# The adjunction between $\mathcal{H}(k)$ and $DM_-^{\text{eff}}(k)$

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**Abstract**

The adjunction between the Morel-Voevodsky $\mathbb{A}^1$-local homotopy category and Voevodsky's triangulated category $DM_-^{\text{eff}}(k)$ of motivic complexes is constructed, when the field $k$ is perfect.

Two issues are involved: firstly the free sheaf with transfers functor is not known to preserve weak equivalences and therefore has to be derived. Secondly the notion of motivic complex is *a priori* weaker than that of $\mathbb{A}^1$-local. The hypothesis that $k$ be perfect is required to identify these two notions.

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# 1 Introduction

There are several variants for constructing a homotopy category of schemes; for concreteness, the approach of [MV] is followed for the definition of the pointed simplicial homotopy category $\mathcal{H}_{s,\bullet}(k)$ and the pointed $\mathbb{A}^1$-local homotopy category $\mathcal{H}_\bullet(k)$. In particular, these categories are localizations of the category $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$ of pointed simplicial sheaves on the (large) smooth Nisnevich site.

Voevodsky's triangulated category $DM_-^{\mathrm{eff}}(k)$ is the full sub-category of motivic complexes in the derived category $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$ of Nisnevich sheaves with transfers [V]. This note provides a proof of the following theorem:

**Theorem 1** *Suppose that $k$ is a perfect field. There is an adjunction:*

$$M(.) : \mathcal{H}_\bullet(k) \rightleftarrows DM_-^{\mathrm{eff}}(k) : K.$$

The importance of this result is that it establishes the equivalence between the cohomology theory represented by motivic Eilenberg-MacLane spaces in the Morel-Voevodsky (and thus in any equivalent) homotopy category and the motivic cohomology theory defined by Voevodsky in the category $DM_-^{\mathrm{eff}}(k)$, when the field $k$ has characteristic zero. (The requirement that the field have characteristic zero is necessary since resolution of singularities is required in order to prove that the motivic Eilenberg-MacLane spaces form an $\Omega_{\mathbb{P}^1}$-spectrum and hence give rise to a cohomology theory).

Theorem 1 is proved by first establishing the following proposition:

**Proposition 2** *There is an adjunction*

$$\mathbb{L}\mathbb{Z}_{\mathrm{tr}}(.) : \mathcal{H}_{s,\bullet}(k) \rightleftarrows \mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}} : K.$$

The technical point in the proof is that the functor $\mathbb{L}\mathbb{Z}_{\mathrm{tr}}(.)$ is a derived functor of $\mathbb{Z}_{\mathrm{tr}}(.) : \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet \to \mathcal{N}_k^{\mathrm{tr}}$, the free sheaf with transfers functor, since the latter is not known to preserve weak equivalences.

The second part of the proof is the passage to $\mathbb{A}^1$-localizations, which requires the comparison between the notion of motivic complex and that of an $\mathbb{A}^1$-local complex. This is the source of the hypothesis that $k$ be perfect.

For applications, the following result is necessary:

**Proposition 3** *Suppose that the morphism $\mathcal{Y} \to \mathcal{X}$ is a simplicial weak equivalence in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$, where in each simplicial degree $n$, the sheaf $\mathcal{Y}_n$ is a coproduct of sheaves represented by smooth schemes. Then there is an isomorphism $\mathbb{L}\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \cong C_*^N\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y})$ in $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$, where $C_*^N$ is the normalized chain complex functor.*

**Sins of Omission:**

1. This version of this note does not introduce the *cdh* topology and hence does not immediately yield representability of the Suslin-Voevodsky motivic cohomology of non-smooth schemes.

2. The author believes that the introduction of the *cdh* topology could provide a more direct (but not necessarily more elegant) proof of Proposition 2.

### 1.1 Conventions and Notation

Throughout the note the following notation and conventions are used. The category $\mathcal{S}m/k$ denotes the category of smooth schemes over $k$ which are of finite type and are separated.

The category of pointed sets is written as $\mathrm{Set}_\bullet$; by logical extension, the subscript $(.)_\bullet$ will be used to indicate any category obtained from the category of pointed sets (for example the category of pointed sheaves of sets) and also homotopy categories derived from a pointed category.

Chain complexes will always be in the sense of algebraic topology: the differentials are of degree $-1$. As usual, indices are written homologically for such complexes; if $C_*$ is a chain complex then there is a complex $C^{-*}$ with differential of degree $+1$. The category of bounded below chain complexes in an abelian category $\mathcal{A}$ will be denoted by $\mathcal{C}_-\mathcal{A}$.

### 1.2 References

The technical results which are used in the paper concerning the categories of sheaves with transfers and the full sub-category of homotopy invariant presheaves are all stated in [V]. Certain of these results are covered in greater detail in [SV]; in particular the paper [SV] gives a careful exposition of the hypercohomology spectral sequences which are used and the details of the construction of the tensor product on the category $DM_-^{\mathrm{eff}}(k)$.

An alternative technical approach is suggested by the preprint [SP], which uses the methods of [Ho2] to construct a model category structure on the category of chain complexes of sheaves with transfers (not bounded below) and thus construct the associated derived category. This approach is appealing, especially when considering the $\mathbb{A}^1$-localization of the derived category.

The early version of Deligne's notes [D] on Voevodsky's lectures at the Institute for Advanced Study, Princeton cover some of the foundational material.

**Acknowledgement:** The author lays little claim to originality in these notes; in particular, the scheme of proof of Proposition 2 was indicated to the author by Fabien Morel, to whom he is extremely grateful.

## 2 The adjunction for abelian sheaves

### 2.1 The Dold-Kan theorem

There is an adjunction

$$C_*^N : \Delta^{\mathrm{op}}\mathcal{A}b \rightleftarrows \mathcal{C}_{\geq 0}\mathcal{A}b : K$$

where $C_*^N$ is the normalized chain complex functor and the right adjoint is the Kan functor. The Dold-Kan theorem states that this is an equivalence of categories [W, §8.4]; under this equivalence the homotopy of a simplicial abelian group (pointed by zero) corresponds to the homology of the associated chain complex. (Observe that a simplicial abelian group is always fibrant (Kan) as a simplicial set).

The full embedding $\mathcal{C}_{\geq 0}\mathcal{A}b \hookrightarrow \mathcal{C}_-\mathcal{A}b$ admits a right adjoint, the truncation functor $\tau_{\geq 0} : \mathcal{C}_-\mathcal{A}b \to \mathcal{C}_{\geq 0}\mathcal{A}b$, which maps $(C_\bullet, d)$ to the evident subcomplex obtained by placing $\ker\{d : C_0 \to C_{-1}\}$ in degree zero. The composite adjunction (no longer an equivalence of categories), will be noted abusively:

$$C_*^N : \Delta^{\mathrm{op}}\mathcal{A}b \rightleftarrows \mathcal{C}_-\mathcal{A}b : K$$

The free abelian group functor, which is left adjoint to the faithful embedding $\mathcal{A}b \hookrightarrow \mathrm{Set}$, induces an adjunction on simplicial objects:

$$\mathbb{Z}[.] : \Delta^{\mathrm{op}}\mathrm{Set} \rightleftarrows \Delta^{\mathrm{op}}\mathcal{A}b.$$

The functor $\mathbb{Z}[.]$ preserves weak equivalences [GJ, III.2.16] and the above adjunction factorizes through the adjunction

$$(.)_+ : \Delta^{\mathrm{op}}\mathrm{Set}_\bullet \rightleftarrows \Delta^{\mathrm{op}}\mathrm{Set}$$

where $\Delta^{\mathrm{op}}\mathrm{Set}_\bullet \to \Delta^{\mathrm{op}}\mathrm{Set}$ is the forgetful functor from the category of pointed simplicial sets and the adjunction

$$\mathbb{Z}(.) : \Delta^{\mathrm{op}}\mathrm{Set}_\bullet \rightleftarrows \Delta^{\mathrm{op}}\mathcal{A}b$$

is given by the functor $\mathbb{Z}(X) := \mathbb{Z}[X]/\mathbb{Z}[*]$, together with the faithful embedding $\mathcal{A}b \hookrightarrow \mathrm{Set}_\bullet$, taking zero as the base point.

