0° Graded Poisson Structure.

$A = \bigoplus_{i \in \mathbb{Z}} A_i$ - $\mathbb{Z}$-graded v. space. char $k = 0$, A_i - composants of degree i

$a \in A \rightsquigarrow |a|$ and if $a \in A_i$ $|a| = i$;

$A[n]$ - shifted graded v. space $A[n] = \bigoplus_{i \in \mathbb{Z}} (A[n])_i = \bigoplus_{i \in \mathbb{Z}} A_{n+i}$

Tensor product: $(A \otimes B)_r = \bigoplus_{p+q=r} A_p \otimes A_q$

A - graded (deg. 0) algebra if $ab = (-1)^{|a||b|} ba$
commutative

A - graded Lie algebra of degree n is a graded v. space A with a bracket on $A[n]$: The degree of the bracket is $-n$: $\{-,-\}: A \otimes A \to A[-n]$ (on A).

$$\{a, b\} = (-1)^{(|a|+n)(|b|+n)+1} \{b, a\}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n)(|b|+n)} \{a, \{b, c\}\}$$

Def. Graded Poisson algebra of degree n or n-Poisson is a graded v. space $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with 0-degree comm. product and with degree $-n$ Lie bracket such that the bracket is a biderivation in the usual sense:

$$\{a, b, c\} = \{a, b\}c + (-1)^{|b|(|a|+n)} b\{a, c\}$$

Remarks 1° graded 0-Poisson = Poisson algebra
graded 1-Poisson = Gerstenhaber algebra
(or Schouten-Nijenhuis?).

Basic examples

DGA_k - category of ass. DG-algebras with diff. degree (-1) has a model structure:
weak equiv. - quasi-isomorphisms
fibrations - surjections
cofibrations - morphisms adm. lifting property along acyclic fibrations
k - initial object 0 - terminal
This is an ex. of the recent.

NC-Poisson algebras (Berest-Crawley-Boevey)

Let A be an ass. $\mathbb{C}$-algebra with 1.

$Der(A)$ - A-module of its derivations.

Def. $Der(A)^{\natural} := \{ d \in Der(A) \mid d(A) \in [A,A], \forall a \in A \}$

Lemma $Der(A)^{\natural} \subset Der(A)$ is a Lie ideal and

$$Der(A)_{\natural} := Der(A) / Der(A)^{\natural} \text{ is a } \mathbb{C}\text{-Lie algebra.}$$

Let V be a $Der(A)_{\natural}$-module: there is a representation $\rho : Der(A)_{\natural} \to End(V)$ - a Lie algebra homomorphism.

Def. a ~~NC~~ Poisson structure on V is a DG-Lie algebra structure on V such that the adjoint representation

$$\begin{tikzcd} V \arrow[r, "ad"] \arrow[dr, "i"'] & End(V) \\ & Der(A)_{\natural} \arrow[u, "\rho"'] \end{tikzcd}$$

factors through ρ: ($\exists\, i$ - morphism of DGLA). $\boxed{ad = \rho \circ i}$

Now if A is a DG-algebra. the natural action of $Der(A)$ on A induces a Lie algebra $Der(A)_{\natural}$-action on $(A)_{\natural}$. NC-Poisson structure is (by definition) a Poisson structure on the representation of $A_{\natural}$:

$$\begin{tikzcd} A_{\natural} \arrow[r, "ad"] \arrow[dr, "i"'] & End(A_{\natural}) \\ & Der(A)_{\natural} \arrow[u, "\rho"'] \end{tikzcd}$$

Let A and B be objects in DGA_k equipped with NC Poiss. str.

$f : A \to B$ is a morphism of NC Poisson algebras if it is a morphism of DGA's $f : A \to B$ s.t.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Downarrow & & \Downarrow \\ A_\natural & \xrightarrow{f_\natural} & B_\natural \end{array}$$

is a morphism of DGLA's.

We have a category $NCPoiss_k$.

