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On a theorem of Rees-Shishikura

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Abstract. — Rees-Shishikura's theorem plays an important role in the study of matings of polynomials. It promotes Thurston's combinatorial equivalence into a semi-conjugacy. In this work we restate and reprove Rees-Shishikura's theorem in a more general form, which can then be applied to a wider class of postcritically finite branched coverings. We provide an application of the restated theorem.

Résumé. — Le théorème de Rees-Shishikura joue un rôle important dans l'étude des accouplements de polynômes. Il permet d'obtenir une semi-conjugaison à partir d'une equivalence combinatoire de Thurston. Dans ce travail, nous reformulons et redémontrons ce théorème dans un cadre plus général. Cette nouvelle version du théorème est applicable à une classe plus large de revêtements ramifiés postcritiquement finis. Nous en fournissons un exemple à la fin de notre article.

1. Introduction

Consider the mating of two polynomials (refer to [4, 10, 11, 12] for the definitions of mating). M. Rees and M. Shishikura [10, 11] proved that if the formal mating of two postcritically finite polynomials is Thurston equivalent to a rational map, then the topological mating is conjugate to the rational map. The main step of the proof is to show the existence of a semi-conjugacy

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from the formal mating to the rational map (refer to Theorem 2.1 in [11] and the theorem below).

Theorem A. — *Suppose that the degenerate mating $F' = (f_1 \perp\!\!\!\perp f_2)'$ of polynomials f_1 and f_2 is Thurston equivalent to a rational map R mapping from the Riemann sphere $\widehat{\mathbb{C}}$ onto itself. Then there exists a continuous map $h: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, satisfying that*
(i) the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{F} & \widehat{\mathbb{C}} \\ h \downarrow & & \downarrow h \\ \widehat{\mathbb{C}} & \xrightarrow{R} & \widehat{\mathbb{C}}, \end{array}$$

where $F = f_1 \perp\!\!\!\perp f_2$ is the formal mating;
(ii) h is a uniform limit of orientation preserving homeomorphisms;
(iii) h is conformal in $\mathrm{int}K_{f_1} \sqcup \mathrm{int}K_{f_2}$ onto $\widehat{\mathbb{C}} \smallsetminus J_R$ and $h^{-1}(\widehat{\mathbb{C}} \smallsetminus J_R) = \mathrm{int}K_{f_1} \sqcup \mathrm{int}K_{f_2}$, where $\mathrm{int}K_{f_i}$ are the interior of the filled-in Julia sets of f_i for $i = 1, 2$ and J_R is the Julia set of R.

M. Rees ([10]) proved that there exists a semi-conjugacy from a general postcritically finite branched covering to a rational map if it is Thurston equivalent to the rational map by a pair of homeomorphisms (ϕ_0, ϕ_1) and $\phi_0 = \phi_1$ near the critical cycles. In fact, the pull-back sequence $\{\phi_n\}$ (see the definition below) of the Thurston equivalence converges uniformly to the semi-conjugacy.

In the proof of Theorem A, under the property that the degenerate mating F' is holomorphic in a neighborhood of the critical cycles, M. Shishikura modified the original Thurston equivalence (θ_0, θ_1) so that $\theta_0 = \theta_1$ near the critical cycles by using Dehn twist near those points.

In this note, we will show that if the Thurston equivalence (ϕ_0, ϕ_1) satisfies that ϕ_0 is a local conjugacy near the critical cycles, then the pull-back sequence $\{\phi_n\}$ of the Thurston equivalence converges uniformly to the semi-conjugacy. Under the assumption that a postcritically finite branched covering is Thurston equivalent to a rational map, when the branched covering is holomorphic in a neighborhood of the critical cycles, then it is easy to show that there exists a Thurston equivalence (ϕ_0, ϕ_1) such that ϕ_0 is a local conjugacy near the critical cycles. Note that in this case ϕ_0 needs not coincide with ϕ_1 near the critical cycles and we do not need Dehn twist as constructed in [11].

Statements: Let F be a branched covering of the Riemann sphere $\widehat{\mathbb{C}}$. We always assume $\deg F \geqslant 2$ in this paper. Denote by Ω_F the set of critical points of F. The *postcritical set* of F is defined by

$$\mathcal{P}_F = \overline{\bigcup_{n\geqslant 0} F^n(\Omega_F)}.$$

The map F is called *postcritically finite* if $\mathcal{P}_F$ is a finite set. Let f be a rational map. We denote by $\mathcal{F}_f$ and $\mathcal{J}_f$ the Fatou set and Julia set of f respectively.