Composing this sequence of adjunctions gives the adjunction:

$$C_*^N\mathbb{Z}(.) : \Delta^{\mathrm{op}}\mathrm{Set}_\bullet \rightleftarrows \mathcal{C}_-\mathcal{A}b : K$$

This adjunction passes to the associated homotopy categories:

$$\mathcal{H}(\Delta^{\mathrm{op}}\mathrm{Set}_\bullet) \rightleftarrows \mathcal{D}_-\mathcal{A}b.$$

## 2.2 The adjunction for abelian sheaves

Suppose that $T$ is a site with enough points and write $\mathrm{Shv}(T)_\bullet$ for the category of sheaves of pointed sets and $\mathcal{A}b(T)$ for the category of sheaves of abelian groups. The faithful embedding $\mathcal{A}b(T) \hookrightarrow \mathrm{Shv}(T)_\bullet$ has left adjoint $\mathbb{Z}(.)$ which sends a sheaf $F$ to the sheaf associated to the presheaf $X \mapsto \mathbb{Z}(F(X))$. This extends to an adjunction:

$$C_*^N\mathbb{Z}(.) : \Delta^{\mathrm{op}}\mathrm{Shv}(T)_\bullet \rightleftarrows \mathcal{C}_-\mathcal{A}b(T) : K.$$

**Lemma 2.2.1** *Suppose that $T$ is a site with enough points.*

1. *The functor $C_*^N\mathbb{Z}(.)$ sends simplicial weak equivalences to quasi-isomorphisms.*
2. *The functor $K$ sends quasi-isomorphisms to simplicial weak equivalences.*

**Proof:** The functors $C_*^N\mathbb{Z}(.)$ and $K$ commute with the passage to points; thus the proposition reduces to the case where $T$ is a point, considered in Section 2.1. ■

The following result is a generalization of Brown's adjoint functor theorem [B].

**Proposition 2.2.2** *Suppose that the site $T$ has enough points and consider $D_* \in \mathcal{C}_-\mathcal{A}b(T)$, $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}(T)_\bullet$, $p \in \mathbb{Z}$. There is a natural isomorphism:*

$$\mathrm{Hom}_{\mathcal{H}(\Delta^{\mathrm{op}}\mathrm{Shv}(T)_\bullet)}(\mathcal{X}, K(D_*[p])) \cong [C_*^N\mathbb{Z}(\mathcal{X}), D_*[p]]_{\mathcal{D}_-\mathcal{A}b(T)}.$$

If the topology $T$ is subcanonical and $\mathcal{X}$ is the sheaf $X_+$ equipped with constant simplicial structure, where $X$ is the sheaf represented by $X \in T$, then $C_*^N\mathbb{Z}(X_+)$ identifies with the free abelian sheaf $\mathbb{Z}[X]$ concentrated in degree zero. The above isomorphism gives:

$$\mathrm{Hom}_{\mathcal{H}(\Delta^{\mathrm{op}}\mathrm{Shv}(T)_\bullet)}(X_+, K(D_*[p])) \cong [\mathbb{Z}[X], D_*[p]]_{\mathcal{D}_-\mathcal{A}b(T)}.$$

The abelian group $[\mathbb{Z}[X], D_*[p]]_{\mathcal{D}_-\mathcal{A}b(T)}$ can be interpreted as a hypercohomology group.

**Example 2.2.3** [MV, 2.1.26] Suppose that $T$ is the site $(\mathcal{S}m/k)_{\mathcal{N}is}$, then the category $\mathcal{H}(\Delta^{\mathrm{op}}\mathrm{Shv}(T)_\bullet)$ is the homotopy category $\mathcal{H}_{s,\bullet}(k)$ of [MV]. The category of abelian sheaves is written $\mathcal{N}_k$ and the abelian group $[\mathbb{Z}[X], D_*[p]]_{\mathcal{D}_-\mathcal{N}_k}$ identifies with the hypercohomology group $\mathbb{H}^p(X, D^{-*})$, which can be calculated on the small Nisnevich site $(X)_{\mathcal{N}is}$. In particular, the above isomorphism reads:

$$\mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(X_+, K(D_*[p])) \cong \mathbb{H}^p(X, D^{-*}).$$

# 3 Transfers

**Definition 3.0.1** The category $\mathcal{S}m\mathcal{C}or/k$ of smooth correspondences is the category with objects $[X]$, $X \in \mathcal{S}m/k$, and morphisms $\mathrm{Hom}_{\mathcal{S}m\mathcal{C}or/k}([X],[Y]) := c(X,Y)$, where $c(X,Y) := \mathbb{Z}[C(X,Y)]$ and $C(X,Y)$ denotes the set of closed integral subschemes (not necessarily smooth) $Z \hookrightarrow X \times Y$ which are finite and surjective over an irreducible component of $X$.

The category $\mathcal{S}m\mathcal{C}or/k$ is additive, in particular the sum $[X] \oplus [Y]$ is given by $[X \amalg Y]$.

**Definition 3.0.2**

1. The category of presheaves with transfers $\mathcal{P}_k^{\mathrm{tr}}$ is the category of additive functors $\mathcal{S}m\mathcal{C}or/k^{\mathrm{op}} \to \mathcal{A}b$.

2. The category of sheaves with transfers $\mathcal{N}_k^{\mathrm{tr}}$ is the full subcategory of $\mathcal{P}_k^{\mathrm{tr}}$ of presheaves with transfers which are abelian sheaves for the Nisnevich topology.

**Notation 3.0.3** Let $\mathcal{S}m/k^{\amalg}$ denote the full subcategory of $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ with objects which are coproducts of representable sheaves.

**Proposition 3.0.4**

1. *The forgetful functor $\mathcal{N}_k^{\mathrm{tr}} \to \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ admits a left adjoint $\mathbb{Z}_{\mathrm{tr}}[.] : \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \mathcal{N}_k^{\mathrm{tr}}$, which identifies on the representable sheaf $X$ with the presheaf $U \mapsto c(U,X)$.*

2. *Suppose that the simplicial sheaf $\mathcal{X}$ is an object in $\Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$, then there is a natural morphism in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$: $\eta_{\mathcal{X}} : \mathbb{Z}[\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}]$.*

**Proof:** 1) The presheaf $U \mapsto c(U,X)$ is a sheaf for the étale topology and in particular is a Nisnevich sheaf; moreover the presheaf is clearly equipped with transfers. The functor $\mathbb{Z}_{\mathrm{tr}}[.]$ is defined as the left Kan extension of the functor $X \mapsto c(.,X)$.

2) There is a natural transformation of functors $\mathcal{S}m/k \to \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, $\mathbb{Z}[-] \to \mathbb{Z}_{\mathrm{tr}}[-]$, which is induced by the diagonal morphism $X \hookrightarrow X \times X$. This extends to give the natural transformation $\eta_{\mathcal{X}}$. ■

**Definition 3.0.5** Let $\mathbb{Z}_{\mathrm{tr}}(.)$ denote the functor $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_{\bullet} \to \mathcal{N}_k^{\mathrm{tr}}$ which is given by: $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) := \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}]/\mathbb{Z}_{\mathrm{tr}}[*]$.