If $A \in NCPoiss_k$ and Ω is the de Rham $\Omega^\bullet(\mathbb{A}^1)$ (a chain complex of k-vector spaces:

$$\mathbb{V} := \{0 \to k\cdot t \xrightarrow{d} k\cdot dt \to 0\} \quad \deg t = 0 \ ;$$

$\Omega = \Lambda^\bullet_\circ(\mathbb{V})$ $\quad \Omega_{ass} = T(V)$ $\quad$ Ω - de Rham compl. of affine line with dif. deg. -1

Then $\underline{A\otimes\Omega \in NCPoiss_k}$:

$$[A\otimes\Omega, A\otimes\Omega] = [A,A]\otimes\Omega \Rightarrow (A\otimes\Omega)_\natural = A_\natural\otimes\Omega$$

$i : A_\natural\otimes\Omega \to Der(A\otimes\Omega)_\natural$ is the composition

$$\begin{tikzcd} A_\natural\otimes\Omega \arrow[r] \arrow[rr, bend right] & Der(A\otimes\Omega)_{A,\natural} \arrow[r] & Der(A\otimes\Omega)_\natural \end{tikzcd}$$

where $Der(A\otimes\Omega)_{A,\natural} = Der_\Omega(A\otimes\Omega) \big/ Der_\Omega(A\otimes\Omega)^\natural$

$\underline{\Omega\text{-linear derivations}}$

$\underline{\text{Lie}}$ ideal with images in $A_\natural\otimes\Omega$

$\underline{\text{Poisson homotopy ~~along the line~~}}$

$f, g \in Mor(NCPoiss_k)$ are Poisson homotopic if $\exists$ a morphism $A \xrightarrow{h} B\otimes\Omega$ such that

$$h(0) = f \quad h(1) = g$$

This is an equivalence relation in $\mathrm{Hom}_{NCPoiss}(A, B)$

$\Rightarrow$ $Ho'(NC\,Poisson)$ - category whose ~~[illegible]~~ objects are cofibrant in DGA_k

Th: Now if we have a representation functor

$$\begin{tikzcd} (-)_V^{GL} : NCPoiss_k \arrow[r] \arrow[d, "\text{forgetful}"] & Poiss_k \arrow[d, "\text{forgetful.}"] \\ DGA_k \arrow[r] & Com\,DGA_k \end{tikzcd}$$

It descends to a functor

$$\begin{tikzcd} L^*(-)_V^{GL} : Ho'(NCPoiss_k) \arrow[r] \arrow[d] & Ho(Poiss_k) \arrow[d] \\ L(-)_V^{GL} : Ho(DGA) \arrow[r] & Ho(Comm\,DGA) \end{tikzcd}$$

Cofibrant -

<u>Example</u> V - graded v. space with a degree 0 antisymmetric bilinear form
$$\omega : V \times V \to k.$$
TV - tensor algebra of V considering as a DG-algebra with 0-differential.

<u>NC - Poisson structure on TV:</u>

$$\{u_1 \otimes \dots \otimes u_k , v_1 \otimes \dots \otimes v_\ell\} := \sum_{i,j} (\pm)\, \omega(u_i, v_j)\, u_{i+1} \otimes \dots \otimes u_k \otimes u_1 \otimes \dots \otimes u_{i-1} \otimes v_{j+1} \otimes \dots \otimes v_\ell \otimes v_1 \otimes \dots \otimes v_{j-1}$$
(Koszul conventions of sign)

Then we have a bracket $\{-,-\}_\natural$ induced on $(TV)_\natural$

<u>Proposition (Ginzburg)</u> • $\{-,-\}_\natural$ is a graded Lie bracket on $(TV)_\natural$

• For any $\underbrace{u_1 \dots u_k}_{\text{"word"}} \in (TV)_\natural$ $\{u_1 \dots u_k , -\}$ is induced by the derivation:
$$\sum_{i=1}^{k} \pm (u_{i+1} \otimes \dots \otimes u_k \otimes u_1 \dots u_{i-1})\, \partial_{u_i} \in \mathrm{Der}(TV)$$
where
$\partial_{u_i}(v) = \omega(u_i, v)$ a unique derivation.

