INTRODUCTION TO ALGEBRAIC TOPOLOGY

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1. General topology

1.1. **Topological spaces.**

Notation 1.1. For X a set, $\mathscr{P}(X)$ denotes the power set of X (the set of subsets of X).

Definition 1.2.
(1) A topological space $(X, \mathscr{U})$ is a set X equipped with a *topology* $\mathscr{U} \subset \mathscr{P}(X)$ such that $\emptyset, X \in \mathscr{U}$ and $\mathscr{U}$ is closed under *finite* intersections and *arbitrary* unions.
(2) A subset $A \subset X$ is *open* for the topology $\mathscr{U}$ if and only if it belongs to $\mathscr{U}$ and is *closed* if the complement $X \backslash A$ is open.
(3) A *neighbourhood* of a point $x \in X$ is a subset $B \subset X$ containing x such that $\exists U \in \mathscr{U}$ such that $x \in U \subset B$.
(4) A subset $\mathscr{B} \subset \mathscr{U}$ is a *basis* for the topology $\mathscr{U}$ if every element of $\mathscr{U}$ can be expressed as the union of elements of $\mathscr{B}$.
(5) A subset $\mathscr{S} \subset \mathscr{U}$ is a *sub-basis* for the topology $\mathscr{U}$ if the set of finite intersections of elements of $\mathscr{S}$ is a basis for $\mathscr{U}$.

If the topology $\mathscr{U}$ is clear from the context, a topological space $(X, \mathscr{U})$ may be denoted simply by X.

Remark 1.3. A given set X can have many different topologies; for example the *coarse topology* on X is $\mathscr{U}_{\mathrm{coarse}} := \{\emptyset, X\}$ and the *discrete topology* is $\mathscr{U}_{\mathrm{discrete}} := \mathscr{P}(X)$. In the coarse topology, the only open sets are $\emptyset$ and X whereas, in the discrete topology, every subset is both open and closed.

More generally, a topology $\mathscr{V}$ on X is finer than $\mathscr{U}$ (or $\mathscr{U}$ is coarser than $\mathscr{V}$) if $\mathscr{U} \subset \mathscr{V}$; this defines a *partial order* on the set of topologies on X. The coarse topology is the minimal element and the discrete topology the maximal element for this partial order.

Recall that a *metric* on a set X is a real-valued function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$:
(1) $d(x, y) \geq 0$ with equality iff $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

A *metric space* is a pair (X, d) with d a metric on X. For $0 < \varepsilon \in \mathbb{R}$ and $x \in X$ the open ball of radius ε centred at x is

$$B_\varepsilon(x) := \{y \in X | d(x, y) < \varepsilon\}.$$

Definition 1.4. For (X, d) a metric space, the *underlying topological space* $(X, \mathscr{U}_d)$ is the topology with basis:

$$\{B_\varepsilon(x) | x \in X,\ 0 < \varepsilon \in \mathbb{R}\}.$$

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Equivalently, a subset $U \subset X$ is open (belongs to $\mathscr{U}_d$) if and only if, $\forall u \in U, \exists \varepsilon > 0$ such that $B_\varepsilon(u) \subset U$.

Example 1.5. Metric spaces give a source of examples of topological spaces; for example, for $n \in \mathbb{N}$, $\mathbb{R}^n$ equipped with the usual Euclidean metric is a metric space; this defines the 'usual' topology on $\mathbb{R}^n$.

In general a subset $A \subset X$ of a topological space $(X, \mathscr{U})$ is neither open nor closed.

Definition 1.6. For $A \subset X$ a subset of a topological space $(X, \mathscr{U})$, the

(1) *interior* $A^\circ \subset A$ is the largest open subset contained in A, so that $A^\circ := \bigcup_{U \subset A, U \in \mathscr{U}} U$;
(2) *closure* $\overline{A} \supset A$ is the smallest closed subset containing A, so that $\overline{A} := \bigcap_{A \subset Z, X \backslash Z \in \mathscr{U}} Z$;
(3) *frontier* $\partial A := \overline{A} \backslash A^\circ$.

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Remark 1.7.

(1) If $A \subset X$ is closed, then the frontier ∂A is the usual notion of *boundary* of A.
(2) The interior (respectively closure) of A can be very different from A; for example, for the coarse topology $(X, \mathscr{U}_{\text{coarse}})$, if A is not open, then $A^\circ = \emptyset$ and $\overline{A} = X$, so that $\partial A = X$.

Definition 1.8. A subset $A \subset X$ is *dense* if $\overline{A} = X$.

The notion of covering of a topological space is fundamental.

Definition 1.9. A *covering* of a topological space X is a family of subsets $\{A_i | i \in \mathscr{I}\}$ such that $\bigcup_{i \in \mathscr{I}} A_i = X$. The covering is *open* (or an *open cover*) if each subset $A_i \subset X$ is open.

A *subcovering* of $\{A_i | i \in \mathscr{I}\}$ is a covering $\{B_j | j \in \mathscr{J}\}$ such that $\mathscr{J} \subset \mathscr{I}$ and, $\forall j \in \mathscr{J}, B_j = A_j$.

Remark 1.10. Intuitively, a topological space X is constructed by gluing together spaces of an open cover.

1.2. **Continuous maps.**

Definition 1.11. For topological spaces $(X, \mathscr{U})$, $(Y, \mathscr{V})$, a map $f : X \to Y$ is *continuous* (or f is a *continuous map*) if $\forall V \in \mathscr{V}$ open in Y, the preimage $f^{-1}(V) \in \mathscr{U}$ is open in X.

Exercise 1.12. For X, Y metric spaces equipped with the underlying topology, show that $f : X \to Y$ is continuous if and only if $\forall x \in X, \forall 0 < \varepsilon \in \mathbb{R}, \exists 0 < \delta \in \mathbb{R}$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. (This is the usual $\varepsilon - \delta$ definition of continuity.)

Proposition 1.13. *For topological spaces $(X, \mathscr{U})$, $(Y, \mathscr{V})$, $(Z, \mathscr{W})$,*

(1) *the identity map Id_X is continuous;*
(2) *the composite $g \circ f$ of continuous maps $f : X \to Y$, $g : Y \to Z$ is a continuous map $g \circ f : X \to Z$;*
(3) *if $(Y, \mathscr{V})$ is the coarse topology, then* every *set map $f : X \to Y$ is continuous;*
(4) *if $(X, \mathscr{U})$ is the discrete topology, then* every *set map $f : X \to Y$ is continuous.*

Proof. Exercise. □

Exercise 1.14. When is the identity map $(X, \mathscr{U}) \to (X, \mathscr{V})$ continuous?

Remark 1.15. Topological spaces and continuous maps form a *category* $\mathfrak{Top}$, with

▷ objects: topological spaces (these form a *class* rather than a *set*);

▷ morphisms: continuous maps, equipped with composition of continuous maps. Explicitly $\mathrm{Hom}_{\mathfrak{Top}}(X, Y)$ is the set of continuous maps from X to Y and composition is a set map

$$\circ : \mathrm{Hom}_{\mathfrak{Top}}(Y, Z) \times \mathrm{Hom}_{\mathfrak{Top}}(X, Y) \to \mathrm{Hom}_{\mathfrak{Top}}(X, Z).$$

These satisfy the *Axioms* of a category: the existence and properties of identity morphisms $\mathrm{Id}_X \in \mathrm{Hom}_{\mathfrak{Top}}(X, X)$ and associativity of composition of morphisms.

Example 1.16. Further examples of categories which are important here are:

(1) the category $\mathfrak{Set}$ of sets and all maps;
(2) the category $\mathfrak{Group}$ of groups and *group homomorphisms*;
(3) the category $\mathfrak{Ab}$ of *abelian* (or commutative) groups and group homomorphisms.

As in any category, there is a natural notion of *isomorphism* of topological spaces:

Definition 1.17. For $(X, \mathscr{U})$, $(Y, \mathscr{V})$ topological spaces,

(1) a continuous map $f : X \to Y$ is a *homeomorphism* if there exists a continuous map $g : Y \to X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. (If g exists, then g is unique, namely the *inverse* of f; moreover g is a homeomorphism.)
(2) Two spaces X, Y are *homeomorphic* if there exists a homeomorphism $f : X \to Y$.

Remark 1.18. Homeomorphic spaces are considered as being *equivalent*. A topological space X usually admits many interesting self-homeomorphisms.

Definition 1.19. A continuous map $f : X \to Y$ is *open* (respectively *closed*) if $f(A)$ is *open* (resp. *closed*) for every *open* (resp. *closed*) subset $A \subset X$.

Remark 1.20. Let $f : X \to Y$ be a continuous map which is a bijection of sets.

(1) In general f is not a homeomorphism. (Give an example.)
(2) The map f is open if and only if it is closed. (Prove this.)
(3) The map f is a homeomorphism if and only if it is open (and closed). (Prove this.)

1.3. **The subspace topology.**

Definition 1.21. For $(X, \mathscr{U})$ a topological space and $A \subset X$, the subspace topology $(A, \mathscr{U}_A)$ is given by $\mathscr{U}_A := \{A \cap U | U \in \mathscr{U}\}$. Thus a subset $V \subset A$ is open if and only if there exists $U \in \mathscr{U}$ such that $A \cap U = V$.

Exercise 1.22. For A, X as above, show that

(1) the inclusion $i : A \hookrightarrow X$ is continuous for the subspace topology $(A, \mathscr{U}_A)$;
(2) the subspace topology is the *coarsest* topology on A for which $i : A \hookrightarrow X$ is continuous;
(3) a map $g : W \to A$ is continuous if and only if the composite $i \circ g : W \to X$ is continuous.

The subspace topology provides many more examples of topological spaces.

Example 1.23.

(1) The usual topology on the interval $I := [0, 1] \subset \mathbb{R}$ is the subspace topology.
(2) The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ can be equipped with the subspace topology (show that this is not homeomorphic to the discrete topology).
(3) The sphere S^n is the subspace $S^n \subset \mathbb{R}^{n+1}$ of points of norm one.

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Remark 1.24. For $f : X \to Y$ a map between topological spaces, the image $f(X) \subset Y$ of f is a subset of Y, which can be equipped with the subspace topology. Then the map f is continuous if and only if the induced map $f : X \twoheadrightarrow f(X)$ is continuous.

1.4. **New spaces from old.**

Definition 1.25. For topological spaces $(X, \mathscr{U})$, $(Y, \mathscr{V})$, the *disjoint union* $X \amalg Y$ is the topological space with underlying set the disjoint union and with basis for the topology given by $\mathscr{U} \amalg \mathscr{V}$ (interpreted via $\mathscr{P}(X), \mathscr{P}(Y) \subset \mathscr{P}(X \amalg Y)$).

This is equipped with continuous inclusions

$$X \overset{i_X}{\hookrightarrow} X \amalg Y \overset{i_Y}{\hookleftarrow} Y.$$

The space $X \amalg Y$ has a universal property (in the terminology of categories, it is a *coproduct*):

Proposition 1.26. *For $f_X : X \to Z$ and $f_Y : Y \to Z$ be continuous maps, there is a unique continuous map $f : X \amalg Y \to Z$ such that $f_X = f \circ i_X$ and $f_Y = f \circ i_Y$.*

Proof. Exercise. □

Definition 1.27. For topological spaces $(X, \mathscr{U})$, $(Y, \mathscr{V})$, the *product* $X \times Y$ is the topological space with underlying set the product $X \times Y$ and with topology defined by the basis $\{U \times V | U \in \mathscr{U}, V \in \mathscr{V}\}$.

The projections

$$X \overset{p_X}{\twoheadleftarrow} X \times Y \overset{p_Y}{\twoheadrightarrow} Y$$

are continuous surjections.

The product space $X \times Y$ also has a universal property (it is a *categorical product*):

Proposition 1.28. *For $g_X : Z \to X$ and $g_Y : Z \to Y$ continuous maps, there is a unique continuous map $g : Z \to X \times Y$ such that $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$.*

Proof. It suffices to show that the set map defined by $g(z) = (g_X(z), g_Y(z))$ is continuous. (Exercise.) □

Exercise 1.29. For X, Y topological spaces, show that the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are open maps.

Proposition 1.30. *For continuous maps $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$, the maps*

$$\begin{aligned} f \times \mathrm{Id}_Y &: X_1 \times Y \to X_2 \times Y \\ \mathrm{Id}_X \times g &: X \times Y_1 \to X \times Y_2 \end{aligned}$$

are continuous.

Proof. Exercise. □

Example 1.31. Let $n \in \mathbb{N}$ be a natural number.

1. (1) The product topology on $\mathbb{R}^n \cong (\mathbb{R})^{\times n}$ is equivalent to the topology associated to the Euclidean metric on $\mathbb{R}^n$. (Prove this.)
2. (2) The n-dimensional solid cube is the product space $I^{\times n}$; this is equivalent to the subspace topology associated to the inclusion $I^{\times n} \subset \mathbb{R}^{\times n} = \mathbb{R}^n$.
3. (3) The *torus* T is defined as a topological space as $T := S^1 \times S^1$.
4. (4) The *cylinder* on a topological space X is, by definition the space $X \times I$. It is equipped with the inclusions $i_0, i_1 : X \rightrightarrows X \times I$ induced by the inclusions of subspaces $\{0\}, \{1\} \subset I$.

Definition 1.32. Let $X \overset{p}{\to} B \overset{q}{\leftarrow} Y$ be continuous maps of topological spaces. The *fibre product* $X \times_B Y$ is the subspace of the product space $X \times Y$ formed by the subspace of elements $(x, y) \in X \times Y$ such that $p(x) = q(y)$ in B.

Remark 1.33. If $*$ is the singleton topological space (which has a unique topology), there are unique continuous maps $X \overset{p}{\to} * \overset{q}{\leftarrow} Y$ and $X \times_* Y \cong X \times Y$ is the product space.

Exercise 1.34. Formulate a *universal property* for the fibre product.

The product of topological spaces allows the introduction of the notion of a *topological group*.

Definition 1.35. A topological group is a group G equipped with a topology such that the structure maps:

$$\begin{aligned} \mu &: G \times G \to G \\ \chi &: G \to G \end{aligned}$$

are continuous maps, where μ is the multiplication $\mu(g, h) = gh$ and χ the inverse $\chi(g) = g^{-1}$.

A homomorphism $\varphi : G \to H$ between topological groups is a group homomorphism which is continuous as a map of topological spaces.

Remark 1.36. If G is a group, then $(G, \mathscr{U}_{\text{discrete}})$ is a topological group.

Example 1.37. The circle S^1 is a subspace of $\mathbb{C}^* := \mathbb{C}\backslash\{0\} \subset \mathbb{C}$. The multiplication of $\mathbb{C}$ provides S^1 with the structure of a topological group.

Definition 1.38. For G a topological group, a (left) G-space is a topological space X equipped with a (left) G-action such that the structure map $\nu : G \times X \to X$ is continuous. (Recall that the axioms of a G-action require that ν is associative (ie $\nu(g, \nu(h, x)) = \nu(\mu(g, h), x)$) and the identity element $e \in G$ acts trivially ($\nu(e, x) = x$).

A morphism of left G-spaces is a continuous map $f : X \to Y$ which is compatible with the respective G-actions.

Example 1.39. The discrete group $\mathbb{Z}/2 = \{1, -1\}$ acts on the sphere $S^n \subset \mathbb{R}^{n+1}$ by the antipodal action $(-1)x = -x$.

Exercise 1.40. For X a G-space and $g \in G$, show that the map $\nu(g, -) : X \to X$ is a homeomorphism.

1.5. **The quotient topology.** Recall that, to give a surjective map of sets $p : X \twoheadrightarrow Y$ is equivalent to defining an equivalence relation R on X together with a bijection $X/\sim_R \cong Y$. Explicitly, the relation associated to p is given by $x \sim y$ if and only if $p(x) = p(y)$. The *fibres* $p^{-1}(y)$ of p are precisely the equivalence classes of R.

Definition 1.41. For $(X, \mathscr{U})$ a topological space and $p : X \twoheadrightarrow Y$ a surjective map of sets, the *quotient topology* $(Y, \mathscr{U}_p)$ on Y is the *finest* topology on Y for which p is a continuous map. Explicitly: $V \subset Y$ is open in Y if and only if $p^{-1}(V) \in \mathscr{U}$.

(Terminology: a continuous surjection $p : X \to Y$ is a *quotient map* if Y has the quotient topology.)

Proposition 1.42. *For X, Z topological spaces and $p : X \twoheadrightarrow Y$ a surjective map of sets, with Y equipped with the quotient topology, a map $g : Y \to Z$ is continuous if and only if the composite $g \circ p : X \to Z$ is continuous.*

Proof. Exercise. □

Exercise 1.43. Let $p : X \twoheadrightarrow Y$ and $q : Y \twoheadrightarrow Z$ be continuous surjections such that p is a quotient map. Show that q is a quotient map if and only if $q \circ p$ is a quotient map.

Example 1.44. Consider the antipodal action of $\mathbb{Z}/2 = \{1, -1\}$ on the sphere S^n. Real projective space of dimension n is the quotient space

$$\mathbb{R}P^n := S^n/\sim$$

where the equivalence relation collapses orbits: $x \sim -x$.

(Observation: S^n is a smooth manifold; the action of $\mathbb{Z}/2$ is free, hence the smooth structure passes to $\mathbb{R}P^n$.)

Remark 1.45. More generally, if X is a left G-space, then the space X/G (the space of G-orbits) is the quotient of X by the relation $x \sim \nu(g, x)\ \forall g \in G, x \in X$.

Example 1.46. The *Möbius band* M is the quotient

$$M := I \times I/\sim$$

where $(s, 0) \sim (1-s, 1)\ \forall s \in I$, which can be embedded in $\mathbb{R}^3$ as a band with a twist.

The projection onto the second factor $p_2 : I \times I \to I$ induces a projection $M \twoheadrightarrow S^1$, which *locally* is a projection from a product.

Example 1.47. The *Klein bottle* K is the quotient

$$K := (S^1 \times I)/\sim$$

where the relation identifies the ends of the cylinder by $(0, x) \sim (1, -x)$, using the antipodal action on S^1.

The projection map $S^1 \times I \twoheadrightarrow I$ induces a projection $K \twoheadrightarrow S^1$, which *locally* is a projection from a product.

Remark 1.48. The projections $M \twoheadrightarrow S^1$ and $K \to S^1$ are examples of *fibre bundles*.

The quotient topology allows the definition of the *cone* and the *suspension* of a space.

Definition 1.49. For X a topological space,

(1) the (unreduced) *cone* on X is the quotient space

$$CX := (X \times I)/\sim$$

where $\sim$ is the equivalence relation $(x, 1) \sim (x', 1)$, $\forall x, x' \in X$, equipped with the continuous map $i : X \hookrightarrow CX$ induced by $i_0 : X \to X \times I$;

(2) the (unreduced) *suspension* of X is the quotient space

$$\widetilde{\Sigma} X := (X \times I)/\sim',$$

where $\sim'$ is the equivalence relation $(x, \varepsilon) \sim' (x', \varepsilon)$, $\forall x, x' \in X$, $\varepsilon \in \{0, 1\}$.

Remark 1.50. By construction, there are continuous maps $X \hookrightarrow CX \twoheadrightarrow \widetilde{\Sigma} X$ and the composite sends X to a point. (More precisely, $X \subset CX$ is the fibre of $CX \twoheadrightarrow \widetilde{\Sigma} X$ over this point.)

Exercise 1.51. For $f : X \to Y$ a continuous map, show that

(1) the continuous map $f \times I : X \times I \to Y \times I$ induces a continuous map $Cf : CX \to CY$ which fits into the commutative diagram

$$\begin{tikzcd} X \times I \arrow[r, "f \times I"] \arrow[d, two heads] & Y \times I \arrow[d, two heads] \\ CX \arrow[r, "Cf"'] & CY; \end{tikzcd}$$

(2) $C(\mathrm{Id}_X) = \mathrm{Id}_{CX}$;

(3) if $g : Y \to Z$ is continuous, then $C(g \circ f) = C(g) \circ C(f)$ as a map $CX \to CZ$.

(These properties correspond to the fact that the cone is a *functor* from $\mathfrak{Top}$ to $\mathfrak{Top}$; this is denoted by $C : \mathfrak{Top} \to \mathfrak{Top}$.)

Establish the analogous properties for $\widetilde{\Sigma} : \mathfrak{Top} \to \mathfrak{Top}$.

The quotient topology provides ways of constructing new topological spaces from old; in particular it is used for *gluing* topological spaces.

Definition 1.52. For continuous maps of topological spaces $X \overset{i}{\leftarrow} A \overset{j}{\rightarrow} Y$, the topological space $X \bigcup_A Y$ is the quotient $X \amalg Y/\sim$ by the relation $i(a) \sim j(a)$ (understood via the inclusions i_X, i_Y).

Remark 1.53. When $A = \emptyset$, $X \cup_\emptyset Y \cong X \amalg Y$.

Exercise 1.54. Formulate a *universal property* of $X \cup_A Y$.

2. Basic properties of topological spaces

2.1. Connectivity.

Definition 2.1. A topological space X is connected if the only subsets of X which are both open and closed are $\emptyset, X$. Equivalently, if $X = U \cup V$, with U, V open and non-empty, then $U \cap V \neq \emptyset$.

Example 2.2. The space $\mathbb{R}$ is connected (prove this!). However, the subspace $\mathbb{Q} \subset \mathbb{R}$ is *not* connected.

Proposition 2.3.

(1) *The continuous image of a connected space is connected.*
(2) *If X and Y are homeomorphic, then X is connected if and only if Y is connected.*

Proposition 2.4. *Let Z be a connected subset of a topological space X; then the closure $\overline{Z}$ is connected.*

Proof. Let A be a closed and open subset of $\overline{Z}$, then $A \cap Z$ is open and closed in Z; since Z is connected, $A \cap Z$ is either Z or $\emptyset$. Since Z is dense in $\overline{Z}$, $A \cap Z \neq \emptyset$, so $A \cap Z = Z$, or equivalently $Z \subset A$; it follows that $\overline{Z} = A$, since A is closed in $\overline{Z}$. □

Theorem 2.5. *A topological space X can be written as a disjoint union*

$$X = \amalg_{i \in \pi(X)} X_i$$

of connected components, where each X_i is a maximal connected subspace of X, in particular is closed in X; $\pi(X)$ is the set of connected components of X.

Example 2.6. For $\mathbb{Q} \subset \mathbb{R}$, equipped with the subspace topology, the connected components are precisely the *points* of $\mathbb{Q}$: the space $\mathbb{Q}$ is *totally disconnected*. Moreover, the set $\pi(X)$ is in bijection with $\mathbb{Q}$ and hence inherits a topology.

This example shows that the connected components of a space are not in general open.

Remark 2.7. The set $\pi(X)$ of connected components of a topological space is a homeomorphism invariant of a space. For example, X is connected if and only if $|\pi(X)| = 1$.

Proposition 2.8. *For connected topological spaces X, Y, the product $X \times Y$ is connected.*

Proof. Exercise. □

2.2. Separation.

The notion of *separation* highlights one of the standard properties of metric spaces.

Definition 2.9. A topological space X is *Hausdorff* (or *separated* or T_2) if, $\forall x \neq y \in X$, $\exists$ open sets $x \in U$, $y \in V$ such that $U \cap V = \emptyset$.

Example 2.10.

(1) The coarse topology on a set X is separated if and only if $|X| \leq 1$.
(2) The set of real numbers $\mathbb{R}$ with the *finite complement topology* (a non-empty subset U is open if and only if $\mathbb{R} \backslash U$ is a finite set) is *not* separated.

Exercise 2.11. For X, Y homeomorphic topological spaces, show that X is Hausdorff if and only if Y is Hausdorff.

Proposition 2.12. *A topological space X is Hausdorff if and only if the diagonal subset $\Delta \subset X \times X$ (of elements of the form (x, x)) is closed.*

Proof. Exercise. □

Proposition 2.13. *For X, Y non-empty topological spaces, the product $X \times Y$ is Hausdorff if and only if X and Y are both Hausdorff.*

Proof. Exercise. □

Exercise 2.14. Show that the subspace $A \subset X$ of a Hausdorff topological space X is Hausdorff.

Passage to a quotient space does *not* in general preserve separation.

Example 2.15. Let $\mathbb{R}^{\ominus}$ denote the quotient of $\mathbb{R} \amalg \mathbb{R}$ which identifies the two subspaces $\mathbb{R}\backslash\{0\}$; thus the underlying set of $\mathbb{R}^{\ominus}$ identifies with $\mathbb{R}\amalg\{0\}$; this is the space of real numbers with two origins $0_1, 0_2$. The space $\mathbb{R}^{\ominus}$ is not Hausdorff, since open neighbourhoods of the distinct points $0_1, 0_2$ always intersect.

2.3. **Compact spaces.**

Definition 2.16. A topological space X is *compact* if every open cover admits a finite subcover

Remark 2.17. In the French literature, this property is called *quasi-compact*; the space is *compact* if it is also separated.

Example 2.18. The interval $I = [0, 1]$ is compact (this is the *Heine-Borel* theorem), whereas $\mathbb{R}$ is *not* compact. Similarly, the open interval $(0, 1) \subset I$ is not compact; this shows that a subspace of a compact space need not be compact.

Proposition 2.19. *For X, Y homeomorphic topological spaces, X is compact if and only if Y is compact.*

Proof. Exercise. □

Proposition 2.20.

(1) *A closed subset of a compact space is compact.*
(2) *The continuous image of a compact space is compact.*

Proof. Exercise. □

In presence of a *separation* hypothesis, the first property has a converse:

Proposition 2.21. *A compact subspace of a Hausdorff topological space is closed.*

Proof. Exercise. □

Proposition 2.22. *Let $p : X \twoheadrightarrow Y$ be a continuous map which is surjective. If X is compact and Y is separated then Y has the quotient topology.*

In particular, if p is a bijection of sets, then p is a homeomorphism.

Proof. It suffices to show that a subset $A \subset Y$ such that $f^{-1}(A)$ is closed, is closed in Y.

Proposition 2.20 implies that the closed subspace $f^{-1}(A)$ is compact in X, since X is compact; moreover, by Proposition 2.20, the image $f(f^{-1}(A)) = A$ is compact. Since Y is Hausdorff, the compact space A is closed by Proposition 2.21, as required. □

Example 2.23.

(1) The surjection $I = [0,1] \twoheadrightarrow S^1 \subset \mathbb{C}$ defined by $t \mapsto e^{2\pi i t}$ is continuous and I is compact and S^1 is separated, hence
$$S^1 \cong I/0 \sim 1.$$
(2) The analogous argument shows that the torus $S^1 \times S^1$ is homeomorphic to the quotient space of $I \times I$ which identifies $(0,s) \sim (1,s)$ and $(t,0) \sim (t,1)$.
(3) Real projective space $\mathbb{R}P^n$ is homeomorphic to the quotient of $\mathbb{R}^{n+1}\backslash\{0\}$ by the group action of $\mathbb{R}\backslash\{0\}$ given by
$$\nu(\lambda, (x_0, \ldots, x_n)) = (\lambda x_0, \ldots, \lambda x_n).$$
The orbit of $(x_0, \ldots, x_n)$ is usually denoted by $[x_0 : \ldots : x_n]$.

2.4. **Locally compact spaces.**

Definition 2.24. A topological space X is *locally compact* if each point has a compact neighbourhood.

Exercise 2.25. Show that
(1) a compact space is locally compact;
(2) a closed subset of a locally compact space is locally compact.

2.5. **Paths.** A point $x \in X$ of a topological space is equivalent to a (continuous) map $* \xrightarrow{x} X$; we consider *deforming points* along *paths*.

Definition 2.26.
(1) A *path* γ in a topological space X is a continuous map $I = [0,1] \xrightarrow{\gamma} X$; this is also referred to as a path *from* $\gamma(0)$ *to* $\gamma(1)$. ✓14/09/13
(2) The *inverse path* $\gamma^{-1} : I \to X$ is the path $\gamma^{-1}(t) = \gamma(1-t)$.
(3) If $\lambda : I \to X$ is a path with $\lambda(0) = \gamma(1)$, the *composite path* $\gamma \cdot \lambda : I \to X$ is given by
$$\gamma \cdot \lambda(t) = \begin{cases} \gamma(2t) & 0 \leq 2t \leq 1 \\ \lambda(2t-1) & 1 \leq 2t \leq 2. \end{cases}$$

Exercise 2.27. Verify that γ^{-1} and $\gamma \cdot \lambda$ are paths.

