Zariski excision and the Steinberg relation

Geoffrey M.L. Powell

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Abstract

This note proves the Steinberg relation in $\mathbb{A}^1$-homotopy theory (first proved by Hu and Kriz) by using Zariski excision together with an explicit $\mathbb{A}^1$-homotopy.

1 Introduction

Let $\mathcal{H}_\bullet(k)$ denote the pointed $\mathbb{A}^1$-local homotopy category with respect to the Zariski topology on the category $\mathcal{S}m/k$ of smooth schemes over a field k [MV]. There is a canonical simplicial structure upon $\mathcal{H}_\bullet(k)$, hence the simplicial spheres S^n, for natural numbers n, define objects of the homotopy category. The multiplicative group $\mathbb{G}_m \in \mathcal{S}m/k$ is pointed canonically by 1 and plays the rôle of a geometric circle. A rational point $\alpha : \mathrm{Spec}(k) \to \mathbb{G}_m$ induces a homotopy class in $[S^0, \mathbb{G}_m]_{\mathcal{H}_\bullet(k)}$; if α, β are two rational points of $\mathbb{G}_m$, then the smash product defines the symbol $\{\alpha, \beta\} \in [S^0, \mathbb{G}_m \wedge \mathbb{G}_m]_{\mathcal{H}_\bullet(k)}$.

The purpose of this note is to give an alternative proof of the Steinberg relation in $\mathbb{A}^1$-homotopy theory, which was first established by Hu and Kriz [HuK]. The functor simplicial suspension, which corresponds to the smash product with S^1, is denoted by Σ in the following statement.

Theorem 1 *Let α be a rational point of $\mathbb{G}_m \backslash \{1\}$, then the suspension $\Sigma\{\alpha, 1 - \alpha\} \in [S^1, \Sigma(\mathbb{G}_m \wedge \mathbb{G}_m)]_{\mathcal{H}_\bullet(k)}$ is trivial.*

The proof relies upon Zariski excision for sheaves, together with the usage of a strict $\mathbb{A}^1$-homotopy. The dependency on the simplicial homotopy structure is subsumed in using sheaves of the form $\mathbb{A}^n/X$, where $X \hookrightarrow \mathbb{A}^n$ is an inclusion of smooth schemes, as a model for the unreduced suspension of X.

Remark 1.0.1 The approach here is motivated by the observation that the object $\mathbb{A}^1 \backslash \{0, 1\}$ is equivalent in the pointed homotopy category $\mathcal{H}_\bullet(k)$ to the wedge of two copies of $\mathbb{G}_m$ after a single simplicial suspension. The result then follows by consideration of the corresponding inclusion of $\mathbb{G}_m \vee \mathbb{G}_m$ in $\mathbb{G}_m \times \mathbb{G}_m$ (still after simplicial suspension).

If the field k admits a complex embedding, then there is a complex realization functor to the homotopy category of spaces. After complex realization, the suspension is no longer necessary, since the morphisms reduce to those appearing in the cofibration sequence $S^1 \vee S^1 \to S^1 \times S^1 \to S^1 \wedge S^1$.

Notation 1.0.2 Throughout the paper 'sheaf' indicates sheaf of sets with respect to the Zariski topology on the category of smooth schemes $\mathcal{S}m/k$. The 'simplicial homotopy category' refers to the Joyal-Jardine model structure on the category of simplicial sheaves, with respect to the Zariski topology, and the $\mathbb{A}^1$-local model structure is the $\mathbb{A}^1$-localization of the Joyal-Jardine structure with respect to the Zariski topology, as constructed in [MV].

2 Zariski excision

Let $X = U \cup V$ be a Zariski open covering of $X \in \mathcal{S}m/k$, then there is a cocartesian diagram in the category of sheaves:

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X. \end{array}$$

All morphisms in the diagram are monomorphisms, hence cofibrations. Thus, the diagram is homotopy cartesian in the simplicial model structure (where each object is equipped with constant simplicial structure). This square is the geometric origin of Mayer-Vietoris sequences in representable cohomology theories.