The following result is a key ingredient to understanding Nisnevich sheaves with transfers.

**Lemma 3.0.6** *[SV, Lemma 1.6] Suppose that $U \to X$ is a Nisnevich covering in $\mathcal{S}m/k$, then the complex*

$$\ldots \to \mathbb{Z}_{\mathrm{tr}}[U_X^{\times n+1}] \to \mathbb{Z}_{\mathrm{tr}}[U_X^{\times n}] \to \ldots \to \mathbb{Z}_{\mathrm{tr}}[U] \to \mathbb{Z}_{\mathrm{tr}}[X] \to 0$$

*induced by alternating sums of projections is exact in $\mathcal{N}_k$.*

This lemma implies the following basic result:

**Proposition 3.0.7** *[V, Lemma 3.1.6]*

1. *The category $\mathcal{N}_k^{\mathrm{tr}}$ has a unique abelian category structure such that the forgetful functor $\Phi : \mathcal{N}_k^{\mathrm{tr}} \to \mathcal{N}_k$ is exact.*
2. *The associated Nisnevich sheaf functor $(.)_{\mathcal{N}is} : \mathcal{P}_k \to \mathcal{N}_k$ induces an exact functor $(.)_{\mathcal{N}is} : \mathcal{P}_k^{\mathrm{tr}} \to \mathcal{N}_k^{\mathrm{tr}}$ which is left adjoint to the full embedding $\mathcal{N}_k^{\mathrm{tr}} \to \mathcal{P}_k^{\mathrm{tr}}$.*

These functors fit into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{P}_k^{\mathrm{tr}} \arrow[r, shift left, "(.)_{\mathcal{N}is}"] \arrow[d, "\Phi"'] & \mathcal{N}_k^{\mathrm{tr}} \arrow[l, shift left] \arrow[d, "\Phi"] \\
\mathcal{P}_k \arrow[r, shift left, "(.)_{\mathcal{N}is}"] & \mathcal{N}_k \arrow[l, shift left]
\end{tikzcd}$$

in which the forgetful functors $\Phi$ are exact and the horizontal arrows from right to left are full embeddings.

**Corollary 3.0.8** *The exact functor $\Phi : \mathcal{N}_k^{\mathrm{tr}} \to \mathcal{N}_k$ induces a functor $\mathcal{D}_-\Phi : \mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}} \to \mathcal{D}_-\mathcal{N}_k$.*

**Definition 3.0.9** Suppose that the simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathcal{S}m/k^{\mathrm{II}}$ and that $C_* \in \mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$, then let $\beta_{\mathcal{X},C_*} : [C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}} \to [C_*^N\mathbb{Z}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k}$ denote the natural morphism given by the composite:

$$[C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}} \stackrel{\mathcal{D}_-\Phi}{\to} [C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k} \stackrel{\eta_{\mathcal{X}}^*}{\to} [C_*^N\mathbb{Z}[\mathcal{X}], C_*]_{\mathcal{D}_-\mathcal{N}_k}.$$

The key result which allows the functor $\mathbb{Z}_{\mathrm{tr}}[.]$ to be derived is the following:

**Proposition 3.0.10** *[SV, Corollary 1.7.1] Suppose that $U \in \mathcal{S}m/k$ and that $C_* \in \mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$, then for all $p \in \mathbb{Z}$, the morphism $\beta_{U,C_*[p]}$ induces an isomorphism:*

$$[\mathbb{Z}_{\mathrm{tr}}[U], C_*[p]]_{\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}} \stackrel{\cong}{\to} [\mathbb{Z}[U], C_*[p]]_{\mathcal{D}_-\mathcal{N}_k}.$$

*In particular, the hypercohomology groups $U \mapsto \mathbb{H}^p(U, C^{-*})$ are presheaves with transfers.*

This result follows from Lemma 3.0.11, which is a consequence of Lemma 3.0.6, by a Cartan-Leray spectral sequence argument. Proposition 3.0.12 is required to ensure convergence of certain hypercohomology spectral sequences (see [SV, §0]).

**Lemma 3.0.11** *[SV, Corollary 1.7] Suppose that $I_{\mathrm{tr}} \in \mathcal{N}_k^{\mathrm{tr}}$ is injective and that $U \in \mathcal{S}m/k$, then for all $i > 0$, $H^i_{\mathrm{Nis}}(U, I_{\mathrm{tr}}) = 0$.*

**Proposition 3.0.12** *Suppose that $X$ is a Noetherian scheme of dimension $d$, then Nisnevich sheaf cohomology has finite cohomological dimension: $H^i_{\mathrm{Nis}}(X, F) = 0$ for $i > d$.*

**Remark 3.0.13** Suppose that $F \in \mathcal{N}_k^{\mathrm{tr}}$ is a Nisnevich sheaf with transfers and that $U \in \mathcal{S}m/k$; the categories $\mathcal{N}_k, \mathcal{N}_k^{\mathrm{tr}}$ have enough injectives so that there exist injective resolutions $F \to I^\bullet$ and $F \to I^\bullet_{\mathrm{tr}}$ in $\mathcal{N}_k$ and $\mathcal{N}_k^{\mathrm{tr}}$ respectively. Moreover, there is an induced morphism of complexes $\iota : I^\bullet_{\mathrm{tr}} \to I^\bullet$ in $\mathcal{C}_-\mathcal{N}_k$.

The isomorphism of Proposition 3.0.10 is induced in cohomology by the composite:

$$\mathrm{Hom}_{\mathcal{N}_k^{\mathrm{tr}}}(\mathbb{Z}_{\mathrm{tr}}[U], I^\bullet_{\mathrm{tr}}) \to \mathrm{Hom}_{\mathcal{N}_k}(\mathbb{Z}_{\mathrm{tr}}[U], I^\bullet_{\mathrm{tr}}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathcal{N}_k}(\mathbb{Z}[U], I^\bullet_{\mathrm{tr}}) \xrightarrow{\iota_*} \mathrm{Hom}_{\mathcal{N}_k}(\mathbb{Z}[U], I^\bullet),$$

where $\beta^*$ is induced by the morphism $\mathbb{Z}[U] \to \mathbb{Z}_{\mathrm{tr}}[U]$. The composite of the first two morphisms is an isomorphism by an adjunction argument, whereas Lemma 3.0.11 implies that Nisnevich cohomology with coefficients in a Nisnevich sheaf with transfers can be calculated via resolutions by injectives in $\mathcal{N}_k^{\mathrm{tr}}$, hence the last morphism induces an isomorphism in cohomology.

# 4 Split simplicial sheaves

Split simplicial sheaves play a technical rôle in the proof of the main result of this section, Proposition 4.0.8, which is a generalization of Proposition 3.0.10.

**Definition 4.0.1**

1. A simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ is split if, for all $n \geq 0$, there exists a set $\{U_i | i \in \mathcal{I}_n\}$ of sheaves represented by smooth schemes $U_i$ such that:
$$\mathcal{X}_n \cong (\mathcal{X}_n)^{\mathrm{deg}} \amalg (\amalg_{i \in \mathcal{I}_n} U_i),$$
where $(\mathcal{X}_n)^{\mathrm{deg}}$ denotes the degenerate part of the simplicial structure.