Proposition 2.28. *For $\gamma : I \to X$ a path in X and $f : X \to Y$ a continuous map, the composite map $f \circ \gamma : I \to Y$ is a path in Y from $f(\gamma(0))$ to $f(\gamma(1))$.*

Proof. Exercise. □

Definition 2.29. Let X be a topological space,
(1) X is *path connected* if, $\forall x, y \in X$, $\exists \gamma$ a path from x to y.
(2) X is *locally path connected* if, $\forall x \in X$ and for every neighbourhood $x \in A \subset X$, there exists an path connected open subspace $x \in V \subset A$. ✓14/09/13

Remark 2.30. A path connected space is *not* necessarily locally path connected. (Give an example.)

Proposition 2.31. *For X a topological space,*
(1) *if X is path connected, then X is connected;*
(2) *if X is locally path connected and connected, then X is path connected.* ✓14/09/13

Proof. Exercise. □

Exercise 2.32. Give an example of a space which is connected but not path connected.

The relation on points of X given by $x \sim y$ if and only if $\exists$ a path from x to y is an equivalence relation (exercise), hence a topological space decomposes as the disjoint union of path-connected components.

Definition 2.33. For X a topological space, $\pi_0(X)$ denotes the *set of path-connected components* of X.

Proposition 2.34. *A continuous map $f : X \to Y$ induces a map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$; the association $f \mapsto \pi_0(f)$ has the following properties:*

(1) $\pi_0(\mathrm{Id}_X) = \mathrm{Id}_{\pi_0(X)}$;

(2) *if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are composable continuous maps, then $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.*

Proof. Exercise. □

Remark 2.35. This is an example of a *functor* from the category $\mathfrak{Top}$ of topological spaces to the category $\mathfrak{Set}$ of sets.

2.6. **Mapping spaces.** The set of paths in a topological space X has a natural topology, which is a particular case of the following:

Definition 2.36. For topological spaces X, Y, let $\mathrm{Map}(X, Y)$ (sometimes written Y^X) denote the set of continuous maps from X to Y equipped with the *compact-open* topology, which is defined by the *sub-basis* of subsets $\langle K, U\rangle$ for $K \subset X$ compact and $U \subset Y$ open, where

$$\langle K, U\rangle := \{f : X \to Y | f(K) \subset U\}.$$

The *path space* of X is the space $\mathrm{Map}(I, X)$ (or X^I).

Proposition 2.37. *Restriction to the endpoints $0, 1 \in I$ induces continuous surjections:*

$$X^I \underset{p_1}{\overset{p_0}{\rightrightarrows}} X.$$

Proof. Exercise. □

3. Homotopy

3.1. **Motivation.** The *category* of topological spaces and continuous maps is very rigid. This is illustrated by considering the paths of a topological space X.

Suppose that λ, μ, ν are three paths $I \to X$ which are *composable* ($\lambda(1) = \mu(0)$ and $\mu(1) = \nu(0)$). Then there there are two *a priori* different composite paths from $\lambda(0)$ to $\nu(1)$

$$(\lambda \cdot \mu) \cdot \nu, \ \lambda \cdot (\mu \cdot \nu) : I \to X;$$

namely, the composition of paths is *not associative*. (Exercise: give a simple example where these paths are different.)

The problem arises from the fact that the composites are defined using two *different* homeomorphisms

$$I_1 \cup_{1_1 \sim 0_2} I_2 \cup_{1_2 \sim 0_3} I_3 \cong I,$$

where I_1, I_2, I_3 are copies of the interval. (Explicitly, the two composites rely on the decompositions $[0,1] = [0, \frac{1}{4}] \cup [\frac{1}{4}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, 1]$, together with the linear homeomorphisms between the sub-intervals and $[0,1]$.)

Similarly, the composite $\lambda \cdot \lambda^{-1}$ is a path from $\lambda(0)$ to $\lambda(0)$ which retraces its steps. One would like to consider this as being 'equivalent' to the constant path $c_{\lambda(0)} : I \to X$, (defined by $c_{\lambda(0)}(t) = \lambda(0)$); however, these are *not equal* as paths, unless λ is itself a constant path.

To get around these problems, one *reparametrizes* paths; this uses the notion of *continuous deformation* or *homotopy*.

3.2. **Homotopy.** The notion of homotopy formalizes the idea of *continuous deformation* corresponding to a *continuous family* of continuous maps f_t, indexed by $t \in \mathbb{R}$.

Definition 3.1. Let $f, g : X \rightrightarrows Y$ be two continuous maps.

(1) A *homotopy* from f to g is a continuous map $H : X \times I \to Y$ which makes the following diagram commute

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & Y. & \end{tikzcd}$$

(2) The maps f, g are *homotopic* if there exists a homotopy from f to g; this will be denoted by $f \sim g$.

Example 3.2. If $X = \{*\}$, then continuous maps $f, g : X \rightrightarrows Y$ correspond to points $f(*)$, $g(*)$ of Y. A homotopy from f to g is a path from $f(*)$ to $g(*)$. In the general case, for each point $x \in X$, the restriction $H(x, -) : I \to Y$ is a path from $f(x)$ to $g(x)$ in Y; the definition of homotopy requires that the set of paths $\{H(x, -) | x \in X\}$ forms a *continuous family*.

Remark 3.3. The homotopy H from f to g is not unique; for example, if $\alpha : I \to I$ is *any* continuous map such that $\alpha|_{\partial I}$ is the identity (ie the endpoints of the interval are fixed), then $H_\alpha := H \circ (\mathrm{Id}_X \times \alpha)$ is a homotopy from f to g. The map α *reparametrizes* the homotopy.

When considering homotopies between paths in X from x_1 to x_2, one wants to consider continuous families of paths from x_1 to x_2. This imposes a restriction on the homotopy in terms of the values on the endpoints $\partial I = \{0, 1\} \subset I$. This corresponds to the general notion of *homotopy relative to a subset* $A \subset X$.

Definition 3.4. For $A \subset X$ and maps $f, g : X \rightrightarrows Y$ such that $f|_A = g|_A : A \rightrightarrows Y$,

(1) a *homotopy relative to* A from f to g is a homotopy $H : X \times I \to Y$ from f to g such that $H(a, t) = f(a) = g(a)\ \forall a \in A, t \in I$;
(2) f and g are *homotopic rel* A if there exists a relative homotopy (rel A) from f to g (this is denoted $f \sim g \text{ rel } A$ or $f \sim_{\text{rel } A} g$).

Example 3.5. For $\gamma_0, \gamma_1 : I \rightrightarrows Y$ two paths in Y such that $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, a homotopy rel ∂I from γ_0 to γ_1 is a continuous family of paths $\{\gamma_t | t \in I\}$ from $\gamma_0(0)$ to $\gamma_0(1)$.

Remark 3.6. The notion of relative homotopy rel $\emptyset$ coincides with the *absolute* version given in Definition 3.1.

3.3. **First properties of homotopy.** The notion of homotopy leads to a *weaker* notion of equivalence between topological spaces than homeomorphism: spaces can be deformed *continuously*. For *surfaces*, this is often referred to as *rubber sheet geometry*.

Notation 3.7. For topological spaces X, Y, $A \subset X$ a subspace and $\psi : A \to Y$ a continuous map, let

$$\mathrm{Hom}_{\mathfrak{Top}}(X, Y)_\psi \subset \mathrm{Hom}_{\mathfrak{Top}}(X, Y)$$

denote the set of continuous maps $f : X \to Y$ such that $f|_A = \psi : A \to Y$.

Proposition 3.8. *In the situation of Notation 3.7, the relation* $\sim_{\text{rel } A}$ *is an equivalence relation on* $\mathrm{Hom}_{\mathfrak{Top}}(X, Y)_\psi$.

Proof.

▷ *reflexivity*: for $f \in \mathrm{Hom}_{\mathfrak{Top}}(X, Y)_\psi$, it suffices to take the homotopy $H(x, t) = f(x)$ (this is a constant family);
▷ *symmetry*: if H is a relative homotopy from f to g, then H' defined by $H'(x, t) := H(x, 1 - t)$ is a relative homotopy from g to f;
▷ *transitivity*: this corresponds to gluing homotopies, which generalizes the composition of paths; if H_1 is a homotopy rel A from f to g and H_2 is a homotopy rel A from g to h, then $H : X \times I \to Y$ defined by

$$H(x, t) := \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a homotopy rel A from f to h.

□

Remark 3.9. It can be useful to represent a homotopy H from f to g by a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y. \end{tikzcd}$$

Then, the transitivity homotopy H corresponds to the *vertical* composition of H_1 and H_2 in the following diagram:

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow H_1"] \arrow[r, phantom, shift right=3, "\Downarrow H_2"] & Y. \end{tikzcd}$$

Use of such diagrams is formalized by the theory of 2-*categories*; this theory will *not* be used here!

Homotopy behaves well with respect to composing maps:

Proposition 3.10. *For continuous maps $\alpha : U \to X$, $f, g : X \rightrightarrows Y$ and $\omega : Y \to Z$, if $f \sim g$ are homotopic then*

$$(\omega \circ f \circ \alpha) \sim (\omega \circ g \circ \alpha).$$

Proof. Let H be a homotopy from f to g; the required homotopy is represented by the following diagram

$$\begin{tikzcd} U \arrow[r, "\alpha"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \arrow[r, "\omega"] & Z. \end{tikzcd}$$

(Exercise: write down this homotopy explicitly.) □

Exercise 3.11.

(1) Suppose that $B \subset U$ such that $\alpha(B) \subset A$; formulate and prove a version of Proposition 3.10 for relative homotopy.
(2) For continuous maps $f, g : X \rightrightarrows Y$ and $\omega, \zeta : Y \rightrightarrows Z$ such that $f \sim g$ and $\omega \sim \zeta$, show that the composites $\omega \circ f,\ \zeta \circ g : X \rightrightarrows Z$ are homotopic.
    More precisely, given homotopies represented by the diagram

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow H"] & Y \arrow[r, "\omega"] \arrow[r, bend right=60, "\zeta"'] \arrow[r, phantom, shift right=3, "\Downarrow K"] & Z, \end{tikzcd}$$

give an explicit homotopy from $\omega \circ f$ to $\zeta \circ g$ by using transitivity from Proposition 3.8 and Proposition 3.10. (The form of the diagram should suggest how to do this.)

Since the homotopy relation $\sim$ is an equivalence relation (by Proposition 3.8), one can pass to homotopy classes.

Notation 3.12. For topological spaces X, Y, write

$$[X, Y] := \mathrm{Hom}_{\mathfrak{Top}}(X, Y)/\sim$$

for the set of *homotopy classes* of continuous maps from X to Y. The homotopy class of a continuous map $f : X \to Y$ will be denoted $[f]$.

Proposition 3.13. *For topological spaces X, Y, Z, the composition of continuous maps induces a composition law:*

$$\begin{aligned} [Y, Z] \times [X, Y] &\longrightarrow [X, Z] \\ [g], [f] &\longmapsto [g \circ f]. \end{aligned}$$

The class $[\mathrm{Id}_X] \in [X, X]$ acts as the identity for this composition and composition is associative.

Proof. Exercise (use Proposition 3.10 and Exercise 3.11). □

Remark 3.14. Proposition 3.13 gives a *category* with objects topological spaces and morphisms homotopy classes of continuous maps. (Exercise: check the axioms of a category - see Section A.1.)

This is *not* the *homotopy category of topological spaces* which is usually studied in algebraic topology. This is given by restricting to a well-behaved class of topological spaces (*CW-complexes*); most spaces arising naturally in geometry can be given the structure of a CW-complex, so this is not a serious restriction.

3.4. **Homotopy equivalence.** The notion of homotopy leads naturally to that of *homotopy equivalence*:

Definition 3.15.

(1) A continuous map $f : X \to Y$ is a *homotopy equivalence* if there exists a *homotopy inverse* $g : Y \to X$ (namely a continuous map such that $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$).
(2) Topological spaces X, Y are *homotopy equivalent* (or *have the same homotopy type*) if there exists a homotopy equivalence $f : X \to Y$; write $X \simeq Y$ in this case.

Remark 3.16.

(1) The simplest topological space is $\emptyset$; however there is *no* map of sets $X \to \emptyset$ unless $X = \emptyset$, in which case the only map is $\mathrm{Id}_\emptyset$ (which is continuous!). Thus the only topological space homotopy equivalent to $\emptyset$ is $\emptyset$ itself.
   (The space $\emptyset$ is in fact the *initial object* of the category of topological spaces, $\mathfrak{Top}$, in the language of category theory. Namely, there is a *unique* continuous map $\emptyset \to X$ to any topological space X.)
(2) The singleton set $*$ has a unique topology (the discrete and coarse topologies coincide) and also plays a special rôle amongst topological spaces: for any topological space, there is a unique (continuous) map $X \to *$. This means that $*$ is the *final object* of $\mathfrak{Top}$.
   A point $x \in X$ corresponds to a continuous map $x : * \to X$ which is the inclusion of the subspace $\{x\}$; the unique map $X \to *$ provides a *retraction* of this inclusion.

It is natural to consider the topological spaces which are homotopically equivalent to $*$.

Definition 3.17.

(1) A topological space X is *contractible* if $X \simeq *$.
(2) A continuous map $f : X \to Y$ is *homotopically trivial* if it is homotopic to a constant map.

Exercise 3.18. Show that a space X is contractible if and only if the identity map Id_X is homotopic to a constant map.

Example 3.19. The following spaces are contractible:

(1) the interval I;
(2) Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$;
(3) the closed ball $e^n \subset \mathbb{R}^n$.

For example, the continuous map $H : I \times I \to I$, $(s,t) \mapsto st$ shows that the identity map Id_I is homotopic to the constant map on I with value $0 \in [0,1]$.

Proposition 3.20. *Let X, Y be topological spaces.*

(1) *If X, Y are homeomorphic then $X \simeq Y$ have the same homotopy type.*
(2) *The relation $\simeq$ is an equivalence relation.*

Proof. Exercise. $\square$

Example 3.21. The spaces $\mathbb{R}^2 \backslash \{0\}$ and S^1 have the same homotopy type. The inclusion $i : S^1 \hookrightarrow \mathbb{R}^2$ admits a *retract* $r : \mathbb{R}^2 \backslash \{0\} \to S^1$ which sends a point $(\rho \cos\theta, \rho \sin\theta) \mapsto (\cos\theta, \sin\theta)$, where $\rho > 0$; the composite $i \circ r : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\}$ is homotopic to the identity map (exercise!).

However, these spaces are *not* homeomorphic. One way of showing this is to observe that, for any point $* \in S^1$, the complement $S^1 \backslash \{*\}$ is homeomorphic to

$(0, 1)$, which is contractible. The space $\mathbb{R}^2\backslash\{0, *\}$ is homotopy equivalent to a figure eight embedded in $\mathbb{R}^2$; we shall be able to show shortly that this space is *not* contractible, hence neither is $\mathbb{R}^2\backslash\{0, *\}$.

Remark 3.22. The example shows that the relation $\simeq$ is *coarser* than the relation $\cong$; one of the aims of algebraic topology is to understand topological spaces *up to homotopy equivalence*.

3.5. **The cone and homotopically trivial maps.** Recall that the cone CX with base a topological space X is the quotient $X \times I/X \times \{1\}$ and that the inclusion $i_0 : X \to X \times I$ induces the inclusion $i : X \to CX$ of the base of the cone, which fits into the commutative diagram

$$\begin{tikzcd} & X \times I \arrow[d] \\ X \arrow[ur, "i_0"] \arrow[r, "i"'] & CX. \end{tikzcd}$$

The *homotopical* importance of the cone CX on a space X is shown by the following result:

Proposition 3.23.

(1) *The cone CX on a topological space X is contractible.*
(2) *A continuous map $f : X \to Y$ is homotopically trivial if and only if it extends to a continuous map $\tilde{f} : CX \to Y$ making the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & CX \arrow[dl, "\tilde{f}"] \\ Y. & \end{tikzcd}$$

Proof. For the first point, define a continuous map $\tilde{H} : (X \times I) \times I \to X \times I$ by $((x, s), t) \mapsto (x, s(1 - t) + t)$. The map $\tilde{H}$ is a homotopy between $\mathrm{Id}_{X\times I}$ and the projection to the top of the cylinder, $(x, s) \mapsto (x, 1)$.

By construction, $\tilde{H}((x, 1), t) = (x, 1)$, hence $\tilde{H}$ induces a continuous map

$$H : CX \times I \to CX$$

(this uses the defining property of the quotient map $X \times I \twoheadrightarrow CX$). Moreover, H is a homotopy between Id_{CX} and the constant map sending CX to the point of the cone, by construction of $\tilde{H}$. This proves that CX is contractible.

For the second point, consider the commutative diagram

$$\begin{tikzcd} & X \times I \arrow[d] \\ X \arrow[ur, "i_0"] \arrow[r, "i"'] \arrow[d, "f"'] & CX \arrow[dl, dotted, "\tilde{f}"] \\ Y. & \end{tikzcd}$$

If $\tilde{f}$ exists, then the composite $X \times I \twoheadrightarrow CX \xrightarrow{\tilde{f}} Y$ defines a homotopy between f and a constant map.

Conversely, let $K : X \times I \to Y$ be a homotopy between the map f and a constant map with value $y \in Y$. The map $\tilde{f} : CX \to Y$ defined by $\tilde{f}([x, t]) = K(x, t)$ is a continuous map (well-defined since $K(x, 1) = y \; \forall x \in X$). $\square$

3.6. **Deformation retracts.** A deformation retract is a special form of homotopy equivalence.

Definition 3.24. Let A be a subspace of X equipped with the inclusion $i : A \hookrightarrow X$.

(1) A is a *retract* of X if there exists a *retraction* $r : X \to A$, namely a continuous map such that $r \circ i = \mathrm{Id}_A$;
(2) A is a *deformation retract* of X if there exists a retraction r such that $i \circ r \sim \mathrm{Id}_X$;
(3) A is a *strong deformation retract* of X if there exists a retraction r such that $i \circ r \sim_{\mathrm{rel}\ A} \mathrm{Id}_X$.

Proposition 3.25. *If $A \subset X$ is a deformation retract with respect to the inclusion i and the retraction r, then i, r are homotopy equivalences, in particular A and X have the same homotopy type.*

Proof. Exercise. □

Example 3.26. For any $n \in \mathbb{N}$, $S^n \hookrightarrow \mathbb{R}^{n+1}\backslash\{0\}$, S^n is a strong deformation retract of $\mathbb{R}^{n+1}\backslash\{0\}$.

Example 3.27. Consider the Möbius band M (recall that this is defined as the quotient of $I \times I$ by the relation $(s, 0) \sim (1 - s, 1)$). There are two natural embeddings of the circle S^1 in M:

(1) The *zero section* (the terminology comes from the theory of vector bundles and is not important for this example) which is induced by the continuous map $I \to I \times I$, $t \mapsto (\frac{1}{2}, t)$.
(2) The *boundary* $\partial M \cong S^1$.

The projection $M \twoheadrightarrow S^1$ induced by $I \times I \to I$, $(s, t) \mapsto t$ is a retract of the zero section, which is a strong deformation retract of M. However, the inclusion $S^1 \cong \partial M \hookrightarrow M$ does not even admit a retract! We will shortly see how to prove this.

Example 3.28. Let X denote the subspace $\{0\} \cup \{\frac{1}{n} | 0 < n \in \mathbb{N}\} \subset \mathbb{R}$. (Note that the topology on X is *not* the discrete topology.) By Proposition 3.23, the space CX is contractible.

Consider $0 \in X \subset CX$; $\{0\} \subset CX$ is a deformation retract of CX but is *not* a strong deformation retract of CX (equivalently, Id_{CX} is not homotopic $\mathrm{rel}\ \{0\}$ to the constant map at $\{0\}$). (Exercise: prove this assertion.) The problem arises from the fact that, for any open neighbourhood of 0 in X, the inclusion $\{0\} \subset U$ is not a homeomorphism and is not even a homotopy equivalence.

Form the space $Y := CX \cup_{\{0\}} CX$ by identifying the two respective points $0 \in CX$. Although each cone is contractible, the space Y is *not*. One cannot simply first collapse one cone and then the other.

Proposition 3.29. *The subspace $\mathrm{Map}(I, I)_{\partial I} \subset \mathrm{Map}(I, I)$ of continuous maps which restrict to the identity on ∂I is contractible and the subspace $\{\mathrm{Id}_I\} \subset \mathrm{Map}(I, I)_{\partial I}$ is a strong deformation retract.*

Proof. Define a homotopy $H : \mathrm{Map}(I, I)_{\partial I} \times I \to \mathrm{Map}(I, I)_{\partial I}$ by

$$H(\varphi, t) = \{s \mapsto st + (1 - t)\varphi(s)\}.$$

Thus $H(\varphi, 0) = \varphi$ and $H(\varphi, 1) = \mathrm{Id}_I$ is a homotopy between the identity map on $\mathrm{Map}(I, I)_{\partial I}$ and the constant map with value Id_I; moreover $H(\mathrm{Id}_I, t) = \mathrm{Id}_I\ \forall t \in I$. This homotopy exhibits $\{\mathrm{Id}_I\}$ as a strong deformation retract of $\mathrm{Map}(I, I)_{\partial I}$. □

Remark 3.30. The space $\mathrm{Map}(I, I)_{\partial I}$ acts as the *space of reparametrizations of homotopies*, via the *evaluation map*:

$$\begin{array}{rcl} \mathrm{eval} : I \times \mathrm{Map}(I, I) & \to & I \\ (s, \varphi) & \mapsto & \varphi(s). \end{array}$$

Namely, if $H : X \times I \to Y$ is a homotopy, then the evaluation map induces the composite:

$$X \times I \times \mathrm{Map}(I, I)_{\partial I} \stackrel{\mathrm{Id}_X \times \mathrm{eval}}{\longrightarrow} X \times I \stackrel{H}{\to} Y.$$

Fixing a reparametrization $\varphi \in \mathrm{Map}(I, I)_{\partial I}$, this gives the homotopy H_φ as in Remark 3.3.

3.7. **Paths again.** With the notion of relative homotopy in hand, we can resolve the problems of Section 3.1:

Notation 3.31. For X a topological space and $x \in X$, let $c_x : I \to X$ denote the *constant path* $t \mapsto x$.

Proposition 3.32. *For X a topological space and composable paths $\lambda, \mu, \nu : I \to X$:*

1. *the composite paths $(\lambda \cdot \mu) \cdot \nu,\ \lambda \cdot (\mu \cdot \nu) : I \to X$ are homotopic* $\mathrm{rel}\ \partial I$*;*
2. *the composite path $\lambda \cdot \lambda^{-1} : I \to X$ is homotopic* $\mathrm{rel}\ \partial I$ *to the constant path $c_{\lambda(0)}$.*

Proof. The results are proved by *reparametrization*. For example, consider the second point.

The continuous map $H : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} \lambda(2st) & 0 \leq s \leq \frac{1}{2} \\ \lambda(2(1-s)t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

is a homotopy $\mathrm{rel}\ \partial I$ between the constant map $c_{\lambda(0)}$ and $\lambda \cdot \lambda^{-1}$.

(Exercise: prove the associativity property.) □

This leads to an important *invariant* of a topological space, the *fundamental groupoid*:

Definition 3.33. For X a topological space, the *fundamental groupoid* $\Pi(X)$ of X is the small category:

- ▷ $\mathrm{Ob}\,\Pi(X) = X$ (the objects are the points of X);
- ▷ $\mathrm{Hom}_{\Pi(X)}(x, y) = \{[\gamma] | \gamma : I \to X, \gamma(0) = x, \gamma(1) = y\}$ is the set of homotopy classes $\mathrm{rel}\ \partial I$ of continuous paths from x to y;

with identity maps $[c_x] \in \mathrm{Hom}_{\Pi(X)}(x, x)$ and composition induced by composition of paths $[\mu] \circ [\lambda] = [\lambda \cdot \mu]$.

The inverse of $[\lambda]$ is $[\lambda^{-1}]$.

Exercise 3.34. Prove that $\Pi(X)$ is a groupoid.

Remark 3.35. For each topological space X, we obtain the fundamental groupoid $\Pi(X)$; this contains important information on the topological space X (as we shall see). Moreover, if $f : X \to Y$ is a continuous map, the fundamental groupoids are related by a *morphism*

$$\Pi(X) \stackrel{\Pi(f)}{\to} \Pi(Y).$$

This is an example of a *functor* from $\mathfrak{Top}$ to *groupoids*. (See Section A.2 for the notion of a functor.)

Exercise 3.36. Show that one can recover the set of path connected components $\pi_0(X)$ of a topological space X from its fundamental groupoid $\Pi(X)$.

4. The fundamental group

4.1. **Path connected components revisited.** Recall that $[X, Y]$ denotes the set of *homotopy classes* of continuous maps from X to Y; a homotopy class is denoted $[f]$, where $f : X \to Y$ is a continuous map, so that $[f] = [g]$ if and only if f is homotopic to g.

Proposition 4.1. *For X a topological space, π_0 defines a functor $\pi_0 : \mathfrak{Top} \to \mathfrak{Set}$. Moreover, there is a* natural *bijection of sets:*

$$\pi_0(X) \cong [*, X].$$

Proof. For X a topological space, we first establish the bijection of sets, by showing that the respective definitions are equivalent.

A continuous map $* \to X$ is equivalent to a *point* x of X. For two points $x, y \in X$, a homotopy from x to y is a continuous map $H : I \cong * \times I \to X$ such that $H(0) = x$ and $H(1) = y$; this is a continuous path from x to y. Hence x and y (considered as maps to X) are homotopic if and only if $x \sim y$ are connected by a path.

If $f : X \to Y$ is a continuous map, the induced map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is defined by

$$\pi_0(f)[x] := [f(x)].$$

This is equivalent to the *composition*

$$[X, Y] \circ [*, X] \to [*, Y]$$

defined on homotopy classes (see Proposition 3.13).

In any category $\mathscr{C}$, for A an object of $\mathscr{C}$, the association $B \mapsto \mathrm{Hom}_{\mathscr{C}}(A, B)$ defines a *representable* functor with values in the category of sets:

$$\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \to \mathfrak{Set}.$$

(Exercise: prove this assertion.) If follows immediately that $\pi_0(X)$ is a functor, by taking for $\mathscr{C}$ the category introduced in Proposition 3.13). $\square$

Remark 4.2. The *naturality* in the statement of Proposition 4.1 corresponds to a *natural equivalence* in category theory (see Section A.3).

4.2. **Groupoids and the fundamental groupoid revisited.** A *small* groupoid is a *small category* $\mathscr{G}$ (so that the objects form a *set*) in which every morphism is invertible. Namely

- ▷ there is an associative composition law $\circ$;
- ▷ every object admits an identity morphism for this composition law;
- ▷ every morphism admits an inverse $f \mapsto f^{-1}$, $\mathrm{Hom}_{\mathscr{G}}(A, B) \stackrel{(-)^{-1}}{\to} \mathrm{Hom}_{\mathscr{G}}(B, A)$.

A morphism of groupoids $\varphi : \mathscr{G}_1 \to \mathscr{G}_2$ is a functor from $\mathscr{G}_1$ to $\mathscr{G}_2$. This is equivalent to

- ▷ a map of sets $\varphi : \mathrm{Ob}\,(\mathscr{G}_1) \to \mathrm{Ob}\,(\mathscr{G}_2)$;
- ▷ for all pairs of objects A, B of $\mathscr{G}_1$, a set map
$$\varphi_{A,B} : \mathrm{Hom}_{\mathscr{G}_1}(A, B) \to \mathrm{Hom}_{\mathscr{G}_2}(\varphi(A), \varphi(B))$$
which is compatible with composition and sends identity maps to identity maps.

Remark 4.3. The behaviour of a morphism of groupoids on inverses follows automatically ($\varphi(f^{-1}) = \varphi(f)^{-1}$), so this is not required in the definition.

Definition 4.4. Let $\mathfrak{Groupoid}$ denote the *category of small groupoids*, with

- ▷ objects: small groupoids
- ▷ morphisms: morphisms of groupoids.

This is a full subcategory of the category $\mathfrak{CAT}$ of small categories (see Definition A.3).

Example 4.5. A discrete group G is equivalent to a groupoid $\underline{G}$ with a single object $*$, by taking $\mathrm{Hom}_{\underline{G}}(*,*) = G$, with composition induced by group multiplication and inverse by group inverse. If G_1, G_2 are discrete groups, a morphism of the associated groupoids $\underline{G_1} \to \underline{G_2}$ is equivalent to a group morphism $G_1 \to G_2$.

Conversely, for any small groupoid $\mathscr{G}$ and object A of $\mathscr{G}$, $\mathrm{Hom}_{\mathscr{G}}(A, A)$ is a group.