The category of sheaves of simplicial sets has a terminal object, namely the constant sheaf, $\star$, taking value the singleton set, with constant simplicial structure. Suppose that $A \hookrightarrow X$ is a cofibration (ie monomorphism) of simplicial sheaves, then the cofibre is defined to be the pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \star & \longrightarrow & X/A \end{array}$$

in the category of simplicial sheaves. Observe that the simplicial sheaf X/A is pointed canonically.

Remark 2.0.3 The above square is homotopy cocartesian with respect to the simplicial model structure, since $A \hookrightarrow X$ was supposed to be a cofibration. Without this hypothesis, the cofibre functor must be derived functor to obtain a functor on the homotopy category.

The following Zariski excision result is standard, by regarding the cofibres of the cocartesian square:

Lemma 2.0.4 *Let U, V be a Zariski covering of $X \in \mathcal{S}m/k$, then there is an isomorphism of pointed sheaves $V/(U \cap V) \stackrel{\cong}{\to} X/U$.*

Proposition 2.0.5 *Let U, V be a Zariski open covering of $X \in \mathcal{S}m/k$, then there is an isomorphism of pointed Zariski sheaves*

$$(X/U) \vee (X/V) \xrightarrow{\cong} X/(U \cap V).$$

Proof: There is a cocartesian square of pointed Zariski sheaves

$$\begin{array}{ccc} \star & \longrightarrow & U/(U \cap V) \\ \downarrow & & \downarrow \\ V/(U \cap V) & \longrightarrow & X/(U \cap V). \end{array}$$

Hence there is a natural isomorphism of pointed Zariski sheaves: $U/(U \cap V) \vee V/(U \cap V) \xrightarrow{\cong} X/(U \cap V)$. The result follows by applying Zariski excision again. ■

Remark 2.0.6 The significance of the above approach is that it yields an explicit morphism of sheaves: $(X/U) \vee (X/V) \to X/(U \cap V)$.

When working with respect to the Nisnevich topology and with the $\mathbb{A}^1$-local model structure, the existence of an $\mathbb{A}^1$-weak equivalence between the two sides above is a direct consequence of the purity theorem of [MV], in the case of pure codimension.

3 Unreduced suspension of subschemes of affine space

Notation 3.0.7 Let $1 \to \tilde{C}_s$ denote a natural transformation of functors from simplicial sets to simplicial sets, where for each $X \in \Delta^{\mathrm{op}} Set$, $X \to \tilde{C}_s X$ is a cofibration (ie monomorphism) and $\tilde{C}_s X$ is weakly equivalent to the terminal object. (For example, use the fibrant resolution functor which is provided by the model structure). Let $\tilde{\Sigma}_s X$ denote the cofibre $\tilde{C}_s(X)/X$. The object $\tilde{C}_s X$ shall be referred to as the unreduced cone on X and the object $\tilde{\Sigma}_s X$ as the unreduced suspension of X.

The functors $\tilde{C}_s, \tilde{\Sigma}_s$ extend canonically to the category of simplicial sheaves.

Proposition 3.0.8 *Let $X \hookrightarrow \mathbb{A}^n$ be an immersion in $\mathcal{S}m/k$, which is regarded as a cofibration in the simplicial model structure. There is an $\mathbb{A}^1$-weak equivalence:*

$$\phi_X : \mathbb{A}^n/X \simeq_{\mathbb{A}^1} \tilde{\Sigma}_s X$$

with respect to the $\mathbb{A}^1$-local model structure.