Two postcritically finite branched coverings F and G are called *Thurston equivalent* through a pair of orientation preserving homeomorphisms $(\phi_0, \phi_1) : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ if ϕ_1 is isotopic to ϕ_0 rel $\mathcal{P}_F$ and $\phi_0 \circ F \circ \phi_1^{-1} = G$. The *pull-back* sequence $\{\phi_n\}_{n\geqslant 1}$ of the Thurston equivalence means that $\{\phi_n\}$ is a sequence of homeomorphisms of $\widehat{\mathbb{C}}$ such that ϕ_{n+1} is isotopic to ϕ_n rel $\mathcal{P}_F$ and $\phi_n \circ F = G \circ \phi_{n+1}$.

A *continuum* is a connected compact subset of $\widehat{\mathbb{C}}$.

Theorem 1.1. — *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering. Suppose that F is Thurston equivalent to a rational map f through a pair of homeomorphisms (ϕ_0, ϕ_1) such that $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F. Let $\{\phi_n\}$ $(n \geqslant 1)$ be a sequence of homeomorphisms of $\widehat{\mathbb{C}}$ such that $\phi_n \circ F = f \circ \phi_{n+1}$ and ϕ_{n+1} is isotopic to ϕ_n rel $\mathcal{P}_F$. Then $\{\phi_n\}$ converges uniformly to a continuous onto map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as $n \to \infty$. Moreover,*
(1) $h \circ F = f \circ h$.
(2) $h^{-1}(w)$ is a single point for $w \in \mathcal{F}_f$ and a full continuum for $w \in \mathcal{J}_f$.
(3) For points $x, y \in \widehat{\mathbb{C}}$ with $f(x) = y$, $h^{-1}(x)$ is a connected component of $F^{-1}(h^{-1}(y))$ and $F(h^{-1}(x)) = h^{-1}(y)$. Moreover, the degree of the map $F : h^{-1}(x) \to h^{-1}(y)$ is equal to $\deg_x f$, precisely speaking, for any $w \in h^{-1}(y)$,

$$\sum_{z\in F^{-1}(w)\cap h^{-1}(x)} \deg_z F = \deg_x f,$$

where $\deg_x f, \deg_z F$ are the local degrees of f, F at x, z respectively.
(4) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.
(5) $h(F^{-1}(E)) = f^{-1}(h(E))$ for any $E \subset \widehat{\mathbb{C}}$.
(6) $F^{-1}(\widehat{E}) = \widehat{F^{-1}(E)}$ for any $E \subset \widehat{\mathbb{C}}$, where $\widehat{E} = h^{-1}(h(E))$.

Corollary 1.2. — *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering which is holomorphic in a neighborhood of the critical cycles. Suppose that F is Thurston equivalent to a rational map f through a pair of* *homeomorphisms* (ϕ_0, ϕ_1). *Then there exists a semi-conjugacy* h *from* F *to* f *in the homotopy class of* ϕ_0 *such that it satisfies the above conditions (1)-(6).*

As in [10, 11], the main idea of the proof is that the rational map f is expanding under the orbifold metric. The only new observation is that the homotopic length of the isotopy for any point is bounded if $\phi_0 \circ F = f \circ \phi_0$ near critical cycles.

Points (4)-(6) are also new but they are not difficult to prove. They are applied in our work [3].

2. Homotopic length of the isotopy

In this section we assume that the reader is familiar with the theory of orbifolds.

Let f be a postcritically finite rational map of $\widehat{\mathbb{C}}$. Denote by $\rho(z)|dz|$ the orbifold metric of f ([5]). Then $||f'|| > 1$ on $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$ with respect to the orbifold metric $\rho(z)|dz|$, and on any compact subset $E \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$, there is a constant $\lambda > 1$ such that $||f'|| > \lambda$. Define the *homotopic length* of a path $\alpha : [0,1] \to \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ by

$$\text{h-length}(\alpha) = \inf\{\text{length of } \alpha' \text{ with metric } \rho\},$$

where the infimum is taken over all the paths α' from $\alpha(0)$ to $\alpha(1)$ and homotopic to α in $\widehat{\mathbb{C}} \backslash \mathcal{P}_f$.

Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite branched covering. Suppose that F is Thurston equivalent to a rational map f via a pair of homeomorphisms (ϕ_0, ϕ_1), i.e., $\phi_0 \circ F = f \circ \phi_1$, and ϕ_1 is isotopic to ϕ_0 rel $\mathcal{P}_F$, that is, there is a continuous map $H_0 : \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$ such that $H_0(\cdot, 0) = \phi_0, H_0(\cdot, 1) = \phi_1$, $H_0(\cdot, t)$ is a homeomorphism for any $t \in (0,1)$ and $H_0(z,t) = \phi_0(z)$ for $z \in \mathcal{P}_F, t \in [0,1]$.

LEMMA 2.1. — *If* $\phi_0 \circ F = f \circ \phi_0$ *in a neighborhood of the critical cycles of* F, *then the homotopic length of* $\{H_0(z,t), 0 \leqslant t \leqslant 1\}$ *is bounded by a constant* $M < \infty$ *for any point* $z \in \widehat{\mathbb{C}} \backslash \mathcal{P}_F$.

Proof. — We only need to show that the homotopic length of $\gamma := \{H_0(z,t), 0 \leqslant t \leqslant 1\}$ is bounded in a neighborhood of each critical cycle of f. Let x be a point in a critical cycle of f. Define the winding angle of the path γ around the point x by:

$$w_x(\gamma) = \frac{1}{2\pi i}\int_{\zeta\in B(\gamma)} \frac{d\zeta}{\zeta},$$

where B is the Böttcher map and ζ is Böttcher's coordinate of f at the point x. It is continuous. On the other hand, since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, we have $\phi_1 \circ \phi_0^{-1}$ is a rotation in Böttcher's coordinates of f at the point x, with angles $2k\pi/d$, where k is an integer and $d = \deg_x f$. Thus $w_x(\gamma) \equiv k/d (\text{mod } 1)$. It follows that $w_x(\gamma)$ is a constant in a neighborhood of x. This implies that the homotopic length of γ is bounded in a neighborhood of the point x. □

Lemma 2.2. — *If $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, then the pull-back sequence $\{\phi_n\}$ converges uniformly to a continuous onto map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as $n \to \infty$.*

Proof. — By lifting the map H_0, for each $n \geqslant 1$, we get a continuous map $H_n : \widehat{\mathbb{C}} \times [0,1] \to \widehat{\mathbb{C}}$ satisfying that $H_n(\cdot, t)$ is a homeomorphism for any $t \in [0,1]$, $H_n(\cdot, 0) = \phi_n$, $H_n(\cdot, 1) = \phi_{n+1}$, $H_n(z,t) = \phi_n(z)$ for $z \in \mathcal{P}_F, t \in [0,1]$ and $H_n(F(z),t) = f(H_{n+1}(z,t))$ for $z \in \widehat{\mathbb{C}}, t \in [0,1]$.

Let U be an open set containing critical cycles of F such that $\phi_0 \circ F = f \circ \phi_0$ in U, $F(\overline{U}) \subset U$ and every component of U contains exactly one point in the critical cycles of F.

Claim. — *For each $n \geqslant 1$, $\phi_n \circ \phi_0^{-1}$ is a rotation in Böttcher coordinates of the critical cycles of f.*

Proof. — Let x be a point in a critical cycle of f. By Böttcher's Theorem, there is a Jordan domain $U_x \subset \phi_0(U)$, $x \in U_x$ and a conformal map $u_x : U_x \to D_x = \{z \in \mathbb{C} : |z| < r_x < 1\}$ such that $f(U_x)$ is compactly contained in $U_{f(x)}$ (denote by $f(U_x) \subset\subset U_{f(x)}$), $u_x(x) = 0$ and

$$u_{f(x)} \circ F \circ u_x^{-1}(z) = z^{d_x},$$

where $d_x = \deg_x f$. In fact u_x is the Böttcher's coordinate of f at the cycle through the point x.