Exercise 4.6. Show that the association $G \mapsto \underline{G}$ defines a fully faithful embedding (see Definition A.9) of the category of groups in the category of small groupoids

$$\mathfrak{Group} \hookrightarrow \mathfrak{Groupoid}.$$

Definition 4.7. For $\mathscr{G}$ a small groupoid, define

(1) the equivalence relation $\sim$ on $\mathrm{Ob}\,\mathscr{G}$ by $A \sim B$ if and only if $\mathrm{Hom}_{\mathscr{G}}(A, B) \neq \emptyset$;
(2) $\pi_0(\mathscr{G}) := \mathrm{Ob}\,\mathscr{G}/\sim$, the set of *connected components* of $\mathscr{G}$;
(3) $\mathscr{G}$ is *connected* if $|\pi_0(\mathscr{G})| = 1$.

Proposition 4.8. *The connected component defines a functor*

$$\pi_0(-) : \mathfrak{Groupoid} \to \mathfrak{Set}.$$

Proof. Exercise. □

Exercise 4.9. For $\mathscr{G}$ a small groupoid and objects $A, B \in \mathrm{Ob}\,\mathscr{G}$ such that $A \sim B$, show that the groups $\mathrm{Hom}_{\mathscr{G}}(A, A)$ and $\mathrm{Hom}_{\mathscr{G}}(B, B)$ are isomorphic.

Recall the definition of the fundamental groupoid of a space X:

Definition 4.10. For X a topological space, the fundamental groupoid $\Pi(X)$ has

- ▷ $\mathrm{Ob}\,\Pi(X) = X$, the set of points of X;
- ▷ $\mathrm{Hom}_{\Pi(X)}(x, y) := \{\alpha : I \to X \mid \alpha(0) = x, \alpha(1) = y\}/\sim \mathrm{rel}\ \partial I$, the set of homotopy classes $\mathrm{rel}\ \partial I$ of continuous paths from x to y;
- ▷ composition induced by composition of paths: $[\beta] \circ [\alpha] := [\alpha \cdot \beta]$;
- ▷ inverse given by $[\alpha]^{-1} := [\alpha^{-1}]$.

Remark 4.11. If a *path* in X from x to y is thought of as a *homotopy* between $x, y : * \rightrightarrows X$, the composite of paths α (from x to y) and β (from y to z) should be thought of as a composite of homotopies:

$$\begin{tikzcd} x \arrow[d, Rightarrow, "\alpha"] \\ y \arrow[d, Rightarrow, "\beta"] \\ z \end{tikzcd}$$

using the diagrammatic representation of Remark 3.9.

A homotopy $\mathrm{rel}\ \partial I$ between paths α_1 to α_2 in X is a map $H : I \times I \to X$ where the first coordinate corresponds to progression along the path and the second progression along the homotopy (H is a homotopy between homotopies).

This should now be represented by

$$\begin{tikzcd} x \arrow[d, Rightarrow, bend right=60, "\alpha_1"'] \arrow[d, Rightarrow, bend left=60, "\alpha_2"] \arrow[d, phantom, "\overset{H}{\Rrightarrow}"] \\ y. \end{tikzcd}$$

There are *two* possible compositions:

▷ *vertical* composition corresponds to composition of paths;
▷ *horizontal* composition corresponds to composition of homotopies $\mathrm{rel}\ \partial I$.

Proposition 4.12. *The fundamental groupoid defines a* functor*:*

$$\Pi(-) : \mathfrak{Top} \to \mathfrak{Groupoid}.$$

Proof. We require to show that a continuous map $f : X \to Y$ induces a morphism of groupoids $\Pi(f)$ such that

▷ $\forall X$, $\Pi(\mathrm{Id}_X)$ is the identity morphism of $\Pi(X)$;
▷ for composable continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\Pi(g \circ f) = \Pi(g) \circ \Pi(f).$$

On the objects of X (that is the points $x \in X$), $\Pi(f)$ is the underlying map of sets $f : X \to Y$. On a morphism $[\alpha]$, represented by a continuous path $\alpha : I \to X$, $\Pi(f)[\alpha] = [f \circ \alpha]$. This defines a morphism of groupoids.

The morphism of groupoids $\Pi(\mathrm{Id}_X) : \Pi(X) \to \Pi(X)$ is clearly the identity and behaviour on composites is easy to check. □

There are two notions of connected component associated to a topological space X: the set of path-connected components $\pi_0(X)$ and the set of components of the fundamental groupoid $\Pi(X)$. These coincide:

Proposition 4.13. *The functors*

$$\begin{aligned} \pi_0(-) : \mathfrak{Top} &\to \mathfrak{Set} \\ \pi_0 \circ \Pi(-) : \mathfrak{Top} &\to \mathfrak{Set} \end{aligned}$$

are naturally isomorphic.

Proof. Exercise. (See Section A.3 for the notion of natural equivalence.) □

Remark 4.14. Proposition 4.12 introduces the morphism of groupoids $\Pi(f)$ associated to a continuous map $f : X \to Y$. What happens for two maps $f, g : X \rightrightarrows Y$ which are homotopic via $H : X \times I \to Y$?

Consider a path α from x to y in X then there is a diagram of composable paths:

$$\begin{tikzcd} f(x) \arrow[r, Rightarrow, "f(\alpha)"] \arrow[d, Rightarrow, "H(x,-)"'] \arrow[dr, phantom, "\Downarrow H"] & f(y) \\ g(x) \arrow[r, Rightarrow, "g(\alpha)"'] & g(y), \arrow[u, Rightarrow, "H(y,-)^{-1}"'] \end{tikzcd}$$

where the square represents two paths from $f(x)$ to $f(y)$ in Y. The path corresponding to the bottom of the square is obtained from $g(\alpha)$ by composing with the path $H(x, -)$ from $f(x)$ to $g(x)$ and the inverse of the path $H(y, -)$ from $f(y)$ to $g(y)$. The homotopy H induces a homotopy $\mathrm{rel}\ \partial I$ between these two paths (exercise).

For points $x, y \in X$, define the map of sets

$$H_{x,y} : \mathrm{Hom}_{\Pi(Y)}(g(x), g(y)) \to \mathrm{Hom}_{\Pi(Y)}(f(x), f(y))$$

by $[\beta] \mapsto [H(y, -)]^{-1} \circ [\beta] \circ [H(x, -)]$, generalizing the construction used above. This provides the compatibility between $\Pi(g)$ and $\Pi(f)$ which is given by the commutative diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\Pi(X)}(x, y) \arrow[dl, "\Pi(g)"'] \arrow[dr, "\Pi(f)"] & \\ \mathrm{Hom}_{\Pi(Y)}(g(x), g(y)) \arrow[rr, "\cong", "H_{x,y}"'] & & \mathrm{Hom}_{\Pi(Y)}(f(x), f(y)). \end{tikzcd}$$

(Exercise: verify that $H_{x,y}$ is a bijection.)

This could be made more conceptual by introducing the general notion of *homotopy* between morphisms of groupoids.

4.3. **The fundamental group.** If the topological space X is path connected, most of the information encoded in the fundamental groupoid $\Pi(X)$ can be recovered by fixing a point $x \in X$ and considering:

$$\mathrm{Hom}_{\Pi(X)}(x, x)$$

which is a *group* (see Example 4.5). Thus, we consider *loops* in X, which start and end at x (loops *based* at x).

Definition 4.15. For X a topological space and $x \in X$, the *fundamental group* $\pi_1(X, x)$ is the group with

- ▷ elements $\{[\alpha] | \alpha(0) = \alpha(1) = x\}$, the set of homotopy classes $\mathrm{rel}\ \partial I$ of continuous paths starting and ending at x;
- ▷ group multiplication $[\alpha][\beta] = [\alpha \cdot \beta]$;
- ▷ inverse $[\alpha]^{-1} = [\alpha^{-1}]$.

Remark 4.16. The group multiplication is defined by the composition of paths. This must not be confused with the group structure $\circ$ on $\mathrm{Hom}_{\Pi(X)}(x, x)$ which is given by $[\alpha] \circ [\beta] = [\beta \cdot \alpha]$; this is the *opposite* group structure.

Exercise 4.17. Show that the fundamental group $\pi_1(X, x)$ depends only upon the path-connected component of X containing x.

The fundamental group is defined in terms of *pointed* topological spaces.

Definition 4.18.

(1) A pointed topological space is a pair (X, x) where X is a topological space equipped with a *basepoint* $x \in X$.
(2) The category of *pointed* topological spaces, $\mathfrak{Top}_\bullet$, has:
  - ▷ objects: pointed topological spaces (X, x)
  - ▷ morphisms: $\mathrm{Hom}_{\mathfrak{Top}_\bullet}((X, x), (Y, y))$ is the set of continuous maps $f : X \to Y$ such that $f(x) = y$.
(3) For (X, x) and (Y, y) pointed topological spaces, let
$$[(X, x), (Y, y)]_{\mathfrak{Top}_\bullet}$$
denote the set of homotopy classes $\mathrm{rel}\ x$ of morphisms $(X, x) \to (Y, y)$.

Example 4.19. The circle S^1 is homeomorphic to the quotient space $[0, 1]/0 \sim 1$, which has a natural choice of basepoint, given by the image of 0. Write this pointed space as $(S^1, *)$.

Proposition 4.20. *The fundamental group defines a functor:*

$$\pi_1(-) : \mathfrak{Top}_\bullet \to \mathfrak{Group}.$$

Proof. (This can be deduced from Proposition 4.12.) If $f : (X, x) \to (Y, y)$ is a morphism of pointed spaces (so that $f(x) = y$), the morphism of groups

$$\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$$

is given by $\pi_1(f)[\alpha] = [f \circ \alpha]$.

It is straightforward to check that $\pi_1(f)$ is a morphism of groups and that this defines a functor (exercise!). □

Exercise 4.21. Show that

(1) a loop based at $x \in X$ is equivalent to a morphism in $\mathfrak{Top}_\bullet$:
$$(S^1, *) \to (X, x);$$

(2) there is a natural isomorphism (of sets)

$$\pi_1(X, x) \cong [(S^1, *), (X, x)]_{\mathfrak{Top}_\bullet}.$$

(3) How can one recover the group structure?

The dependency on the choice of basepoint (within a path-connected component) is explained by the following.

Proposition 4.22. *For X a topological space and $[\gamma] \in \mathrm{Hom}_{\Pi(X)}(x, y)$, there is an isomorphism of groups*

$$\Phi_{[\gamma]} : \pi_1(X, y) \to \pi_1(X, x)$$

defined for α a loop based at y by $\Phi_{[\gamma]}[\alpha] = [\gamma \cdot \alpha \cdot \gamma^{-1}]$.

Proof. Exercise (hint: draw a picture - compare also Exercise 4.9). □

Proposition 4.23. *For $H : X \times I \to Y$ a homotopy from f to g, where $f, g : X \rightrightarrows Y$ are continuous map, and $x \in X$ a basepoint, the associated path $H(x, -) : I \to Y$ induces a group isomorphism $\Phi_{[H(x,-)]}$ which fits into the commutative diagram:*

$$\begin{tikzcd} & \pi_1(X,x) \arrow[dl, "\pi_1(g)"'] \arrow[dr, "\pi_1(f)"] & \\ \pi_1(Y, g(x)) \arrow[rr, "\cong", "\Phi_{[H(x,-)]}"'] & & \pi_1(Y, f(x)). \end{tikzcd}$$

Proof. Exercise. (This corresponds to the discussion in Remark 4.14 for the fundamental groupoid; the proof is a good exercise in understanding *based* homotopies.) □

Corollary 4.24. *For $f : X \to Y$ a homotopy equivalence, the induced map*

$$\pi_1(f) : \pi_1(X, x) \stackrel{\cong}{\to} \pi_1(Y, f(x))$$

is an isomorphism of groups.

In particular, if X is contractible, then $\pi_1(X, x) \cong \{e\}$, the trivial group.

Proof. Exercise. □

Remark 4.25. This shows that the fundamental group $\pi_1(X, x)$ is a *pointed homotopy invariant* of (X, x).

Remark 4.26. There are many interesting path-connected spaces X such that $\pi_1(X, x) = \{e\}$ but which are *not contractible*. For example, $X = S^2$.

4.4. **Interlude on groups.** In order to motivate the constructions used in the following section, we recall the construction of the *free product* $G \star H$ of groups G, H and, more generally, the *pushout* $G \star_K H$ associated to the diagram of solid arrows:

(1)
$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, dotted] \\ H \arrow[r, dotted] & G \star_K H. \end{tikzcd}$$

Definition 4.27. Let $\mathscr{F} : \mathfrak{Set} \to \mathfrak{Group}$ denote the *free group* functor, which associates to a set S the group freely generated by the elements of S.

Exercise 4.28. For S a set,

(1) give an explicit description of the free group $\mathscr{F}(S)$, in particular, show that there is a natural inclusion of sets $S \hookrightarrow \mathscr{F}(S)$ (which will be denoted here by $s \mapsto [s]$);

(2) show that $\mathscr{F}(S)$ satisfies the following *universal property*: $\forall G \in \mathrm{Ob}\,\mathfrak{Group}$, there is a *natural* bijection

$$\mathrm{Hom}_{\mathfrak{Group}}(\mathscr{F}(S), G) \cong \mathrm{Hom}_{\mathfrak{Set}}(S, G),$$

where, on the right hand side, G is considered as a set;

(3) describe the natural morphism of groups $\mathscr{F}(G) \to G$ corresponding to the identity map (of sets!) of G.

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Remark 4.29. To give a group G by a *presentation* by generators and relations is equivalent to giving a set S of generators, which induces a *surjective* group morphism

$$\mathscr{F}(S) \twoheadrightarrow G$$

and a set of relations, $R \subset \mathscr{F}(S)$, which generate a normal subgroup $N \lhd \mathscr{F}(S)$ such that $G \cong \mathscr{F}(S)/N$.

Another way of representing this is by the *presentation*

$$\mathscr{F}(R) \to \mathscr{F}(S) \twoheadrightarrow G.$$

Note that the image of $\mathscr{F}(R)$ in $\mathscr{F}(S)$ is not in general a normal subgroup.

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Example 4.30. Every group G has a *canonical* presentation, taking G as the set of generators and the set of relations $\{[g_1][g_2][g_1g_2]^{-1} | (g_1, g_2) \in G \times G\}$. This involves no arbitrary choices, hence a morphism of groups $\varphi : G \to H$ induces a commutative diagram:

$$\begin{tikzcd}
\mathscr{F}(G \times G) \arrow[r] \arrow[d, "\mathscr{F}(\varphi\times\varphi)"'] & \mathscr{F}(G) \arrow[r, two heads] \arrow[d, "\mathscr{F}(\varphi)"] & G \arrow[d, "\varphi"] \\
\mathscr{F}(H \times H) \arrow[r] & \mathscr{F}(H) \arrow[r, two heads] & H.
\end{tikzcd}$$

This explains the adjective *canonical*.

Definition 4.31. For groups G, H, the *free product* of G and H is the group

$$G \star H := \mathscr{F}(G \amalg H)/N$$

where N is the normal subgroup $N \lhd \mathscr{F}(G \amalg H)$ generated by the following relations ✓03/10/13

(1) $[g_1][g_2][g_1g_2]^{-1}$;
(2) $[h_1][h_2][h_1h_2]^{-1}$,

$\forall g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Exercise 4.32. Using the notation of the definition, check that the relations imply that $[e_G] = e = [e_H]$, where e_G, e_H and e are the respective identities of $G, H, \mathscr{F}(G \amalg H)$. Show that $[g^{-1}] = [g]^{-1}$ and $[h^{-1}] = [h]^{-1}$ $\forall g \in G, h \in H$.

Exercise 4.33. For groups G, H, show that the set maps $G \to \mathscr{F}(G) \subset \mathscr{F}(G \amalg H)$ and $H \to \mathscr{F}(H) \subset \mathscr{F}(G \cup H)$ induce group morphisms:

$$G \to G \star H \leftarrow H.$$

Show that group morphisms $\varphi : G \to Q$, $\psi : H \to Q$ induce a *unique* morphism of groups: $G \star H \to Q$.

Definition 4.34. For group morphisms $H \overset{j}{\leftarrow} K \overset{i}{\rightarrow} G$, let $G \star_K H$ denote the quotient of $G \star H$ by the normal subgroup generated by ✓03/10/13

$$[i(\kappa)][j(\kappa)]^{-1}\ \forall \kappa \in K.$$

Remark 4.35. Taking $K = \{e\}$ (which is the *initial object* of the category of groups - and also the *final object*), $G \star_{\{e\}} H = G \star H$.

Exercise 4.36. Establish the *universal property* of $G \star_K H$: given morphisms $\varphi : G \to Q$, $\psi : H \to Q$ of groups which make the outer square commute,

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d] \arrow[ddr, bend left, "\varphi"] & \\
H \arrow[r] \arrow[drr, bend right, "\psi"'] & G \star_K H \arrow[dr, dotted] & \\
 & & Q
\end{tikzcd}$$

there is a *unique* morphism of groups $G \star_K H \to Q$ (indicated by the dotted arrow) which makes the diagram commute.

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Exercise 4.37. For G a group, show that there exist sets R, S and a group homomorphism $\mathscr{F}(R) \to \mathscr{F}(S)$ such that

$$G \cong \mathscr{F}(S) \star_{\mathscr{F}(R)} \{e\}.$$

4.5. **The Seifert-van Kampen theorem for the fundamental groupoid.** We require techniques for calculating the fundamental group $\pi_1(X, x)$; for instance, suppose that $X = U \bigcup V$, where U and V are open subsets of X, how can we calculate $\pi_1(X, x)$ for $x \in U \cap V$, from information given by U and V? It turns out to be more natural to consider the fundamental groupoid $\Pi(X)$. The key technical ingredient is that I and $I \times I$ are compact metric spaces; this allows Lebesgue's theorem to be applied:

Proposition 4.38. *For M a compact metric space and $\mathscr{U} = \{U_i | i \in \mathscr{I}\}$ an open cover of M, there exists a* Lebesgue number $0 < \varepsilon \in \mathbb{R}$ *such that, $\forall m \in M$ $\exists i_m \in \mathscr{I}$ such that*

$$B_\varepsilon(m) \subset U_{i_m}.$$

Proof. Exercise. □

Proposition 4.38 is applied to the spaces I, $I \times I$ to *decompose* paths and homotopies between paths.

Corollary 4.39. *Let $\mathscr{V} = \{V_j | j \in \mathscr{J}\}$ be an open cover of a topological space X and $\alpha : I \to X$, $H : I \times I \to X$ be continuous maps, then $\exists N \in \mathbb{N}$ such that*

(1) *$\forall 0 \leq a < N$, $\exists j_a \in \mathscr{J}$ such that $\alpha([\frac{a}{N}, \frac{a+1}{N}]) \subset V_{j_a}$;*
(2) *$\forall 0 \leq a, b < N$, $\exists j_{a,b} \in \mathscr{J}$ such that $H([\frac{a}{N}, \frac{a+1}{N}] \times [\frac{b}{N}, \frac{b+1}{N}]) \subset V_{j_{a,b}}$.*

Proof. Let ε_I be the Lebesgue number provided by Proposition 4.38 for the open cover $\{f^{-1}(U_j)\}$ of I and $\varepsilon_{I \times I}$ the Lebesgue number for the open cover $\{H^{-1}(U_j)\}$ of $I \times I$. Set $\varepsilon := \min\{\varepsilon_I, \varepsilon_{I \times I}\}$, then taking $0 < N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$, the result follows. □

For simplicity, we now limit discussion to an open cover $X = U \bigcup V$. The inclusions of subspaces

$$\begin{tikzcd}
U \cap V \arrow[r, hook, "i_U"] \arrow[d, hook, "i_V"'] & U \arrow[d, hook, "j_U"] \\
V \arrow[r, hook, "j_V"'] & X
\end{tikzcd}$$

induce a commutative diagram of morphisms of groupoids:

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r, "\Pi(i_U)"] \arrow[d, "\Pi(i_V)"'] & \Pi(U) \arrow[d, "\Pi(j_U)"] \\
\Pi(V) \arrow[r, "\Pi(j_V)"'] & \Pi(X).
\end{tikzcd}$$

Remark 4.40. Although the underlying maps on *objects* are the inclusions, the morphisms of groupoids are not in general injective on the *morphisms*. (Two paths can become homotopic $\mathrm{rel}\ \partial I$ in a larger space.)

The Seifert-van Kampen theorem shows that $\Pi(X)$ is obtained by *gluing* the groupoids $\Pi(U)$ and $\Pi(V)$ together, building in compatibility via $\Pi(i_U)$ and $\Pi(i_V)$. The starting point is the following observation:

Lemma 4.41. *Let $\alpha : I \to X$ be a path in X, then there exists a decomposition*

$$[\alpha] = [\alpha_N] \circ \ldots \circ [\alpha_1]$$

in $\Pi(X)$ such that, $\forall t, 1 \leq t \leq N$, one of the following holds:

(1) $[\alpha_t] = \Pi(j_U)[\alpha_t^U]$ *or*
(2) $[\alpha_t] = \Pi(j_V)[\alpha_t^V]$,

for some $\alpha_t^U : I \to U$ or $\alpha_t^V : I \to V$.

Proof. An immediate consequence of Corollary 4.39. □

Definition 4.42. For an open cover $X = U \cup V$, let $\Pi^{U,V}(X)$ denote the groupoid with

- ▷ objects: $\mathrm{Ob}\,\Pi^{U,V}(X) = X$;
- ▷ morphisms: composable sequences generated by $\{[\alpha^U] \in \mathrm{Mor}\Pi(U)\}$ and $\{[\alpha^V] \in \mathrm{Mor}\Pi(V)\}$ subject to the following relations:
  (1) $[\alpha_2^U] \circ [\alpha_1^U] = [\alpha_1^U \cdot \alpha_2^U]$
  (2) $[\alpha_2^V] \circ [\alpha_1^V] = [\alpha_1^V \cdot \alpha_2^V]$
  (3) $[i_U(\alpha^{U\cap V})] = [i_V(\alpha^{U\cap V})]$
  for paths $\alpha_s^U : I \to U$, $\alpha_s^V : I \to V$ and $\alpha^{U\cap V} : I \to U \cap V$.

Exercise 4.43. Check that $\Pi^{U,V}(X)$ is a groupoid and that the inclusions $U, V \subset X$ induce a morphism of groupoids:

$$\Pi^{U,V}(X) \to \Pi(X)$$

which is the identity map on objects and which fits into the commutative diagram:

$$\begin{tikzcd}
\Pi(U \cap V) \arrow[r, "\Pi(i_U)"] \arrow[d, "\Pi(i_V)"'] & \Pi(U) \arrow[d] \arrow[ddr, bend left, "\Pi(j_U)"] & \\
\Pi(V) \arrow[r] \arrow[drr, bend right, "\Pi(j_V)"'] & \Pi^{U,V}(X) \arrow[dr] & \\
 & & \Pi(X)
\end{tikzcd}$$

Theorem 4.44. *For U, V an open cover of X,*

$$\Pi^{U,V}(X) \to \Pi(X)$$

is an isomorphism of groupoids.

Proof. On the level of objects, the morphism is the identity and Lemma 4.41 shows that it is *surjective* on morphisms: to a path $\alpha : I \to X$, one associates the morphism of $\Pi^{U,V}(X)$:

$$[[\alpha]] := [\alpha_N^{Z_N}] \circ \ldots \circ [\alpha_1^{Z_1}]$$

where $Z_t \in \{U, V\}$, as in the decomposition of Lemma 4.41; by construction, this maps to $[\alpha]$. The compatibility with composition in U and V built into the definition of $\Pi^{U,V}(X)$ shows that the element $[[\alpha]]$ does not depend on N.

For injectivity on morphisms, suppose that $[\alpha] = [\beta]$ in $\Pi(X)$, we require to show that $[[\alpha]] = [[\beta]]$ in $\Pi^{U,V}(X)$. Hence fix a homotopy $\mathrm{rel}\ \partial I$ from α to β and choose $N \in \mathbb{N}$ as in Corollary 4.39; in particular this value of N can be used to construct $[[\alpha]]$ and $[[\beta]]$. Thus $I \times I$ is subdivided into N^2 squares such that H maps each small square either to U or to V; restricting to a small square, H induces a homotopy between the composites corresponding to the edges.

The heart of the argument is to analyse what happens when adjacent squares map to different opens; this means that, on their intersection, they map to $U \cap V$. For example, consider the following situation:

$$\begin{tikzcd}
\bullet \arrow[r, "\beta^U"] & \bullet \arrow[r, "\beta^V"] & \bullet \\
\bullet \arrow[u, "\gamma^U"] \arrow[r, "\alpha^U"'] \arrow[ur, phantom, "H^U"] & \bullet \arrow[u, "\gamma"] \arrow[r, "\alpha^V"'] \arrow[ur, phantom, "H^V"] & \bullet, \arrow[u, "\gamma^V"']
\end{tikzcd}$$

representing a diagram of composable paths and homotopies, where the superscript V in α^V indicates a path $I \to V$, for example; γ is a path $I \to U \cap V$, hence can be considered as a path to *both* U and V.

The right hand square gives the identity in $\Pi^{U,V}(X)$:

$$[\gamma^V] \circ [\alpha^V] = [\alpha^V \cdot \gamma^V] = [(i_V\gamma) \cdot \beta^V] = [\beta^V] \circ [i_V(\gamma)]$$

and the left hand square :

$$[i_U(\gamma)] \circ [\alpha^U] = [\alpha^U \cdot (i_U\gamma)] = [\gamma^U \cdot \beta^U] = [\beta^U] \circ [\gamma^U].$$

Using the identity $[i_U(\gamma)] = [i_V(\gamma)]$, it follows that

$$[\gamma^V] \circ [\alpha^V] \circ [\alpha^U] = [\beta^V] \circ [i_V(\gamma)] \circ [\alpha^U] = [\beta^V] \circ [i_U(\gamma)] \circ [\alpha^U] = [\beta^V] \circ [\beta^U] \circ [\gamma^U].$$

Thus, in $\Pi^{U,V}(X)$, the composite around the top of the rectangle is equal to the composite around the bottom.

The argument can be carried out for any choices of U, V in the above diagram. A straightforward recursive argument then shows that $[[\alpha]] = [[\beta]]$ as required. □

Remark 4.45. Working with the fundamental groupoid $\Pi(X)$ has the technical advantage that it is not necessary to impose any extra hypotheses on U, V and $U \cap V$. The price to pay is the introduction (given here in an *ad hoc* manner) of the groupoid $\Pi^{U,V}(X)$.

As an immediate application, we get our first non-trivial calculation:

Theorem 4.46. *There is an isomorphism of groups*

$$\pi_1(S^1, *) \cong \mathbb{Z}.$$

Proof. Fix $0 < \varepsilon < 1$ (to improve intuition, take ε *very* close to 0). Take the open cover of $S^1 \subset \mathbb{R}^2$ by $U := S^1 \cap \mathbb{R}^2_{y > -\varepsilon}$ and $V := S^1 \cap \mathbb{R}^2_{y < \varepsilon}$, so that U, V are contractible (homeomorphic to open intervals) and $U \cap V$ is homeomorphic to a disjoint union of two (small!) open intervals. Consider the calculation of $\mathrm{Hom}_{\Pi(S^1)}(*, *)$ using the Seifert-van Kampen theorem, where $* = (0, 1) \in S^1$. Choose distinguished paths $\gamma^U : I \to U$ and $\gamma^V : I \to V$ so that $\mathrm{Hom}_{\Pi(U)}(*, -*) =$

$\{[\gamma^U]\}$ and $\mathrm{Hom}_{\Pi(V)}(*, -*) = \{[\gamma^V]\}$ (exercise: why is this possible?). This gives the loops $[\gamma^U \cdot (\gamma^V)^{-1}]$ and $[\gamma^V \cdot (\gamma^U)^{-1}]$ which are mutually inverse.

Using the relations in the definition of $\Pi^{U,V}(S^1)$, because $U \cap V$ is the disjoint union of two contractible spaces, it is straightforward to see that, for $[\alpha] \in \mathrm{Hom}_{\Pi(S^1)}(*, *)$, there is a *unique* integer n such that

$$[\alpha] = \big([\gamma^V]^{-1} \circ [\gamma^U]\big)^{\circ n}$$

(exercise: prove this assertion!). The result follows. $\square$

Remark 4.47. It is *not* possible to prove this result using the fundamental group version of the Seifert-van Kampen theorem, (see Theorem 4.49 below). Why?