Proof: For later consideration of morphisms, it is necessary to make the construction functorial, using the functorality of $\tilde{C}_s, \tilde{\Sigma}_s$. There is a strictly commutative diagram

$$\begin{array}{ccccc}
X & \longrightarrow & \tilde{C}_s X & \longrightarrow & \tilde{\Sigma}_s X \\
\| & & \downarrow & & \downarrow \\
X & \longrightarrow & \tilde{C}_s \mathbb{A}^n & \longrightarrow & \tilde{C}_s(\mathbb{A}^n)/X \\
\| & & \uparrow & & \uparrow \\
X & \longrightarrow & \mathbb{A}^n & \longrightarrow & \mathbb{A}^n/X
\end{array}$$

in which the rows are cofibre sequences, using the fact that the left hand morphism in each row is a cofibration, by hypothesis.

The middle column consists of $\mathbb{A}^1$-weak equivalences, hence the gluing lemma implies that the morphisms induced on the cofibre are $\mathbb{A}^1$-weak equivalences. ■

Proposition 3.0.9 *Let $f : X \to Y$ be a morphism in $\mathcal{S}m/k$ which fits into a commutative diagram*

$$\begin{array}{ccc}
X & \xrightarrow{i_X} & \mathbb{A}^m \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\
Y & \xrightarrow[i_Y]{} & \mathbb{A}^n
\end{array}$$

of morphisms in $\mathcal{S}m/k$. Then the induced morphism on cofibres $\mathbb{A}^m/X \to \mathbb{A}^n/Y$ identifies, via the $\mathbb{A}^1$-weak equivalences ϕ_X, ϕ_Y, with the unreduced suspension of f, $\tilde{\Sigma}_s f : \tilde{\Sigma}_s X \to \tilde{\Sigma}_s Y$.

Proof: Construct the strictly commutative diagram of morphisms of sheaves:

$$\begin{array}{ccccc}
\mathbb{A}^m/X & \longrightarrow & \tilde{C}_s(\mathbb{A}^m)/X & \longleftarrow & \tilde{C}_s(X)/X \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{A}^n/Y & \longrightarrow & \tilde{C}_s(\mathbb{A}^n)/Y & \longleftarrow & \tilde{C}_s(Y)/Y
\end{array}$$

The proposition implies that the horizontal morphisms are $\mathbb{A}^1$-weak equivalences, hence the result follows. ■

3.1 Unreduced suspension and smash product

Consider the smash product of pointed representable sheaves. Let $*_X \hookrightarrow X$, $*_Y \hookrightarrow Y$ be smooth schemes pointed by rational points; the smash product $X \wedge Y$ is the cofibre in the pointed homotopy category of the morphism $X \vee Y \hookrightarrow$

$X \times Y$ in the category of pointed (simplicial) sheaves. In the case that X, Y are immersed in affine spaces, at the level of the unreduced suspension, there is a geometric model.

Let X, Y be pointed smooth schemes as above and suppose that there exists immersions $X \overset{i_X}{\hookrightarrow} \mathbb{A}^m$ and $Y \overset{i_Y}{\hookrightarrow} \mathbb{A}^n$. The closed points of X, Y induce closed points of $\mathbb{A}^m, \mathbb{A}^n$ respectively. Moreover, there is an immersion $X \times Y \to \mathbb{A}^{m+n}$ induced by the product. There are induced morphisms of pointed sheaves:

$$\begin{array}{rcl} j_X & : & \mathbb{A}^m/X \to \mathbb{A}^{m+n}/(X \times Y) \\ j_Y & : & \mathbb{A}^n/Y \to \mathbb{A}^{m+n}/(X \times Y) \end{array}$$

which are induced by the rational points.

Proposition 3.1.1 *Let X, Y be pointed smooth schemes which satisfy the above hypotheses, then there is a homotopy cofibre sequence*

$$(\mathbb{A}^m/X) \vee (\mathbb{A}^n/Y) \overset{j_X \vee j_Y}{\to} \mathbb{A}^{m+n}/(X \times Y) \to \tilde{\Sigma}_s(X \wedge Y)$$

in the pointed $\mathbb{A}^1$-local homotopy category.