Fix $n \geqslant 1$. We may assume that $f^n(U_x) \subset\subset U_{f^n(x)}$ and $\phi_n\phi_0^{-1}(U_x) \subset\subset U_x$. Since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F and $\phi_0 \circ F^n = f^n \circ \phi_n$ on $\widehat{\mathbb{C}}$, we have the following commutative diagrams.

$$\begin{array}{ccccccccc}
D_x & \xleftarrow{u_x} & \phi_n(\phi_0^{-1}(U_x)) & \xleftarrow{\phi_n} & \phi_0^{-1}(U_x) & \xrightarrow{\phi_0} & U_x & \xrightarrow{u_x} & D_x \\
P\downarrow & & f^n\downarrow & & F^n\downarrow & & f^n\downarrow & & \downarrow P \\
D_{f^n(x)} & \xleftarrow{u_{f^n(x)}} & U_{f^n(x)} & \xleftarrow{\phi_0} & \phi_0^{-1}(U_{f^n(x)}) & \xrightarrow{\phi_0} & U_{f^n(x)} & \xrightarrow{u_{f^n(x)}} & D_{f^n(x)},
\end{array}$$

where $P(z) = z^{d_x d_{f(x)} \cdots d_{f^n(x)}}$. It follows easily that $\phi_n \circ \phi_0^{-1}$ is a rotation in Böttcher coordinates of the critical cycles of f. $\square$

By the claim, we may take a compact subset $E \subset \widehat{\mathbb{C}} \backslash \mathcal{P}_f$ such that $\widehat{\mathbb{C}} \backslash \phi_n(U) \subset E$ for all $n \geqslant 0$. Then there exists a constant $\lambda > 1$ such that $||f'|| > \lambda$ on E. Let $d(\cdot, \cdot)$ denote the spherical metric of $\widehat{\mathbb{C}}$.

Fix $n \geqslant 1$.

If $z \in \widehat{\mathbb{C}} \backslash F^{-n}(U \cup \mathcal{P}_F)$, then the path $\{H_n(z,t), 0 \leqslant t \leqslant 1\} \subset \widehat{\mathbb{C}} \backslash (\phi_n(F^{-n}(U)) \cup \mathcal{P}_f) \subset \widehat{\mathbb{C}} \backslash (\phi_n(U) \cup \mathcal{P}_f) \subset E$. Thus $F(z) \in \widehat{\mathbb{C}} \backslash F^{-(n-1)}(U \cup \mathcal{P}_F)$ and

$$\begin{aligned} \text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \frac{1}{\lambda} \text{h-length}\big(f(\{H_n(z,t), 0 \leqslant t \leqslant 1\})\big) \\ &= \frac{1}{\lambda} \text{h-length}(\{H_{n-1}(F(z),t), 0 \leqslant t \leqslant 1\}). \end{aligned}$$

Note that by Lemma 2.1, for all $z \in \widehat{\mathbb{C}} \backslash \mathcal{P}_F$,

$$\text{h-length}(\{H_0(z,t), 0 \leqslant t \leqslant 1\}) \leqslant M.$$

Hence for $z \in \widehat{\mathbb{C}} \backslash F^{-n}(U \cup \mathcal{P}_F)$,

$$\begin{aligned} d(\phi_n(z), \phi_{n+1}(z)) &= d(H_n(z,0), H_n(z,1)) \\ &\leqslant \text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) \\ &\leqslant \frac{1}{\lambda^n} \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) \\ &\leqslant M\lambda^{-n}. \end{aligned}$$

If $z \in F^{-n}(\mathcal{P}_F)$, then it follows from the relation $H_n(F(z),t) = f(H_{n+1}(z,t))$ that $d(\phi_n(z), \phi_{n+1}(z)) = 0$.

If $z \in F^{-n}(U) \backslash F^{-n}(\mathcal{P}_F)$, then $f^n(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) = \{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}$ and $F^n(z) \in U \backslash \mathcal{P}_F$. Let p be the least common multiple of the periods of all critical cycles of F, l be the minimal of $\frac{p}{p'}$, where p' is the period of a critical cycle of F, and D be the minimal of the product of local degrees of all critical points in C, where C is a critical cycle of F.

We may assume $n \geqslant p$. If $z, F(z), \cdots, F^n(z) \in U$, then there is a critical cycle of F such that $F^m(z) \in U_0$, $\forall m \geqslant 0$, where U_0 is the union of components of U containing that cycle. Let p_0 be the period of that cycle, $l_0 := \frac{p}{p_0}$, D_0 be the product of the local degrees of all critical points in that cycle.