Corollary 4.48. *The circle S^1 is not contractible.*

4.6. **The Seifert-van Kampen theorem for the fundamental group.** Consider U, V an open cover of X and choose a basepoint $* \in U \cap V$ (which also gives a basepoint for U, V and X). The inclusions induce a commutative diagram of morphisms between fundamental groups:

$$\begin{tikzcd}
\pi_1(U \cap V, *) \arrow[r] \arrow[d] & \pi_1(U, *) \arrow[d] \\
\pi_1(V, *) \arrow[r] & \pi_1(X, *).
\end{tikzcd}$$

This induces a unique group morphism (by Exercise 4.36):

$$\pi_1(U, *) \star_{\pi_1(U \cap V, *)} \pi_1(V, *) \to \pi_1(X, *).$$

Theorem 4.49. *For U, V an open cover of X and basepoint $* \in U \cap V$, if $U \cap V$, U, V are all path-connected, then the group morphism $\pi_1(U, *) \star_{\pi_1(U \cap V, *)} \pi_1(V, *) \to \pi_1(X, *)$ is an isomorphism.*

Proof. This result is a *consequence* of the result for the fundamental groupoid, Theorem 4.44, which did not require the path-connected hypothesis. It can be proved by adapting the proof of Theorem 4.44 to use loops. (Exercise: prove this result!) $\square$

Exercise 4.50. Calculate $\pi_1(S^1 \vee S^1, *)$, where $S^1 \vee S^1$ is the *wedge* of two copies of $(S^1, *)$, obtained by identifying the basepoints.

As a consequence, calculate $\pi_1(\mathbb{R}^2 \backslash \{0, *\}), \bullet)$, for any basepoint $\bullet$.

Remark 4.51. Theorem 4.49 admits the conceptual interpretation that $\pi_1(-,-)$ *preserves pushouts*. Namely, the open cover U, V allows X to be recovered by gluing along $U \cap V$, which can be interpreted as saying that X is the *pushout* in pointed topological spaces of the diagram $V \leftarrow U \cap V \to U$.

Similarly, the group $\pi_1(U, *) \star_{\pi_1(U \cap V, *)} \pi_1(V, *)$ is the *pushout* of the diagram $\pi_1(V, *) \leftarrow \pi_1(U \cap V, *) \to \pi_1(U, *)$ (this is the universal property established in Exercise 4.36).

Note that the path-connected hypothesis has to be imposed on the spaces $U, V, U \cap V$.

Example 4.52. The conclusion of Theorem 4.49 is *false* for the open cover of the circle S^1 used in Theorem 4.46. (Exercise: check this.) The theorem is not violated, however, since $U \cap V$ is not path connected.

4.7. **First applications of the Seifert-van Kampen theorem.**

Definition 4.53. A topological space X is *simply connected* if it is path connected and, for any choice of basepoint, $\pi_1(X, x) = \{e\}$.

Proposition 4.54. *For $2 \leq n \in \mathbb{Z}$, the sphere S^n is simply connected.*

Proof. The sphere S^n is path connected if $n > 0$; choose an open cover U^+, U^- of S^n by the northern and southern hemispheres, so that $U^+ \cap U^-$ is homeomorphic to $S^{n-1} \times (-\varepsilon, \varepsilon)$. Since $n \geq 2$, these spaces are all path connected.

Choosing a basepoint $* \in U^+ \cap U^-$. The spaces U^+, U^- are contractible, hence the Seifert-van Kampen theorem implies that

$$\pi_1(S^n, *) \cong \{e\} \star_{\pi_1(S^{n-1},*)} \{e\} \cong \{e\}.$$

□

In the category of pointed topological spaces, $\mathfrak{Top}_\bullet$, one replaces disjoint union $\amalg$ by the *wedge product*.

Definition 4.55. For (X, x), (Y, y) pointed topological spaces, the *wedge* $X \vee Y$ is the quotient

$$X \vee Y := \big(X \amalg Y\big)/x \sim y$$

which glues the basepoints together, pointed by the image of x and y.

Remark 4.56. To ensure that basepoints behave well in *homotopy theory*, one requires a hypothesis which ensures that there is a *nice* neighbourhood of the basepoint.

The following condition is introduced in [FT10]; it is slightly weaker than the condition *well pointed* which is frequently used.

Definition 4.57. A pointed topological space (X, x) is *correctly pointed* if there is an open neighbourhood $x \in V$ such that $V \simeq x \text{ rel } \{x\}$ (x is a strong deformation retract of V - see Section 3.6).

Proposition 4.58. *For (X, x), (Y, y) path-connected, correctly-pointed topological spaces,*

$$\pi_1(X \vee Y, *) \cong \pi_1(X, x) \star \pi_1(Y, y).$$

Proof. Let $x \in V_x \subset X$ and $y \in V_y \subset Y$ be open neighbourhoods which are strong deformation retracts of the basepoint. Then $U_Y := V_x \vee Y$ and $U_X := X \vee V_y$ give an open cover of $X \vee Y$ with $U_Y \simeq Y$, $U_X \simeq X$ and $U_Y \cap U_X = V_x \vee V_y \simeq *$. The conclusion follows from Theorem 4.49. □

Example 4.59. For $n \in \mathbb{N}$, $\pi_1(\bigvee_n S^1, *) \cong \mathbb{Z}^{\star n}$ is the free group on n generators.

Exercise 4.60. Give an example of a wedge $X \vee Y$ for which the conclusion of Proposition 4.58 is *false*.

4.8. **Attaching cells.** A fundamental way of *building* topological spaces is by *gluing* on cells. Here a cell of dimension n is $e^n \subset \mathbb{R}^n$, the closed Euclidean ball, which has boundary $\partial e^n = S^{n-1}$. The cell is *glued* to a topological space along its boundary.

Notation 4.61. For $f : S^n \to X$ a continuous map, let C_f denote the quotient space

$$C_f := \big(X \amalg e^{n+1}\big)/s \sim f(s) \in X, \forall s \in \partial e^{n+1}.$$

This is equipped with the natural inclusion $X \hookrightarrow C_f$.

Remark 4.62. The notation reflects the fact that this is a special case of the construction of the *mapping cone* of a continuous map (note that e^{n+1} is homeomorphic to the cone CS^n).

Example 4.63. The projective plane $\mathbb{R}P^2$ is homeomorphic to the mapping cone $C_{[2]}$ of the continuous map $S^1 \stackrel{[2]}{\to} S^1$, $z \mapsto z^2$ (considering $S^1 \subset \mathbb{C}$).

Proposition 4.64. *For $f : S^n \to X$ a continuous map (with $n \geq 1$) and C_f the mapping cone of f, pointed by $x = f(*) \in X \subset C_f$, for $* \in S^n$:*

(1) *if $n = 1$, the inclusion $X \hookrightarrow C_f$ induces an isomorphism*
$$\pi_1(C_f, x) \cong \pi_1(X, x)/[f]$$
where $[f] \lhd \pi_1(X, x)$ is the normal subgroup generated by the image of $\pi_1(f) : \pi_1(S^1, *) \cong \mathbb{Z} \to \pi_1(X, x)$;

(2) *if $n > 1$, $\pi_1(X, x) \cong \pi_1(C_f, x)$.*

Proof. We may assume that X is path connected, since the image of f lies in a single path component of X.

By construction, C_f is a quotient of $X \amalg e^{n+1}$. Take an open cover U_∂, U of e^{n+1} such that $S^n \subset U_\partial$ is homeomorphic to $S^n \times [0, 1)$, $U \cong (e^{n+1})^\circ$ and $U_\partial \cap U \cong S^n \times (0, 1)$. Write V for the image of $X \amalg U_\partial$ in C_f and U for the image of U. Hence, by construction, $U, V, U \cap V$ are path connected and U is contractible, $U \cap V \simeq S^n$ and $V \simeq X$ (exercise: check this!).

Applying the Seifert-van Kampen theorem gives
$$\pi_1(C_f, x) \cong \pi_1(X, x) \star_{\pi_1(S^n,*)} \{e\}.$$
For $n \geq 2$, $\pi_1(S^n, *) \cong \{e\}$ and the result is clear. In the case $n = 1$, the right hand side is, by construction, isomorphic to the stated quotient. □

Exercise 4.65. What happens when $n = 0$?

Example 4.66. Recall from Example 4.63 that $\mathbb{R}P^2$ is homeomorphic to the mapping cone $C_{[2]}$ of $S^1 \stackrel{[2]}{\to} S^1$. Hence
$$\pi_1(\mathbb{R}P^2, *) \cong \mathbb{Z}/2.$$
($\mathbb{R}P^2$ is path connected, hence the choice of basepoint is unimportant.)

Moreover, considering the wedge product:
$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *) \cong \mathbb{Z}/2 \star \mathbb{Z}/2.$$

✓10/10/13

4.9. **Products.** The product of two topological spaces is a *categorical product*: to give a continuous map to the product is equivalent to specifying the components. This means that the calculation of the fundamental group of a product is straightforward.

Proposition 4.67. *For (X, x), (Y, y) pointed topological spaces, the projections $X \stackrel{p_X}{\leftarrow} X \times Y \stackrel{p_Y}{\to} Y$ induce an isomorphism of groups*
$$\pi_1(X \times Y, (x, y)) \stackrel{\cong}{\to} \pi_1(X, x) \times \pi_1(Y, y).$$

Proof. The projections are pointed maps, hence induce morphisms of groups
$$\pi_1(X, x) \stackrel{\pi_1(p_X)}{\leftarrow} \pi_1(X \times Y, (x, y)) \stackrel{\pi_1(p_Y)}{\to} \pi_1(Y, y),$$
which induce the given morphism of groups.

Consider the based circle $(S^1, *)$ a loop based at (x, y) in $X \times Y$ is a continuous *pointed* map $\alpha : (S^1, *) \to (X \times Y, (x, y))$. This is equivalent to giving the two component maps, which are loops $p_X \circ \alpha : (S^1, *) \to (X, x)$ and $p_Y \circ \alpha : (S^1, *) \to (Y, y)$.

Similarly, a *based* homotopy $H : S^1 \times I \to X \times Y$ between two loops based at (x, y) is equivalent to giving the component based homotopies $H_X : S^1 \times I \to X$ and $H_Y : S^1 \times I \to X$.

It follows that the map $[\alpha] \mapsto ([p_X \circ \alpha], [p_Y \circ \alpha]$ is a bijection, as required. □

Example 4.68. The fundamental group of the torus $S^1 \times S^1$ is

$$\begin{aligned}\pi_1(S^1 \times S^1, (*, *)) &\cong \pi_1(S^1, *) \times \pi_1(S^1, *) \\ &\cong \mathbb{Z} \times \mathbb{Z},\end{aligned}$$

the free abelian group on two generators.

Remark 4.69. There is an alternative method for calculating $\pi_1(S^1 \times S^1, (*, *))$ by using Proposition 4.64. The torus is a quotient $I \times I \twoheadrightarrow S^1 \times S^1$ of the square. Under this map, the boundary $\partial(I \times I)$ is sent to $S^1 \vee S^1$ and filling in the square is equivalent to adding a two-cell, which is glued along an attaching mapping

$$f : \partial e^2 \cong S^1 \to S^1 \vee S^1.$$

The map f has class $[f] \in \pi_1(S^1 \vee S^1, *) \cong \langle\alpha\rangle \star \langle\beta\rangle$; this class identifies as $\alpha\beta\alpha^{-1}\beta^{-1}$, the *commutator* on the generators of the free group (prove this!). ✓24/10/13

Proposition 4.64 gives

$$\pi_1(S^1 \times S^1, (*, *)) \cong \langle\alpha\rangle \star \langle\beta\rangle/(\alpha\beta\alpha^{-1}\beta^{-1})$$

which is isomorphic to the free abelian group on α, β.

Exercise 4.70. Calculate the fundamental group of the Klein bottle K (see Example 1.47). Deduce that the torus $S^1 \times S^1$ and K do not have the same homotopy type.

4.10. **Groups as fundamental groups.**

Theorem 4.71. *For G a discrete group, there exists a pointed topological space (X_G, x) such that*

$$\pi_1(X_G, x) \cong G.$$

Moreover, this construction is functorial*: $G \mapsto X_G$ is a functor*

$$\mathfrak{Group} \to \mathfrak{Top}_\bullet.$$

Proof. If G admits a *finite presentation* (finitely many generators and relations), the existence of such a X_G follows easily from Proposition 4.64. (Exercise!)

However, the construction can be made *functorial* by using the canonical presentation

$$\mathscr{F}(G \times G) \to \mathscr{F}(G) \twoheadrightarrow G$$

of Example 4.30. Namely, X_G is built from the (in general infinite) wedge $\bigvee_{g\in G} S^1$ by attaching, for each pair $(g_1, g_2) \in G \times G$ a two cell along the loop $S^1 \to \bigvee_{g\in G} S^1$ which is given by the path

$$\alpha_{g_1} \cdot \alpha_{g_2} \cdot \alpha_{g_1g_2}^{-1},$$

where $\alpha_g : S^1 \to \bigvee_{g\in G} S^1$ is the inclusion of the circle indexed by g, which can be interpreted as a *loop*. It is clear that this construction is functorial (no arbitrary choices have been made).

The proof of Proposition 4.64 generalizes to this setting, which shows that the space X_G has the required fundamental group. (Exercise: fill in the details.) □

Example 4.72. Carrying out this construction for the group $\mathbb{Z}/2$, one has

$$\begin{aligned}\mathbb{Z}/2 &= \{0, 1\} \\ \mathbb{Z}/2 \times \mathbb{Z}/2 &= \{(0, 0), (0, 1), (1, 0), (1, 1)\}\end{aligned}$$

so that $X_{\mathbb{Z}/2}$ is built from $S^1 \vee S^1$ by attaching four 2-cells.

This space is *homotopy equivalent* to $\mathbb{R}P^2$; the cells and relations associated to $0 \in \mathbb{Z}/2$ are redundant.

Remark 4.73. Theorem 4.71 shows that algebraic topology *contains* group theory! ✓11/12/13

Remark 4.74. There is a *better* construction, which is given by the *classifying space* of a group. This is a special case of the construction of the *classifying space $B\mathscr{C}$* of a small category $\mathscr{C}$. See Example 6.42 and Remark 6.47.

4.11. **Addendum: mapping cylinders and mapping cones.** The mapping cone is a very important construction in homotopy theory; it was introduced in Section 4.8 in the special case of attaching cells. To introduce the general construction, we first consider the *mapping cylinder*. ✓10/10/13

Recall that the cylinder on a topological space X is the space $X \times I$; the inclusion $X \hookrightarrow X \times I$, $x \mapsto (x, 0)$ gives the subspace $X_0 \subset X \times I$, which is a strong deformation retract of $X \times I$ (see Definition 3.24), with retract the projection $X \times I \overset{p_X}{\to} X$. In particular, X and $X \times I$ have the same homotopy type. The same construction works replacing I with the open subspaces $[0, \frac{2}{3})$, for example.

Definition 4.75. For $f : X \to Y$ a continuous map, the *mapping cylinder* M_f is the quotient space of $(X \times I) \amalg Y$

$$M_f := (X \times I) \cup_{(x,1) \sim f(x)} Y$$

which corresponds to gluing the end of the cylinder $X \times \{1\}$ to Y using the map f.

The inclusion $Y \hookrightarrow M_f$ and the retraction $M_f \to Y$ induced by the projection $X \times I \to X$ exhibits Y as a deformation retract of M_f and the inclusion $X_0 \subset X \times I$ provides an inclusion $X \overset{i}{\hookrightarrow} M_f$.

Lemma 4.76. *For $f : X \to Y$, the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[dr, "f"'] & M_f \arrow[d, "\simeq"] \\ & Y. \end{tikzcd}$$

Proof. Exercise. □

Remark 4.77. The mapping cylinder M_f has a standard open cover:

$$\begin{aligned} U &:= X \times [0, \frac{2}{3}) \\ V &:= \big(X \times (\frac{1}{3}, 1]\big) \cup_f Y \end{aligned}$$

so that X is a deformation retract of U and $V \simeq Y$. The intersection $U \cap V$ identifies with the subspace $X \times (\frac{1}{3}, \frac{2}{3})$ of the cylinder, and has the homotopy type of X.

The factorization of f given by Lemma 4.76 replaces f by the inclusion $X \hookrightarrow M_f$ followed by the homotopy equivalence $M_f \overset{\simeq}{\to} Y$. The open neighbourhood U of X means that the inclusion has *good homotopical properties*.

Definition 4.78. For $f : X \to Y$ a continuous map, the *mapping cone* C_f is the quotient

$$C_f := M_f/(x, 0) \sim (x', 0)$$

which collapses the subspace $X \times \{0\}$ of the attached cylinder to a point. This is homeomorphic to the space obtained by attaching the cone CX to Y using the map f to glue the base.

Remark 4.79. The open cover $M_f = U \cup V$ induces an open cover $\underline{U}, \underline{V}$ by passage to the quotient. Here $\underline{V}$ is homeomorphic to V and $\underline{U} \cap \underline{V} \cong X \times (\frac{1}{3}, \frac{2}{3})$ has the homotopy type of X. However, the space $\underline{U}$ is *contractible* (it is a cone).

Remark 4.80. There is a quotient map

$$C_f \cong CX \cup_f Y \twoheadrightarrow Y/\mathrm{Image} f$$

given by collapsing the cone CX to a point.

In general there spaces are *not* homotopy equivalent.

Example 4.81.

(1) Consider $f : X = \{*_1, *_2\} \to \{*\} = Y$. The space $Y/\mathrm{Image} f$ is simply $\{*\}$, whereas the cone C_f is homeomorphic to S^1. These spaces do not have the same homotopy type.

(2) Consider $f, g : \partial I \rightrightarrows I$, where f is the constant map at 0 and g is the inclusion of the boundary. Since I is contractible, f, g are homotopic. However:

$$\begin{aligned} I/\mathrm{Image} f &\cong I \\ I/\mathrm{Image} g &\cong S^1, \end{aligned}$$

in particular, these two spaces do not have the same homotopy type. Hence the quotient by the image of a map does *not* behave well in homotopy theory.

Remark 4.82. The construction of the mapping cone is *homotopy invariant*: if $f \simeq g$, then $C_f \simeq C_g$.

5. Covering spaces

5.1. **Slice categories.**

Definition 5.1. For $\mathscr{C}$ a category and $X \in \mathrm{Ob}\,\mathscr{C}$, let $\mathscr{C} \downarrow X$ denote the category of objects over X:

- ▷ objects: morphisms with range X in $\mathscr{C}$: $E \xrightarrow{f} X$;
- ▷ a morphism from $E \xrightarrow{f} X$ to $E' \xrightarrow{f'} X$ is a morphism $g : E \to E'$ which makes the following diagram commute:

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "f"'] & & E' \arrow[dl, "f'"] \\ & X. & \end{tikzcd}$$

Exercise 5.2. For $\mathscr{C}, X$ as above,

(1) check that $\mathscr{C} \downarrow X$ is a category; if $\mathscr{C}$ is small, show that $\mathscr{C} \downarrow X$ is small;
(2) show that a morphism $\beta : X \to Y$ induces a functor $\mathscr{C} \downarrow \beta : \mathscr{C} \downarrow X \to \mathscr{C} \downarrow Y$;
(3) deduce that, if $\mathscr{C}$ is small, $\mathscr{C} \downarrow -$ defines a functor $\mathscr{C} \to \mathfrak{CAT}$ to the category of small categories.

Example 5.3. The case of interest here is where $\mathscr{C} = \mathfrak{Top}$ is the category of topological spaces and continuous maps; the category $\mathfrak{Top} \downarrow B$ is the category of topological spaces over B.

Definition 5.4. For B a topological space and $E_1 \xrightarrow{f_1} B$, $E_2 \xrightarrow{f_2} B$

(1) the *coproduct* $(E_1 \xrightarrow{f_1} B) \amalg (E_2 \xrightarrow{f_2} B)$ in $\mathfrak{Top} \downarrow B$ is the topological space $E_1 \amalg E_2$ equipped with the continuous map $E_1 \amalg E_2 \xrightarrow{f_2 \amalg f_2} B$; ✓07/11/13
(2) the *fibre product* in $\mathfrak{Top} \downarrow B$ is the topological space $E_1 \times_B E_2$ (the subspace $\{(e_1, e_2) \in E_1 \times E_2 | f_1(e_1) = f_2(e_2)\}$), equipped with the continuous map $E_1 \times_B E_2 \to B$ induced by $(e_1, e_2) \mapsto f_1(e_1) = f_2(e_2)$.

Proposition 5.5. *For $A \to B$ a continuous map, the fibre product $- \times_B A$ induces the* base change functor

$$\begin{array}{rcl} \mathfrak{Top} \downarrow B & \to & \mathfrak{Top} \downarrow A \\ (E \to B) & \mapsto & (E \times_B A \xrightarrow{p_A} A) \end{array}$$

where p_A is induced by the projection $E \times A \to A$.

Proof. Exercise. □

Remark 5.6. Base change or *pull back* is a fundamental construction; frequently we are interested in studying objects $E \xrightarrow{f} B$ where f has specified properties which are preserved under pullback.

5.2. **Local homeomorphisms, locally trivial maps and covering maps.**

Covering spaces arise naturally in algebraic topology. The first non-trivial examples occur in considering the circle; here S^1 is considered as the subspace of $\mathbb{C}$ $\{z | \, |z| = 1\}$.

Example 5.7. Consider the following continuous maps

(1) $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi t}$;
(2) $[n] : S^1 \to S^1$, for $0 \neq n \in \mathbb{Z}$, given by $z \mapsto z^n$.

Both p and $[n]$ are surjective (this is why $n = 0$ is excluded); for instance, the inverse image $p^{-1}(1)$ is $\mathbb{Z} \subset \mathbb{R}$, whereas the inverse image $[n]^{-1}(1)$ is the multiplicative subgroup of nth roots of unity $\in \mathbb{C}$, which is isomorphic to $\mathbb{Z}/n$. More is true: there is nothing special about the choice of the point $1 \in S^1$ above.

Moreover, as remarked above, $p(0) = 1$; restricting the map p to the open interval $(-\frac{1}{2}, \frac{1}{2})$, p defines a *homeomorphism*

$$\mathbb{R} \supset (-\frac{1}{2}, \frac{1}{2}) \xrightarrow{p,\cong} S^1 \backslash \{-1\} \subset S^1$$

from an open neighbourhood of $0 \in \mathbb{R}$ to an open neighbourhood of $1 \in S^1$. Better still everything can be shifted by an integer k. (Exercise: exhibit a similar property of $[n] : S^1 \to S^1$.)

The property indicated above is that of a local homeomorphism:

Definition 5.8. A continuous map $f : X \to Y$ is a *local homeomorphism* if, for every point $x \in X$, there exists an open neighbourhood $x \in U_x \subset X$ such that $f(U_x)$ is open in Y and $f|_{U_x} : U_x \stackrel{\cong}{\to} f(U_x)$ is a homeomorphism.

Example 5.9.

(1) Every homeomorphism is a local homeomorphism.
(2) The maps $p, [n] : S^1 \to S^1$ ($n \neq 0$) are local homeomorphisms.
(3) The inclusion $(0, 1) \hookrightarrow \mathbb{R}$ is a local homeomorphism. In particular, an open homeomorphism is not necessarily surjective.
(4) Recall (Example 2.15) that $\mathbb{R}^{\ominus}$ is the real line with the origin doubled. Identifying the two origins gives a continuous surjection
$$q : \mathbb{R}^{\ominus} \twoheadrightarrow \mathbb{R}.$$
This is a quotient map, which is a local homeomorphism. The inverse image $q^{-1}(t)$, for $t \in \mathbb{R}$ is a single point everywhere, *except* at $t = 0$.

Lemma 5.10. *A local homeomorphism $f : X \to Y$ is an* open *map (for every open subset $U \subset X$, $f(U)$ is open in Y).*

Proof. Exercise. □

Exercise 5.11. Give an example of an open map which is *not* a local homeomorphism.

Proposition 5.12. *For $X \xrightarrow{f} Y \xrightarrow{g} Z$ continuous maps,*

(1) *if f, g are both local homeomorphisms, then so is $g \circ f$;*
(2) *if g and $g \circ f$ are both local homeomorphisms, then so is f.*

Proof. Suppose that f, g are local homeomorphisms, thus there exist open neighbourhoods $x \in U_x \subset X$ and $f(x) \in V_{f(x)} \subset Y$ which satisfy the conditions of Definition 5.8 for f and g respectively. Then $f(U_x) \cap V_{f(x)} \subset Y$ is open and contains $f(x)$; the subset $W_x := f^{-1}\big(f(U_x) \cap V_{f(x)}\big) \cap U_x \subset X$ is an open neighbourhood of x. By construction, $g \circ f(W_x)$ is an open subset of $g(V_{f(x)})$, hence is open in Z; moreover, $(g \circ f)|_{W_x}$ is a homeomorphism onto $g \circ f(W_x)$. This proves the first point.

For the second point, since $g \circ f$ is a local homeomorphism by hypothesis, $\forall x \in X$ there exists an open neighbourhood $x \in U'_x$ such that $g \circ f(U'_x)$ is open in Z and the restriction of $g \circ f$ to U'_x is a homeomorphism. Similarly, there is an open neighbourhood $f(x) \in V_{f(x)}$ as above. Consider the open subspace $A_{g \circ f(x)} := g \circ f(U'_x) \cap g(V_{f(x)}) \subset Z$, which contains $g \circ f(x)$. The inverse image of $A_{g \circ f(x)}$ under the homeomorphism $g \circ f|_{U'_x}$ is the required open neighbourhood of x for the continuous map f. □

Exercise 5.13. Give examples of continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that

(1) $g \circ f$ is a local homeomorphism (such as the identity map!) but f (hence g also) is not a local homeomorphism;
(2) $f, g \circ f$ are local homeomorphisms but g is not. (Hint: f can be the inclusion of an open subset.)

The local homeomorphisms of Example 5.7 have a further property: they are *locally trivial*. The following is the general definition of local triviality.

Definition 5.14. A continuous map $p : E \to B$ is

(1) *trivial* (or a *projection*) if there exists a space F and a homeomorphism $E \stackrel{\cong}{\to} F \times B$ which makes the following diagram commute:

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[d] & F \times B \arrow[dl, "\mathrm{pr}_B"] \\ B, & \end{tikzcd}$$

where pr_B is the projection onto B;
(2) *locally trivial* if there exists an open cover (a *trivializing cover*) $\mathscr{U} := \{U_i | i \in \mathscr{I}\}$ such that the restriction $p^{-1}(U_i) \to U_i$ is trivial $\forall i \in \mathscr{I}$.

Exercise 5.15. When is a locally trivial continuous map a local homeomorphism?

Remark 5.16. Frequently when studying locally trivial maps, one imposes a condition on how the spaces $p^{-1}(U_i)$ can be glued together to form E. This is considered in the general theory of *fibre bundles*.

Definition 5.17. A continuous map $p : E \to B$ is

(1) a *covering* (of B) if it is locally trivial and, $\forall b \in B$, the fibre $p^{-1}(b)$ is a **non-empty** discrete topological subspace of E;
(2) a *finite covering* (of B) if, in addition, each $|p^{-1}(b)| < \infty$ $\forall b \in B$. If the cardinality $|p^{-1}(b)| = n$ is constant, p is a covering with n-sheets (or leaves).

Remark 5.18. The non-empty hypothesis ensures that a covering map is surjective.

Example 5.19.

(1) If B is a topological space and K a set (considered as a topological space with the discrete topology), the projection $\mathrm{pr}_B : K \times B \to B$ is a covering map.
(2) The map $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi t}$ is a covering map with fibre $p^{-1}(z) \cong \mathbb{Z}$, $\forall z \in S^1$.
(3) The map $[n] : S^1 \to S^1$, $z \mapsto z^n$ ($n \neq 0$) is a finite covering with n sheets.

Proposition 5.20. *If $p : E \to B$ is a covering, then p is a local homeomorphism.*

Proof. Exercise. □

Exercise 5.21. Show that

(1) the inclusion of an open subspace $U \subset X$ (which is a local homeomorphism) is a covering if and only if $U = X$;
(2) the quotient map $\mathbb{R}^{\ominus} \twoheadrightarrow \mathbb{R}$ is not a covering.

Proposition 5.22. *For $p : E \to B$ a covering map, the map*

$$B \to \mathbb{N}, b \mapsto |p^{-1}(b)|$$

is continuous, where $\mathbb{N}$ is given the discrete topology. In particular, the cardinality of fibres is constant on each connected component of B.

Proof. By hypothesis, $\forall b \in B$ there is an open neighbourhood $b \in U \subset B$ such that $p^{-1}(U) \cong K \times U$, for some set K and the restriction $p|_{p^{-1}(U)}$ identifies the projection. In particular, the cardinality of the fibres is constant on U. □

Proposition 5.23. *For $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ two covering maps,*

(1) *the map $p_1 \amalg p_2 : E_1 \amalg E_2 \to B$ is a covering, which is finite if and only if both p_1 and p_2 are finite;*
(2) *the fibre product $E_1 \times_B E_2$, equipped with the induced map $p : E_1 \times_B E_2 \to B$, $(e_1, e_2) \mapsto p_1(e_1)$ is a covering, which is finite if and only if both p_1 and p_2 are finite.*

Proof. Exercise. □

5.3. **Morphisms.**

Definition 5.24. For $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ covering spaces, a morphism of covering spaces $(E_1 \to B) \to (E_2 \to B)$ is a morphism in $\mathfrak{Top} \downarrow B$, namely a continuous map $f : E_1 \to E_2$ which makes

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

commute.