Proof: There is a cofibration sequence $X \vee Y \to X \times Y \to X \wedge Y$ in the category of pointed sheaves. The unreduced suspension functor $\tilde{\Sigma}_s$ converts this to a (homotopy) cofibration sequence. The result follows by Proposition 3.0.9. ■

Example 3.1.2 This result may be applied in the case of $\mathbb{G}_m$ pointed by the identity. The morphism $j_{\mathbb{G}_m} : \mathbb{A}^1/\mathbb{G}_m \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m)$ is induced by the rational point $\{1\} \in \mathbb{G}_m \subset \mathbb{A}^1$. Thus, there is an explicit homotopy cofibre sequence:

$$(\mathbb{A}^1/\mathbb{G}_m) \vee (\mathbb{A}^1/\mathbb{G}_m) \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m) \to \tilde{\Sigma}_s(\mathbb{G}_m \wedge \mathbb{G}_m).$$

4 Geometry and the Steinberg relation

4.1 Geometric construction

Let $h : \mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ be the closed immersion which is given by $x \mapsto (x, 1-x)$. The open immersion $\mathbb{G}_m \times \mathbb{G}_m \hookrightarrow \mathbb{A}^2$, given by the product of the immersions $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$, induces a cartesian diagram:

$$\begin{array}{ccc} \mathbb{A}^1\backslash\{0,1\} & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow[h]{} & \mathbb{A}^2 \end{array}$$

in which the horizontal morphisms are closed immersions and the vertical morphisms are open immersions. The morphism h is the restriction to $t = 1$ of the morphism $H : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^2$ given by $(t, x) \mapsto (x, 1 - tx)$.

There is a cartesian diagram of smooth schemes

$$\begin{array}{ccc} \mathbb{A}^2\backslash(Z_1 \amalg Z_2) & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow[H]{} & \mathbb{A}^2 \end{array}$$

where $Z_1 \cong \mathbb{A}^1$ is the closed subscheme $\{x = 0\}$ and $Z_2 \cong \mathbb{G}_m$ is the closed subsheme $\{1 - tx = 0\}$.

Remark 4.1.1 The morphism $\mathbb{A}^2\backslash(Z_1 \amalg Z_2) \to \mathbb{G}_m \times \mathbb{G}_m$ is an isomorphism of schemes. The key point here is that the closed subschemes Z_1, Z_2 only intersect 'at infinity'.

By construction, the morphism $\mathbb{A}^2 \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m)$ gives rise to a morphism $\mathbb{A}^2\backslash Z_2/(\mathbb{A}^2\backslash(Z_1 \amalg Z_2)) \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m)$ of Zariski sheaves. Zariski excision with respect to the open covering of $\mathbb{A}^2$ by $\mathbb{A}^2\backslash Z_1$ and $\mathbb{A}^2\backslash Z_2$ induces a morphism of Zariski sheaves:

$$\mathbb{A}^2/(\mathbb{A}^2\backslash Z_1) \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m).$$

Lemma 4.1.2 *There is an isomorphism of Zariski sheaves:*

$$\mathbb{A}^2/(\mathbb{A}^2\backslash Z_1) \cong \mathbb{A}^1{}_+ \wedge \mathbb{A}^1/\mathbb{A}^1\backslash\{0\}.$$

The restrictions induced by the rational points $\{0, 1\}$ of $\mathbb{A}^1$ induce morphisms

$$\mathbb{A}^1/\mathbb{A}^1\backslash\{0\} \overset{H_1}{\underset{H_0}{\rightrightarrows}} \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m).$$

The morphism H_1 is the extension via Zariski excision of the morphism:

$$\mathbb{A}^1\backslash\{1\}/(\mathbb{A}^1\backslash\{0,1\}) \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m)$$

which is induced by h and H_0 is the morphism which is induced by $\mathbb{A}^1 \to \mathbb{A}^2$, $x \mapsto (x, 1)$.

