\right\}\to\left\{\frac{1}{5},\frac{2}{5}\right\}+\left\{\frac{1}{5},\frac{3}{5}\right\}$

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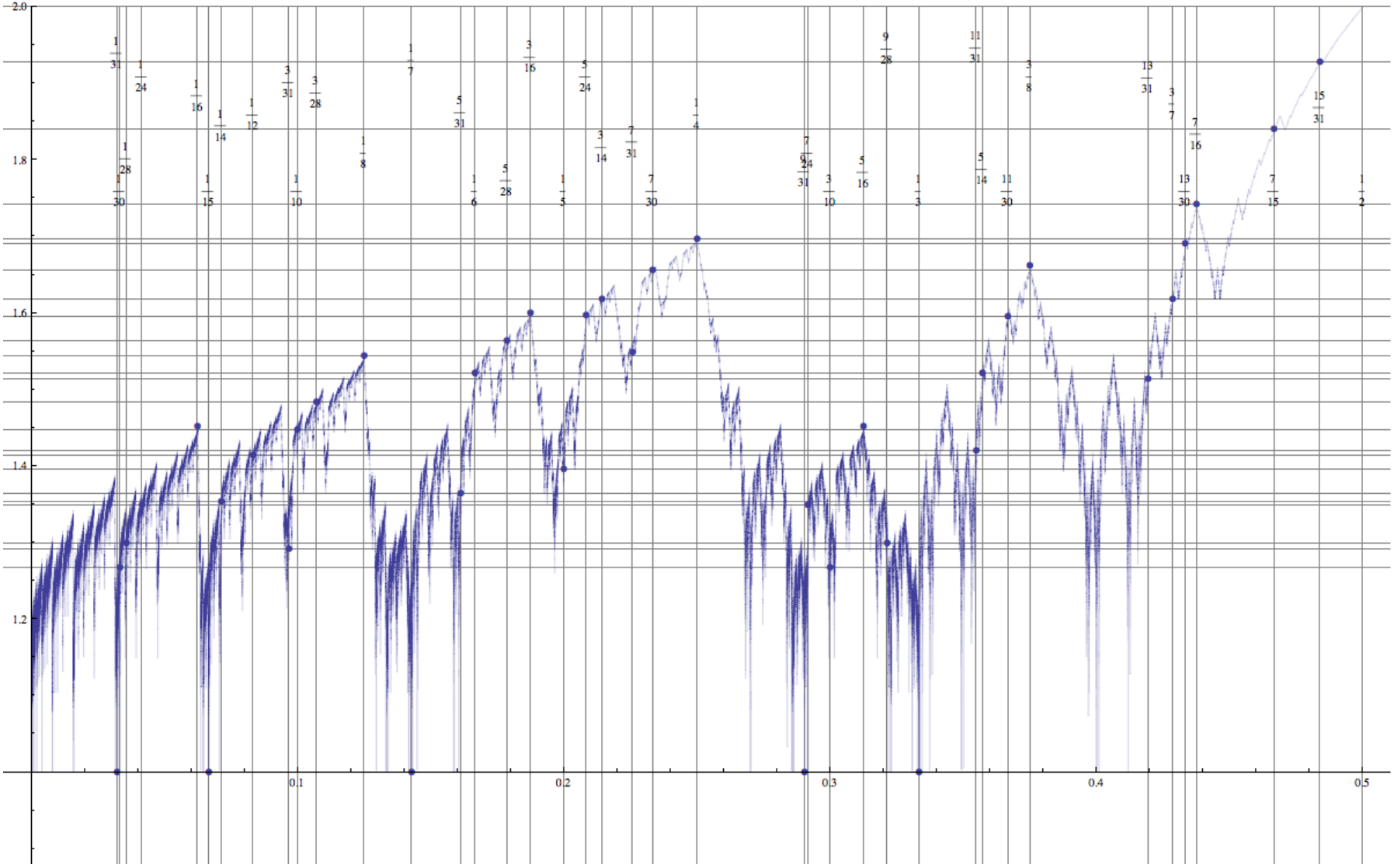
A similar algorithm works for any primitive major of any degree.

$\mathbb{Q} \ni \theta \mapsto \lambda_{leading}\Gamma(\frac{\theta}{2}, \frac{\theta+1}{2})$, plot of W. Thurston



Is this function continuous ? Dyadic angles seem to be local maxima, true ? It is l.s.c. (by W. Jung using results of Tiozzo about $\mathcal{M}$). See also Bruin-Schleicher's arxiv paper.

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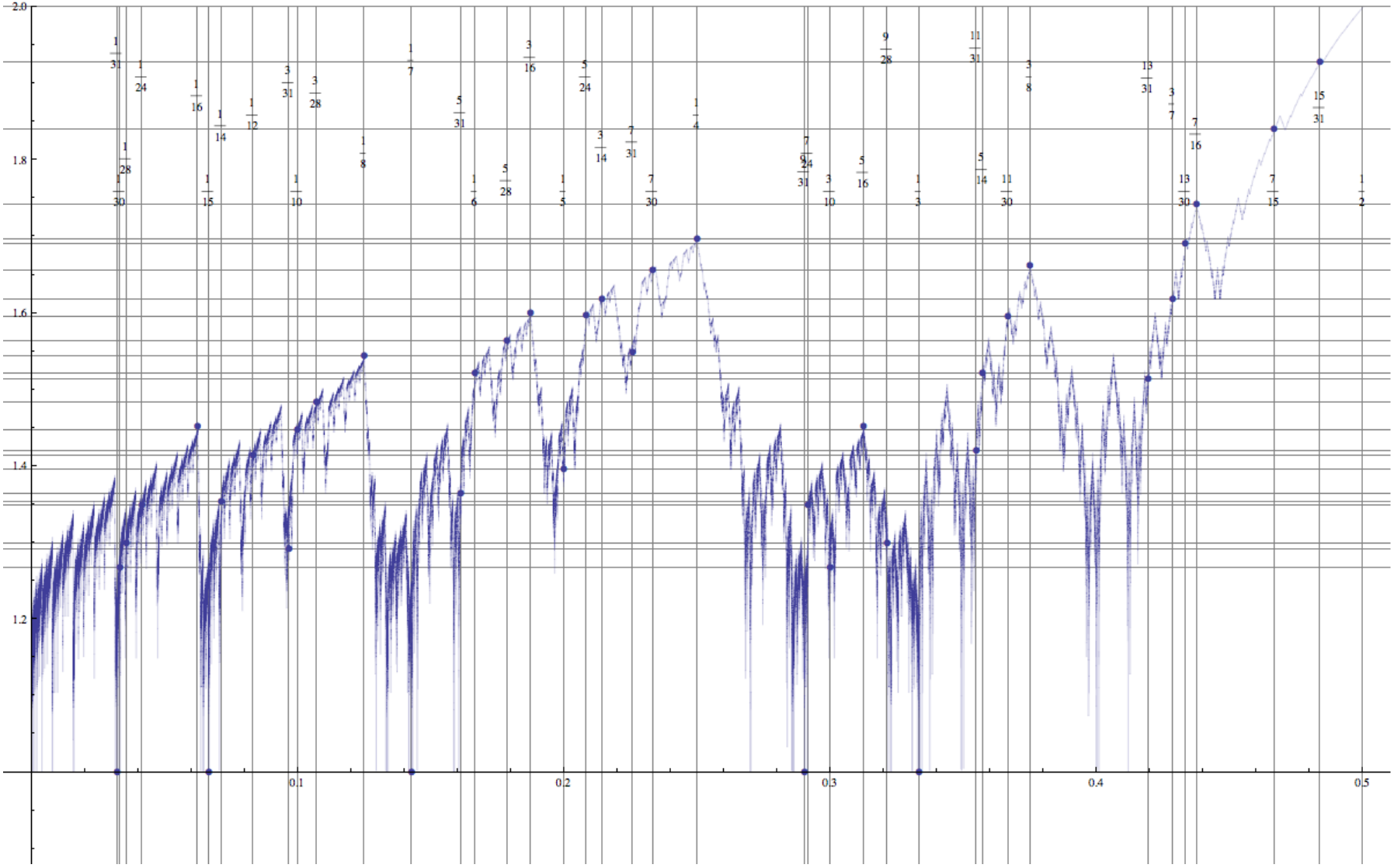