Remark 5.25. Proposition 5.12 implies that the continuous map $f : E_1 \to E_2$ of Definition 5.24 is a local homeomorphism.

Proposition 5.26. *Covering spaces over B and morphisms of covering spaces over B form a category $\mathfrak{Cover}(B)$.*

Proof. Exercise. □

Remark 5.27. Proposition 5.26 implies that there is a natural notion of *isomorphism* of covering spaces; this is simply a morphism of covering spaces corresponding to a *homeomorphism* $f : E_1 \to E_2$ (as in Definition 5.24).

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Example 5.28. For B a connected topological space and sets K, L (discrete topological spaces), consider a morphism between the trivial coverings

$$\begin{tikzcd} B \times K \arrow[rr, "f"] \arrow[dr] & & B \times L \arrow[dl] \\ & B. & \end{tikzcd}$$

By commutativity of the diagram, f is of the form $(b, k) \mapsto (b, g(b, k))$ for a continuous map $g : B \times K \to L$. In particular, for fixed $k \in K$, $g(-, k) : B \to L$ is continuous, hence is constant, since L is a discrete topological space and B is connected, by hypothesis.

It follows that the morphism of coverings is equivalent to a set map $\overline{f} : K \to L$. In particular $f : B \times K \to B \times L$ is a covering map if and only if $\overline{f}$ is surjective.

Proposition 5.29. *For B a locally connected topological space and a morphism of covering spaces*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B, & \end{tikzcd}$$

(1) *the image of f defines covering spaces* $\mathrm{Image}(f) \to B$ *and* $E_2 \backslash \mathrm{Image}(f) \to B$ *and the induced morphism* $E_1 \to \mathrm{Image}(f)$ *is a morphism of covering spaces over* B*;*
(2) *there is an isomorphism of covering spaces* $E_2 \cong \mathrm{Image}(f) \amalg (E_2\backslash \mathrm{Image}(f))$ *over* B*;*
(3) *the morphism* $E_1 \to \mathrm{Image}(f)$ *is a covering.*

In particular, if f is surjective (for instance if E_2 is connected), then $f : E_1 \to E_2$ is a covering.

Proof. The result is proved by reducing to the case where both p_1 and p_2 are trivial coverings over a connected topological space; this case follows using the analysis of Example 5.28.

For the general case, for any $b \in B$, there exists a connected open neighbourhood $b \in U \subset B$ such that p_1, p_2 are trivial when restricted to U, since B is locally connected, by hypothesis. □

Proposition 5.30. *Let $p : E \to B$ be a covering. Then the associated covering $p^{[2]} : E \times_B E \to B$ is isomorphic to*

$$\big(E \to B\big) \amalg \big(E' \to B\big)$$

where $E' \subset E \times_B E$ is the subspace of points (e_1, e_2) such that $e_1 \neq e_2$ and $E \to B$ the diagonal subspace $E_{\mathrm{diag}} \subset E \times_B E$ of points (e, e).

Proof. As sets it is clear that $E \times_B E = E' \amalg E_{\mathrm{diag}}$; by considering local behaviour it follows that this is a homeomorphism of topological spaces and that the projections are coverings. □

Remark 5.31. If B is locally connected, one can deduce the result from Proposition 5.29, since the diagonal map induces a morphism of covering spaces:

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr] & & E \times_B E \arrow[dl] \\ & B. & \end{tikzcd}$$

5.4. **Lifting maps.**

Definition 5.32. For $p : E \to B$ a covering and $g : X \to B$ a continuous map, a *lifting* of g is a continuous map $\tilde{g} : X \to E$ which defines a morphism of $\mathfrak{Top}/B$, so that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "\tilde{g}"] \arrow[r, "g"'] & B. \end{tikzcd}$$

Example 5.33. Liftings do not always exist; for example:

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^1 \arrow[ur, dotted, "\nexists"] \arrow[r, "\mathrm{Id}"'] & S^1. \end{tikzcd}$$

This can be proved using the fundamental group: the space $\mathbb{R}$ is contractible, hence has $\pi_1(\mathbb{R}, *) = \{e\}$, whereas $\pi_1(S^1, *) \cong \mathbb{Z}$. The group $\mathbb{Z}$ is not a retract of the trivial group $\{e\}$.

Under a connectivity hypothesis, liftings (when they exist) are unique:

Proposition 5.34. *For $p : E \to B$ a covering and $g : X \to B$ a continuous map, where X is connected, two liftings $\tilde{g}_1, \tilde{g}_2 : X \rightrightarrows E$ of g coincide if and only if $\exists x \in X$ such that $\tilde{g}_1(x) = \tilde{g}_2(x)$.*

Proof. The liftings $\tilde{g}_1, \tilde{g}_2$ induce a continuous map $G : X \to E \times_B E$, by $x \mapsto (\tilde{g}_1(x), \tilde{g}_2(x))$ which fits into a commutative diagram

$$\begin{tikzcd} & E \times_B E \arrow[r, "\cong"] \arrow[d, "p^{[2]}"] & E_{\mathrm{diag}} \amalg E' \arrow[dl] \\ X \arrow[ur, "G"] \arrow[r, "g"'] & B, & \end{tikzcd}$$

where the isomorphism of coverings is provided by Proposition 5.30.

The hypothesis that X is connected implies that, under the isomorphism, G maps either to E_{diag} or to E'. In the first case $\tilde{g}_1 = \tilde{g}_2$ and in the second, $\forall x \in X$, $\tilde{g}_1(x) \neq \tilde{g}_2(x)$. $\square$

Example 5.35. Recall that a covering map is a local homeomorphism. Unicity of liftings does not hold in general for local homeomorphisms: for instance, consider the quotient map

$$q : \mathbb{R}^{\ominus} \to \mathbb{R}$$

which identifies the 'two origins' (see Example 2.15 for $\mathbb{R}^{\ominus}$). As observed in Example 5.9, this is a local homeomorphism. However, there are two *sections* of q (that is *lifts* of the identity $\mathrm{Id}_{\mathbb{R}}$), corresponding to which of the 'two origins' is chosen. These coincide for all $0 \neq t \in \mathbb{R}$.

The behaviour of the morphism of fundamental groupoids $\Pi(p) : \Pi(E) \to \Pi(B)$ associated to a covering p determines the covering under suitable hypotheses. The key ingredient is the lifting of paths and homotopies.

Theorem 5.36. *For $p : E \to B$ a covering and $\alpha, \beta : I \rightrightarrows B$ two paths such $\alpha \sim_{\mathrm{rel}\ \partial I} \beta$ via a homotopy $H : I \times I \to B$,*

(1) *for any $e \in p^{-1}(\alpha(0))$, there exist unique lifts $\tilde{\alpha}_e, \tilde{\beta}_e : I \to E$ of α, β respectively such that $\tilde{\alpha}_e(0) = e = \tilde{\beta}_e(0)$;*
(2) *there exists a unique lift $\tilde{H} : I \times I \to B$ of H which is a homotopy* rel ∂I *between $\tilde{\alpha}$ and $\tilde{\beta}$; in particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$.*

Proof. Consider the path $\alpha : I \to B$ (the argument for β is identical) and take an open cover $\mathscr{U} := \{U_i | i \in \mathscr{I}\}$ which trivializes the covering, so that $p^{-1}(U_i) \cong U_i \times K_i$, for some discrete topological space K_i. By Lebesgue's theorem, Proposition 4.38, $\exists 0 < N \in \mathbb{N}$ such that $\forall 0 \leq s < N$, $\exists i_s \in \mathscr{I}$ such that $\alpha([\frac{s}{N}, \frac{s}{N}] \subset U_{i_s}$.

The lifting $\tilde{\alpha}$ is constructed inductively on s for each sub-interval of the form $[0, \frac{s}{N}]$. To start the process, one takes $\tilde{\alpha}(0) = x$. Suppose $\tilde{\alpha}$ constructed on $[0, \frac{s}{N}]$ and consider the extension to $[0, \frac{s+1}{N}] = [0, \frac{s}{N}] \cup [\frac{s}{N}, \frac{s+1}{N}]$. By hypothesis of the trivializing cover, $p^{-1}(U_{i_s})$ is a trivial covering, hence there exists a *unique* lifting of $\alpha|_{[\frac{s}{N}, \frac{s+1}{N}]}$ to $p^{-1}(U_{i_s}) \subset E$ with value at $\frac{s}{N}$ equal to $\tilde{\alpha}(\frac{s}{N})$. By construction this gives an extension to a continuous map $\tilde{\alpha}$ defined on $[0, \frac{s+1}{N}]$.

Consider the homotopy $H : I \times I \to B$; the lifting to $\tilde{H}$ uses a generalization of the previous argument. Lebesgue's theorem provides a decomposition of $I \times I$ into squares of edge $\frac{1}{N}$ each of which maps to just one open of $\mathscr{U}$. As before, one constructs a lifting by adding the little squares; the unicity result of Proposition 5.34 implies that the individual lifts can be glued along the edges to define the continuous map $\tilde{H}$.

It remains to show that $\tilde{H}$ is a homotopy rel ∂I between $\tilde{\alpha}$ and $\tilde{\beta}$. By construction $\tilde{H}|_{\{0\} \times I}$ and $\tilde{H}|_{\{1\} \times I}$ are lifts of the respective constant maps at $\alpha(0)$ and $\alpha(1)$.

By unicity of liftings (Proposition 5.34), these liftings are respectively the constant maps at $\tilde{\alpha}(0) = x$ and at $\widetilde{\alpha(1)}$.

Similarly, by uniqueness of liftings, $\tilde{H}|_{I\times\{0\}} = \tilde{\alpha}$ and $\tilde{H}|_{I\times\{1\}} = \tilde{\beta}$. It follows, since $\tilde{H}|_{\{1\}\times I}$ is constant at $\tilde{\alpha}(1)$, that $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{H}$ is a relative homotopy, as required. □

Corollary 5.37. *For $p : E \to B$ a covering,*

(1) *the groupoid morphism $\Pi(p) : \Pi(E) \to \Pi(B)$ is faithful ($\forall e_1, e_2 \in E$, the map $\mathrm{Hom}_{\Pi(E)}(e_1, e_2) \to \mathrm{Hom}_{\Pi(B)}(p(e_1), p(e_2))$ is injective);*
(2) *$\forall e \in E$, the group morphism $\pi_1(p) : \pi_1(E, e) \to \pi_1(B, p(e))$ is injective.*

Proof. Consider two morphisms $[\tilde{\alpha}], [\tilde{\beta}] \in \mathrm{Hom}_{\Pi(E)}(e_1, e_2)$, represented by paths $\tilde{\alpha}, \tilde{\beta} : I \rightrightarrows E$. If $\Pi(p)[\tilde{\alpha}] = \Pi(p)[\tilde{\beta}]$, then $\alpha := p(\tilde{\alpha})$ and $\beta := p(\tilde{\beta})$ are homotopic $\mathrm{rel}\ \partial I$. By construction, $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of α and β respectively, with the same starting point, e_1. By the lifting of homotopies, Theorem 5.36, it follows that $\tilde{\alpha}$ and $\tilde{\beta}$ are homotopic $\mathrm{rel}\ \partial I$.

The second statement follows from the first. □

Theorem 5.38. *For a covering $p : E \to B$ and $g : X \to B$ a continuous map, where X is connected and locally path connected, there exists a lifting*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "x\mapsto e", "\tilde{g}"'] \arrow[r, "g"'] & B \end{tikzcd}$$

(where $p(e) = g(x)$) if and only if $g_*(\pi_1(X, x)) \subset \mathrm{Image}\{\pi_1(E, e) \stackrel{p_*}{\to} \pi_1(B, g(x))\}$.

Proof. The condition on π_1 is clearly necessary.

For the converse, the construction is carried out in two steps: first a map $\tilde{g}$ is constructed using path lifting and then the local path connectivity is used to show that it is continuous. The hypotheses imply that X is path connected.

Fix $x \in X$ and $e \in E$ such that $p(e) = g(x)$; for a point $y \in X$, define $\tilde{g}(y) := \widetilde{g(\alpha)}_e(1)$, where α is any path from x to y in X. By Theorem 5.36, $\tilde{g}(y)$ only depends on $[\alpha] \in \mathrm{Hom}_{\Pi(X)}(x, y)$.

Consider the hypothesis on π_1; by lifting of homotopies (Theorem 5.36), this implies that any composite pointed map $(S^1, *) \to (X, x) \to (B, g(x))$ lifts to a map $(S^1, *) \to (E, *)$. In particular, if α, β are two paths from x to y in X, the composite path $\beta^{-1} \circ \alpha$ defines a loop $(S^1, *) \to (X, x)$ which lifts to a *loop* as above. By construction this provides lifts $\widetilde{g(\alpha)}$ and $\widetilde{g(\beta)}$ which compose to form a loop - hence have the same endpoints, as required.

It remains to prove that $\tilde{g}$ is continuous. The space E has a basis of open subsets given by pairs (U, k) where $U \subset B$ is open such that $p^{-1}(U) \cong U \times K$ is a trivial covering and $k \in K$. It suffices to show that $\tilde{g}^{-1}(U, k)$ is open in X.

Consider $y \in \tilde{g}^{-1}(U, k) \subset X$; since X is locally path connected, there exists a path connected open subset $y \in V \subset X$ such that $g(V) \subset U$. By unicity of path lifting and the fact that $p^{-1}(U)$ is a trivial cover, it follows that $\tilde{g}$ maps V to (U, k). This completes the proof. □

Recall from Definition 4.18 the category $\mathfrak{Top}_\bullet$ of pointed (or based) topological spaces and the notation for the set of based homotopy classes $[(X, x), (Y, y)]_{\mathfrak{Top}_\bullet}$ of based maps from X to Y.

Definition 5.39. For (X, x) a pointed topological space and $0 < n \in \mathbb{N}$, the *nth homotopy group* is

$$\pi_n(X, x) := [(S^n, *), (X, x)]_{\mathfrak{Top}_\bullet}$$

where the addition of morphisms is induced by the *pinch map* $S^n \to S^n \vee S^n$ which collapses the equator to a point. (Exercise: prove that $\pi_n(X, x)$ is a group and that, for $n = 1$, this recovers the fundamental group.)

Remark 5.40.

(1) The nth homotopy group defines a functor
$$\pi_n(-) : \mathfrak{Top}_\bullet \to \mathfrak{Group}.$$
(Exercise: check this.)

(2) For $n \geq 2$, $\pi_n(X, x)$ is an *abelian* group. (Slightly harder exercise: why?)

Corollary 5.41. *For $p : E \to B$ a covering, basepoints $b \in B$, $e \in p^{-1}(b)$ and $2 \leq n \in \mathbb{N}$, the covering map p induces an isomorphism:*
$$\pi_n(p) : \pi_n(E, e) \xrightarrow{\cong} \pi_n(B, b).$$

Proof. For $n \geq 1$, S^n is connected and locally path connected and, for $n \geq 2$, Proposition 4.54 shows that $\pi_1(S^n, *) = 0$. Hence Theorem 5.38 implies that any based continuous map $(S^n, *) \to (B, b)$ lifts to a continuous map $(S^n, *) \to (E, e)$, which shows surjectivity of $\pi_n(p)$. To show injectivity, one uses the lifting of homotopies, as in Theorem 5.36. (Exercise: provide the details.) □

Remark 5.42. One way of understanding this result is by the *much more* general theory of *fibrations*, which is based on the lifting property of homotopies. In particular, a covering $p : E \to B$ is a fibration, with fibre $F := p^{-1}(b)$ (in the case of coverings, F is a discrete space). In general, a fibration $F \to E \to B$ gives a relationship between the homotopy groups $\pi_n(F)$, $\pi_n(E)$ and $\pi_n(B)$.

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5.5. **Morphisms between coverings and the universal cover.** The general lifting result, Theorem 5.38, has an immediate application to the study of the category of coverings $\mathfrak{Cover}(B)$ over B.

Proposition 5.43. *For B a connected and locally path connected topological space and two coverings $p_1 : E_1 \to B$, $p_2 : E_2 \to B$, where E_1 is connected, and points $b \in B$, $e_1 \in p_1^{-1}(b)$, $e_2 \in p_2^{-1}(b)$, there exists a morphism of coverings*
$$\begin{tikzcd} E_1 \arrow[rr, "e_1 \mapsto e_2"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$
if and only if $(p_1)_*\pi_1(E_1, e_1) \subset (p_2)_*\pi_1(E_2, e_2) \subset \pi_1(B, b)$.

Proof. An immediate consequence of Theorem 5.38. □

Recall from Definition 4.53 that a space X is simply connected if $|\pi_0(X)| = 1$ and $\pi_1(X, x) = \{e\}$, for any choice of basepoint.

Corollary 5.44. *For B a connected and locally path connected topological space and two coverings $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ and points $b \in B$, $e_1 \in p_1^{-1}(b)$, $e_2 \in p_2^{-1}(b)$,*

(1) *if E_1 is simply-connected, there is a unique morphism of coverings $E_1 \to E_2$ which sends e_1 to e_2;*

(2) *if E_2 is simply-connected and E_1 is connected but $\pi_1(E_1, e_1) \neq \{e\}$ there is* no *morphism of coverings from E_1 to E_2;*

(3) *if E_1, E_2 are both simply-connected, then any morphism of coverings from E_1 to E_2 is an isomorphism.*

Proof. The first two statements are straightforward applications of Proposition 5.43. The first statement and unicity of lifts implies the final statement. □

This shows the interest of the following definition:

Definition 5.45. A *universal cover* of a topological space B is a covering $p : E \to B$ such that E is simply-connected.

Remark 5.46.

(1) A universal cover need not exist.
(2) If B is connected and locally path connected, Corollary 5.44 implies that (if it exists) a universal cover is unique up to isomorphism (but *not* unique isomorphism, in general). Thus one can talk about *the* universal cover, without ambiguity; this is frequently denoted
$$\tilde{p} : \tilde{B} \to B.$$
(3) If B admits a universal cover, B is path connected.

Example 5.47.

(1) The universal cover of the circle is the covering $\mathbb{R} \to S^1$.
(2) The universal cover of the torus $S^1 \times S^1$ is $\mathbb{R}^2 \to S^1 \times S^1$, given by the product of two copies of the universal covering of S^1.
(3) The universal covering of $\mathbb{R}P^2$ is the quotient map $S^2 \to \mathbb{R}P^2$, associated to the antipodal action (see Example 1.44.)

To prove the *existence* of a universal cover, the following condition is introduced.

Definition 5.48. A topological space B is *semi-locally simply-connected* if, $\forall b \in B$, $\exists b \in U \subset B$ an open neighbourhood such that the inclusion $U \hookrightarrow B$ induces the trivial group homomorphism
$$\pi_1(U, b) \stackrel{\{e\}}{\to} \pi_1(B, b).$$
(Every loop in U based at b is based homotopic *in* B to the constant loop.)

This condition is *necessary* for the existence of a universal cover. Here it is not necessary to assume that B is locally path connected (but see the discussion in Remark 5.46).

Proposition 5.49. *For B a topological space, if a universal cover $\tilde{p} : \tilde{B} \to B$ exists, then B is semi-locally simply-connected.*

Proof. For $b \in B$, there exists an open neighbourhood $b \in U \subset B$ such that $\tilde{p}^{-1}(U)$ is a trivial covering. This satisfies the required hypothesis, since a loop α in U based at b lifts to a *loop* $\tilde{\alpha}$ in $\tilde{p}^{-1}(U) \subset \tilde{B}$, since the open U trivializes the covering. By hypothesis, the space $\tilde{B}$ is simply-connected, hence the loop $\tilde{\alpha}$ is based homotopic to a constant loop in $\tilde{B}$ (*not* necessarily in $\tilde{p}^{-1}(U)$), by a homotopy H. The based homotopy $\tilde{p} \circ H$ shows that α is homotopic to the constant loop at b in B, as required. □

Remarkably, this condition is also *sufficient* when B is connected and locally path connected. The key point is the following Lemma.

Lemma 5.50. *For B a topological space and U a neighbourhood of b in B such that $\pi_1(U, b) \stackrel{\{e\}}{\to} \pi_1(B, b)$ is the trivial morphism, the image of $[\gamma] \in \mathrm{Hom}_{\Pi(U)}(u, b)$ in $\mathrm{Hom}_{\Pi(B)}(u, b)$ depends only upon $u \in U$.*

Proof. Suppose that γ_1, γ_2 are two paths in U from u to b; then the composite path $\gamma_2 \circ \gamma_1^{-1}$ is a loop in U based at b. By the hypothesis, this becomes based homotopically trivial in B; thus, in the fundamental groupoid $\Pi(B)$:
$$[\gamma_2] \circ [\gamma_1]^{-1} = \mathrm{Id}_b.$$
This implies that $[\gamma_1] = [\gamma_2]$ in $\Pi(B)$, as required. □

Theorem 5.51. *For B a topological space which is connected and locally path connected, B admits a universal cover if and only if B is semi-locally simply-connected.*

Proof. (Sketch.) The condition is necessary, by Proposition 5.49. Hence it suffices to show existence, under the hypotheses.

Fix a basepoint $* \in B$ and define the underlying set of $\tilde{B}$ to be

$$\tilde{B} := \amalg_{b \in B} \mathrm{Hom}_{\Pi(B)}(b, *)$$

equipped with the projection $\tilde{p} : \tilde{B} \to B$ which sends $[\gamma] \mapsto \gamma(0)$. Thus, the fibre at $b \in B$ is the set $\mathrm{Hom}_{\Pi(B)}(b, *)$.

It remains to

(1) define the topology on $\tilde{B}$, using the fact that each $b \in B$ admits a path-connected open neighbourhood $b \in U$ such that $\pi_1(U, b) \to \pi_1(B, b)$ is the trivial morphism (using Lemma 5.50);
(2) show that $\tilde{p}$ is continuous;
(3) show that $\tilde{p}$ is a covering;
(4) show that $\tilde{B}$ is path connected and deduce that $\tilde{B}$ is a universal cover.

□

5.6. **Coverings from group actions.** In general, if a group G acts continuously on a topological space X, the quotient map

$$X \twoheadrightarrow G \backslash X$$

(which sends a point x to its G-orbit) is not a covering map.

Example 5.52. The action of $\mathbb{Z}/2$ on $\mathbb{R}$ by $t \mapsto -t$ defines a quotient map

$$\mathbb{R} \twoheadrightarrow [0, \infty) \subset \mathbb{R}$$

which is not a covering.

Definition 5.53. A left action of the (discrete) group G on a topological space X is *totally discontinuous* if, $\forall x \in X$, $\exists x \in U_x$ an open neighbourhood such that $gU_x \cap U_x \neq \emptyset$ if and only if $g = e$.

Remark 5.54. A totally discontinuous action $G \times X \to X$ is necessarily *free*.

Proposition 5.55. *For a totally discontinuous action $G \times X \to X$ of a discrete group on the topological space X, the quotient map*

$$X \twoheadrightarrow G \backslash X$$

is a covering, with fibre G.

Moreover, the left action of the group G on X is by automorphisms of the covering.

Proof. For a trivializing cover, take $\{U_x | x \in X\}$, where U_x is as in Definition 5.53.

It is clear that the homeomorphism $g : X \to X$, for $g \in G$, defined by the left action of G, induces an automorphism of the covering. □

Example 5.56. The antipodal action of $\mathbb{Z}/2$ on the sphere S^n is totally discontinuous. The associated covering (Example 1.44) is

$$S^n \twoheadrightarrow \mathbb{R}P^n.$$

For $n > 1$, by Proposition 4.54 the sphere S^n is simply-connected and locally path connected, so this is the universal cover.

Proposition 5.57. *For $\tilde{p} : \tilde{B} \to B$ the universal cover of a connected and locally path connected space B and $b \in B$, the group $\pi_1(B, b)$ acts naturally on the left by automorphisms of the covering $\tilde{p}$ and the group action is totally discontinuous. The universal covering is isomorphic to the quotient covering*

$$\tilde{B} \twoheadrightarrow \pi_1(B, b) \backslash \tilde{B}.$$

Proof. (Indication.) The result can be proved from the construction of the universal cover indicated in the proof of Theorem 5.51. □

5.7. **Monodromy and classification.** The lifting of paths gives rise to the monodromy action of the fundamental group on the fibre of a covering. Recall that, for G a group, a right G-set is a set X equipped with an action:

$$\begin{array}{rcl} X \times G & \to & X \\ (x, g) & \mapsto & xg \end{array}$$

which is associative ($(xg)h = x(gh)$) and unital ($xe = x$).

Notation 5.58. For G a group, write $\mathfrak{Set}_G$ for the category of *right G-sets* and G-equivariant morphisms and $\overline{\mathfrak{Set}}_G$ for the full subcategory of *non-empty* right G-sets.

Theorem 5.59. *For $p : E \to B$ a covering and $b \in B$, there exists a natural right action of $\pi_1(B, b)$ on the fibre $p^{-1}(b)$. This corresponds to a functor:*

$$\begin{array}{rcl} \mathfrak{Cover}(B) & \to & \overline{\mathfrak{Set}}_{\pi_1(B,b)} \\ p & \mapsto & p^{-1}(b). \end{array}$$

Moreover,

(1) *the stabilizer in $\pi_1(B, b)$ of a point $e \in p^{-1}(b)$ is the subgroup $p_*(\pi_1(E, e)) \subset \pi_1(B, b)$;*
(2) *E is path connected if and only if the $\pi_1(B, b)$ action is transitive.*

Proof. Recall that, by definition of a covering, the fibre $p^{-1}(b)$ is non-empty.

Consider $e \in p^{-1}(b)$ and a loop α based at b which defines an element $[\alpha] \in \pi_1(B, b)$. Define:

$$e[\alpha] := \tilde{\alpha}_e(1),$$

where $\tilde{\alpha}_e$ is the lift of α with $\alpha_e(0) = e$. By Theorem 5.36, $e[\alpha]$ depends only upon the class $[\alpha]$.

It is straightforward to check that this defines a *right* action of $\pi_1(B, b)$ on the fibre $p^{-1}(b)$ (this is where the convention used in defining the group structure of $\pi_1(B, b)$ intervenes). Moreover, the action is *natural* with respect to covering spaces.

An element $[\alpha]$ stabilizes e (that is $e[\alpha] = e$) if and only if the lift satisfies $\tilde{\alpha}_e(1) = e$, so that $\tilde{\alpha}_e$ is a loop based at $e \in E$. Thus $[\alpha]$ is in the image of the monomorphism $p_* : \pi_1(E, e) \hookrightarrow \pi_1(B, b)$.

If E is path connected, for any two points $e_1, e_2 \in p^{-1}(b)$ of the fibre, there is a path γ from e_1 to e_2 in E. The projection $p(\gamma)$ is a loop in B based at b, which has lift $\tilde{\gamma}_{e_1} = \gamma$. It follows that $e_1[p(\gamma)] = e_2$, thus the action is transitive.

It is straightforward to prove the converse: if the $\pi_1(B, b)$-action is transitive, then E is path connected. (Exercise.) □

When B is connected and locally path connected and admits a universal cover, the above provides an equivalence of categories. This shows the power of the fundamental group.

Theorem 5.60. *For B a connected and locally path connected space which admits a universal cover $\tilde{p} : \tilde{B} \to B$, the functor*

$$\begin{array}{rcl} \mathfrak{Cover}(B) & \to & \overline{\mathfrak{Set}}_{\pi_1(B,b)} \\ p & \mapsto & p^{-1}(b). \end{array}$$

is an equivalence of categories.

Proof. (Indications.) For F a non-empty right $\pi_1(B,b)$-set, define the topological space
$$E_F := F \times_{\pi_1(B,b)} \tilde{B}$$
using the left action of $\pi_1(B,b)$ on $\tilde{B}$ (see Proposition 5.57). This is the quotient of the product space $F \times \tilde{B}$ by the relation
$$(yg, x) = (y, gx)$$
$\forall y \in F, g \in \pi_1(B,b), x \in \tilde{B}$.

This construction is *functorial in F*, in particular the map $F \to *$ of $\pi_1(B,b)$-sets induces
$$p_F : E_F \to * \times_{\pi_1(B,b)} \tilde{B} \cong B,$$
by Proposition 5.57. The continuous map p_F is a covering map, with fibre F (exercise: prove this!).