Proposition 4.1.3 *The morphism H defines an $\mathbb{A}^1$-homotopy between the extension H_0 of h and the morphism H_1.*

Proof: This is immediate from Lemma 4.1.2 together with the identification of H_1, H_0. ■

Remark 4.1.4 The above argument is based on excision of the point $\{1\}$. A symmetric argument (by change of variables) applies to the consideration of the analogous morphism

$$\mathbb{A}^1/(\mathbb{A}^1\backslash\{1\}) \to \mathbb{A}^2/(\mathbb{G}_m \times \mathbb{G}_m).$$

4.2 The Steinberg relation

A rational point $\mathrm{Spec}(k) \to \mathbb{G}_m$ induces a homotopy class in $[S^0, \mathbb{G}_m]_{\mathcal{H}_\bullet(k)}$. Let α be a rational point of $\mathbb{G}_m \backslash \{1\}$, then there is a composite morphism:

$$\mathrm{Spec}(k) \xrightarrow{\alpha} \mathbb{A}^1 \backslash \{0,1\} \xrightarrow{h} \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$$

in the category of simplicial sheaves. This construction yields the symbol $\{\alpha, 1-\alpha\} \in [S^0, \mathbb{G}_m \wedge \mathbb{G}_m]_{\mathcal{H}_\bullet(k)}$.

Theorem 4.2.1 *The suspension $\Sigma\{\alpha, 1-\alpha\} \in [\tilde{\Sigma}_s S^0, \tilde{\Sigma}_s(\mathbb{G}_m \wedge \mathbb{G}_m)]_{\mathcal{H}_\bullet(k)}$ is trivial in the pointed $\mathbb{A}^1$-local homotopy category.*

This result is a consequence of the following.

Theorem 4.2.2 *Let $\delta : \mathbb{A}^1 \backslash \{0,1\} \to \mathbb{G}_m \wedge \mathbb{G}_m$ denote the composite $\mathbb{A}^1 \backslash \{0,1\} \to \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m$. Then the unreduced suspension $\tilde{\Sigma}_s \delta$ is trivial in the pointed $\mathbb{A}^1$-local homotopy category.*

The Zariski excision argument of Proposition 2.0.5 yields the following:

Proposition 4.2.3 *There is an isomorphism in the category of pointed Zariski sheaves:*

$$\mathbb{A}^1/(\mathbb{A}^1 \backslash \{0\}) \vee \mathbb{A}^1/(\mathbb{A}^1 \backslash \{1\}) \to \mathbb{A}^1/(\mathbb{A}^1 \backslash \{0,1\})$$

where the morphisms $\mathbb{A}^1/(\mathbb{A}^1 \backslash \{0\}) \to \mathbb{A}^1/(\mathbb{A}^1 \backslash \{0,1\}) \leftarrow \mathbb{A}^1/(\mathbb{A}^1 \backslash \{1\})$ are induced by Zariski excision from the natural inclusions $\mathbb{A}^1 \backslash \{1\} \hookrightarrow \mathbb{A}^1 \hookleftarrow \mathbb{A}^1 \backslash \{0\}$.

Hence, to prove Theorem 4.2.2, it is sufficient to show that the morphisms on each wedge factor

$$\mathbb{A}^1/(\mathbb{A}^1 \backslash \{0\}) \to \mathbb{A}^1/(\mathbb{A}^1 \backslash \{0,1\}) \to \tilde{\Sigma}_s(\mathbb{G}_m \wedge \mathbb{G}_m)$$
$$\mathbb{A}^1/(\mathbb{A}^1 \backslash \{1\}) \to \mathbb{A}^1/(\mathbb{A}^1 \backslash \{0,1\}) \to \tilde{\Sigma}_s(\mathbb{G}_m \wedge \mathbb{G}_m)$$

are trivial in the pointed $\mathbb{A}^1$-homotopy category.

This follows from the $\mathbb{A}^1$-homotopy argument of Proposition 4.1.3, together with the homotopy cofibre sequence which is provided by Proposition 3.1.1, as applied in Example 3.1.2.

References

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