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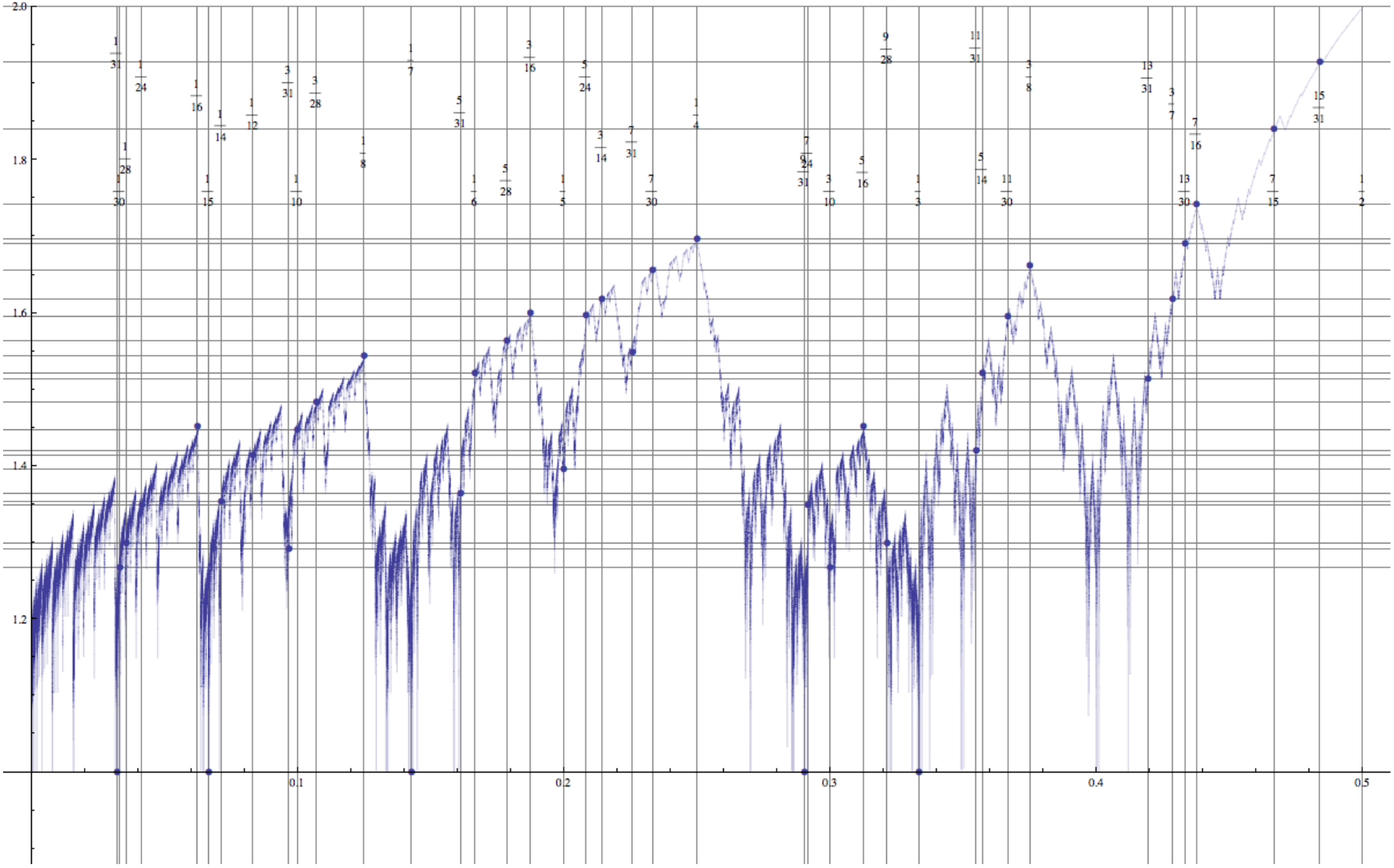