Thus, the above defines a functor:
$$\begin{array}{ccc} \overline{\mathfrak{Set}}_{\pi_1(B,b)} & \to & \mathfrak{Cover}(B). \\ F & \mapsto & p_F : E_F \to B. \end{array}$$
It is straightforward to show that the fibre $p_F^{-1}(b)$ identifies with F with the given $\pi_1(B,b)$-action. Hence, to complete the proof, it suffices to show that there is a natural isomorphism of coverings
$$E_{p^{-1}(b)} \xrightarrow{\cong} \big(p : E \to B\big),$$
for any covering p. This is constructed by using the lifting result, Proposition 5.43. □

Remark 5.61.
(1) The universal cover corresponds to the free $\pi_1(B,b)$-set $\pi_1(B,b)$.
(2) The space E_F is path connected if and only if F is a transitive $\pi_1(B,b)$-set. The covering associated to a transitive $\pi_1(B,b)$ set $H\backslash\pi_1(B,b)$ (for H a subgroup of $\pi_1(B,b)$) is:
$$H\backslash\tilde{B} \to B$$
where $H\backslash\tilde{B}$ is the quotient of $\tilde{B}$ by the action of H on $\tilde{B}$. In particular, the universal covering map $\tilde{p}$ factorizes as covering maps:
$$\tilde{B} \to H\backslash\tilde{B} \to B.$$

It is essential to study examples in order to get a good understanding of coverings.

Example 5.62. Theorem 5.60 applies to the wedge $S^1 \vee S^1$ of two circles, which is usually pointed by the common point.
$$\pi_1(S^1 \vee S^1, *) \cong \mathbb{Z} \star \mathbb{Z}$$
is the free group on two generators. The category $\mathfrak{Set}_{\mathbb{Z}\star\mathbb{Z}}$ is *very* rich.

Exercise 5.63.
(1) Describe the universal cover of $S^1 \vee S^1$.
(2) Classify the two-sheeted covers of $S^1 \vee S^1$ (up to isomorphism).
(3) Slightly harder: classify the three-sheeted covers of $S^1 \vee S^1$.

Theorem 5.60 means that, to understand coverings, one must understand the category of G-sets.

Exercise 5.64. Let $H, K \leq G$ be subgoups of the group G and consider the transitive right G-sets $H\backslash G$ and $K\backslash G$.
(1) Show that

   (a) to give a map of G-sets $f : H\backslash G \to K\backslash G$, it suffices to specify the element $f(He)$ (the image of the coset of the identity $e \in G$);
   (b) there exists a map of G sets with $f(He) = K\gamma$ if and only if, $\forall h \in H$, $\gamma h \gamma^{-1} \in K$ (or $H \leq \gamma^{-1} K \gamma$).
(2) When are the G-sets $H\backslash G$ and $K\backslash G$ isomorphic?
(3) Describe the group of automorphisms of $H\backslash G$.

5.8. **Naturality.** The conclusion of Theorem 5.60 is *natural* in the base space B.

Lemma 5.65. *A group morphism $\varphi : G \to H$ induces a functor*

$$\varphi^* : \mathfrak{Set}_H \to \mathfrak{Set}_G$$

by restriction along φ, such that $\varphi^* X$ has underlying set X with H action given by $xg := x(\varphi(g))$.

Proof. Exercise. $\square$

Proposition 5.66. *For $f : B \to C$ a continuous map and basepoint $b \in B$, the following diagram commutes (up to natural isomorphism)*

$$\begin{tikzcd} \mathfrak{Cover}(C) \arrow[r, "p \mapsto p^{-1}(f(b))"] \arrow[d, "f^*"'] & \overline{\mathfrak{Set}}_{\pi_1(C, f(b))} \arrow[d, "\pi_1(f)^*"] \\ \mathfrak{Cover}(B) \arrow[r, "q \mapsto q^{-1}(b)"'] & \overline{\mathfrak{Set}}_{\pi_1(B,b)}, \end{tikzcd}$$

where f^* denotes the pullback of coverings along f.

Proof. Consider the behaviour on objects: for a covering $p : E \to C$; the pullback along f gives the commutative square of continuous maps

$$\begin{tikzcd} E \times_C B \arrow[r, "\overline{f}"] \arrow[d, "q := f^* p"'] & E \arrow[d, "p"] \\ B \arrow[r, "f"'] & C, \end{tikzcd}$$

in which the vertical morphisms are coverings.

The fibre $q^{-1}(b)$ of $q := f^* p$ over b identifies with $p^{-1}(c)$ via $\overline{f}$, hence it remains to consider the action of the fundamental groups. Consider a based loop α in B, based at b and an element x of the fibre $q^{-1}(b)$. The lift $\tilde{\alpha}$ of α such that $\tilde{\alpha}(0) = x$ defines a lift $\overline{f} \circ \tilde{\alpha}$ of $f \circ \alpha$ starting at $\overline{f}(x)$.

By the definition of the Monodromy action and unicity of path lifts, this shows that the $\pi_1(B, b)$ action on $q^{-1}(b)$ identifies with the restricted action $\pi_1(f)^* \big(p^{-1}(f(b))\big)$. To complete the proof, it remains to check that this identification is *natural* with respect to morphisms of covering spaces (exercise). $\square$

Corollary 5.67. *Let B, C be connected and locally path connected spaces which admit a universal cover. If $f : B \to C$ is a continuous map which induces an isomorphism $\pi_1(f) : \pi_1(B, b) \xrightarrow{\cong} \pi_1(C, f(b))$ for $b \in B$, then*

$$f^* : \mathfrak{Cover}(C) \to \mathfrak{Cover}(B)$$

is an equivalence of categories.

Proof. The result is a straightforward consequence of Proposition 5.66. $\square$

Remark 5.68. This result shows that, in order to understand covering spaces (for *nice* spaces, such as CW-complexes) it is sufficient to consider spaces which can be built from cells of dimension at most two. *Exercise: why?*

5.9. **Galois coverings.**

Notation 5.69. For $p : E \to B$ a covering, let $\mathrm{Aut}(p)$ denote the group of automorphisms (in $\mathfrak{Cover}(B)$) of p. (Elements are homeomorphisms $E \to E$ which commute with p.)

Lemma 5.70. *For $p : E \to B$ a covering and $b \in B$, the group $\mathrm{Aut}(p)$ acts (on the left) on the fibre $p^{-1}(b)$.*

Moreover, for any $e \in p^{-1}(b)$ the set map

$$\begin{array}{rcl} \mathrm{Aut}(p) & \hookrightarrow & p^{-1}(b) \\ g & \mapsto & g(e) \end{array}$$

is injective.

Proof. The first statement is clear and the second follows by unicity of liftings (Proposition 5.34). □

Definition 5.71. A covering $p : E \to B$ is *Galois* if E is path connected and the action of the automorphism group $\mathrm{Aut}(p)$ on the fibres is transitive.

Example 5.72. For X a path-connected topological space and a totally-discontinuous action $G \times X \to X$, the associated covering (cf. Proposition 5.55)

$$X \twoheadrightarrow G\backslash X$$

is Galois.

The classification result, Theorem 5.60, means that the Galois condition can be detected in terms of the associated $\pi_1(B, b)$-set.

Theorem 5.73. *For B a connected, locally path connected space which admits a universal cover $\tilde{p} : \tilde{B} \to B$, the covering $p_F : E_F \to B$ is Galois if and only if F is isomorphic to a transitive right $\pi_1(B, b)$-set of the form*

$$N\backslash\pi_1(B, b),$$

where $N \lhd \pi_1(B, b)$ is a normal subgroup.

Proof. (Indications.) The result can either be proved directly as a consequence of Proposition 5.43 or as a Corollary of Theorem 5.60. (Cf. Exercise 5.64.) □

Corollary 5.74. *For B a connected, locally path connected space which admits a universal cover, if $\pi_1(B, b)$ is abelian then every covering of B is Galois.*

Proof. Exercise. □

Example 5.75. The circle S^1 is connected, locally path connected with universal cover $\mathbb{R} \to S^1$ and fundamental group $\mathbb{Z}$ (thus abelian!). Hence every covering of S^1 is Galois.

Using Theorem 5.60, one shows that the examples of Example 5.7 give the set of isomorphism classes of coverings of S^1. (Exercise.)

Exercise 5.76. What can be said about coverings of the torus, $S^1 \times S^1$?

Exercise 5.77. Give an example of a covering which is *not* Galois.

6. HOMOLOGY

We have seen that the fundamental group is a useful invariant of path-connected spaces. However, simply connected spaces are *invisible* to the fundamental group (by definition!); for example, π_1 cannot see the difference between the spheres S^m and S^n for $m, n \geq 2$.

The higher homotopy groups $\pi_n(X, x)$ $n \geq 2$, are *much* more powerful - but are very difficult to calculate. For $n \in \mathbb{N}$, the nth *homology group* $H_n(X)$ of a space X is an abelian group which provides useful invariants of the space. In particular

- ▷ homology ($n \in \mathbb{N}$) is a functor:
$$H_n : \mathfrak{Top} \to \mathfrak{Ab}$$
with values in abelian groups which satisfies the **homotopy axiom**: if $f \sim g$ are homotopic continuous maps from X to Y, then $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$;
- ▷ the homology of a point is $H_n(*) = 0$ except for $H_0(*) = 0$; moreover, homology satisfies the **dimension axiom**:
$$H_n(S^t) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = t = 0 \\ \mathbb{Z} & n = 0 < t \\ \mathbb{Z} & n = t > 0; \end{cases}$$
in particular homology sees *higher dimensional phenomena* than the fundamental group;
- ▷ homology is *calculable*: for example
  (1) the homology of a disjoint union is the direct sum: $H_n(X \amalg Y) \cong H_n(X) \oplus H_n(Y)$;
  (2) if X admits a *triangulation*, $H_n(X)$ can be calculated in terms of the combinatorial data defining the triangulation.

Remark 6.1. The homology groups of spheres are much simpler that the *higher homotopy groups*, $\pi_t(S^n)$ (which, for $n \geq 2$, are unknown for $t \gg 0$!). For example, $\pi_3(S^2) \cong \mathbb{Z}$, generated by the *Hopf map* η, which is represented by the Hopf fibration $S^3 \simeq \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}P^1 \simeq S^2$, which sends $0 \neq (z_1, z_2) \in \mathbb{C}^2 \mapsto [z_1 : z_2] \in \mathbb{C}P^1$. ✓15/11/13

As a first example of the usefulness of homology, consider the following:

Definition 6.2. The degree of a continuous map $f : S^n \to S^n$ ($0 < n \in \mathbb{N}$) is the integer $\deg(f)$ which defines the morphism of abelian groups
$$H_n(f) : H_n(S^n) \cong \mathbb{Z} \to \mathbb{Z} \cong H_n(S^n).$$
(A morphism $\mathbb{Z} \to \mathbb{Z}$ is uniquely determined by the image of a chosen generator.)

Remark 6.3. It is a fundamental result (the Hurewicz theorem) that the degree of f determines the homotopy class of f.

To construct homology, we need some knowledge of homological algebra.

6.1. **Exact sequences.** In this section, $\mathscr{A}$ denotes the category of modules over a fixed commutative ring R. For example, taking $R = \mathbb{Z}$, this is the category of abelian groups.

Remark 6.4. More generally, $\mathscr{A}$ can be taken to be any *abelian category*. (See [Wei94], for example, for a definition.)

Recall the definition of the *kernel* and the *image* of a morphism $f : M \to N$ of $\mathscr{A}$:

- ▷ the *kernel* $\ker(f) := \{m \in M | f(m) = 0\} \subseteq M$;
- ▷ the *image* $\mathrm{image}(f) := \{n \in N | \exists m \in M \mathrm{s.t.} f(m) = n\} \subseteq N$.

The important properties of $\mathscr{A}$ are the following:

(1) $\mathscr{A}$ is an additive category:

(a) 0 is both *initial* and *final* in $\mathscr{A}$;
(b) $\mathrm{Hom}_{\mathscr{A}}(M, N)$ is an abelian group and composition is biadditive;
(c) $\mathrm{Hom}_{\mathscr{A}}(X, M \oplus N) \cong \mathrm{Hom}_{\mathscr{A}}(X, M) \oplus \mathrm{Hom}_{\mathscr{A}}(X, N)$ and $\mathrm{Hom}_{\mathscr{A}}(M \oplus N, X) \cong \mathrm{Hom}_{\mathscr{A}}(M, X) \oplus \mathrm{Hom}_{\mathscr{A}}(N, X)$ (in categorical language $M \oplus N$ is both the *product* and the *coproduct* of M and N).
(d) The zero morphism from M to N is the *trivial morphism* $M \to 0 \to N$.

(2) For $f : M \to N$ a morphism of $\mathscr{A}$,
(a) there is a natural isomorphism
$$\mathrm{image}(f) \cong M/\ker(f)$$
where $M/\ker(f)$ is the quotient by the sub-object $\ker(f) \subseteq M$;
(b) any morphism $g : N \to Q$ such that the composite $g \circ f$ is zero factorizes canonically as
$$\begin{tikzcd} N \arrow[r, two heads] \arrow[d, "g"'] & N/\mathrm{image}(f) \arrow[dl, dotted, "\exists!"] \\ Q & \end{tikzcd}$$
(in categorical language, $N \twoheadrightarrow N/\mathrm{image}(f)$ is the *cokernel* $\mathrm{coker}(f)$ of f). Moreover, the kernel of $N \to N/\mathrm{image}(f)$ is $\mathrm{image}(f)$.

Remark 6.5. The category $\mathfrak{Group}$ does *not* satisfy all these properties.

(1) The *free product* (coproduct) of groups $\star$ does not coincide with the *product* $\times$.
(2) The relationship between the cokernel and the image is more delicate; this is why the notion of *normal* subgroup is important. Consider an inclusion of groups $H \hookrightarrow G$, then the cokernel is the quotient $G/N_G H$, where $N_G H$ is the *normalizer* of H in G (the smallest normal subgroup containing H). The kernel of $G \twoheadrightarrow G/N_G H$ is $N_G H$, which coincides with H if and only if H is normal.

The following definition is fundamental:

Definition 6.6. Let $M \xrightarrow{f} N \xrightarrow{g} Q$ be morphisms of $\mathscr{A}$.

(1) g, f form a *sequence* if $g \circ f = 0$ (equivalently $\mathrm{image}(f) \subseteq \ker(g)$);
(2) a sequence is *exact* (implicitly at N) if $\mathrm{image}(f) = \ker(g)$.

A set of morphisms $\{f_n : M_n \to M_{n+1} | n \in \mathbb{Z}\}$, forms a *sequence* (respectively an *exact sequence*) if and only if each
$$M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1}$$
is a sequence (resp. exact sequence).

Remark 6.7. A set of morphisms $\{f_n : M_n \to M_{n+1} | n \in \mathscr{I} \subset \mathbb{Z}\}$ indexed by a subset $\mathscr{I} \subset \mathbb{Z}$ can always be completed to a set indexed by $\mathbb{Z}$ by replacing the missing objects by 0 and the missing morphisms by zero. (Exercise: why?)

Definition 6.8. A *short exact sequence* in $\mathscr{A}$ is an exact sequence of the form:
$$0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0.$$

Exercise 6.9. Check the following assertions.

(1) For $A \subset B$ in $\mathscr{A}$, there is a short exact sequence:
$$0 \to A \to B \to B/A \to 0.$$

(2) For $f : M \to N$ a morphism of $\mathscr{A}$, there are short exact sequences:

$$0 \to \ker(f) \to M \to \mathrm{image}(f) \to 0$$
$$0 \to \mathrm{image}(f) \to N \to \mathrm{coker}(f) \to 0.$$

(3) For $M, N \in \mathrm{Ob}\,\mathscr{A}$, there is a short exact sequence

$$0 \to M \to M \oplus N \to N \to 0.$$

(This is known as a *split short exact sequence*.)

(4) $\forall 0 \neq n \in \mathbb{N}$, multiplication by n induces an exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0.$$

Exercise 6.10. For $0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ a short exact sequence in $\mathscr{A}$, show that there exists a *section* $s : Q \to N$ (a morphism of $\mathscr{A}$ such that $g \circ s = \mathrm{Id}_Q$) if and only if there exists an isomorphism making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & M \oplus Q \arrow[r] \arrow[d, "\cong"'] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] & N \arrow[r, "g"'] & Q \arrow[r] & 0,
\end{tikzcd}$$

where the top row is the *split short exact sequence* of the previous exercise.

6.2. **Chain complexes and homology.**

Definition 6.11.

(1) A *complex* (or *chain complex*) in $\mathscr{A}$ is a sequence $\{d_n : C_n \to C_{n-1} | n \in \mathbb{Z}\}$ (equivalently $d_n \circ d_{n-1} = 0\ \forall n$). This will be denoted by $(C_\bullet, d)$ or simply C when no confusion is likely.

(2) A *morphism of chain complexes* $\varphi : C \to D$ is a set of morphisms $\{\varphi_n : C_n \to D_n | n \in \mathbb{Z}\}$ such that $\forall n \in \mathbb{Z}$ the following diagram commutes:

$$\begin{tikzcd}
C_n \arrow[r, "d_n^C"] \arrow[d, "\varphi_n"'] & C_{n-1} \arrow[d, "\varphi_{n-1}"] \\
D_n \arrow[r, "d_n^D"'] & D_{n-1}.
\end{tikzcd}$$

Remark 6.12. As in Remark 6.7, one can consider chain complexes indexed by $\mathbb{N}$ (by extending by zero).

Proposition 6.13. *Chain complexes in $\mathscr{A}$ form a category $\mathfrak{Ch}(\mathscr{A})$ which contains $\mathbb{N}$-graded chain complexes as a full subcategory $\mathfrak{Ch}_{\geq 0}(\mathscr{A})$.*

Proof. Exercise. □

Remark 6.14. More is true: the category $\mathfrak{Ch}(\mathscr{A})$ is *abelian*. (Optional exercise: make sense of this statement and prove it!)

The *homology* of a complex measures its *failure to be exact*. The terminology below is inspired by the *geometric* origin of chain complexes, for instance from *simplicial homology*.

Definition 6.15. For $(C_\bullet, d)$ a chain complex and $n \in \mathbb{Z}$:

(1) the *n-cycles* $Z_n \subseteq C_n$ is the subobject $Z_n := \ker(d_n)$;

(2) the *n-boundaries* $B_n \subseteq Z_n$ is the subobject $\mathrm{image}(d_{n+1})$: (which is contained in $\ker(d_n)$, since $d_n \circ d_{n+1} = 0$, by the hypothesis that C_n is a chain complex);

(3) the nth *homology* is the quotient $H_n(C_\bullet, d) := Z_n/B_n$.

Exercise 6.16. For $(C_\bullet, d)$ a chain complex, show that $C_{n+1} \to C_n \to C_{n-1}$ is exact if and only if $H_n(C) = 0$.

Proposition 6.17. *For $n \in \mathbb{Z}$, the nth homology defines a functor:*

$$H_n(-) : \mathfrak{Ch}(\mathscr{A}) \to \mathscr{A}.$$

Proof. For $\varphi : C \to D$ a morphism of chain complexes, the morphism $\varphi_n : C_n \to D_n$ restricts to morphisms forming a commutative diagram

$$\begin{tikzcd} B_n(C) \arrow[r, hook] \arrow[d] & Z_n(C) \arrow[r, hook] \arrow[d] & C_n \arrow[d, "\varphi_n"] \\ B_n(D) \arrow[r, hook] & Z_n(D) \arrow[r, hook] & D_n \end{tikzcd}$$

since the morphisms φ_n are compatible with the differentials. (Exercise: check this).

The morphism $H_n(\varphi) : H_n(C) \to H_n(D)$ sends an element $[z]$ represented by an n-cycle $z \in Z_n(C)$ to $[\varphi_n(z)] \in H_n(D)$. By the left hand commutative square, this is independent of the choice of z.

Exercise: check that this defines a functor. □

Definition 6.18. A morphism of chain complexes $\varphi : C \to D$ is a *quasi-isomorphism* (or *homology equivalence*) if $H_n(\varphi)$ is an isomorphism $\forall n \in \mathbb{Z}$.

Example 6.19. The chain complex of abelian groups with $d_0 : C_0 = \mathbb{Z} \stackrel{\mathrm{Id}}{\to} \mathbb{Z} = C_1$ and all other terms zero has homology $H_n(C) = 0 \ \forall n \in \mathbb{Z}$. The unique morphism of chain complexes $0 \to C$ is a quasi-isomorphism.

Remark 6.20. The *derived category* $\mathfrak{D}\mathscr{A}$ of $\mathscr{A}$ is obtained from $\mathfrak{Ch}(\mathscr{A})$ by *inverting* the quasi-isomorphisms (the precise construction is slightly delicate). The category $\mathfrak{D}\mathscr{A}$ is very important, since it all of the information of $\mathfrak{Ch}(\mathscr{A})$ which can be seen by homology.

6.3. **Chain homotopy.** There is a notion of *homotopy* for morphisms between chain complexes in $\mathscr{A}$.

Definition 6.21. For $f, g : C \rightrightarrows D$ morphisms of chain complexes in $\mathscr{A}$, a *chain homotopy* from f to g is a set of morphisms $\{h_n : C_n \to D_{n+1} | n \in \mathbb{Z}\}$ such that $f_n - g_n = d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n$

$$\begin{tikzcd} & C_n \arrow[r, "d^C_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n+1} \arrow[dl, "h_{n-1}"] \\ D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n. & \end{tikzcd}$$

Morphisms f, g are *chain homotopic* if there exists a chain homotopy from f to g, denoted by $f \sim g$ (or $f \sim_h g$), if the homotopy is given.

Proposition 6.22.

(1) *The chain homotopy relation $\sim$ is an equivalence relation on morphism of chain complexes $\in \mathrm{Hom}_{\mathfrak{Ch}(\mathscr{A})}(C, D)$.*

(2) *For morphisms of chain complexes*

$$B \xrightarrow{u} C \underset{g}{\overset{f}{\rightrightarrows}} D \xrightarrow{v} E,$$

if $f \sim_h g$ then $v \circ f \circ u \sim_{v \circ h \circ u} v \circ g \circ u$.

Proof. The relation $\sim$ is reflexive (take $h = 0$) and symmetric (replace h by $-h$). For transitivity, if $\alpha \sim_h \beta$ and $\beta \sim_k \gamma$, then

$$\begin{aligned}\alpha_n - \gamma_n &= (\alpha_n - \beta_n) + (\beta_n - \gamma_n)\\ &= (d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n) + (d^D_{n+1} \circ k_n + k_{n-1} \circ d^C_n)\\ &= d^D_{n+1} \circ (h_n + k_n) + (h_{n-1} + k_{n-1}) \circ d^C_n\end{aligned}$$

where the additivity of $\mathscr{A}$ is used. Hence $\{h_n + k_n | n \in \mathbb{Z}\}$ forms a chain homotopy from α to γ.

The second statement is left as an exercise. (Use the fact that u, v are chain maps.) □

This allows the usual notion of *homotopy equivalence* to be introduced:

Definition 6.23. Chain complexes C, D have the *same chain homotopy type* if there exist morphisms $\varphi : C \to D$ and $\psi : D \to C$ of chain complexes such that $\psi \circ \varphi \sim \mathrm{Id}_C$ and $\varphi \circ \psi \sim \mathrm{Id}_D$; denote the associated relation $C \simeq D$.

A chain complex C is *homotopically trivial* if $C \simeq 0$.

Exercise 6.24. Show that $\simeq$ is an equivalence relation on the objects of $\mathfrak{Ch}(\mathscr{A})$.

Example 6.25. For $0 \neq d \in \mathbb{N}$, consider the chain complexes C, D of $\mathfrak{Ab}$ with $C_n = 0 = D_n$ if $n \notin \{0, 1\}$ and

$$C_1 = \mathbb{Z} \xrightarrow{\times d} C_0 = \mathbb{Z}$$

$$D_1 = 0 \longrightarrow D_0 = \mathbb{Z}/d.$$

Show that

(1) The morphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/d$ induces a morphism of chain complexes $C \to D$ which induces an isomorphism in homology.
(2) There is *no* non-zero morphism of chain complexes $D \to C$.

Deduce that C, D do not have the same chain homotopy type.

Part of the interest of chain homotopy is through the following:

Proposition 6.26. *If $f, g : C \rightrightarrows D$ are chain homotopic, then $H_n(f) = H_n(g) : H_n(C) \rightrightarrows H_n(D)$, $\forall n \in \mathbb{Z}$.*

Proof. Consider a homology class $[z] \in H_n(C)$, represented by a cycle $z \in Z_n(C)$ (so that $d^C_n z = 0$). By definition, $H_n(f)[z] = [f_n(z)]$ and $H_n(g)[z] = [g_n(z)]$. Suppose $f \sim_h g$, so that $f_n - g_n = d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n$; applied to z, since $d^C_n z = 0$, this gives

$$f_n(z) - g_n(z) = d^D_{n+1} h_n(z).$$

Thus, the cycles (of D_n) $f_n(z)$ and $g_n(z)$ differ by a boundary. Hence, their classes in homology coincide. □

6.4. **Simplicial objects.** If one forgets the condition $d^2 = 0$, a chain complex in $\mathscr{A}$ can be viewed as a *functor*. Consider $\mathbb{Z}$ as a category $\underline{\mathbb{Z}}$, with objects $\{n | n \in \mathbb{Z}\}$ and

$$\mathrm{Hom}_{\underline{\mathbb{Z}}}(a, b) = \begin{cases} * & a \leq b \\ \emptyset & a > b. \end{cases}$$

(This construction is general: a *partially ordered set* can be considered as a category in which $|\mathrm{Hom}(a, b)| \leq 1$ and $\mathrm{Hom}(a, b) \neq \emptyset$ if and only if $a \leq b$.)

Recall that the opposite category is obtained by 'reversing' the direction of the arrows; thus the opposite category $\underline{\mathbb{Z}}^{\mathrm{op}}$ has objects $\{n | n \in \mathbb{Z}\}$ and

$$\mathrm{Hom}_{\underline{\mathbb{Z}}^{\mathrm{op}}}(a, b) = \begin{cases} * & a \geq b \\ \emptyset & a < b. \end{cases}$$

A *functor* from $\underline{\mathbb{Z}}^{\mathrm{op}}$ to $\mathscr{A}$ is equivalent to a set of morphisms $\{d_n : C_n \rightarrow C_{n-1} | n \in \mathbb{Z}\}$. These functors form a category $\mathscr{A}_{\underline{\mathbb{Z}}}$, with objects the natural transformations of functors. Concretely, a morphism from (C, d^C) to (D, d^D) is a set of morphisms $\{f_n : C_n \rightarrow D_n | n \in \mathbb{Z}\}$ which make the squares commute as in definition 6.11.

Remark 6.27. From this viewpoint, the category $\mathfrak{Ch}(\mathscr{A})$ of chain complexes is the *full subcategory* of $\mathscr{A}_{\underline{\mathbb{Z}}}$ of objects such that $d_{n-1} \circ d_n = 0, \forall n$.

The above used the following general notation.

Notation 6.28. For $\mathscr{I}$ a small category (the *indexing* category) and $\mathscr{C}$ any category,

(1) the category of functors from $\mathscr{I}$ to $\mathscr{C}$ is written $\mathscr{C}^{\mathscr{I}}$;
(2) the category of functors from $\mathscr{I}^{\mathrm{op}}$ to $\mathscr{C}$ is written $\mathscr{C}_{\mathscr{I}}$.

The category $\mathscr{I}$ should be visualized as the category of *diagrams of shape* $\mathscr{I}$ in $\mathscr{C}$.

Exercise 6.29. For $\mathscr{I}$ the category with three objects and non-identity morphisms $\bullet \leftarrow \bullet \rightarrow \bullet$, give an explicit description of the objects and morphisms of the categories $\mathfrak{Top}^{\mathscr{I}}$ and $\mathfrak{Top}_{\mathscr{I}}$.

Exercise 6.30. (Optional.) If $\mathscr{A}$ is an abelian category and $\mathscr{I}$ is a small indexing category, show that $\mathscr{A}^{\mathscr{I}}$ and $\mathscr{A}_{\mathscr{I}}$ are abelian categories. (Note: since $\mathscr{A}_{\mathscr{I}} = \mathscr{A}^{\mathscr{I}^{\mathrm{op}}}$, it suffices to treat one case.)

In the case $\mathscr{I} = \underline{\mathbb{Z}}$, show that $\mathfrak{Ch}(\mathscr{A})$ is an abelian subcategory of $\mathscr{A}_{\mathbf{\Delta}}$.

Remark 6.31. Chain complexes are not suitable for general categories $\mathscr{C}$:

(1) a zero morphism is required (the category $\mathscr{C}$ must be pointed) for the condition $d^2 = 0$ to have a meaning;
(2) the notion of *chain homotopy* requires the addition and subtraction of morphisms.

A more general approach is to use *simplicial objects*. These can again be defined as functors; first we need to introduce the *indexing* category.