<u>Case</u>: $V = k\cdot x \oplus k\cdot y \oplus k\cdot t$, ~~$d(x) = dt$~~ $\deg(x) = \deg(y) = 0$, $\deg(t) = 1$

define $\omega : V \times V \to k$ $\omega(x,y) = 1$
$\omega(x,t) = \omega(y,t) = 0$

then $R := (TV, d)$ such that $dx = dy = 0$ $dt = [x,y]$ is NC Poisson

Since R is a free resolution of $k[x,y] = A$

then the bracket $\{-,-\}_\natural$ induces a Poisson structure on
$$HC_0(A) \times HC_0(A) \to HC_0(A).$$

Double Poisson DG-algebras

$A \in DGA_k$ $\quad A \otimes A$ admits two bimodule structures:

outer: $a(u \otimes v)b = au \otimes vb$ and inner

$$a(u \otimes v) \otimes b = (-1)^{(|a|+|v|)(|b|+|u|)} ub \otimes av$$

A double n-braket is

$\{\!\{-,-\}\!\}\ A \otimes A \to A \otimes A$ which is a degree n

derivation of the outer structure in the second argument

and $\{\!\{a, b\}\!\} = (-1)^{(|a|+n)(|b|+n)} \{\!\{b, a\}\!\}^{op}$

$$(u \otimes v)^{op} = (-1)^{|u||v|} v \otimes u$$

left bracket $\{\!\{-,-\}\!\}_L$ is defined

$$\{\!\{a, b \otimes c\}\!\}_L := \{\!\{a, b\}\!\} \otimes c$$

The Jacobi identity:

$$\{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + (-1)^{(|a|+n)(|b|+|c|)} \sigma \{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L + (-1)^{(|c|+n)(|a|+|b|)} \sigma^2 \{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L = 0.$$

Thm. (V.d. Bergh – Berest + C°)

$\mu \circ \{\!\{-,-\}\!\}$ induces a NCn-Poisson structure on A

($n=0$ Poisson NC) ($n=1$ NC Gerstenhaber).

Cyclic graded algebras & D-Poisson n-brackets

$\langle a, bc \rangle = \pm \langle ca, b \rangle$ $\quad a, b, c \in$ graded f.d. ass. A

$\hookrightarrow$ symmetric inner product of degree n.

then the dual space $C = \mathrm{Hom}(A, k)$ is a coalgebra with a symmetric bilinear pairing degree $(-n)$

The symmetric inner product on A is dual to the identity

$$\langle v', w \rangle \cdot v'' = \pm \langle v, w'' \rangle w' \qquad \forall\, v, w \in C$$

C is a <u>cyclic</u> $(-n)$-coalgebra; it is $\underbrace{\text{DG coalgebra}}_{\text{cyclic}}$ with $\langle du, v \rangle \pm \langle u, dv \rangle = 0 \quad \forall\, u, v \in C$

<u>Thm.</u> If C is cyclic coassociative coalgebra and $\Omega(C)$ its cobar construction then $\Omega(C)$ has DG $(n+2)$-Double Poisson brackets:

$$\{\!\{ v, w \}\!\} := \sum_{i,j} \pm \langle v_i, w_j \rangle (w_1, \dots, w_{j-1}, v_{i+1}, \dots, v_n) \otimes (v_1, \dots, v_{i-1}, w_{j+1} \dots w_m)$$

<u>Rem.</u> $\Omega(C)$ – cofibrant DG – algebra

C – coaugmented coalgebra $\quad \bar{C} = \mathrm{Coker}(\eta : k \to C)$

coaugmentation coideal

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{q \otimes q} \bar{C} \otimes \bar{C}$$

q – quotient

$$\Omega(C) := (T(s^{-1}\bar{C}), d_\Omega) \qquad d_\Omega s^{-1} = -s^{-1}d + (s^{-1} \otimes s^{-1})\bar{\Delta}$$

$$d_\Omega(s^{-1}c) = -s^{-1}(dc) + (-1)^{\deg c_i} s^{-1}c_i\, s^{-1}c^i$$

$$\bar{\Delta}(c) = c_i \otimes c^i$$