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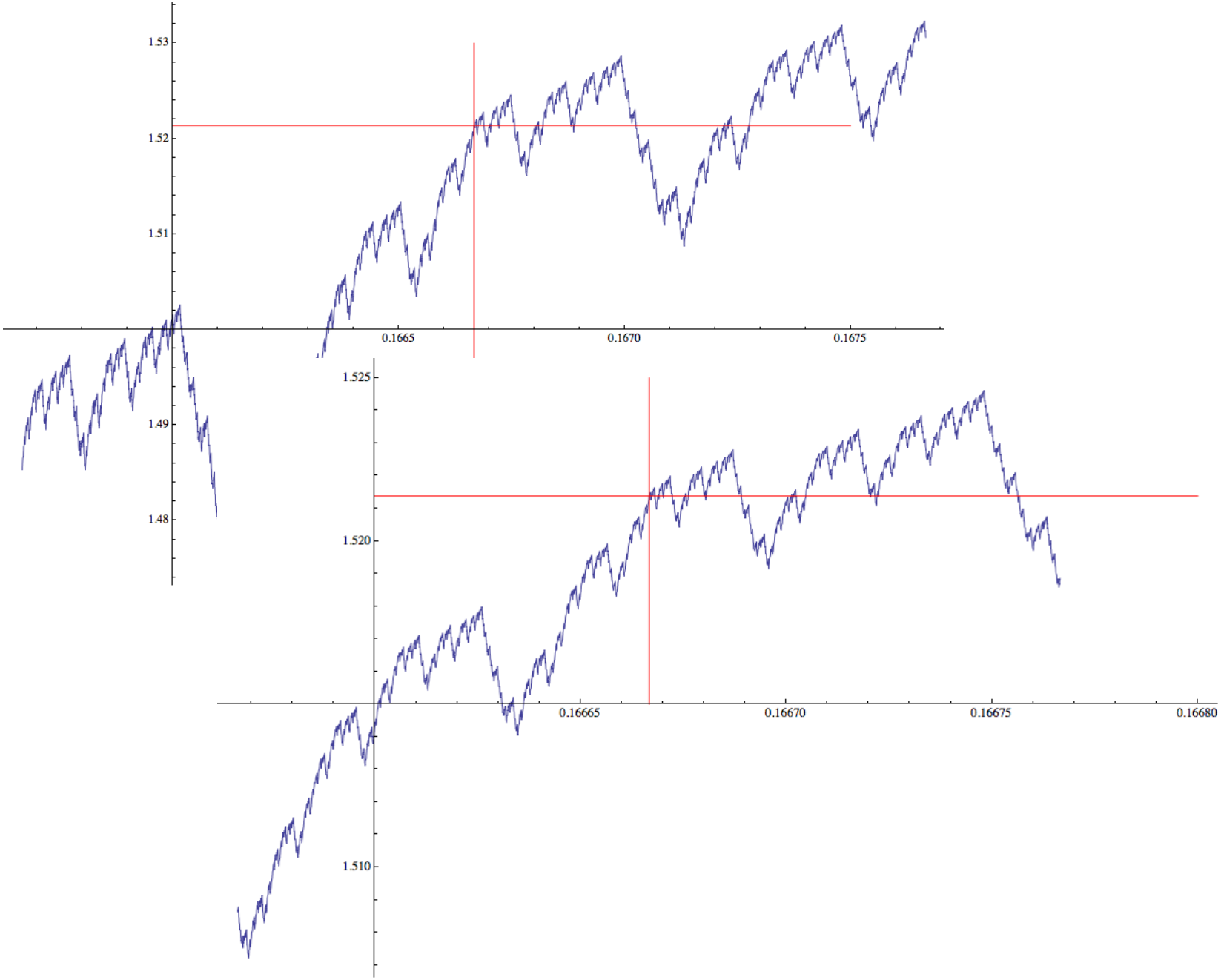
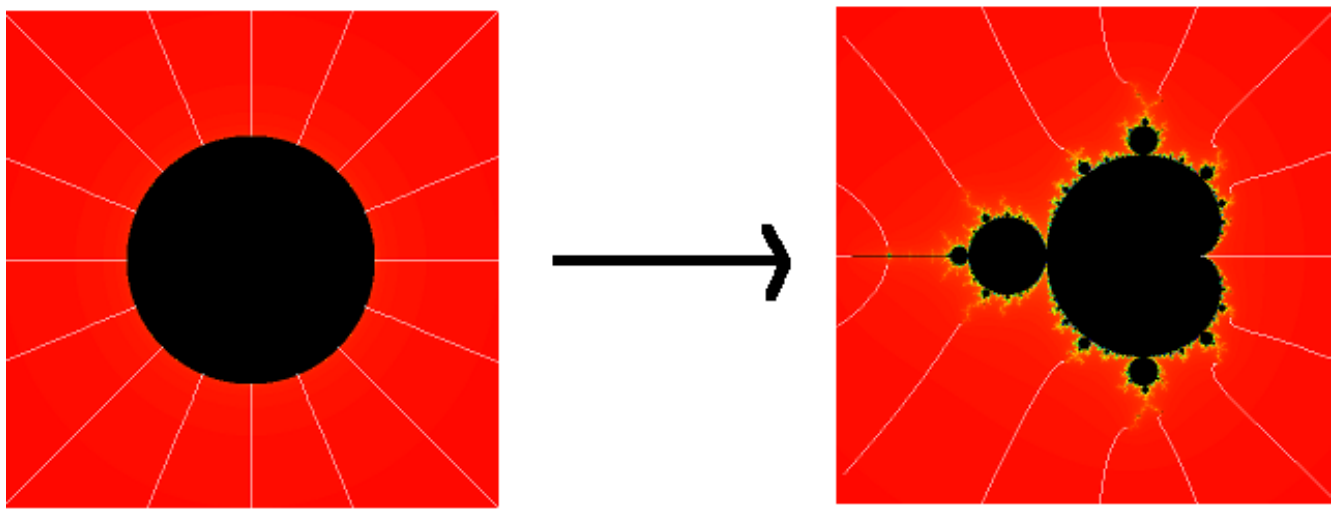
two zooms at 1/6



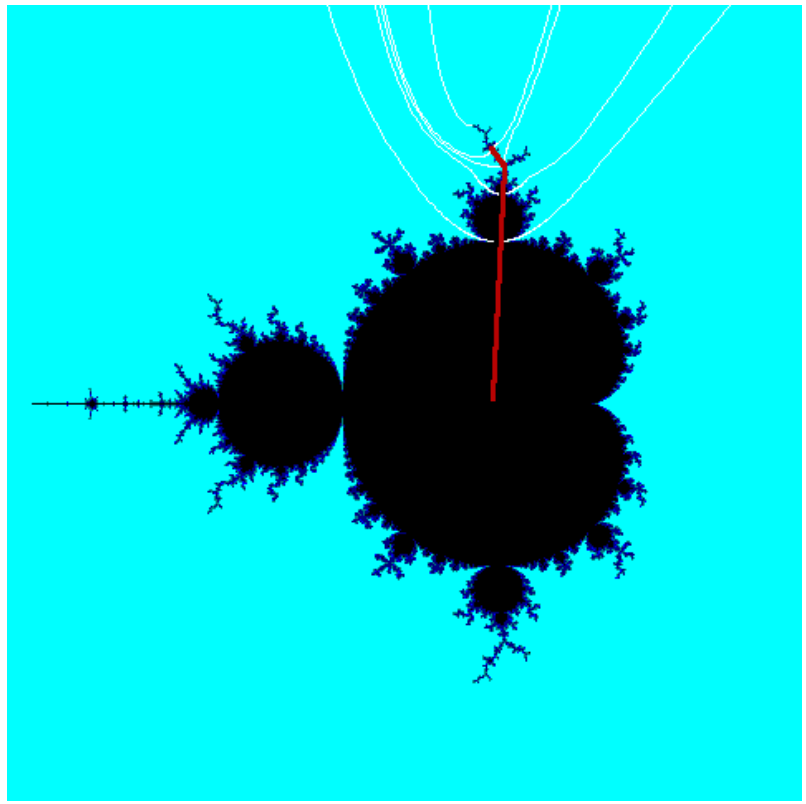
Seeing Core-entropy from inside of

$$\mathcal{M} = \{c \in \mathbb{C} \ : \ f_c^{\circ n}(0) \nrightarrow \infty\}$$



The Mandelbrot set has a tree-like structure of veins, with the real segment as a particular vein.