2. A split simplicial sheaf is finite if the set $\bigcup_n \mathcal{I}_n$ is finite.

**Remark 4.0.2** Suppose that $\mathcal{X}$ is a split simplicial sheaf, with $U_i \in \mathcal{S}m/k$ as in the definition, then for each $n$ there is a cocartesian diagram in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$:

$$\begin{tikzcd}
\amalg_{i \in \mathcal{I}_n} U_i \times \partial\Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(\mathcal{X}) \arrow[d] \\
\amalg_{i \in \mathcal{I}_n} U_i \times \Delta^n \arrow[r] & \mathrm{sk}_n(\mathcal{X}),
\end{tikzcd}$$

where $\mathrm{sk}_j(\mathcal{X})$ denotes the $j$-skeleton of the simplicial sheaf $\mathcal{X}$ (see [GJ, Section VII.1] for basic notions concerning the skeleton and coskeleton of a simplicial object). This presentation makes it intuitively clear how split simplicial sheaves are constructed by attaching 'cells' which are labelled by smooth schemes. Moreover the 'attaching' morphisms $U_i \times \partial\Delta^n \to \mathrm{sk}_{n-1}\mathcal{X}$ are adjoint to morphisms $U_i \to (\mathrm{cosk}_{n-1}(\mathcal{X}))_n$; this establishes the connection with other treatments of split simplicial objects.

Finite split simplicial sheaves are compact objects in the category $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, as a consequence of the lemma below.

**Lemma 4.0.3**

1. *The set of representable sheaves corresponding to a small skeleton of $\mathcal{S}m/k$ forms a set of generators of the category $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$.*
2. *The sheaf represented by $X \in \mathcal{S}m/k$ is a compact object in the category $\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$.*

**Definition 4.0.4** A morphism $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ is said to be a local fibration if $x^*\mathcal{X} \to x^*\mathcal{Y}$ is a Kan fibration for any point $x^*$ of the Nisnevich topology.

Resolutions can be formed on the left by split simplicial sheaves, using the following variant of [MV, Lemma 1.16]:

**Proposition 4.0.5** *There exists a functor $\Psi : \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ and a natural transformation $\Psi \to 1$ such that, for all $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$,*

1. *$\Psi\mathcal{X}$ is a split simplicial sheaf.*
2. *The morphism $\Psi\mathcal{X} \to \mathcal{X}$ is a simplicial weak equivalence and a local fibration.*

Direct sums in the derived category of an abelian category are treated by the following standard result:

**Lemma 4.0.6**

1. *Suppose that $\mathcal{A}$ is an abelian category in which arbitrary direct sums exists and are exact, then arbitrary direct sums exist in $\mathcal{D}\mathcal{A}$ and are represented by the direct sums of the underlying complexes.*
2. *Suppose further that objects $C_* \in \mathcal{D}_-\mathcal{A}$ are represented by complexes in $\mathcal{C}_{\geq 0}\mathcal{A}$, then the direct sum $\bigoplus C_*$ represents the direct sum in $\mathcal{D}_-\mathcal{A}$.*

The following is clear:

**Lemma 4.0.7** *Suppose that $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ is a split simplicial sheaf, then*

1. $C_n^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \cong \bigoplus_{i \in \mathcal{I}_n} \mathbb{Z}_{\mathrm{tr}}[U_i]$

2. $C_n^N \mathbb{Z}[\mathcal{X}] \cong \bigoplus_{i \in \mathcal{I}_n} \mathbb{Z}[U_i]$

The proof of the following result is analogous to the proof in algebraic topology that a natural transformation between cohomology theories is an isomorphism if it induces an isomorphism on the cohomology of a point.

**Proposition 4.0.8** *Suppose that $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ is a split simplicial sheaf and that $C_* \in \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$, then the morphism $\beta_{\mathcal{X},C_*}$ induces an isomorphism:*

$$[C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} \xrightarrow{\cong} [C_*^N \mathbb{Z}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k}.$$

**Proof:** The split simplicial sheaf is filtered by its skeletal filtration $\mathrm{sk}_n\mathcal{X} \subset \mathrm{sk}_{n+1}\mathcal{X} \subset \ldots \mathcal{X}$. There are induced inclusions $C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathrm{sk}_{n-1}\mathcal{X}] \hookrightarrow C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathrm{sk}_n\mathcal{X}]$ so that $C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}]$ is the colimit of this direct system in the category $\mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$. A corresponding statement holds in $\mathcal{C}\_\mathcal{N}_k$ for $C_*^N\mathbb{Z}[\mathcal{X}]$.

Since $\mathcal{X}$ is split, Lemma 4.0.7 implies that there are induced short exact sequences of complexes:

$$\begin{array}{ccccc} C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathrm{sk}_{n-1}\mathcal{X}] & \to & C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathrm{sk}_n\mathcal{X}] & \to & \bigoplus_{i\in\mathcal{I}_n}\mathbb{Z}_{\mathrm{tr}}[U_i][n] \\ C_*^N\mathbb{Z}[\mathrm{sk}_{n-1}\mathcal{X}] & \to & C_*^N\mathbb{Z}[\mathrm{sk}_n\mathcal{X}] & \to & \bigoplus_{i\in\mathcal{I}_n}\mathbb{Z}[U_i][n] \end{array}$$

in $\mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ and $\mathcal{C}\_\mathcal{N}_k$ respectively. The functors $[-, C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}$ and $[-, C_*]_{\mathcal{D}\_\mathcal{N}_k}$, applied to these short exact sequences respectively, induce long exact sequences of abelian groups and the morphism $\beta$ of Definition 3.0.9 induces a morphism between the two exact sequences. Hence, an induction upon $n$ establishes the Proposition for $\mathrm{sk}_n(\mathcal{X})$ using Proposition 3.0.10 and Lemma 4.0.6, via a five-lemma argument.

The proof is completed by using the Milnor exact sequences (obtained from Lemma 4.0.6) associated to the skeletal filtration of $C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}]$ and $C_*^N\mathbb{Z}[\mathcal{X}]$ in $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$ and $\mathcal{D}\_\mathcal{N}_k$ respectively. ■

# 5 The derived functor of $\mathbb{Z}_{\mathrm{tr}}$

It is technically more straightforward to use resolutions by split simplicial sheaves in the proof of Proposition 2 of the Introduction, which is proved as Theorem 5.1.5 below. However, for applications, it is necessary to know that resolutions by objects in $\Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$ are sufficient to calculate the derived functor; this is established in Proposition 5.3.1.

## 5.1 Deriving using split simplicial sheaves

**Proposition 5.1.1** *Suppose that $\mathcal{X}, \mathcal{Y} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ are split simplicial sheaves.*

1. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a morphism such that $\mathbb{Z}[\mathcal{X}] \to \mathbb{Z}[\mathcal{Y}]$ is a simplicial weak equivalence, then $\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ is a simplicial weak equivalence.*

2. *If $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$, then there is a natural isomorphism (on the full subcategory of split simplicial sheaves)*

$$[\mathcal{X}, KC_*]_{\mathcal{H}_s(k)} \cong [C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}.$$

The following elementary lemma is included to highlight the reduction step in the proof of Proposition 5.1.1.

**Lemma 5.1.2** *Suppose that $\Phi : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories and write $\Phi : \mathcal{D}\_\mathcal{A} \to \mathcal{D}\_\mathcal{B}$ for the induced functor. Suppose that $X \to Y$ is an isomorphism in $\mathcal{D}\_\mathcal{A}$, then $\Phi X \to \Phi Y$ is an isomorphism in $\mathcal{D}\_\mathcal{B}$.*

**Proof of Proposition 5.1.1:** (1) It is sufficient to show that the morphism $\tilde{f} : C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \to C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ is an isomorphism in $\mathcal{D}\_\mathcal{N}_k$, by the Dold-Kan theorem. The morphism $\tilde{f}$ is the image of a morphism in $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$ under the exact forgetful functor $\Phi : \mathcal{N}_k^{\mathrm{tr}} \to \mathcal{N}_k$; hence it is sufficient to establish that $\tilde{f}$ is an isomorphism in $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$, by Lemma 5.1.2.