First we consider the case that $p_0 = 1$, that is U_0 contains a critical fixed point q and $D_0 = \deg_q F$. Since $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F, the point $\phi_0(q)$ is a critical fixed point of f and $\deg_{\phi_0(q)} f = \deg_q F$. Let B be the Böttcher map f at the point $\phi_0(q)$ and we define $w_{\phi_0(q)}(\{H_m(\alpha, t), 0 \leqslant t \leqslant 1\})$ as in Lemma 2.1 for all $0 \leqslant m \leqslant n$ and $\alpha \in U_0$. Fix $0 \leqslant m \leqslant n-1$. Set $\gamma_{m+1} := \{H_{m+1}(z,t), 0 \leqslant t \leqslant 1\}$ and $\gamma_m := \{H_m(F(z),t), 0 \leqslant t \leqslant 1\}$. Then

$$w_{\phi_0(q)}(\gamma_{m+1}) = \frac{1}{2\pi i}\int_{\xi \in B(\gamma_{m+1})} \frac{d\xi}{\xi}$$

and

$$w_{\phi_0(q)}(\gamma_m) = \frac{1}{2\pi i}\int_{\eta \in B(\gamma_m)} \frac{d\eta}{\eta},$$

where $\eta = \xi^{D_0}$. An easy calculation shows that

$$w_{\phi_0(q)}(\gamma_m) = D_0 \cdot w_{\phi_0(q)}(\gamma_{m+1}).$$

This implies that

$$\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) \leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_0^{-n}.$$

For the general case, the assumption $n \geqslant p$ implies that there is an integer $k \geqslant 1$ such that $kl_0p_0 \leqslant n \leqslant (k+1)l_0p_0$. Then

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_0^{-(l_0k)}\\ &\leqslant MD^{-(lk)},\end{aligned}$$

where M is the constant obtained as in Lemma 2.1. Note that as $n \to \infty$, k tends to infinity linearly with l, in particular the bound $MD^{-(lk)}$ has a finite sum over n.

Now we suppose $z \notin U, F(z) \notin U, \cdots, F^{i-1}(z) \notin U, F^i(z) \in U, \cdots, F^n(z) \in U$ for some $i \geqslant 1$. Then similarly to the previous case, there is a critical cycle of F such that $F^m(z) \in U_1$, $\forall m \geqslant n$, where U_1 is the union of components of U containing that cycle. Let p_1 be the period of that cycle, $p = l_1p_1$, D_1 be the product of the local degrees of all critical points in that cycle.

If $n - i < p = l_1p_1$, then there is some integer $0 \leqslant j \leqslant l_1 - 1$, such that $jp_1 \leqslant n - i \leqslant (j+1)p_1$ and

$$\begin{aligned}\text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_0(F^n(z),t), 0 \leqslant t \leqslant 1\}) D_1^{-j}\\ &\leqslant M.\end{aligned}$$

Thus

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\})\lambda^{-i} \\ &\leqslant M\lambda^{-i}.\end{aligned}$$

Noticing that $n-i<p$, we have as $n \to \infty$, the bound $M\lambda^{-i}$ has a finite sum over n.

Otherwise, there is some $s \geqslant 1$ such that $sp \leqslant n-i \leqslant (s+1)p$. Then

$$\text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\}) \leqslant MD_1^{-(l_1 s)} \leqslant MD^{-(ls)}$$

So

$$\begin{aligned}\text{h-length}(\{H_n(z,t), 0 \leqslant t \leqslant 1\}) &\leqslant \text{h-length}(\{H_{n-i}(F^i(z),t), 0 \leqslant t \leqslant 1\})\lambda^{-i} \\ &\leqslant M\lambda^{-i}D^{-(ls)}.\end{aligned}$$

As $n \to \infty$, either i or s tends to infinity.

Combining the conclusions of the above paragraphs together, we get the uniform convergence of ϕ_n with respect to the spherical metric of $\widehat{\mathbb{C}}$. The continuity and surjectivity of h follow directly from the property that it is a uniform limit of a sequence of homeomorphisms. □

Proof of Corollary 1.2. — By Böttcher's theorem, we may modify the Thurston equivalence (ϕ_0, ϕ_1) such that $\phi_0 \circ F = f \circ \phi_0$ in a neighborhood of the critical cycles of F. Now it follows by Theorem 1.1. □

3. Quotient maps

Let $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous onto map. We call it a *quotient map* if $h^{-1}(y)$ is a full continuum for any point $y \in \widehat{\mathbb{C}}$, i.e. $\widehat{\mathbb{C}} \backslash h^{-1}(y)$ is a simply connected domain.

LEMMA 3.1. — *Let $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a continuous onto map. Then the following conditions are equivalent.*
(a) The map h is a quotient map.
(b) $h^{-1}(E)$ is a continuum if $E \subset \widehat{\mathbb{C}}$ is a continuum.
(c) $h^{-1}(E)$ is a full continuum if $E \subset \widehat{\mathbb{C}}$ is a full continuum.
(d) There exists a sequence of homeomorphisms $h_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\{h_n\}$ converges uniformly to h.