Theorem (Milnor-Thurston, Penrose,Li,Tiozzo,Jung, with a contribution of T.L.). Core-entropy is wake-monotone and continuous along the veins.



W. Jung used this to prove the l.s.c. of $\theta \mapsto \lambda_{leading}(\Gamma_{m_\theta})$. Entropy conjecture of Tiozzo.

Other models encoding core entropy

Bartholdi-Dudko-Nekrashevych kneading automata, Tiozzo's extension of Milnor-Thurston's kneading determinant and uniform-expanding tree-maps, etc.

See also works of Alsedà-Fagella, Branner-Hubbard, Dujardin-Favre, Milnor-Tesser, Penrose, Poirier, among others...

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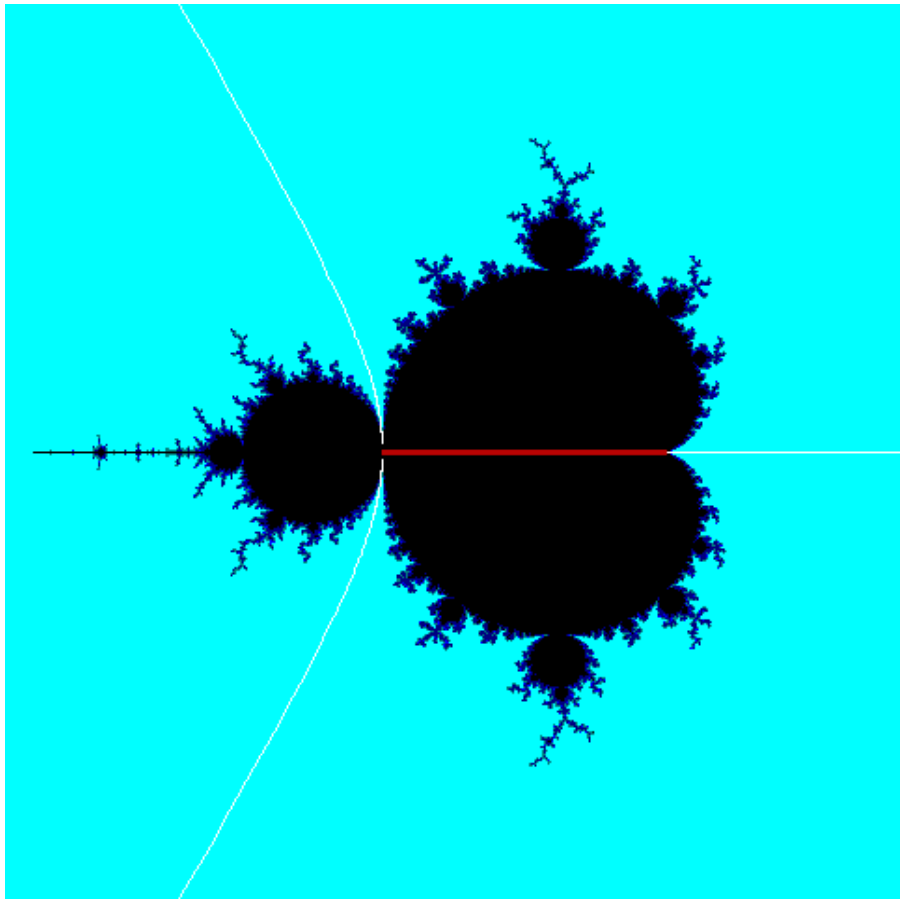
Tiozzo's section theorem (2012)

Set $P_c := \{\theta \in S^1 : \text{ the parameter } \theta\text{-ray lands on } [c, \frac{1}{4}[\}$.



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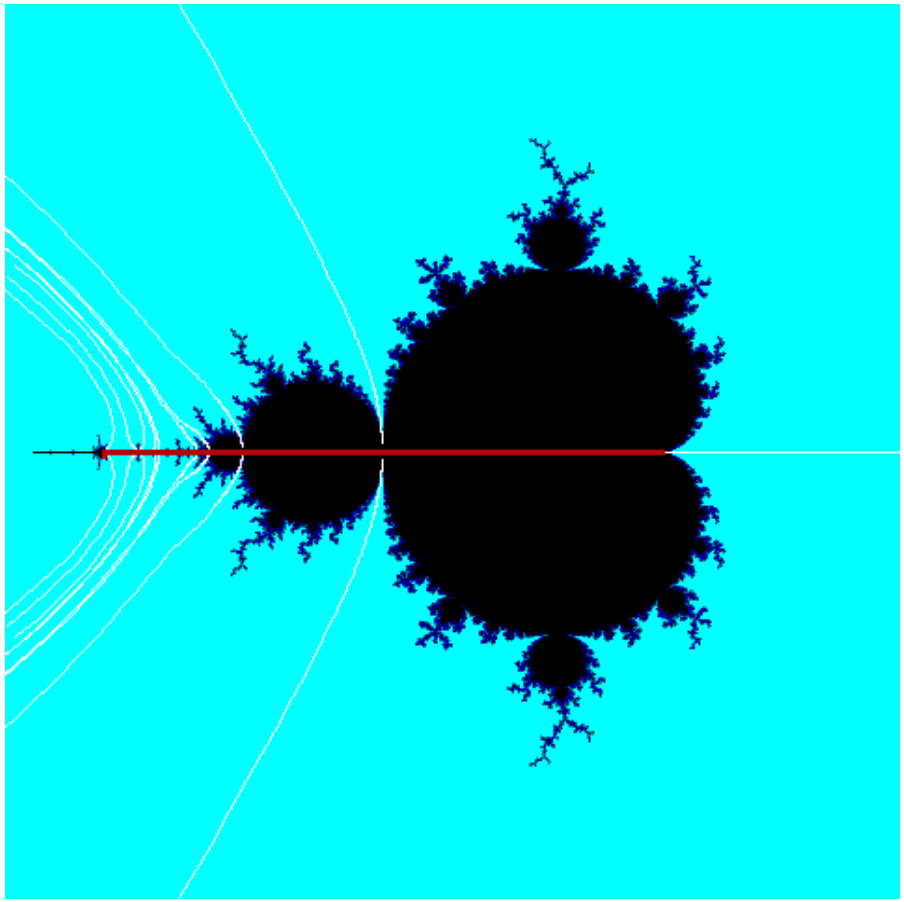