Write $M$ for the cone of $\tilde{f}$ in $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$; it is necessary to show that $M = 0$. It is therefore sufficient to show that $[M, M]^*_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} = 0$, so consider the cohomological functor $[., M]^*_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}$. The cohomology long exact sequence yields the first row of the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & {[M, M]^*_{\mathcal{D}_\mathcal{N}_k^{\mathrm{tr}}}} \arrow[r] & {[C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}], M]^*_{\mathcal{D}_\mathcal{N}_k^{\mathrm{tr}}}} \arrow[r] \arrow[d] & {[C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], M]^*_{\mathcal{D}_\mathcal{N}_k^{\mathrm{tr}}}} \arrow[r] \arrow[d] & {} \\
 & & {[C_*^N \mathbb{Z}[\mathcal{Y}], M]^*_{\mathcal{D}_\mathcal{N}_k}} \arrow[r, "\hat{f}"] & {[C_*^N \mathbb{Z}[\mathcal{X}], M]^*_{\mathcal{D}_\mathcal{N}_k}}. &
\end{tikzcd}$$

The vertical morphisms are isomorphisms by Proposition 4.0.8, whereas the morphism labelled $\hat{f}$ is an isomorphism since $\mathbb{Z}[\mathcal{X}] \to \mathbb{Z}[\mathcal{Y}]$ is a simplicial weak equivalence, by hypothesis. It follows that $[M, M]^*_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} = 0$, which proves the first statement.

(2) Suppose that $C_* \in \mathcal{C}_{\geq 0}\mathcal{N}_k^{\mathrm{tr}}$, then Proposition 2.2.2 implies that there is a natural isomorphism $\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)}(\mathcal{X}, KC_*) \cong [C_*^N \mathbb{Z}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k}$. Since $\mathcal{X}$ is a split simplicial sheaf by hypothesis, there is an isomorphism: $[C_*^N \mathbb{Z}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k} \cong [C_*^N \mathbb{Z}_{\mathrm{tr}}[\mathcal{X}], C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}$ by Proposition 4.0.8; the composite is the required isomorphism. The statement concerning naturality follows from the first part of the Proposition.

**Corollary 5.1.3** *Suppose that $\mathcal{X}, \mathcal{Y} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ are split simplicial sheaves and that $f : \mathcal{X} \to \mathcal{Y}$ is a simplicial weak equivalence, then $\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ is a simplicial weak equivalence.*

**Proof:** The morphism $f$ satisfies the hypothesis of the first part of Proposition 5.1.1, by Lemma 2.2.1. ■

**Corollary 5.1.4** *Suppose that $\Psi$ is the resolution functor of Proposition 4.0.5, then the composite functor $C_*^N\mathbb{Z}_{\mathrm{tr}}\Psi$ sends simplicial weak equivalences to quasi-isomorphisms.*

The above arguments extend to the category of pointed simplicial sheaves and the correponding adjunctions. The following is a formal consequence of the previous results.

**Theorem 5.1.5** *The functors $C_*^N\mathbb{Z}_{\mathrm{tr}}\Psi$ and $K$ induce an adjunction of homotopy categories:*

$$\mathbb{L}\mathbb{Z}_{\mathrm{tr}} : \mathcal{H}_{s,\bullet}(k) \rightleftarrows \mathcal{D}_{-}\mathcal{N}_k^{\mathrm{tr}} : K.$$

## 5.2 Freely adjoining degeneracies

The ordinal number category $\Delta$ has the structure of a Reedy category [Ho, Definition 5.2.1]; in particular, there are faithful sub-categories:

$$\Delta_{\mathrm{mono}} \hookrightarrow \Delta \hookleftarrow \Delta_{\mathrm{surj}}$$

where the morphisms in $\Delta_{\mathrm{mono}}$ are the monomorphisms in $\Delta$ and the morphisms in $\Delta_{\mathrm{surj}}$ are the surjections in $\Delta$. The fact that $\Delta$ is a Reedy category simply corresponds to the fact that any morphism in $\Delta$ factors uniquely as a surjection followed by a monomorphism. The inclusion functor $\Delta_{\mathrm{mono}} \hookrightarrow \Delta$ will be denoted by $\alpha^{-1}$, by analogy with presheaf theory.

This structure makes the definition of the left Kan extension of a functor $\Delta_{\mathrm{mono}}^{\mathrm{op}} \to \mathcal{C}$ very explicit [W, Definition 8.1.9]; such considerations arise in the proof of the Dold-Kan theorem [W, Exercice 8.4.3]. In the current context, this has the following consequence:

**Proposition 5.2.1** *The restriction functor*

$$\alpha_* : \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \Delta_{\mathrm{mono}}^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$$

*admits a left adjoint $\alpha^* : \Delta_{\mathrm{mono}}^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$. Suppose that $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, then :*

1. *The adjunction morphism: $\alpha^*\alpha_*\mathcal{X} \to \mathcal{X}$ induces natural weak equivalences in $\mathcal{C}_{-}\mathcal{N}_k$:*

$$\begin{aligned} C_*^N\mathbb{Z}[\alpha^*\alpha_*\mathcal{X}] &\to C_*^N\mathbb{Z}[\mathcal{X}] \\ C_*^N\mathbb{Z}_{\mathrm{tr}}[\alpha^*\alpha_*\mathcal{X}] &\to C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \end{aligned}$$

2. *Suppose further that $\mathcal{X} \in \Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$, then $\alpha^*\alpha_*\mathcal{X}$ is a split simplicial sheaf.*

**Proof:** Suppose that $\mathcal{Y} \in \Delta^{\mathrm{op}}_{\mathrm{surj}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, then the simplicial sheaf $\alpha^*\mathcal{Y}$ is defined by left Kan extension. Explicitly:

$$(\alpha^*\mathcal{Y})_n = \amalg_{\beta:\mathbf{n}\to\mathbf{k}\in\Delta_{\mathrm{surj}}}(\mathcal{Y}_k)$$

where the (finite) coproduct is taken in the category of sheaves. The simplicial structure is obtained as in [W, Definition 8.1.9]. (Indeed, the most straightforward way to define $\alpha^*$ is to form the left adjoint in the category of simplicial presheaves, which is performed sectionwise; the functor $\alpha^*$ is the degreewise sheafification of this functor.)

Suppose that $\mathcal{Z} : \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \mathcal{A}$ is a functor to an abelian category which commutes with finite coproducts, then

$$\mathcal{Z}(\alpha^*\mathcal{Y})_n \cong \bigoplus_{\beta:\mathbf{n}\to\mathbf{k}\in\Delta_{\mathrm{surj}}} \mathcal{Z}(\mathcal{Y}_k).$$

One checks readily that there is an isomorphism:

$$C^N_*\mathcal{Z}(\alpha^*\mathcal{Y}) \cong C_*\mathcal{Z}(\mathcal{Y}),$$

where $C_*\mathcal{Z}(\mathcal{Y})$ is the un-normalized chain complex in $\mathcal{C}_{\geq 0}\mathcal{A}$ associated to $\mathcal{Z}(\mathcal{Y})$. In particular, if $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, then the adjunction morphism induces a morphism:

$$C_*\mathcal{Z}(\mathcal{X}) \cong C^N_*\mathcal{Z}(\alpha^*\alpha_*\mathcal{X}) \to C^N_*\mathcal{Z}(\mathcal{X}),$$

which identifies with the canonical morphism: $C_*\mathcal{Z}(\mathcal{X}) \to C^N_*\mathcal{Z}(\mathcal{X})$. The latter is a weak equivalence in $\mathcal{C}_-\mathcal{A}$.