Definition 6.32.

(1) For $n \in \mathbb{N}$ let $[n]$ be the category associated to the partially ordered set $(\{0, \ldots, n\}, \leq)$.
(2) The *ordinal category* $\mathbf{\Delta}$ is the full subcategory of $\mathfrak{CAT}$ (the category of small categories) with objects $\{[n] | n \in \mathbb{N}\}$.

Explicitly, $\mathbf{\Delta}$ has objects $\{[n] | n \in \mathbb{N}\}$ and a morphism from $[m]$ to $[n]$ is a non-decreasing map $f : \{0, \ldots m\} \rightarrow \{0, \ldots, n\}$ (i.e. $i \leq j \Rightarrow f(i) \leq f(j)$).

The morphisms of $\mathbf{\Delta}$ can be expressed as composites of the following *generators*:

Definition 6.33. For $0 < n \in \mathbb{N}$,

(1) the *coface* maps $\varepsilon_i : [n-1] \rightarrow [n]$, $0 \leq i \leq n$ are given by:

$$\varepsilon_i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

(the order-preserving inclusion for which i is not in the image);

(2) the *codegeneracy maps* $\sigma_i : [n] \rightarrow [n-1]$, $0 \leq i \leq n-1$:

$$\sigma_i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

(the order-preserving surjection with two elements mapping to j).

Example 6.34. Consider the morphisms between $[0]$ and $[1]$ in $\mathbf{\Delta}$; $[0]$ corresponds to the one-point set $\{0\}$ and $[1]$ to $\{0, 1\}$, Hence, the morphisms are generated by:

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\varepsilon_0"] \arrow[r, shift right, "\varepsilon_1"'] & {[1]}. \arrow[l, bend right, "\sigma_0"'] \end{tikzcd}$$

This diagram is analogous to the diagram of continuous maps:

$$\begin{tikzcd} \mathrm{pt.} \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & I. \arrow[l, bend right, "p"'] \end{tikzcd}$$

Exercise 6.35.

(1) Show that any morphism $[m] \to [n]$ of $\mathbf{\Delta}$ factors *uniquely* as a composite $[m] \twoheadrightarrow [i] \hookrightarrow [n]$ of a surjective order-preserving map followed by an injective order-preserving map.
(2) Identify the sets of injections $[n-1] \hookrightarrow [n]$ and of surjections $[n] \twoheadrightarrow [n-1]$. Deduce that any map can be written as a composite of morphisms of the form ε_i and σ_j.
(3) Verify the *cosimplicial identities*:
    (a) $\varepsilon_j \varepsilon_i = \varepsilon_i \varepsilon_{j-1}$, $i < j$ (this is *essential*);
    (b) $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$, $i \leq j$;
    (c) $\sigma_j \varepsilon_i = \begin{cases} \varepsilon_i \sigma_{j-1} & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \varepsilon_{i-1} \sigma_j & i > j+1. \end{cases}$
(4) Deduce that the category $\mathbf{\Delta}$ is *generated* by the morphisms of the form ε_i and σ_j, subject to the *cosimplicial relations*. (This is analogous to the presentation of a group by generators and relations.)

The *coface* and *codegeneracy* maps can be represented by the diagram:

$$\begin{tikzcd} {[0]} \arrow[r, shift left, "\varepsilon_0"] \arrow[r, shift right, "\varepsilon_1"'] & {[1]} \arrow[l, bend right, "\sigma_0"'] \arrow[r, shift left=2, "\varepsilon_0"] \arrow[r, "\varepsilon_1"] \arrow[r, shift right=2, "\varepsilon_2"'] & {[2]} \arrow[l, bend right, shift right, "\sigma_0"'] \arrow[l, bend right=50, "\sigma_1"'] \arrow[r, shift left=3, "\varepsilon_0"] \arrow[r, shift left=1, "\varepsilon_1"] \arrow[r, shift right=1, "\varepsilon_2"] \arrow[r, shift right=3, "\varepsilon_3"'] & {[3]} \ldots \arrow[l, bend right, "\sigma_0"'] \arrow[l, bend right=50, "\sigma_1"'] \arrow[l, bend right=70, "\sigma_2"'] \end{tikzcd}$$

As in Example 6.34, this has a *topological model*.

Definition 6.36. For $n \in \mathbb{N}$, let Δ_n^{top} denote the n-dimensional *topological simplex*:

$$\Delta_n^{\mathrm{top}} := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1,\ x_i \geq 0 \forall i\} \subset \mathbb{R}^{n+1}.$$

Thus Δ_n^{top} is the convex hull of its set of vertices, $\{v_i | 0 \leq i \leq n\}$, where $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in the $(i+1)$st coordinate. (The v_i form the standard basis of $\mathbb{R}^{n+1}$.)

Proposition 6.37. *The topological simplexes form a functor $\Delta_\bullet^{\mathrm{top}} : \mathbf{\Delta} \to \mathfrak{Top}$, where the map $f : [m] \to [n]$ induces the continuous map $\Delta_m^{\mathrm{top}} \to \Delta_n^{\mathrm{top}}$ which is the restriction of the linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ defined by $v_i \mapsto v_{f(i)}$.*

Proof. This is an exercise in using the definitions; since Δ_m^{top} is the convex hull of its vertices to define a map to Δ_n^{top}, it suffices to specify the image of the vertices. $\square$

The topological simplexes give a first example of a *cosimplicial object*.

Definition 6.38. For $\mathscr{C}$ a category,

(1) the *category of cosimplicial objects* $\mathscr{C}^{\mathbf{\Delta}}$ is the category with objects functors from $\mathbf{\Delta}$ to $\mathscr{C}$ and morphisms natural transformations;
(2) the *category of simplicial objects* $\mathscr{C}_{\mathbf{\Delta}}$ is the category with objects functors from $\mathbf{\Delta}^{\mathrm{op}}$ to $\mathscr{C}$ and morphisms natural transformations (these are contravariant functors from $\mathbf{\Delta}$ to $\mathscr{C}$).

In particular, the *category of simplicial sets* is the category $\mathfrak{Set}_{\mathbf{\Delta}}$.

Remark 6.39.

(1) A simplicial object $X_\bullet$ of a category $\mathscr{C}$ is a sequence of objects $\{X_n | n \in \mathbb{N}\}$ of $\mathscr{C}$ equipped (for $0 < n \in \mathbb{N}$) with
  (a) *face maps* $\partial_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, induced by the ε_i;
  (b) *degeneracies* $s_i : X_{n-1} \to X_n$, $0 \leq i \leq n-1$, induced by the σ_i,
  which satisfy *simplicial identities* (dual to the cosimplicial identities).

$$\begin{tikzcd} X_0 \arrow[r, "s_0", bend left] & X_1 \arrow[l, shift right, "\partial_0"'] \arrow[l, shift left, "\partial_1"] \arrow[r, "s_0", bend left] \arrow[r, "s_1", bend left=20] & X_2 \arrow[l, shift right=2, "\partial_0"'] \arrow[l, "\partial_1"] \arrow[l, shift left=2, "\partial_2"] \arrow[r, "s_0", bend left] \arrow[r, "s_1", bend left=25] \arrow[r, "s_2", bend left=15] & X_3 \ldots \arrow[l, shift right=3, "\partial_0"'] \arrow[l, shift right=1, "\partial_1"] \arrow[l, shift left=1, "\partial_2"] \arrow[l, shift left=3, "\partial_3"] \end{tikzcd}$$

(2) A morphism of simplicial objects $f_\bullet : X_\bullet \to Y_\bullet$ is a sequence of morphisms $\{f_n : X_n \to Y_n | n \in \mathbb{N}\}$ such that, for $0 < n \in \mathbb{N}$,
  (a) $\partial_i f_n = f_{n-1}\partial_i : X_n \to Y_{n-1}$;
  (b) $s_j f_{n-1} = f_n s_j : X_{n-1} \to Y_n$.

Example 6.40. For $n \in \mathbb{N}$, the object $[n]$ of $\mathbf{\Delta}$ defines a simplicial set

$$\Delta_n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]).$$

(Exercise: verify this assertion.)

Exercise 6.41. (Yoneda's Lemma - for simplicial sets). For $n \in \mathbb{N}$ and $X_\bullet$ a simplicial set, prove that there is a natural isomorphism:

$$\mathrm{Hom}_{\mathfrak{Set}_{\mathbf{\Delta}}}(\Delta_n, X) \cong X_n.$$

(Cf. Section A.4.)

Example 6.42. The ordinal category $\mathbf{\Delta}$ is, by definition, a full subcategory of $\mathfrak{CAT}$; the inclusion defines a functor $\mathbf{\Delta} \to \mathfrak{CAT}$, so the categories $[n]$ can be considered as a *cosimplicial category*, or an object of $\mathfrak{CAT}^{\mathbf{\Delta}}$ (!).

This observation leads to the definition of the *nerve* of a category. For $\mathscr{C}$ a small category, the *nerve* $N\mathscr{C}$ is the simplicial set with:

$$(N\mathscr{C})_n := \mathrm{Hom}_{\mathfrak{CAT}}([n], \mathscr{C})$$

and with simplicial structure induced by the *cosimplicial structure* of $\mathbf{\Delta}$.

For example, $(N\mathscr{C})_0$ is the set $\mathrm{Ob}\,\mathscr{C}$ of objects of $\mathscr{C}$ and $(N\mathscr{C})_1$ the set $\mathrm{Mor}\mathscr{C}$ of morphisms of $\mathscr{C}$. The simplicial structure maps are given by the *identity* morphism map, and the *source* and *target*:

$$\begin{tikzcd} (N\mathscr{C})_0 \arrow[r, "\mathrm{id}", bend left] & (N\mathscr{C})_1. \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \end{tikzcd}$$

For $n > 1$, the set $(N\mathscr{C})_n$ is given by the set of composable sequences of n morphisms in $\mathscr{C}$.

An important example is when $\mathscr{C}$ is the category with one object associated to a discrete group G. The nerve NG leads directly to the *classifying space* BG of the group G; this is a path-connected space with $\pi_1(BG, *) \cong G$ and $\pi_n(BG) = 0$ for $n > 0$.

Exercise 6.43.

(1) For $\mathscr{C}$ a small category, describe the face and degeneracy maps (in general) for the nerve $N\mathscr{C}$.
(2) Show that the nerve defines a functor $N : \mathfrak{CAT} \to \mathfrak{Set}_{\mathbf{\Delta}}$ with values in simplicial sets.

6.5. **Singular simplices.** The idea of singular simplices generalizes the consideration of *paths* of a topological space, allowing the interval $I \cong \Delta_1^{\mathrm{top}}$ to be replaced by the higher topological simplices Δ_n^{top}.

Definition 6.44. For X a topological space, let $\mathfrak{Sing}_\bullet(X)$ denote the *singular simplicial set* with

$$\mathfrak{Sing}_n(X) := \mathrm{Hom}_{\mathfrak{Top}}(\Delta_n^{\mathrm{top}}, X)$$

and with simplicial structure induced by the *cosimplicial structure* of $\Delta_\bullet^{\mathrm{top}}$.

Example 6.45. For X a topological space, $\mathfrak{Sing}_0(X)$ is the underlying set of points of X and $\mathfrak{Sing}_1(X)$ the set of paths in X. The simplicial structure maps

$$\begin{tikzcd} \mathfrak{Sing}_0(X) \arrow[r, bend left, "c"] & \mathfrak{Sing}_1(X) \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd}$$

correspond to the *endpoints* of a path and the *constant* path.

Proposition 6.46. *The singular simplicial set defines a functor*

$$\mathfrak{Sing}_\bullet : \mathfrak{Top} \to \mathfrak{Set}_{\mathbf{\Delta}}.$$

Proof. Exercise. □

Remark 6.47. There is an associated functor, *geometric realization* (in the language of category theory, it is the *left adjoint* to $\mathfrak{Sing}$)

$$|-| : \mathfrak{Set}_{\mathbf{\Delta}} \to \mathfrak{Top}.$$

The idea of the definition of this functor is very simple: one *extends* the association $\Delta_n \mapsto \Delta_n^{\mathrm{top}}$, using the fact that any simplicial set can be built out of Δ_n. (The explicit definition is not difficult; see for example [GJ99].)

The importance of the functors

$$|-| : \mathfrak{Set}_{\mathbf{\Delta}} \rightleftarrows: \mathfrak{Top} : \mathfrak{Sing}_\bullet$$

is that they allow simplicial sets to be used as a *combinatorial model* for (nice) topological spaces. A model for the *homotopy category* of topological spaces can be given using simplicial sets.

For example, the *classifying space* BG of a discrete group G (cf. Example 6.42 is, by definition,

$$BG := |NG|,$$

so that this defines a functor $B(-) : \mathfrak{Group} \to \mathfrak{Top}$.

6.6. **Simplicial chains.** Simplicial abelian groups arise from simplicial sets by applying the following general result.

Proposition 6.48. *A functor $F : \mathscr{C} \to \mathscr{D}$ induces functors*

$$\begin{aligned} F^{\mathbf{\Delta}} &: \mathscr{C}^{\mathbf{\Delta}} \to \mathscr{D}^{\mathbf{\Delta}} \\ F_{\mathbf{\Delta}} &: \mathscr{C}_{\mathbf{\Delta}} \to \mathscr{D}_{\mathbf{\Delta}} \end{aligned}$$

between the respective categories of cosimplicial objects and of simplicial objects.

Proof. The result is proved by unravelling definitions. For example, if $(X_\bullet, \partial_\bullet, s_\bullet)$ is a simplicial object of $\mathscr{C}$, then its image in $\mathscr{D}_{\mathbf{\Delta}}$ is $(F(X_\bullet), F(\partial_\bullet), F(s_\bullet))$. It is straightforward to verify that this defines a functor (exercise!). □

Exercise 6.49. Prove the analogue of Proposition 6.48 for *any* indexing category $\mathscr{I}$. Namely, $F : \mathscr{C} \to \mathscr{D}$ induces functors

$$\begin{aligned} F^{\mathscr{I}} &: \mathscr{C}^{\mathscr{I}} \to \mathscr{D}^{\mathscr{I}} \\ F_{\mathscr{I}} &: \mathscr{C}_{\mathscr{I}} \to \mathscr{D}_{\mathscr{I}}. \end{aligned}$$

Example 6.50. The free abelian group functor $\mathbb{Z}[-] : \mathfrak{Set} \to \mathfrak{Ab}$ induces a functor

$$\mathbb{Z}[-]_{\mathbf{\Delta}} : \mathfrak{Set}_{\mathbf{\Delta}} \to \mathfrak{Ab}_{\mathbf{\Delta}}$$

from the category of simplicial sets to *simplicial abelian groups*. This functor is fundamental in studying homology.

Recall that $\mathscr{A}$ denotes the category of modules over a fixed commutative ring (or, more generally, an abelian category).

Definition 6.51. For $M_\bullet \in \mathrm{Ob}\, \mathscr{A}_{\mathbf{\Delta}}$ a simplicial object in $\mathscr{A}$, let $C_*(M)$ denote the *associated chain complex* with $C_n(M) = M_n$ and differential $d_n : C_n(M) \to C_{n-1}(M)$ given by

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i.$$

The fact that $d^2 = 0$ is a consequence of the *simplicial identity* $\partial_i \partial_j = \partial_{j-1}\partial_i$ for $i < j$.

Exercise 6.52. Prove that $d^2 = 0$ in $C_*(M)$.

Proposition 6.53. *The associated chain complex defines a functor $C_* : \mathscr{A}_{\mathbf{\Delta}} \to \mathfrak{Ch}_{\geq 0}(\mathscr{A})$ from the category of simplicial objects in $\mathscr{A}$ to non-negatively graded chain complexes in $\mathscr{A}$.*

Proof. It is straightforward to verify that the definition of C_* is natural in $M_\bullet$. (Exercise.) □

Remark 6.54. The Dold-Kan theorem (see [Wei94], for example) states that the categories $\mathscr{A}_{\mathbf{\Delta}}$ and $\mathfrak{Ch}_{\geq 0}(\mathscr{A})$ are *equivalent*. The equivalence is proved using a refinement of the functor C_*, the *normalized chain complex* functor. Thus, the passage from $\mathscr{A}_{\mathbf{\Delta}}$ to $\mathfrak{Ch}_{\geq 0}(\mathscr{A})$ does not lose any information.

Example 6.55. The composite of the functor $\mathbb{Z}[-] : \mathfrak{Set}_{\mathbf{\Delta}} \to \mathfrak{Ab}_{\mathbf{\Delta}}$ with $C_*(-) : \mathfrak{Ab}_{\mathbf{\Delta}} \to \mathfrak{Ch}_{\geq 0}(\mathscr{A})$ gives a functor

$$C_*(\mathbb{Z}[-]) : \mathfrak{Set}_{\mathbf{\Delta}} \to \mathfrak{Ch}_{\geq 0}(\mathfrak{Ab}).$$

The homology of a simplicial set, $K_\bullet$, is (by definition) the homology of $C_*(\mathbb{Z}[K_\bullet])$.

Exercise 6.56. For G a discrete group, describe the chain complex $C_*(\mathbb{Z}[NG])$.

Definition 6.57. The *singular chains* functor

$$C^{\mathrm{Sing}} : \mathfrak{Top} \to \mathfrak{Ch}_{\geq 0}(\mathfrak{Ab})$$

is the composite functor $X \mapsto C_*^{\mathrm{Sing}}(X) := C_*(\mathbb{Z}[\mathfrak{Sing}_\bullet(X)])$.

Exercise 6.58. For X a topological space, describe the

(1) 0-chains=0-cycles
(2) 0-boundaries
(3) 1-cycles
(4) 1-boundaries

of $C_*^{\mathrm{Sing}}(X)$.

Definition 6.59. The *singular homology* $H_*(X)$ of a topological space X is the homology of the chain complex $C_*^{\mathrm{Sing}}(X)$.

Proposition 6.60. *Singular homology defines functors (for $n \in \mathbb{N}$):*

$$H_n(-) : \mathfrak{Top} \to \mathfrak{Ab}.$$

Proof. Clear. □

Exercise 6.61. What are the relationships between

(1) $\pi_0(X)$ and $H_0(X)$;
(2) $\pi_1(X, *)$ and $H_1(X)$ (if X is path connected)?

It remains to show that singular homology has *good properties*, so that it is *calculable*.

6.7. **Algebraic interlude: the long exact sequence associated to a short exact sequence of complexes.**

Definition 6.62. A *short exact sequence of chain complexes* in $\mathscr{A}$ is a sequence of morphisms of chain complexes $(A, d^A) \xrightarrow{f} (B, d^B) \xrightarrow{g} (C, d^C)$ such that, $\forall n \in \mathbb{Z}$,

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

is a short exact sequence in $\mathscr{A}$.

Remark 6.63. The category $\mathfrak{Ch}(\mathscr{A})$ is abelian; the above definition coincides with the usual notion of short exact sequence with respect to the abelian structure.

Theorem 6.64. *A short exact sequence of chain complexes* $(A, d^A) \xrightarrow{f} (B, d^B) \xrightarrow{g} (C, d^C)$ *induces a natural long exact sequence in homology:*

$$\ldots \to H_n(A) \stackrel{H_n(f)}{\to} H_n(B) \stackrel{H_n(g)}{\to} H_n(C) \stackrel{\delta_n}{\to} H_{n-1}(A) \to \ldots;$$

the morphism δ_n is termed the connecting morphism.

Proof. The morphisms $H_n(f)$ and $H_n(g)$ are given by functoriality of homology. Start by defining the connecting morphism (for this proof, suppose that $\mathscr{A}$ is the category of modules over a commutative ring R, so we can work in terms of elements).

Consider a homology class $[z] \in H_n(C)$, represented by a cycle $z \in Z_n(C)$; by surjectivity of g_n, there is an element $\tilde{z} \in B_n$ such that $g_n(\tilde{z}) = z$. Since z is a cycle and g is a morphism of chain complexes, $g_{n-1}(d_n^B \tilde{z}) = d_n^C g_n(\tilde{z}) = d_n^C z = 0$, hence the element $y := d_n^B(\tilde{z})$ is in the image of $f_{n-1} : A_{n-1} \to B_{n-1}$, by exactness at B_{n-1}. Moreover, $d_{n-1}^B y = d_{n-1}^B d_n^B(\tilde{z}) = 0$ implies that $d_{n-1}^A y = 0$ (since f_{n-2} is injective). Hence, $y \in Z_{n-1}(A)$ is a cycle; moreover, it is straightforward to check that the homology class $[y] \in H_{n-1}(A)$ is independent of the choice of z and of $\tilde{z}$ (exercise!). Set $\delta_n[z] := [y]$; this defines a morphism in $\mathscr{A}$; this construction is *natural*.

The composite $\partial_n H_n(g)$ is trivial, since if $[z] = H_n(g)[w] = [g_n(w)]$, one can take $\tilde{z} = w$, which is a cycle, so that $y = 0$ (as in the above construction). Similarly, the composite $H_{n-1}(f)\partial_n$ is trivial, since the cycle $y \in H_{n-1}(A)$ maps under f_{n-1} to the boundary $d_n^B(\tilde{z})$. The composite $H_n(g)H_n(f)$ is trivial, since $g_n \circ f_n = 0$, thus we have a *sequence* of homology groups; it remains to check exactness.

Exactness at $H_n(C)$: consider a class $[z] \in H_n(C)$ and suppose that $\delta_n[z] = 0$ in $H_{n-1}(A)$; thus, in the above construction, the element y is a boundary $y = d_n^A \overline{y}$, for some $\overline{y} \in A_n$. Consider $\tilde{z}' := \tilde{z} - f_n(\overline{y}) \in B_n$; clearly $g_n(\tilde{z}') = z$ and, since f is a morphism of chain complexes, $d_n^B(\tilde{z}') = 0$. It follows that $[z] = H_n(g)[\tilde{z}']$, as required.

Exactness at $H_n(A)$: consider an element $[v] \in H_n(A)$ which lies in the kernel of $H_n(f)$, represented by a cycle $v \in Z_n(A)$. Hence $f_n(v)$ is a boundary in B, so there exists $\tilde{\alpha} \in B_{n+1}$ such that $d_{n+1}^B \tilde{\alpha} = f_n(v)$. Moreover, the image $\alpha := g_{n+1}(\tilde{\alpha}) \in C_{n+1}$ is a cycle, since $d_{n+1}^C \alpha = g_n d_{n+1}^B \tilde{\alpha} = g_n f_n(v) = 0$. It is straightforward to check that $[v] = \delta_{n+1}[\alpha]$.

Exactness at $H_n(B)$: consider an element $[w] \in H_n(B)$ which lies in the kernel of $H_n(g)$, represented by a cycle $w \in B_n$; this means that $g_n(w) \in C_n$ is a boundary, say $g_n(w) = d^C_{n+1}u$. Surjectivity of g_{n+1} gives an element $\tilde{u} \in B_{n+1}$ with $g_{n+1}(\tilde{u}) = u$. The cycle $w' := w - d^B_{n+1}\tilde{u}$ represents the same homology class as w; moreover, by construction $g_n(w') = 0$ (since g is a morphism of chain complexes). Thus $w' \in A_n$ (more precisely is in the image of f_n), by exactness at B_n, and represents a homology class $[w'] \in H_n(A)$. By construction $[w] = H_n(f)[w']$, as required. $\square$

6.8. **Relative homology.** The definition of singular homology extends easily to *pairs* of topological spaces, to define *relative homology groups*.

Definition 6.65. The category $\mathfrak{Top}^{[2]}$ of *pairs of topological spaces* has objects (X, A), where X is a topological space and A a subspace, and morphisms $f : (X, A) \to (Y, B)$ given by a continuous map $f : X \to Y$ such that $f(A) \subset B$.

Exercise 6.66.
(1) Check that $\mathfrak{Top}^{[2]}$ is a category.
(2) Show that $X \mapsto (X, \emptyset)$ defines a functor $\mathfrak{Top} \to \mathfrak{Top}^{[2]}$, which identifies $\mathfrak{Top}$ as a full subcategory of $\mathfrak{Top}^{[2]}$.
(3) Show that $\mathfrak{Top}_\bullet$ is a full subcategory of $\mathfrak{Top}^{[2]}$.

If (X, A) is a pair of topological spaces, the inclusion $A \subset X$ induces an inclusion of singular simplicial sets $\mathfrak{Sing}(A) \hookrightarrow \mathfrak{Sing}(X)$ and hence a monomorphism of chain complexes:

$$C^{\mathrm{Sing}}(A) \hookrightarrow C^{\mathrm{Sing}}(X).$$

Passage to the quotient gives a chain complex.

Definition 6.67. For (X, A) a pair of topological spaces, let
(1) $C^{\mathrm{Sing}}(X, A) := C^{\mathrm{Sing}}(X)/C^{\mathrm{Sing}}(A)$ denote the *relative singular chain complex*;
(2) $H_n(X, A)$, the nth relative homology denote the nth homology group of the complex $C^{\mathrm{Sing}}(X, A)$.

Lemma 6.68. *For X a topological space, the chain complexes $C^{\mathrm{Sing}}(X)$ and $C^{\mathrm{Sing}}(X, \emptyset)$ are naturally isomorphic, hence there is a natural identification $H_n(X, \emptyset) \cong H_n(X)$, $\forall n \in \mathbb{N}$.*

Proof. Clear, since $\mathfrak{Sing}(\emptyset) = \emptyset$, the empty simplicial set, and $\mathbb{Z}[\emptyset] = 0$. $\square$

Theorem 6.69. *Relative homology defines functors $\forall n \in \mathbb{N}$:*

$$\begin{array}{rcl} H_n(-,-) : \mathfrak{Top}^{[2]} & \to & \mathfrak{Ab} \\ (X, A) & \mapsto & H_n(X, A). \end{array}$$

Moreover, associated to a pair of topological spaces (X, A), there is a natural long exact sequence in homology:

$$\ldots \to H_n(A) \xrightarrow{H_n(i)} H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \ldots$$

where $i : A \hookrightarrow X$ denotes the inclusion of the subspace.

Proof. A morphism of pairs of topological spaces $(X, A) \to (Y, B)$ induces a commutative diagram of chain complexes:

$$\begin{tikzcd} C^{\mathrm{Sing}}(A) \arrow[r, hook] \arrow[d] & C^{\mathrm{Sing}}(X) \arrow[d] \\ C^{\mathrm{Sing}}(B) \arrow[r, hook] & C^{\mathrm{Sing}}(Y). \end{tikzcd}$$

This induces a morphism of the relative chain complexes $C^{\mathrm{Sing}}(X, A) \to C^{\mathrm{Sing}}(Y, B)$ and this construction is functorial. In homology this induces $H_n(X, A) \to H_n(Y, B)$, $n \in \mathbb{N}$, which gives the functoriality of relative homology.

The long exact sequence is simply that associated to the defining short exact sequence of chain complexes:

$$0 \to C^{\mathrm{Sing}}(A) \to C^{\mathrm{Sing}}(X) \to C^{\mathrm{Sing}}(X, A) \to 0,$$

given by Theorem 6.64. □

Remark 6.70. The relative homology groups $H_*(X, A)$ give a measure of the *difference* between the homology of X and of its subspace A.

6.9. **Homotopy invariance of homology.** The homotopy invariance of singular homology is a consequence of the following result, by Proposition 6.26.

Theorem 6.71. *For $f, g : X \rightrightarrows Y$ continuous maps which are homotopic via a homotopy $H : X \times I \to Y$, the homotopy H induces (by a natural construction) a chain homotopy between $C^{\mathrm{Sing}}(f), C^{\mathrm{Sing}}(g) : C^{\mathrm{Sing}}(X) \to C^{\mathrm{Sing}}(Y)$.*

Proof. The construction of the chain homotopy follows by the method of the *universal example*. It is necessary to construct the sequence of morphisms

$$h_n : C_n^{\mathrm{Sing}}(X) \to C_{n+1}^{\mathrm{Sing}}(Y).$$

A generator of $C_n^{\mathrm{Sing}}(X) = \mathbb{Z}[\mathfrak{Sing}_n(X)]$ (as a free abelian group) is given by a continuous map $\alpha : \Delta_n^{\mathrm{top}} \to X$. This (as in Yoneda's lemma - see Section A.4), is the image under

$$C^{\mathrm{Sing}}(\alpha) : C_n^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}}) \to C_n^{\mathrm{Sing}}(X)$$

of the generator given by the identity map $1_{\Delta_n^{\mathrm{top}}}$. Moreover, the morphism $C_*^{\mathrm{Sing}}(\alpha)$ of chain complexes is determined by this element.