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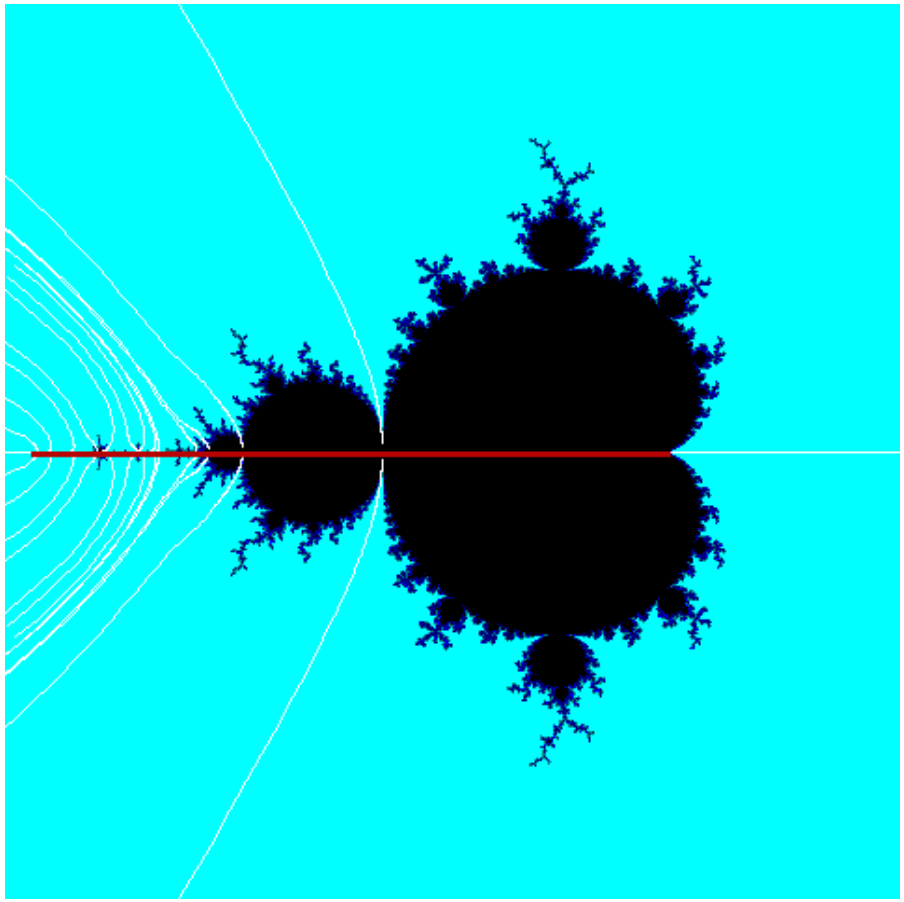

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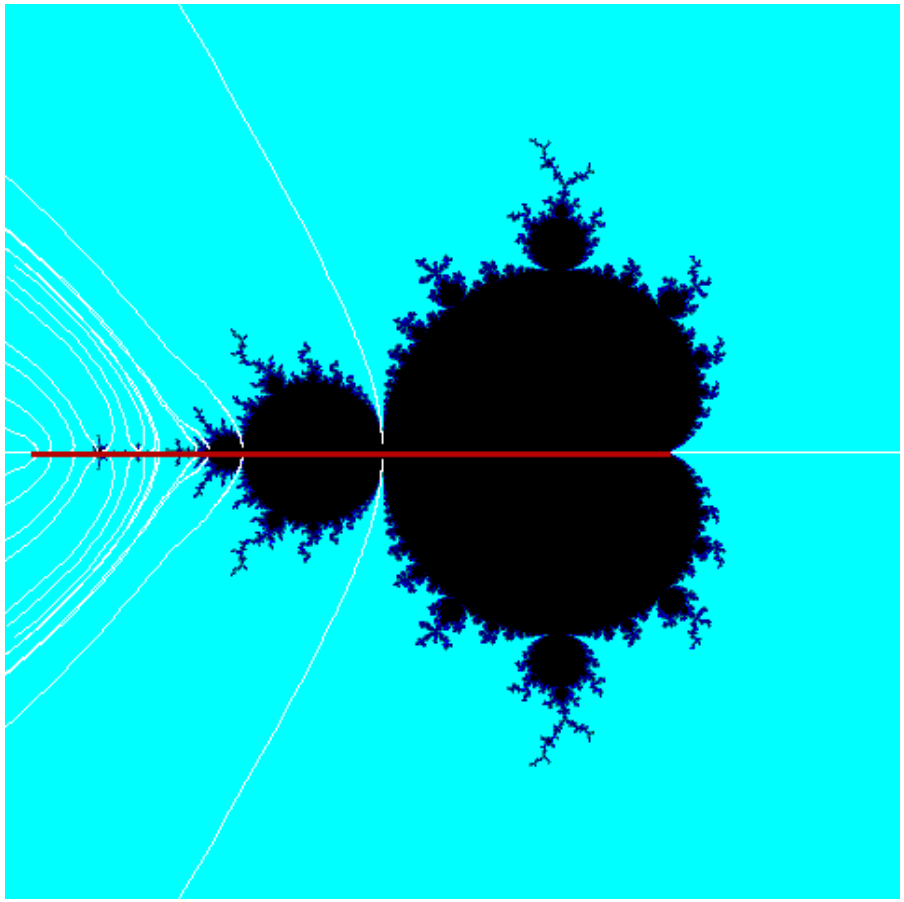

Then,

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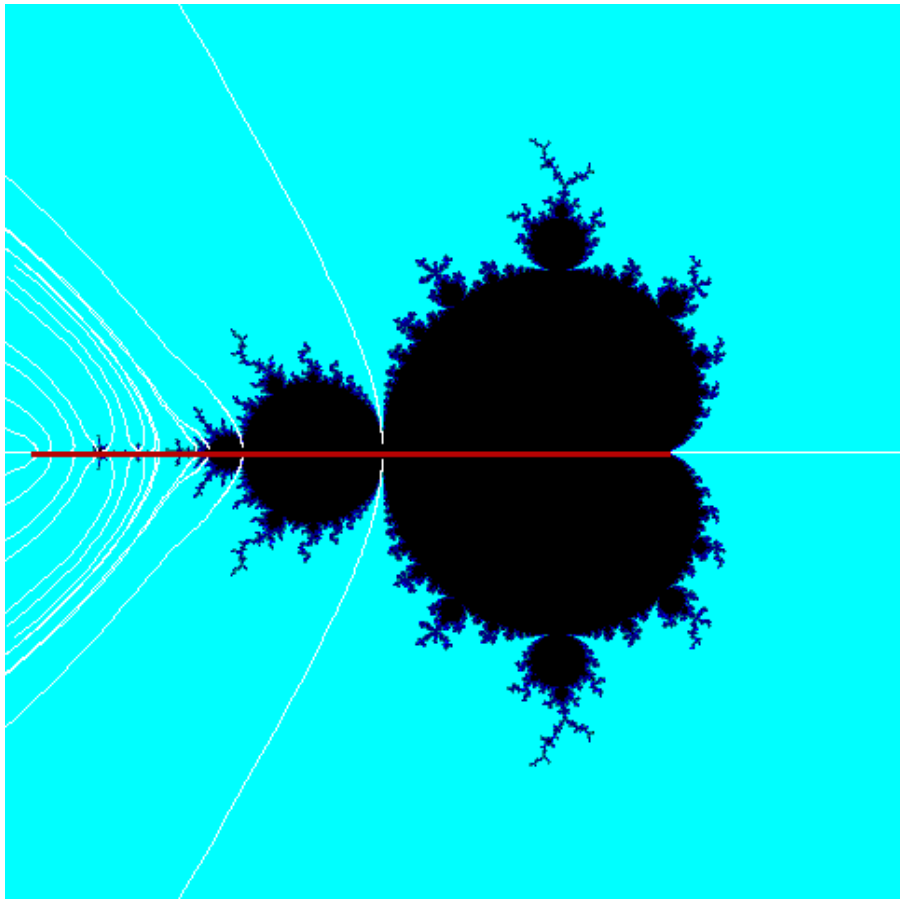

Then, $e^{\text{core-entropy}(f_c)} = 2^{\dim P_c}$.