In particular, this applies to the functors $\mathbb{Z}[.] : \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \mathcal{N}_k$ and $\mathbb{Z}_{\mathrm{tr}}[.] : \mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k) \to \mathcal{N}^{\mathrm{tr}}_k$ respectively, since both functors are defined as left adjoints.

The final statement concerning the case $\mathcal{X} \in \Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$ is clear. ■

**Lemma 5.2.2** *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a morphism in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$, then:*

1. *The morphism $\mathbb{Z}[\mathcal{X}] \to \mathbb{Z}[\mathcal{Y}]$ is a simplicial weak equivalence if and only if $\mathbb{Z}[\alpha^*\alpha_*\mathcal{X}] \to \mathbb{Z}[\alpha^*\alpha_*\mathcal{Y}]$ is a simplicial weak equivalence.*

2. *The morphism $\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ is a simplicial weak equivalence if and only if $\mathbb{Z}_{\mathrm{tr}}[\alpha^*\alpha_*\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\alpha^*\alpha_*\mathcal{Y}]$ is a simplicial weak equivalence.*

**Proof:** (The proofs are formally similar; only the proof of the first statement is given). The counit of the adjunction induces a commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}[\alpha^*\alpha_*\mathcal{X}] \arrow[r] \arrow[d] & \mathbb{Z}[\mathcal{X}] \arrow[d] \\
\mathbb{Z}[\alpha^*\alpha_*\mathcal{Y}] \arrow[r] & \mathbb{Z}[\mathcal{Y}].
\end{tikzcd}$$

Proposition 5.2.1 implies that the horizontal morphisms are simplicial weak equivalences, hence the result follows by the two-out-of-three property for weak equivalences. ■

**Theorem 5.2.3** *Suppose that $\mathcal{X}, \mathcal{Y} \in \Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$ and that $f : \mathcal{X} \to \mathcal{Y}$ is a simplicial weak equivalence in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$. Then the induced morphism $\mathbb{Z}_{\mathrm{tr}}[\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ is a simplicial weak equivalence in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$.*

**Proof:** Lemma 5.2.2 implies that it is sufficient to establish that the morphism $\mathbb{Z}_{\mathrm{tr}}[\alpha^*\alpha_*\mathcal{X}] \to \mathbb{Z}_{\mathrm{tr}}[\alpha^*\alpha_*\mathcal{Y}]$ is a simplicial weak equivalence. Morevoer, Lemma 5.2.2 implies that $\mathbb{Z}[\alpha^*\alpha_*\mathcal{X}] \to \mathbb{Z}[\alpha^*\alpha_*\mathcal{Y}]$ is a simplicial weak equivalence, since $\mathbb{Z}[f]$ is a simplicial weak equivalence, by Lemma 2.2.1 and the hypothesis on $f : \mathcal{X} \to \mathcal{Y}$. Hence, the result follows from part (1) of Proposition 5.1.1. ■

## 5.3 Independence of resolutions

The following result states that the derived functor of $\mathbb{Z}_{\mathrm{tr}}$ can be calculated by using resolutions by objects in $\Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$. Recall that $\Psi$ denotes the resolution functor by split simplicial sheaves.

**Proposition 5.3.1** *Suppose that $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$ and that $\mathcal{Y} \to \mathcal{X}$ is a simplicial weak equivalence, where $\mathcal{Y} \in \Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$, then the objects $C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}]$ and $C_*^N\mathbb{Z}_{\mathrm{tr}}[\Psi\mathcal{X}]$ are isomorphic in $\mathcal{D}_{-}\mathcal{N}_k^{\mathrm{tr}}$.*

**Proof:** Form the cartesian diagram in $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)$:

$$\begin{tikzcd} \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \Psi\mathcal{X} \arrow[r] & \mathcal{X} \end{tikzcd}$$

The morphism $\Psi\mathcal{X} \to \mathcal{X}$ is a weak equivalence and a local fibration, hence the pull-back morphism $\mathcal{Y}' \to \mathcal{Y}$ is a weak equivalence and a local fibration. It follows that $\mathcal{Y}' \to \Psi\mathcal{X}$ is a weak equivalence.

Form the resolution $\Psi\mathcal{Y}' \to \mathcal{Y}'$, then there are composite morphisms

$$\begin{tikzcd} \Psi\mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{Y} \\ \Psi\mathcal{X} & \end{tikzcd}$$

which are weak equivalences. Moreover, the simplicial sheaves appearing in this diagram are in $\Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$, hence Theorem 5.2.3 gives a diagram of weak equivalences in $\mathcal{C}_{-}\mathcal{N}_k^{\mathrm{tr}}$:

$$C_*^N\mathbb{Z}_{\mathrm{tr}}[\Psi\mathcal{X}] \longleftarrow C_*^N\mathbb{Z}_{\mathrm{tr}}[\Psi\mathcal{Y}'] \longrightarrow C_*^N\mathbb{Z}_{\mathrm{tr}}[\mathcal{Y}].$$

$\blacksquare$

**Remark 5.3.2** Observe that the above proof does not use the fact that $\Psi\mathcal{X}$ is a resolution by *split* simplicial sheaves; the proof requires only that the morphism $\Psi\mathcal{X} \to \mathcal{X}$ is a weak equivalence and a local fibration and that $\Psi\mathcal{X}$ is an object in $\Delta^{\mathrm{op}}\mathcal{S}m/k^{\amalg}$.

# 6 $\mathbb{A}^1$-localization

## 6.1 $\mathbb{A}^1$-localization of derived categories

To motivate the consideration of the $\mathbb{A}^1$-localization of the derived categories, recall:

**Definition 6.1.1** *[MV]* A simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$ is $\mathbb{A}^1$-local if, for every $\mathcal{Y} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$, the projection morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism:

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, \mathcal{X}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

**Lemma 6.1.2** *[MV, 2.2.8, 2.3.19] The following conditions are equivalent on a simplicial sheaf $\mathcal{X} \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$:*

1. *$\mathcal{X}$ is $\mathbb{A}^1$-local.*
2. *For all $U \in \mathcal{S}m/k$, for all non-negative integers $i$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism:*

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(S^i \wedge U_+, \mathcal{X}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{H}_s(k)}(S^i \wedge (U \times \mathbb{A}^1)_+, \mathcal{X}),$$

   *where $S^i$ denotes a simplicial model for the $i$-sphere and $(-)_+$ indicates the addition of a disjoint base point.*
3. *For any fibrant model $\mathcal{X} \xrightarrow{\simeq} \hat{\mathcal{X}}$ and for all $U \in \mathcal{S}m/k$, the morphism $\hat{\mathcal{X}}(U) \to \hat{\mathcal{X}}(U \times \mathbb{A}^1)$ is a simplicial homotopy equivalence.*

Similar definitions apply in the derived categories $\mathcal{D}\_\mathcal{N}_k$ and $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$, using the derived tensor product in the category $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$:

**Proposition 6.1.3** *[SV] There is a derived tensor product $\otimes : \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}} \times \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}} \to \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$ which is induced by defining $\mathbb{Z}_{\mathrm{tr}}[X] \otimes \mathbb{Z}_{\mathrm{tr}}[Y] := \mathbb{Z}_{\mathrm{tr}}[X \times Y]$ for $X, Y \in \mathcal{S}m/k$.*

**Definition 6.1.4**

1. A complex $D_* \in \mathcal{C}\_\mathcal{N}_k$ is $\mathbb{A}^1$-local if, for all $N \in \mathcal{C}\_\mathcal{N}_k$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism

$$[N, D_*]_{\mathcal{D}\_\mathcal{N}_k} \xrightarrow{\cong} [N \otimes \mathbb{Z}[\mathbb{A}^1], D_*]_{\mathcal{D}\_\mathcal{N}_k}.$$

2. A complex $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if, for all $M \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism

$$[M, C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} \xrightarrow{\cong} [M \otimes \mathbb{Z}_{\mathrm{tr}}[\mathbb{A}^1], C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}.$$

The following analogue of Lemma 6.1.2 is obtained by a straightforward dévissage argument:

**Lemma 6.1.5**

1. *A complex $D_* \in \mathcal{C}\_\mathcal{N}_k$ is $\mathbb{A}^1$-local if and only if, for all $U \in \mathcal{S}m/k$ and for all $p \in \mathbb{Z}$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism*

$$[\mathbb{Z}[U], D_*[p]]_{\mathcal{D}\_\mathcal{N}_k} \xrightarrow{\cong} [\mathbb{Z}[U \times \mathbb{A}^1], D_*[p]]_{\mathcal{D}\_\mathcal{N}_k}.$$

2. *A complex $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if and only if, for all $U \in \mathcal{S}m/k$ and for all $p \in \mathbb{Z}$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism*

$$[\mathbb{Z}_{\mathrm{tr}}[U], C_*[p]]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} \xrightarrow{\cong} [\mathbb{Z}_{\mathrm{tr}}[U \times \mathbb{A}^1], C_*[p]]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}.$$

It is convenient to introduce the notion of homotopy invariance of presheaves here:

**Definition 6.1.6** A presheaf $F \in \mathcal{P}_k$ is homotopy invariant (denoted $F \in H\mathcal{P}_k$) if, for all $U \in \mathcal{S}m/k$, the morphism $\mathbb{A}^1 \to \mathrm{Spec}(k)$ induces an isomorphism $F(U) \to F(U \times \mathbb{A}^1)$.

If $F$ is an object of one of the categories $\mathcal{P}_k^{\mathrm{tr}}, \mathcal{N}_k^{\mathrm{tr}}, \mathcal{N}_k$, then $F$ is said to be homotopy invariant if the underlying presheaf is homotopy invariant. The corresponding full subcategories are written respectively $H\mathcal{P}_k^{\mathrm{tr}} \subset \mathcal{P}_k^{\mathrm{tr}}$, $H\mathcal{N}_k^{\mathrm{tr}} \subset \mathcal{N}_k^{\mathrm{tr}}$ and $H\mathcal{N}_k \subset \mathcal{N}_k$.

The following result is clear:

**Proposition 6.1.7**

1. *A complex $D_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if and only if the hypercohomology presheaf $U \mapsto \mathbb{H}^i(U, D^{-*})$ is a homotopy invariant presheaf, for all $i$.*

2. *A complex $D_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if and only if for all $p \in \mathbb{Z}$, the simplicial sheaf $K(D_*[p])$ is $\mathbb{A}^1$-local, regarded as an object of $\Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$.*

3. *A complex $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if and only if its image $\Phi C_* \in \mathcal{C}\_\mathcal{N}_k$ is $\mathbb{A}^1$-local.*

4. *A complex $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$ is $\mathbb{A}^1$-local if and only if the hypercohomology presheaf $U \mapsto \mathbb{H}^i(U, C^{-*})$ is a homotopy invariant presheaf with transfers, for all $i$.*

**Remark 6.1.8** The crucial part of the final statement is the observation that $U \mapsto \mathbb{H}^i(U, C^{-*})$ is a presheaf with transfers.

**Remark 6.1.9** It is possible to construct the $\mathbb{A}^1$-localization of the derived category $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$; this fits into the framework of Bousfield localization of model categories when the derived category $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$ is constructed using a suitable model structure.

## 6.2 Motivic complexes

**Definition 6.2.1** A complex $C_* \in \mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$ is motivic if the homology sheaves $H_i(C_*)$ are homotopy invariant, hence belong to the category $H\mathcal{N}_k^{\mathrm{tr}}$.

The following theorem establishes certain permanence properties of homotopy invariance:

**Theorem 6.2.2** *[V, Theorem 3.1.12] Suppose that $F$ is a homotopy invariant presheaf with transfers, then:*

1. *The associated Nisnevich sheaf $F_{\mathrm{Nis}}$ is homotopy invariant, hence belongs to $H\mathcal{N}_k^{\mathrm{tr}}$. Morevoer there is an identification of presheaves: $F_{\mathrm{Nis}} \cong F_{\mathrm{Zar}}$.*
2. *Suppose that the field $k$ is perfect. Then*
   (a) *$U \mapsto H^i_{\mathrm{Nis}}(U, F_{\mathrm{Nis}})$ is a homotopy invariant presheaf with transfers.*
   (b) *There is an identification $H^i_{\mathrm{Nis}}(U, F_{\mathrm{Nis}}) \cong H^i_{\mathrm{Zar}}(U, F_{\mathrm{Zar}})$*

**Corollary 6.2.3** *[V, Proposition 3.1.13] The category $H\mathcal{N}_k^{\mathrm{tr}}$ is abelian and there exists a commutative diagram of exact functors between abelian categories:*

$$\begin{tikzcd} H\mathcal{P}_k^{\mathrm{tr}} \arrow[r, "(\cdot)_{\mathrm{Nis}}"] \arrow[d] & H\mathcal{N}_k^{\mathrm{tr}} \arrow[d] \\ \mathcal{P}_k^{\mathrm{tr}} \arrow[r, "(\cdot)_{\mathrm{Nis}}"'] & \mathcal{N}_k^{\mathrm{tr}} \end{tikzcd}$$

*in which the vertical morphisms are full embeddings.*

The following result relates the notions of $\mathbb{A}^1$-local and motivic complex.

**Proposition 6.2.4** *Suppose that $C_* \in \mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$;*

1. *If $C_*$ is $\mathbb{A}^1$-local, then $C_*$ is motivic.*
2. *Suppose that the field $k$ is perfect and that $C_*$ is motivic, then $C_*$ is $\mathbb{A}^1$-local.*

**Proof:** (1) Without loss of generality, we may suppose that the complex $C_*$ is concentrated in non-negative degrees. It suffices to show that the homotopy sheaves of $KC_* \in \Delta^{\mathrm{op}}\mathrm{Shv}_{\mathcal{N}is}(\mathcal{S}m/k)_\bullet$ are homotopy invariant. Take a fibrant model $KC_* \overset{\simeq}{\to} \widehat{KC_*}$ for $KC_*$, then a standard adjunction argument shows that, for all $U \in \mathcal{S}m/k$,

$$\pi_i(\widehat{KC_*}(U)) \cong \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(S^i \wedge (U_+), KC_*).$$

Hence, by hypothesis, $\pi_i(\widehat{KC_*}(U))$ is a homotopy invariant presheaf. Moreover, it has transfers by Proposition 5.1.1(2), since $S^i \wedge (U_+)$ is a split simplicial sheaf. Theorem 6.2.2(1) implies that the associated Nisnevich sheaf is a homotopy invariant sheaf with transfers. The isomorphism $(\pi_*(\widehat{KC_*}))_{\mathrm{Nis}} \cong (\pi_*(KC_*))_{\mathrm{Nis}}$ completes the proof.