There is a similar statement in [8], see Lemma 2.3 and Theorem 2.12 in [8]. In the following, we will first prove (a), (b) and (c) are equivalent and then prove $(d) \Rightarrow (b)$. For $(a) \Rightarrow (d)$, the reader may refer to [8] for its proof. In the proof of Theorem 1.1, we will not use (a), (b) or $(c) \Rightarrow (d)$, but $(d) \Rightarrow (a)$, (b) and (c).

Proof of Lemma 3.1.— $(a) \Rightarrow (b)$. Let $E \subset \widehat{\mathbb{C}}$ be a continuum. If $h^{-1}(E)$ is not connected, then there are two disjoint open sets U and V in $\widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U \cup V$ and both $K_1 = U \cap h^{-1}(E)$ and $K_2 = V \cap h^{-1}(E)$ are not empty. Note that both K_1 and K_2 are closed since $h^{-1}(E)$ is closed. Thus both $h(K_1)$ and $h(K_2)$ are closed. On the other hand, $h(K_1)$ and $h(K_2)$ are disjoint by (a). This contradicts the condition that E is connected.

$(b) \Rightarrow (c)$. We only need to show that $h^{-1}(E)$ is full. Otherwise, $\widehat{\mathbb{C}} \backslash h^{-1}(E)$ is disconnected. Thus there are two distinct points $x, y \in \widehat{\mathbb{C}} \backslash h^{-1}(E)$ such that they are contained in different domains in $\widehat{\mathbb{C}} \backslash h^{-1}(E)$. Since $h(x), h(y) \in \widehat{\mathbb{C}} \backslash E$ and E is full, there exists an arc $\alpha \subset \widehat{\mathbb{C}} \backslash E$ which connects $h(x)$ with $h(y)$. Thus $h^{-1}(\alpha) \subset \widehat{\mathbb{C}} \backslash h^{-1}(E)$ is a continuum which contains x with y. This is a contradiction.

$(c) \Rightarrow (a)$. This is obvious.

$(d) \Rightarrow (b)$. Suppose that there exists a sequence of homeomorphisms $h_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\{h_n\}$ converges uniformly to h. Then $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a continuous onto map. Thus $h^{-1}(E)$ is closed for any continuum $E \subset \widehat{\mathbb{C}}$. Now assume that $h^{-1}(E)$ is not connected, i.e., there are two disjoint open sets $U, V \subset \widehat{\mathbb{C}}$ such that $h^{-1}(E) \subset U \cup V$ and both U and V intersect with $h^{-1}(E)$. Then $K := h(\widehat{\mathbb{C}} \backslash (U \cup V))$ is a compact set disjoint from E. Let $W \supset E$ be a connected domain such that $\overline{W} \cap K = \emptyset$. Since h_n converges uniformly to h, there exists some $n > 0$ such that

$$d(h, h_n) = \sup_{z \in \widehat{\mathbb{C}}} d(h(z), h_n(z)) < \min\{d(E, \partial W), d(\overline{W}, K)\},$$

where $d(\cdot,\cdot)$ denotes the spherical distance. It follows that $h_n(\widehat{\mathbb{C}} \backslash (U \cup V)) \cap \overline{W} = \emptyset$, hence $h_n^{-1}(W) \subset U \cup V$. It follows from $d(h, h_n) < d(E, \partial W)$ that $h_n(h^{-1}(E)) \subset W$. Thus both U and V intersect with $h_n^{-1}(W)$. This contradicts the fact that $h_n^{-1}(W)$ is connected. □

Proof of Theorem 1.1.— The sequence $\{\phi_n\}$ converges uniformly to a continuous onto map h by Lemma 2.1 and Lemma 2.2. Point (1) follows easily from the fact that $f \circ \phi_{n+1} = \phi_n \circ F$ and h is a uniform limit of ϕ_n. Point (4) follows from Lemma 3.1. Now we want to show the remaining points.