Theorem (W. Thurston, 2011). The space $PM(d)$ *embeds in the space of monic centered degree d (non-dynamical) polynomials as a spine for the set of polynomials with distinct roots, that is, the complement of the discriminant locus. The spine consists of polynomials whose critical values are all on the unit circle.* Thus $\pi_1(PM(d))$ is the d-strand braid group and all higher homotopy groups are trivial (i.e. $PM(d)$ is a $K(B_d, 1)$ space).

Proof. Take any degree d polynomial P with no multiple zeros, and look at $\log(P)$, thought of as a map from $\mathbb{C} \setminus$ *roots* to an infinite cylinder. For each critical point, draw the two separatrices going upward (i.e., this is the curve through each critical point of P that maps by P to a vertical half-line on the cylinder $\leftrightarrow$ a ray in $\mathbb{C}$ pointed opposite the direction to the origin). Then make the finite lamination in a disk that joins the ending angles of these separatrices. This is a degree-d major set. Conversely, for each major set, there is a contractible family of polynomials whose separatrices end at the corresponding pairs of angles. To pick a canonical representative of each of these families : look at polynomials whose critical values are on the unit circle. This forms a spine for the complement of the discriminant locus for degree d polynomials.

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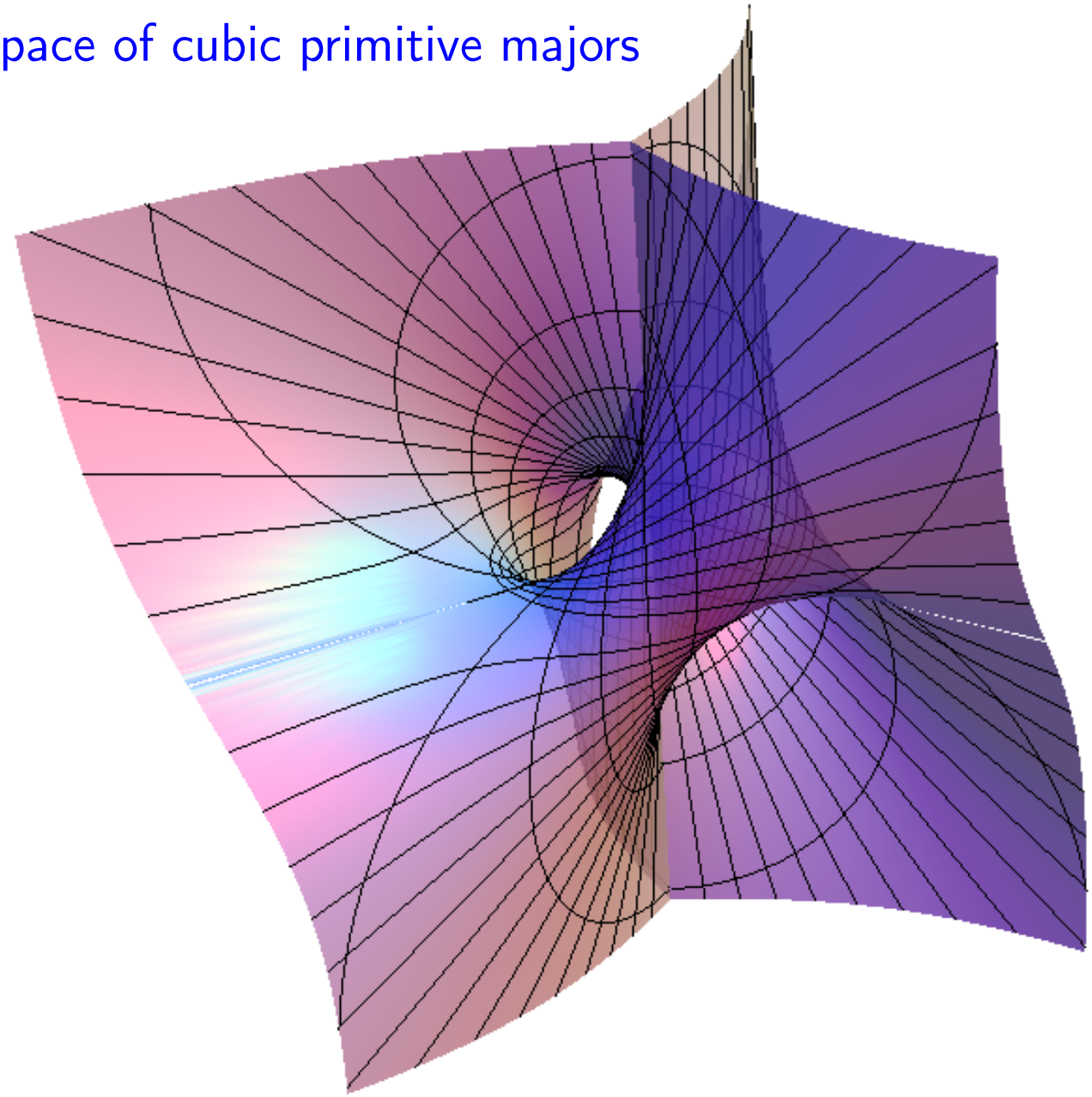
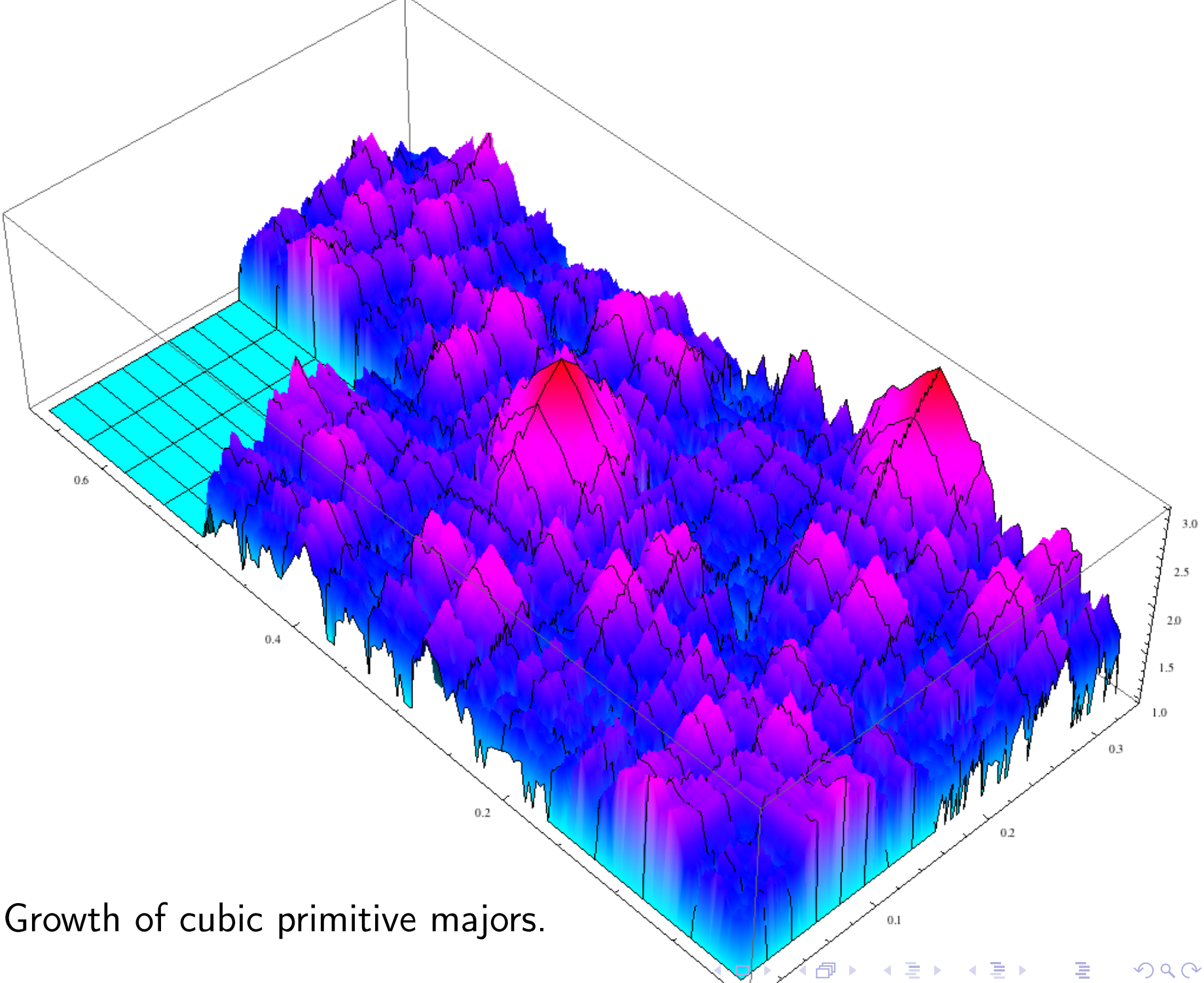
The space of cubic primitive majors