(2) There is a presheaf of convergent hypercohomology spectral sequences:

$$U \mapsto \{\bigoplus H^i(U, H^j(C^{-*})) \rightsquigarrow \mathbb{H}^{i+j}(U, C^{-*})\}.$$

The Nisnevich sheaves with transfers $H^j(C^{-*})$ are homotopy invariant, by hypothesis, hence (assuming that the field $k$ is perfect) Theorem 6.2.2(2) implies that $U \mapsto \bigoplus H^i(U, H^j(C^{-*}))$ is a homotopy invariant presheaf (with transfers). The category of homotopy invariant presheaves is abelian, so it follows that $U \mapsto \mathbb{H}^*(U, C^{-*})$ is a homotopy invariant presheaf, using the strong convergence of the hypercohomology spectral sequence. ■

## 6.3 The category $DM_-^{\mathrm{eff}}(k)$

**Definition 6.3.1** The category $DM_-^{\mathrm{eff}}(k)$ is the full sub-category of $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$ consisting of the objects represented by motivic complexes.

**Definition 6.3.2** A morhism $A_* \to B_*$ in $\mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$ is an $\mathbb{A}^1$-quasi-isomorphism if, for all $\mathbb{A}^1$-local objects $C_* \in \mathcal{C}_-\mathcal{N}_k^{\mathrm{tr}}$, the induced morphism:

$$[B_*, C_*]_{\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}} \to [A_*, C_*]_{\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}}$$

is an isomorphism.

The following result is best proved by using model category theoretic techniques.

**Proposition 6.3.3** *(Compare [MV, 2.2.5].) The $\mathbb{A}^1$-localization of $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$, obtained as the localization of the category $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$ with respect to the $\mathbb{A}^1$-quasi-isomorphisms is equivalent to the full sub-category of $\mathcal{D}_-\mathcal{N}_k^{\mathrm{tr}}$ generated by the $\mathbb{A}^1$-local complexes.*

When the field $k$ is perfect, one has the following result:

**Theorem 6.3.4** *[V, Proposition 3.2.3] Suppose that $k$ is a perfect field, then the category $DM_-^{\mathrm{eff}}(k)$ identifies with the $\mathbb{A}^1$-localization of $\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}$. In particular, there is a commutative diagram:*

$$\begin{tikzcd} DM_-^{\mathrm{eff}}(k) \arrow[r, hook] \arrow[rr, bend right, "1_{DM_-^{\mathrm{eff}}(k)}"'] & \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}} \arrow[r, "L_{\mathbb{A}^1}"] & DM_-^{\mathrm{eff}}(k) \end{tikzcd}$$

**Remark 6.3.5** For the purposes of this note, it is sufficient to know that there exists an $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1}$. One can show that the $\mathbb{A}^1$-localization functor is the derived functor of the *singular complex functor* [V, Proposition 3.2.3]; moreover, [SV, Lemma 1.4] shows that it is not necesary to derive this functor.

# 7 The proof of Theorem 1

The Morel-Voevodsky $\mathbb{A}^1$-local homotopy category $\mathcal{H}_\bullet(k)$ identifies with the full sub-category $\mathcal{H}_{\mathbb{A}^1,\bullet}(k)$ of $\mathbb{A}^1$-local objects in $\mathcal{H}_{s,\bullet}(k)$, by [MV, Theorem 2.2.5]. Write the $\mathbb{A}^1$-localization adjunction as:

$$l : \mathcal{H}_{s,\bullet}(k) \rightleftarrows \mathcal{H}_{\mathbb{A}^1,\bullet}(k) : r.$$

**Lemma 7.0.1** *Suppose that the field $k$ is perfect. The composite functor*

$$\mathcal{H}_{s,\bullet}(k) \stackrel{\mathbb{LZ}_{\mathrm{tr}}(.)}{\rightarrow} \mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}} \stackrel{L_{\mathbb{A}^1}}{\rightarrow} DM_-^{\mathrm{eff}}(k)$$

*factorizes through the $\mathbb{A}^1$-localization $\mathcal{H}_{s,\bullet}(k) \to \mathcal{H}_{\mathbb{A}^1,\bullet}(k)$.*

**Proof:** It is sufficient to show that an $\mathbb{A}^1$-weak equivalence in $\mathcal{H}_{s,\bullet}(k)$ is sent to an $\mathbb{A}^1$-quasi isomorphism in $\mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$, by Lemma 6.3.3. Hence, it is sufficient to show that, for all $\mathbb{A}^1$-local complexes $C_* \in \mathcal{C}\_\mathcal{N}_k^{\mathrm{tr}}$, the induced morphism

$$[\mathbb{LZ}_{\mathrm{tr}}(\mathcal{Y}), C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}} \to [\mathbb{LZ}_{\mathrm{tr}}(\mathcal{X}), C_*]_{\mathcal{D}\_\mathcal{N}_k^{\mathrm{tr}}}$$

is an isomorphism. By adjunction, this morphism identifies with

$$\mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(\mathcal{Y}, KC_*) \to \mathrm{Hom}_{\mathcal{H}_{s,\bullet}(k)}(\mathcal{X}, KC_*).$$

This is an isomorphism since $\mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$-weak equivalence by hypothesis and $KC_*$ is $\mathbb{A}^1$-local by Proposition 6.1.7(2). ■

Suppose that the field $k$ is perfect, then there is a commutative diagram of functors :

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{A}^1,\bullet}(k) \arrow[r, "r"'] \arrow[rr, bend left, "1_{\mathcal{H}_{\mathbb{A}^1,\bullet}(k)}"] & \mathcal{H}_{s,\bullet}(k) \arrow[r, "l"'] \arrow[d, shift right, "\mathbb{L}\mathbb{Z}_{\mathrm{tr}}(.)"'] & \mathcal{H}_{\mathbb{A}^1,\bullet}(k) \arrow[ddl, dashed, "F"] \\
 & \mathcal{D}_\mathcal{N}_k^{\mathrm{tr}} \arrow[u, shift right, "K"'] \arrow[d, shift right, "L_{\mathbb{A}^1}"'] & \\
 & DM_-^{\mathrm{eff}}(k) \arrow[u, shift right] \arrow[uul, dashed, "G"] &
\end{tikzcd}$$

The factorization $G$ exists since $C_*$ motivic implies that $KC_*$ is $\mathbb{A}^1$-local by Proposition 6.2.4(2), using the hypothesis that the field $k$ is perfect. The factorization $F$ exists by Lemma 7.0.1.

**Theorem 7.0.2** *Suppose that the field $k$ is perfect. The functors $F, G$ induce an adjunction which is written*

$$M(.) : \mathcal{H}_\bullet(k) \rightleftarrows DM_-^{\mathrm{eff}}(k) : K.$$

The Theorem is proved by applying the following elementary result:

**Lemma 7.0.3** *Suppose that there is a commutative diagram of functors :*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "r"'] \arrow[rr, bend left, "1_{\mathcal{C}'}"] & \mathcal{C} \arrow[r, "l"'] \arrow[d, shift right, "L"'] & \mathcal{C}' \arrow[dl, "F"] \\
 & \mathcal{D} \arrow[u, shift right, "R"'] \arrow[ul, "G"] &
\end{tikzcd}$$

*in which $(L, R)$ and $(l, r)$ are adjoint functors then the functor $F$ is left adjoint to $G$.*

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