(2) It follows directly from Lemma 3.1 that for any $w \in \widehat{\mathbb{C}}$, $h^{-1}(w)$ is a full continuum. Since $\phi_0 \circ F = f \circ \phi_0$ near the critical cycles of F, $\phi_n \circ \phi_0^{-1}$ is a rotation in the Böttcher coordinates of the the critical cycles of f. It follows that there is a neighbourhood U of critical cycles of f such that $h^{-1}(q)$ is a single point for any $q \in U$. For any $w \in \mathcal{F}_f$, there is an integer $n \geqslant 1$ such that $f^n(w) \in U$. Since $h^{-1} \circ f^n(w) = F^n \circ h^{-1}(w)$, $h^{-1}(f^n(w))$ is a single point and $h^{-1}(w)$ is connected, we get that $h^{-1}(w)$ is a single point.

(3) Clearly $h(F(h^{-1}(x))) = f(h(h^{-1}(x))) = f(x) = y$. So $F(h^{-1}(x)) \subset h^{-1}(y)$. By Point (2), $h^{-1}(x)$ is connected. Let L be the connected component of $F^{-1}(h^{-1}(y))$ containing $h^{-1}(x)$. Then $h(L)$ is connected and $f(h(L)) = h(F(L)) \subset h(h^{-1}(y)) = y$. So $h(L) \subset f^{-1}(y)$. Notice that $x \in h(h^{-1}(x) \cap L) \subset h(L)$, that $f^{-1}(y)$ is a finite set, and that $h(L)$ is connected. We have therefore $h(L) = \{x\}$ and $L \subset h^{-1}(x)$. Consequently $h^{-1}(x) = L$. Notice that $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering. It follows easily from a property of a branched covering that $F(h^{-1}(x)) = h^{-1}(y)$ (see a proof in [1] §5.4).

Suppose $f^{-1}(y)$ has m preimages denoted by $x_1 := x, x_2, \cdots, x_m$. By the previous paragraph, we know that each $h^{-1}(x_i)$ is a connected component of $F^{-1}(h^{-1}(y))$ for $1 \leqslant i \leqslant m$. We claim that they are all the connected components of $F^{-1}(h^{-1}(y))$. In fact, let E be a connected component of $F^{-1}(h^{-1}(y))$. Since $f(h(E)) = h(F(E)) = h(h^{-1}(y)) = y$, we have $h(E) = x_j$ for some $1 \leqslant j \leqslant m$. Noticing that $E \subset h^{-1}(h(E)) = h^{-1}(x_j)$ and both E and $h^{-1}(x_j)$ are connected components of $F^{-1}(h^{-1}(y))$, we get $E = h^{-1}(x_j)$.

Since $\deg_q F = \deg_{\phi_1(q)} f$ for any critical point q of F and $h = \phi_n$ on $\mathcal{P}_F$ for all $n \geqslant 0$, we can conclude that for any critical point c of f, $h^{-1}(c)$ contains a critical point of F with local degree $\deg_c f$. Denote by $\deg F|_{h^{-1}(x_i)}$ the degree of the map $F : h^{-1}(x_i) \to h^{-1}(y)$. It follows that for each $1 \leqslant i \leqslant m$, $\deg F|_{h^{-1}(x_i)} \geqslant \deg_{x_i} f$. But $\sum_{i=1}^m \deg F|_{h^{-1}(x_i)} = \sum_{i=1}^m \deg_{x_i} f = d$, where d is the degree of F and f on $\widehat{\mathbb{C}}$. Thus $\deg F|_{h^{-1}(x_i)} = \deg_{x_i} f$.

(5) From $f \circ h(F^{-1}(E)) = h \circ F(F^{-1}(E)) = h(E)$, we have $h(F^{-1}(E)) \subset f^{-1}(h(E))$. Conversely, for any point $w \in f^{-1}(h(E))$, $f(w) \in h(E)$. So there is a point $z_0 \in E$ such that $f(w) = h(z_0)$. In Point (3), we have shown that $F(h^{-1}(w)) = h^{-1}(f(w))$. Noticing that $z_0 \in h^{-1}(f(w))$, there is a point $z_1 \in h^{-1}(w)$ such that $F(z_1) = z_0$. So $w = h(z_1) \in h(F^{-1}(z_0)) \subset h(F^{-1}(E))$. Therefore, $f^{-1}(h(E)) \subset h(F^{-1}(E))$.