Growth of cubic primitive majors.

Bowen's definition of the relative entropy $h_{\mathbb{T}^2}(F,B)$.

Consider finite open covers $\mathcal{A}$ of $\mathbb{T}^2$ and countable covers $\mathcal{E}$ of B.

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\begin{aligned}
L_{\mathcal{A}}(E) &:= d^{-n_E := \text{separation time}} := d^{-\min\{n,\ F^n(E) \not\subset \text{ a member of } \mathcal{A}\}} \\
\dim_L B &:= \sup_{\mathcal{A}} \inf \left\{ \delta \mid \lim_{r\to 0} \inf_{\mathcal{E}, L_{\mathcal{A}}(E_i)<r} \sum L_{\mathcal{A}}(E_i)^{\delta} = 0 \right\} \\
\dim B &:= \inf \left\{ \delta \mid \lim_{r\to 0} \inf_{\mathcal{E}, |E_i|<r} \sum |E_i|^{\delta} = 0 \right\} \overset{\text{easy}}{=} \dim_L B, \text{ since} \\
|E| &\asymp d^{-\min\{n, |F^n(E)| \geq 1/d\}} \asymp d^{-n_E}. \\
D_{\mathcal{A}}(E) &:= e^{-n_E} \\
h(F,B) &:= \sup_{\mathcal{A}} \inf \left\{ h, \lim_{r\to 0} \inf_{\mathcal{E}, D_{\mathcal{A}}(E_i)<r} \sum D_{\mathcal{A}}(E_i)^{h} = 0 \right\} \\
\frac{h(F,B)}{\log d} &= \dim_L B (= \dim B) \text{ since } D_{\mathcal{A}}(E) = L_{\mathcal{A}}(E)^{1/\log d}.
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Bowen's proof : $F : T \to T$ compact, $h_{top}(F,T) \le h_T(F,T)$

Let $\mathcal{A}$: finite open cover of T,
$\lambda : \lim_{r\to 0} \inf_{\mathcal{E} \text{ cover of } T, D_{\mathcal{A}}(E_i)<r} \sum D_{\mathcal{A}}(E_i)^\lambda = 0.$
$\Longrightarrow \exists$ countable cover $\mathcal{E}$ of T s.t. $\sum D_{\mathcal{A}}(E_i)^\lambda < 1.$
$\Longrightarrow$ finite open cover $\{D_i\}$ s.t. $\sum_{i=1}^{m} e^{-\lambda n_i} < 1$; $n_i = n_{D_i}$ separation time of D_i.
$\Longrightarrow M := \max n_i$, let function system $\{(D_i, F^{n_i})\}$, and its induced deeper pieces $D_{j_1 j_2 \cdots j_s}$ with separating time $\ge n_{j_1} + \cdots + n_{j_s} =: n_{D_*}$.
Now $\mathcal{A}$ induces also puzzle pieces $A_{ij\ldots}$ indicating itineraries.

$\forall\, n$, $\{D_*, n_{D_*} \in [n, n+M[\} =$ a cover $\mathcal{D}_n$ of T and each piece is contained in a A_* of level-n. A minimal cardinality sub cover among the level-n A_*'s gives $N(\mathcal{A}_n)$ the cardinality.

$$N(\mathcal{A}_n)e^{-\lambda n} \le \sum_{D_* \in \mathcal{D}_n} e^{-\lambda n} = \sum_{D_* \in \mathcal{D}_n} e^{\lambda(n_{D_*} - n)} e^{-\lambda n_{D_*}} \le e^{M\lambda} \sum_{\text{all } D_*} e^{-\lambda n_{D_*}} < \infty.$$

Thank you !

Merci !

Relate to Milnor-Thurston's definition

For $n \geq 0$, $\theta(f^n|_{x^+}) := \mathrm{sgn}(f^n)'|_{x^+}$ $(f^0 = id)$. Generating function

$$(\kappa_A, \kappa_B, \kappa_C)_{x^+}(t) := \Big(\sum_{f^n(x^+)\in A} \theta(f^n|_{x^+})\, t^n \ , \text{ in } B, \text{in } C \Big).$$

$$(1 - s_A\, t)\kappa_A = \sum_{f^n(x^+)\in A} \theta(f^n|_{x^+})\, t^n - \sum_{f^n(x^+)\in A} \theta(f^{n+1}|_{x^+})\, t^{n+1}$$

etc., adding the three : $\quad (\kappa_A, \kappa_B, \kappa_C) \begin{pmatrix} 1 - s_A t \\ 1 - s_B t \\ 1 - s_C t \end{pmatrix} = \sum_{n\geq 0} - \sum_{n\geq 1} = 1.$

knead. matrix $\mathcal{N}(t) = (\tilde{A}, \tilde{B}, \tilde{C}) = \begin{pmatrix} (\kappa_A, \kappa_B, \kappa_C)_{c_1^+} - (\kappa_A, \kappa_B, \kappa_C)_{c_1^-} \\ (\kappa_A, \kappa_B, \kappa_C)_{c_2^+} - (\kappa_A, \kappa_B, \kappa_C)_{c_2^-} \end{pmatrix}$

Above (kernel vector)+Cramer gives

$$\frac{|\tilde{B}\tilde{C}|}{1 - s_A t} = -\frac{|\tilde{A}\tilde{C}|}{1 - s_B t} = \frac{|\tilde{A}\tilde{B}|}{1 - s_C t} = \textbf{kneading determinant } D(t).$$

$D_{old} = D_{new}$ in the unimodal case, with $s_A < 0, s_B > 0$

$$\mathcal{N}(t) = (D_B, D_A)$$
$$:= (-1, +1) + \Big(\sum_{n\geq 1, f^n(0^{\pm}) \in A} \pm 2t^n, \sum_{n\geq 1, f^n(0^{\pm}) \in B} \pm 2t^n \Big)$$

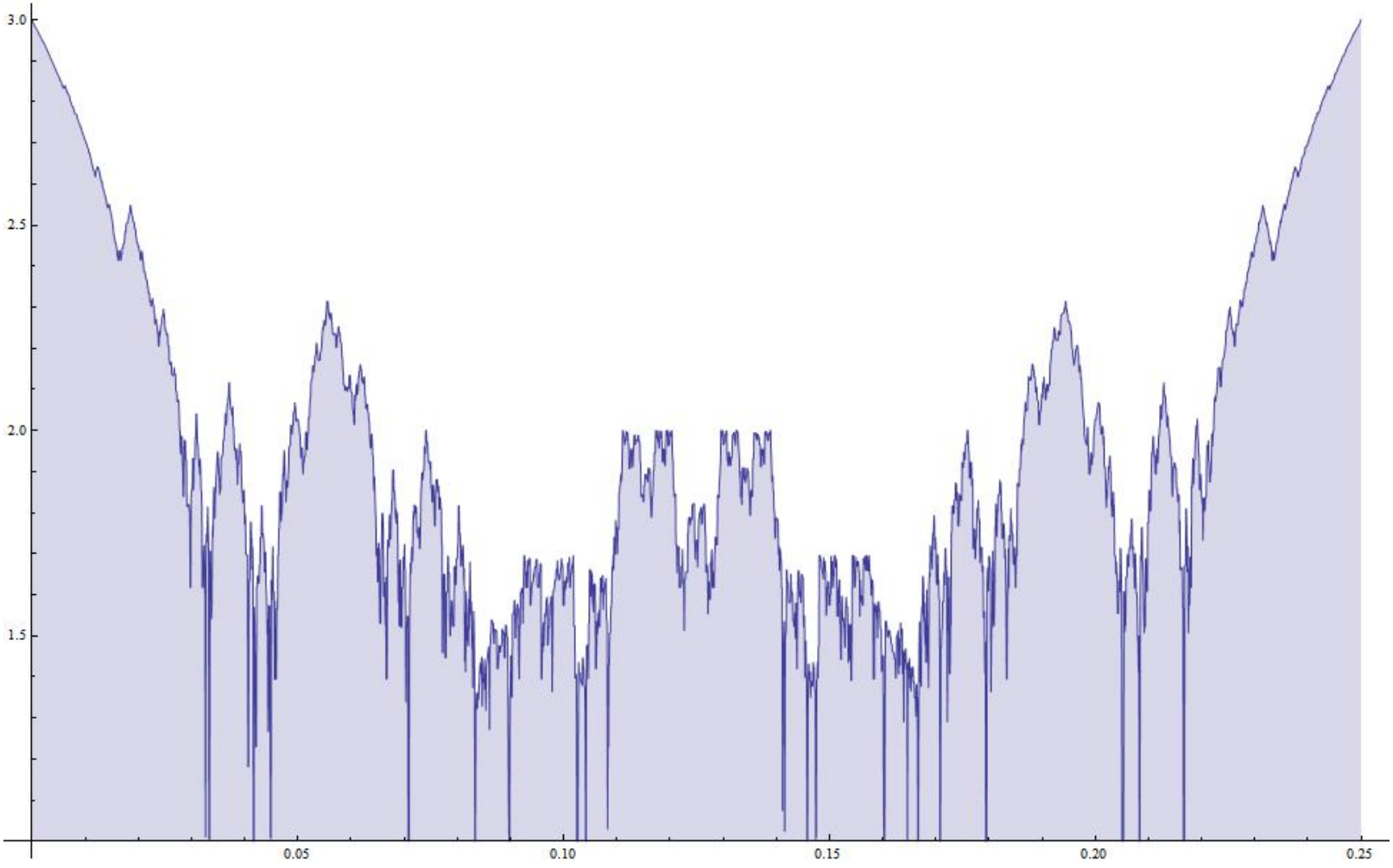
with $+$ if f^n has a local minimum and $-$ if local maximum. Then $D_B = -(1 - s_B t)D_{new}$, $D_A = (1 - s_A t)D_{new}$.

Since $\theta_n = \text{sgn}(f^{n+1})' = s_{interval} \cdot \text{sgn}(f^n)'$,

$$2tD_{old} = (D_B, D_A) \begin{pmatrix} s_A t \\ s_B t \end{pmatrix} \overset{kernel\ vec.}{=} (D_B, D_A) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= (-1 + s_B t + 1 - s_A t)D_{new} = 2tD_{new}.$$

Or $\frac{1}{2}\mathcal{R}(t) = \frac{1}{2}(D_A - D_B) = \frac{1}{2}(D_B, D_A) \begin{pmatrix} s_A \\ s_B \end{pmatrix} = D_{old}$

$= 1 + \sum_{n\geq 1} \pm\varepsilon(n)t^n$ with $\varepsilon(n) = \begin{cases} 1 & \text{if } f^n(0^{\pm}) > 0 \\ -1 & \text{if } f^n(0^{\pm}) < 0 \end{cases}$. Clearly $\varepsilon(n)$ determines whether f changes the type of extrema of $f^n(0)$.

Journées dynamiques holomorphes, 28-29 Mai, Angers

Confirmed speakers : R. Dujardin, N. Mihalache,
P. Roesch, D.Thurston, G. Tiozzo, J. Tomasini