(6) $F^{-1}(\widehat{E}) = F^{-1}(h^{-1}(h(E))) = h^{-1}(f^{-1}(h(E)))$. From Point (5), we obtain

$$F^{-1}(\widehat{E}) = h^{-1}(h(F^{-1}(E))) = \widehat{F^{-1}(E)}.$$

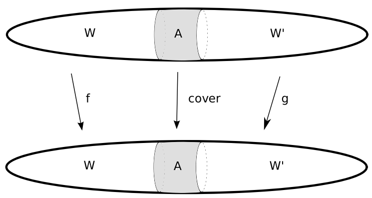
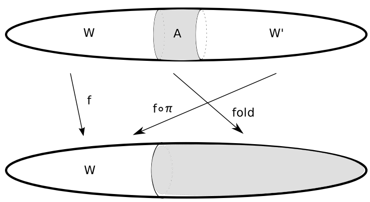
□

4. An application

In [3] a new type of surgery on polynomials, called 'foldings', is constructed. One can compare it with matings as follows: Set

- $\overline{W} = \mathbb{C} \cup \{\infty \cdot e^{2i\pi\theta}, \theta \in \mathbb{R}\}, \quad \overline{W'} = \mathbb{C}' \cup \{(\infty \cdot e^{2i\pi\theta})', \theta \in \mathbb{R}\}$,
- $A = [-1, 1] \times S^1$,
- $S = \overline{W} \sqcup A \sqcup \overline{W'} / \sim$,
  with $\infty \cdot e^{2\pi i\theta} \sim (-1, e^{2\pi i\theta})$ and $(+1, e^{2\pi i\theta}) \sim (\infty \cdot e^{-2\pi i\theta})'$,
- $\pi = id : \overline{W'} \to \overline{W}$.

Let f, g be monic postcritically finite polynomials of degree d. The mating M and a folding F are defined by :

f, g monic mating (f, g) : M :	folding (f, fold) $F^2(\Omega_F \cap K_f)$ (pre)periodic
	

More precisely $M|_W = f$, $M|_{W'} = g$ and $M : A \to A$ is a degree d covering matching the boundary values. This M is automatically postcritically finite and its Thurston equivalence class is uniquely determined (if one does not introduce twist in A). On the other hand, $F|_W = f$, $F|_{W'} = f \circ \pi$ and $F : A \mapsto A \cup \overline{W'}$ is a branched covering matching the boundary values. In order for F to be postcritically finite, we also require that $F^2(\Omega_F \cap A)$ to be contained in the set of preperiodic points of f. The Thurston equivalence class of F depends on the choices of F on A.

The multicurve consisting of the single Jordan curve $\gamma = \partial W$ behaves quite differently under the mating M and the folding F : the set $M^{-1}(\gamma)$ is again a single Jordan curve, and is homotopic to γ rel $\mathcal{P}_M$, whereas $F^{-1}(\gamma)$ has two connected components, and each of them are homotopic rel $\mathcal{P}_F$ to γ.

Just as in the mating case, we have shown in [3] cases of foldings that are Thurston equivalent to a rational map and cases of foldings that are not.

Assume that a folding F is Thurston equivalent to a rational map R. Then there is a pair of homeomorphisms (h_0, h_1) making the following diagram commutative:

$$\begin{array}{ccc} S & \xrightarrow[\approx]{h_1} & \widehat{\mathbb{C}} \\ F\Big\downarrow & & \Big\downarrow R \\ S & \xrightarrow[h_0]{\approx} & \widehat{\mathbb{C}}\,. \end{array}$$

We may then apply Rees-Shishikura's theorem, in the form of Theorem 1.1 and Corollary 1.2, to promote this diagram into a semi-conjugacy diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & \widehat{\mathbb{C}} \\ F\Big\downarrow & & \Big\downarrow R \\ S & \xrightarrow[h]{} & \widehat{\mathbb{C}}\,. \end{array}$$

Note that if F were a mating of polynomials, then h would reduce the annular space between K_f and K_g to a space with empty interior. The folding case is quite the opposite. We have actually proved, using Theorem 1.1 (see [3] for details) :

Proposition 4.1. — *In the above setting, the set $h(A)$ contains a non-empty annulus $\mathcal{A}$ s.t.*

- *$\mathcal{A}$ separates $h(\overline{W})$ and $h(\overline{W'})$,*
- *$\mathcal{A}$ contains two essential annuli A_1, A_2 satisfying that $R : A_1 \to \mathcal{A}$ and $R : A_2 \to \mathcal{A}$ are coverings, and $\partial\mathcal{A} \subset \partial(A_1 \cup A_2)$.*

An interesting consequence is that the folding rational map R has a polynomial renormalization. Moreover it has wandering continua in its Julia set (as in [9]). Such phenomenon does not exist for polynomials ([2, 6, 13]).

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