Now, suppose that $h_n : C_n^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}}) \to C_{n+1}^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}} \times I)$ is constructed, corresponding to the case $X = \Delta_n^{\mathrm{top}}$, $Y = \Delta_n^{\mathrm{top}} \times I$ and H the identity map. The image of $\alpha \in C_n^{\mathrm{Sing}}(X)$ is defined to be the image of the identity map under the composite

$$C_n^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}}) \stackrel{h_n}{\to} C_{n+1}^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}} \times I) \stackrel{C^{\mathrm{Sing}}(\alpha \times 1_I)}{\to} C_{n+1}^{\mathrm{Sing}}(X \times I) \stackrel{C_{n+1}^{\mathrm{Sing}}(H)}{\to} C_{n+1}^{\mathrm{Sing}}(Y).$$

(In fact this definition is *imposed* if the chain homotopy h depends naturally on H.)

Thus, the problem reduces to constructing $h_n : C_n^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}}) \to C_{n+1}^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}} \times I)$ with the requisite properties. This is equivalent to giving an element of $C_{n+1}^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}} \times I) = \mathbb{Z}[\mathfrak{Sing}_{n+1}(\Delta_n^{\mathrm{top}} \times I)]$; the construction is based on the *geometric decomposition* of the topological space $\Delta_n^{\mathrm{top}} \times I$ as a union of $n+1$ topological simplices (homeomorphic to $\Delta_{n+1}^{\mathrm{top}}$) with pairwise disjoint interiors. (This should be seen as an $n+1$-dimensional triangulation of $\Delta_n^{\mathrm{top}} \times I$.) The construction requires the ordering of vertices to be taken into account, together with the resulting orientations of the simplices.

Writing the vertices of Δ_n^{top} as $v_0, \ldots, v_n$ and the endpoints of I as $0, 1$, the product $\Delta_n^{\mathrm{top}} \times I$ is the convex hull of the set of vertices $\mathscr{V}_n := \{(v_i, \varepsilon) | \varepsilon \in \{0, 1\}\}$, considered as a subspace of $\mathbb{R}^{n+2}$. The set of vertices is equipped with the associated partial order $(v_i, \varepsilon) \leq (v_j, \eta)$ if and only if $i \leq j$ and $\varepsilon \leq \eta$.

For $0 \leq t \leq n$, define the continuous map $\sigma_t : \Delta_{n+1}^{\mathrm{top}} \to \Delta_n^{\mathrm{top}} \times I$ by the linear extension of

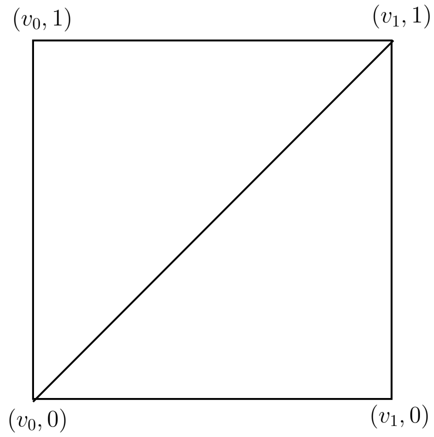
$$v_i(n+1) \mapsto \begin{cases} (v_i, 0) & i \leq t \\ (v_{i-1}, 1) & i > t. \end{cases}$$

This satisfies $\Delta_n^{\mathrm{top}} \times I = \bigcup_{t=0}^n \mathrm{image}(\sigma_t)$ and the interiors $(\mathrm{image}(\sigma_t))^\circ$ are pairwise disjoint.

For example, for $n = 0$, Δ_0^{top} is a point, and there is a homeomorphism $\Delta_1^{\mathrm{top}} \cong \Delta_0^{\mathrm{top}} \times I$, which corresponds to σ_0.

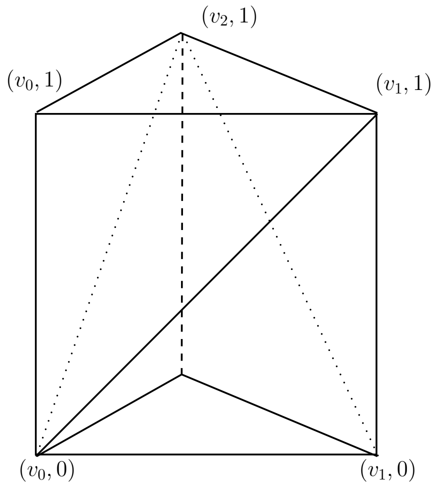
The case $n = 1$ already illustrates the salient features (see Figure 1): the common face corresponds to the face included by ε_1 applied to the two topological 2-simplices. However, the simplices have different orientations.

FIGURE 1. Decomposition of $\Delta_1^{\mathrm{top}} \times I$



In dimension 2 a similar phenomenon occurs (see Figure 2).

FIGURE 2. Decomposition of $\Delta_2^{\mathrm{top}} \times I$



The morphism $h_n : C_n^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}}) \to C_{n+1}^{\mathrm{Sing}}(\Delta_n^{\mathrm{top}} \times I)$ is defined by the element

$$\sum_{t=0}^{n} (-1)^t \sigma_t \in \mathbb{Z}[\mathfrak{Sing}_n(\Delta_n^{\mathrm{top}} \times I)].$$

The alternating sign ensures that, when applying the differential d, the only non-trivial n-simplices which appear are those which correspond to the boundary (in the geometric sense!) of $\Delta_n^{\mathrm{top}} \times I$.

Exercise: show that this gives the required chain homotopy. (This follows from the fact that the decomposition of the boundary $\partial(\Delta_n^{\mathrm{top}} \times I)$ into n-simplices which is induced by the above decomposition, is compatible with that used for the definition of h_{n-1}.) □

Remark 6.72. The geometric decomposition of $\Delta_n^{\mathrm{top}} \times I$ above can be understood entirely in terms of simplicial sets $\mathfrak{Set}_{\mathbf{\Delta}}$. Moreover, this gives the notion of *simplicial homotopy*. A simplicial homotopy induces a chain homotopy on applying the functor $C_*(\mathbb{Z}[-])$; the great advantage of *simplicial* homotopy is that it makes sense in *any* category.

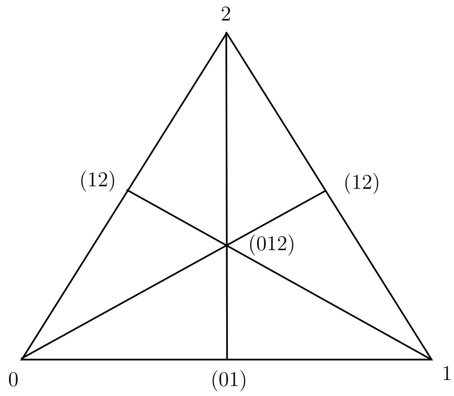
6.10. **Barycentric subdivision and little chains.** The standard techniques for calculating homology rely upon its *local* nature; this is a consequence of the procedure of *subdividing* topological simplexes.

Definition 6.73. The *barycentre* of the topological simplex Δ^{top}_n is the point:

$$\sum_{i=0}^{n} \frac{1}{n+1} v_i.$$

By induction upon the dimension n of the topological simplices, this leads to the *barycentric subdivision* of Δ^{top}_n. (See Figure 3 for an illustration of the barycentric subdivision of the topological 2-simplex.)

FIGURE 3. Barycentric decomposition of Δ^{top}_2



Recall that the topological simplices form a cosimplicial object in $\mathfrak{Top}$. In particular, the inclusion of a k-dimensional face is determined by its set of vertices (equivalently is defined by an order-preserving injection $\{0, \ldots, k\} \hookrightarrow \{0, \ldots, n\}$). The faces of Δ^{top}_n form a partially-ordered set under inclusion of faces.

In the barycentric decomposition of Δ^{top}_n, the vertices are the barycentres of the faces, hence are indexed by the faces. More generally, the topological k-simplices appearing in the decomposition are chains of strict face inclusions of length $k+1$. This is equivalent to giving an *ordering* of the set of vertices of Δ^{top}_n, $i_0, \ldots, i_n$, so that the face inclusions are given by

$$v_{i_0} \subset (v_{i_0} v_{i_1}) \subset (v_{i_0} v_{i_1} v_{i_2}) \ldots \subset (v_{i_0} v_{i_1} \ldots v_{i_n}).$$

This ordering of $\{0, \ldots, n\}$ corresponds to an element $\pi \in \mathfrak{S}_{n+1}$ of the symmetric group on $n+1$ letters.

Thus the barycentric decomposition gives

$$\Delta^{\mathrm{top}}_n \cong \bigcup_{\pi \in \mathfrak{S}_{n+1}} \mathrm{image}(\pi)$$

where π defines the map $\pi : \Delta^{\mathrm{top}}_n \to \Delta^{\mathrm{top}}_n$ which sends the jth vertex to the barycentre of the jth face in the chain of faces corresponding to π. By construction, the interiors $\mathrm{image}(\pi)^\circ$ are pairwise disjoint.

Definition 6.74. For $\mathscr{U} := \{U_i | i \in \mathscr{I}\}$ a family of subsets of a topological space X such that $X = \bigcup_{i \in \mathscr{I}} U_i$, define the *subcomplex of $\mathscr{U}$-little chains*

$$C^{\mathrm{Sing}, \mathscr{U}}(X) \subset C^{\mathrm{Sing}}(X)$$

to be the subcomplex generated by singular simplices $\Delta^{\mathrm{top}}_n \to X$ ($n \in \mathbb{N}$) which factor across some U_i, $i \in \mathscr{I}$.

Theorem 6.75. *For $\mathscr{U} := \{U_i | i \in \mathscr{I}\}$ a family of subsets of a topological space X such that $X = \bigcup_{i \in \mathscr{I}} U_i^\circ$, the inclusion*

$$C^{\mathrm{Sing}, \mathscr{U}}(X) \hookrightarrow C^{\mathrm{Sing}}(X)$$

induces an isomorphism in homology.

The proof of the theorem is based upon the (natural) subdivision operator:

$$S : C_*^{\mathrm{Sing}}(X) \to C_*^{\mathrm{Sing}}(X).$$

In the following definition, the notation introduced in describing the barycentric subdivision of Δ_n^{top} is used.

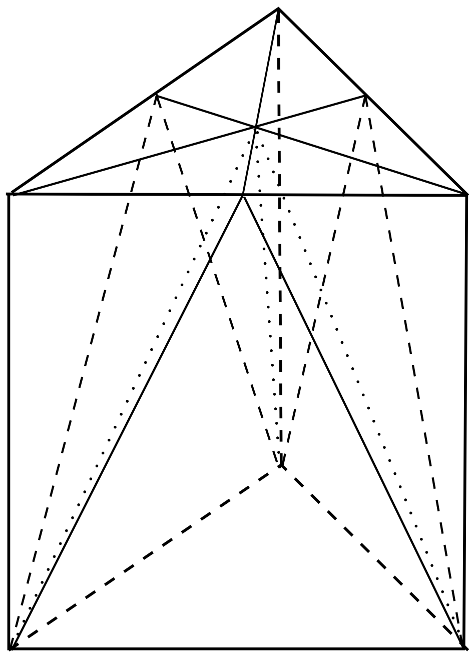
Definition 6.76. For X a topological space, let $S : C_*^{\mathrm{Sing}}(X) \to C_*^{\mathrm{Sing}}(X)$ be the chain map which sends a generator $[f] \in C_n^{\mathrm{Sing}}(X) = \mathbb{Z}[\mathfrak{Sing}_n(X)]$ given by a singular simplex $f : \Delta_n^{\mathrm{top}} \to X$ to

$$S([f]) := \sum_{\pi \in \mathfrak{S}_{n+1}} (-1)^{\mathrm{sign}(\pi)} [f \circ \pi].$$

Exercise 6.77. Show that

(1) S is a chain map;
(2) S defines a natural transformation $C_*^{\mathrm{Sing}}(-) \to C_*^{\mathrm{Sing}}(-)$ of functors from $\mathfrak{Top}$ to $\mathfrak{Ch}_{\geq 0}(\mathfrak{Ab})$.

FIGURE 4. The barycentric cylinder decomposition $\Delta_2^{\mathrm{top}} \times I$



Lemma 6.78. *For X a topological space, the chain map $S : C_*^{\mathrm{Sing}}(X) \to C_*^{\mathrm{Sing}}(X)$ is chain homotopic to the identity.*

Proof. (Indications.) The proof is similar in spirit to that of Theorem 6.71: one uses a subdivision of the space $\Delta_n^{\mathrm{top}} \times I$ related to barycentric subdivision. These subdivisions are constructed recursively on n; supposing the subdivision constructed for $\Delta_n^{\mathrm{top}} \times I$, this yields a subdivision of $(\partial \Delta_n^{\mathrm{top}} \times \{0\} \cup \Delta_{n+1}^{\mathrm{top}}) \times I$. The subdivision of $\Delta^{\mathrm{top}}_{n+1}$ is given by forming the cone with respect to the barycentre of $\Delta^{\mathrm{top}}_{n+1} \times \{1\}$. This is illustrated in Figure 4. □

Proof of Theorem 6.75. Without loss of generality, one may suppose that $\mathscr{U}$ is an open cover of X. Recall that $C^{\mathrm{Sing},\mathscr{U}}(X)$ is a subcomplex of $C^{\mathrm{Sing}}(X)$; first show that this induces a surjection in homology.

Consider an element Φ of $C^{\mathrm{Sing}}_n(X)$; since Δ^{top}_n is a compact metric space, Lebesgue's theorem 4.38 implies that there exists a natural number N such that the iterated subdivision $S^N(\Phi)$ lies in $C^{\mathrm{Sing},\mathscr{U}}_n(X)$ (exercise: prove this assertion). Moreover, if Φ is a cycle, then so is $S^N(\Phi)$, since S^N is a chain map. Since S^N is chain homotopic to the identity map (by Lemma 6.78 and Proposition 6.22), $[S^N(\Phi)] = [\Phi]$, proving surjectivity.

A similar argument establishes that a cycle of $C^{\mathrm{Sing},\mathscr{U}}(X)$ which is a boundary in $C^{\mathrm{Sing}}(X)$ is also a boundary in $C^{\mathrm{Sing},\mathscr{U}}(X)$, noting that S restricts to a chain map $S : C^{\mathrm{Sing},\mathscr{U}}(X) \to C^{\mathrm{Sing},\mathscr{U}}(X)$ which is chain homotopic to the identity. This completes the proof. □

6.11. **First calculations.**

Proposition 6.79. *The singular chain complex $C^{\mathrm{Sing}}(*)$ identifies as follows: $C^{\mathrm{Sing}}_n(*) = \mathbb{Z}$ ($\forall n \in \mathbb{N}$) and $d_{n+1} : C^{\mathrm{Sing}}_{n+1}(*) \to C^{\mathrm{Sing}}_n(*)$ is zero if $n \equiv 0 \mod (2)$ and is the identity morphism if $n \equiv 1 \mod (2)$.*

In particular, the singular homology of a point is

$$H_n(*) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

Proof. Since $*$ is the final object of the category of topological spaces, $\mathfrak{Sing}_n(*)$ is a singleton set, $\forall n \in \mathbb{N}$, hence $C^{\mathrm{Sing}}_n(*) = \mathbb{Z}$. The differential d_{n+1} identifies as multiplication by $\sum_{i=0}^{n+1}(-1)^i$; this is 0 if n is even and 1 for n odd. □

Exercise 6.80. Show that the projection $X \to *$ from a contractible space to a point induces an isomorphism in homology, hence determining $H_n(X)$, by Proposition 6.79.

Proposition 6.81. *For X, Y topological spaces, the inclusions $X \hookrightarrow X \amalg Y \hookleftarrow Y$ induce a natural isomorphism of chain complexes*

$$C^{\mathrm{Sing}}(X \amalg Y) \cong C^{\mathrm{Sing}}(X) \oplus C^{\mathrm{Sing}}(Y)$$

and hence a natural isomorphism in homology, $\forall n \in \mathbb{N}$

$$H_n(X \amalg Y) \cong H_n(X) \oplus H_n(Y).$$

Proof. Since Δ^{top}_n is a connected topological space, it is clear that $\mathfrak{Sing}_n(X \amalg Y) \cong \mathfrak{Sing}_n(X) \amalg \mathfrak{Sing}_n(Y)$ (in fact, this corresponds to an isomorphism of simplicial sets $\mathfrak{Sing}(X \amalg Y) \cong \mathfrak{Sing}(X) \amalg \mathfrak{Sing}(Y)$). The free abelian group functor sends a disjoint union of sets to a direct sum of abelian groups (*for the cognoscenti:* this is actually a formal consequence of the fact that $\mathbb{Z}[-]$ is a *left adjoint*, which implies that it preserves coproducts).

It is straightforward to check that one obtains a natural isomorphism $C^{\mathrm{Sing}}(X \amalg Y) \cong C^{\mathrm{Sing}}(X) \oplus C^{\mathrm{Sing}}(Y)$ of chain complexes (exercise).

For the conclusion, it suffices to observe that the homology of a direct sum of chain complexes is the direct sum of their homologies (exercise). □

Example 6.82. Propositions 6.79 and 6.81 allow the calculation of the homology of S^0:

$$H_n(S^0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & n > 0. \end{cases}$$

For pointed topological spaces, it is convenient to use *reduced homology groups*.

Definition 6.83. For $(X,*)$ a pointed space, the *reduced homology* of X is the relative homology group:

$$\tilde{H}_*(X) := H_*(X,*).$$

Lemma 6.84. *For $(X,*)$ a pointed topological space, the reduced homology is isomorphic to the quotient of the split inclusion $H_*(*) \to H_*(X)$ induced by the inclusion of the basepoint. In particular, for $n > 0$, there is a canonical isomorphism $\tilde{H}_n(X) \cong H_n(X)$.*

Proof. The projection $X \to *$ provides a retract of the inclusion of the basepoint, so $H_*(*) \hookrightarrow H_*(X)$ is a split inclusion. The reduced homology is identified by the long exact sequence in homology for a pair of topological spaces. □

Exercise 6.85. For pointed spaces $(X,*)$, $(Y,*)$ calculate the reduced homology groups $\tilde{H}_*(X \vee Y)$ of the wedge of X and Y in terms of their reduced homology.

Theorem 6.86. *For X a locally path-connected space, the continuous projection $X \to \pi_0(X)$ onto the set of path connected components induces an isomorphism*

$$H_0(X) \cong \mathbb{Z}[\pi_0(X)].$$

Proof. The group of singular chains $C_0^{\mathrm{Sing}}(X)$ identify as the free abelian group on the underlying set of X and $C_1^{\mathrm{Sing}}(X)$ as the free abelian group on the set of paths of X. The differential d_1 sends a generator $[\alpha]$ corresponding to a path $\alpha : I \to X$ to $[\alpha(1)] - [\alpha(0)]$. It follows easily that elements $[x]$, $[y]$ of $C_0^{\mathrm{Sing}}(X)$ define the same homology class if and only if they belong to the same path-connected component. The result follows. □

Exercise 6.87. Prove that two topological spaces with the same homotopy type $X \simeq Y$ have isomorphic homology groups.

6.12. **Mayer-Vietoris.** The little chains theorem, Theorem 6.75, can be applied when $\mathscr{U}$ is a (suitable) cover with just two subspaces.

Theorem 6.88. *For $\mathscr{U} = \{A, B\}$ a cover of a topological space X such that $X = A^\circ \cup B^\circ$, the inclusions*

$$\begin{tikzcd} A \cap B \arrow[r, "i_A"] \arrow[d, "i_B"'] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"'] & X, \end{tikzcd}$$

induce a long exact sequence (the Mayer-Vietoris sequence*) in homology:*

$$\ldots \to H_n(A \cap B) \stackrel{i_A - i_B}{\longrightarrow} H_n(A) \oplus H_n(B) \stackrel{j_A + j_B}{\longrightarrow} H_n(X) \to H_{n-1}(A \cap B) \to \ldots.$$

Proof. The singular chain complexes $C^{\mathrm{Sing}}(A)$ and $C^{\mathrm{Sing}}(B)$ can be considered as subcomplexes of $C^{\mathrm{Sing}}(X)$. The intersection of these subcomplexes is again a subcomplex, which identifies with $C^{\mathrm{Sing}}(A \cap B)$, since a singular chain $\alpha : \Delta^{\mathrm{top}}_n \to X \in \mathfrak{Sing}_n(A) \cap \mathfrak{Sing}_n(B)$ necessarily maps to $A \cap B$. The sum of the two subcomplexes $C^{\mathrm{Sing}}(A) + C^{\mathrm{Sing}}(B)$ identifies with $C^{\mathrm{Sing},\mathscr{U}}(X)$, by definition of the latter.

This implies that there is a short exact sequence of chain complexes:

$$0 \to C^{\mathrm{Sing}}(A \cap B) \stackrel{C^{\mathrm{Sing}}(i_A) - C^{\mathrm{Sing}}(i_B)}{\longrightarrow} C^{\mathrm{Sing}}(A) \oplus C^{\mathrm{Sing}}(B) \to C^{\mathrm{Sing},\mathscr{U}}(X) \to 0.$$

(To see this, consider the kernel of the map $C^{\mathrm{Sing}}(A) \oplus C^{\mathrm{Sing}}(B) \to C^{\mathrm{Sing},\mathscr{U}}(X)$.) By construction, the composite of the surjection with the canonical inclusion $C^{\mathrm{Sing},\mathscr{U}}(X) \hookrightarrow C^{\mathrm{Sing}}(X)$ identifies with the morphism

$$C^{\mathrm{Sing}}(A) \oplus C^{\mathrm{Sing}}(B) \stackrel{C^{\mathrm{Sing}}(j_A) + C^{\mathrm{Sing}}(j_B)}{\longrightarrow} C^{\mathrm{Sing}}(X)$$

of chain complexes.

Theorem 6.64 gives a long exact sequence in homology and the little chains theorem for $\mathscr{U}$ (which is where the hypothesis on the interiors is required) identifies the homology of $C^{\mathrm{Sing},\mathscr{U}}(X)$ with $H_*(X)$. The identification of the morphisms in the long exact sequence is straightforward. $\square$

The Mayer-Vietoris long exact sequence allows for the first non-trivial calculations.

Proposition 6.89. *For $1 \leq n \in \mathbb{N}$, the homology of the sphere S^n is*

$$H_t(S^n) \cong \begin{cases} \mathbb{Z} & t \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The result is proved by induction upon n, using the homology of S^0 given by Example 6.82 to calculate the homology of S^1. For $n \geq 1$, as in Proposition 4.54 (without the restriction $n \geq 2$), take an open cover $S^n = U^+ \cup U^-$ by Northern and Southern hemispheres, so that U^+, U^- are contractible and $U^+ \cap U^- \simeq S^{n-1}$.

For $n = 1$, the Mayer-Vietoris sequence for S^1 immediately shows that $H_t(S^1) = 0$ for $t > 1$ (why?). Using the homotopy invariance of homology (Theorem 6.71) and the fact that $H_1(S^0) = 0 = H_1(U^+) = H_1(U^-)$, the non-trivial part of the long exact sequence is:

$$0 \to H_1(S^1) \to H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_0(U^+) \oplus H_0(U^-) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^1) \to 0.$$

The inclusion $S^0 \hookrightarrow U^+$ identifies as $\mathbb{Z} \oplus \mathbb{Z} \stackrel{\mathrm{id}+\mathrm{id}}{\to} \mathbb{Z}$, likewise for U^-. It follows that $H_1(S^1) \cong \mathbb{Z}$, which embeds diagonally in $H_0(S^0)$ via the connecting morphism. Similarly $H_0(S^1) \cong \mathbb{Z}$ (this fact can also be deduced from Theorem 6.86).

For $n > 1$, the argument is simpler still; for H_n, the relevant portion of the Mayer-Vietoris sequence is

$$0 \to H_n(S^n) \to H_{n-1}(S^{n-1}) \cong \mathbb{Z} \to 0,$$

providing the required isomorphism.

The identification of $H_0(S^n)$ is straightforward (or apply Theorem 6.86). $\square$

6.13. **Excision.** Excision is a powerful tool for understanding relative homology groups.

The proof of the excision theorem is best carried out using the five-lemma, which is a fundamental result of homological algebra. In the following, $\mathscr{A}$ could be any abelian category.

Proposition 6.90. *For a commutative diagram in $\mathscr{A}$*

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "g_0"'] & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "g_3"'] & A_4 \arrow[d, "g_4"'] \\
B_0 \arrow[r, "h_0"'] & B_1 \arrow[r, "h_1"'] & B_2 \arrow[r, "h_2"'] & B_3 \arrow[r, "h_3"'] & B_4
\end{tikzcd}$$

in which the rows are exact, if the morphisms g_0, g_1, g_3, g_4 are isomorphisms, then so is g_2.

Proof. It suffices to show that g_2 is both injective and surjective.

First show injectivity: suppose that $x \in A_2$ lies in the kernel of g_2 then, since g_3 is an isomorphism (in particular injective), x lies in the kernel of f_2, hence $\exists y \in A_1$ such that $x = f_1(y)$. By commutativity of the diagram, $h_1 g_1(y) = 0$, so that $g_1(y)$ lies in the kernel of h_1, thus in the image of h_0, say $g_1(y) = h_0(z)$. Since g_0 is an isomorphism, there exists a unique $\tilde{z} \in A_0$ such that $g_0(\tilde{z}) = z$; moreover, since g_1 is injective and $g_1 f_0(\tilde{z}) = h_0 g_0(\tilde{z}) = g_1(y)$, $f_0(\tilde{z}) = y$. Thus $x = f_1(y) = f_1 f_0(\tilde{z}) = 0$, as required.

The proof of surjectivity is *dual*: consider $u \in B_2$, then $h_2(u) \in B_3$ lies in the image of the isomorphism g_3, say $h_2(u) = g_3(v)$. Arguing as above, v lies in the kernel of f_3, hence $\exists w \in A_2$ such that $f_2(w) = v$, so that $h_2 g_2(w) = h_2(u)$. It follows that $h_2(u - g_2(w)) = 0$, so that $\exists \alpha \in B_1$ with $h_1(\alpha) = u - g_2(w)$. Since g_1 is an isomorphism, $\exists! \tilde{\alpha} \in A_1$ such that $g_1(\tilde{\alpha}) = \alpha$. Now $g_2 f_1(\tilde{\alpha}) = u - g_2(w)$ by commutativity, giving that $g_2(f_1(\tilde{\alpha}) + w) = u$, so g_2 is surjective. $\square$

Theorem 6.91. *For $(X, A) \in \mathrm{Ob}\, \mathfrak{Top}^{[2]}$ and $U \subset X$ a subspace such that $\overline{U} \subset A^\circ$, the inclusion $(X \backslash U, A \backslash U)$ induces an isomorphism on relative homology groups:*

$$H_*(X \backslash U, A \backslash U) \stackrel{\cong}{\rightarrow} H_*(X, A).$$

Proof. Set $\mathscr{U} := \{A, X \backslash U\}$; the hypothesis that $\overline{U} \subset A^\circ$ implies that $A^\circ \cup (X \backslash U)^\circ = X$, so that the little chains theorem can be applied. As in the proof of the Mayer-Vietoris theorem, $C^{\mathrm{Sing}, \mathscr{U}}(X)$ identifies as $C^{\mathrm{Sing}}(A) + C^{\mathrm{Sing}}(X \backslash U)$, in particular, $C^{\mathrm{Sing}}(A)$ is a subcomplex of $C^{\mathrm{Sing}, \mathscr{U}}(X)$. Moreover, one has $C^{\mathrm{Sing}}(A) \cap C^{\mathrm{Sing}}(X \backslash U) = C^{\mathrm{Sing}}(A \backslash U)$. It follows that there is an isomorphism of chain complexes

$$C^{\mathrm{Sing}, \mathscr{U}}(X)/C^{\mathrm{Sing}}(A) \cong C^{\mathrm{Sing}}(X \backslash U)/C^{\mathrm{Sing}}(A \backslash U) = C^{\mathrm{Sing}}(X \backslash U, A \backslash U).$$

(Exercise: prove this!)

This gives rise to a morphism between short exact sequences of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C^{\mathrm{Sing}}(A) \arrow[r] \arrow[d, equal] & C^{\mathrm{Sing}, \mathscr{U}}(X) \arrow[r] \arrow[d, hook] & C^{\mathrm{Sing}}(X \backslash U, A \backslash U) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C^{\mathrm{Sing}}(A) \arrow[r] & C^{\mathrm{Sing}}(X) \arrow[r] & C^{\mathrm{Sing}}(X, A) \arrow[r] & 0
\end{tikzcd}$$

where the right hand vertical morphism is induced by $(X \backslash U, A \backslash U) \to (X, A)$.

Applying homology, the rows induce long exact sequences and the vertical morphisms induce a morphism between the long exact sequences (by the *naturality* of the long exact sequence in Theorem 6.64). The inclusion $C^{\mathrm{Sing}, \mathscr{U}}(X) \hookrightarrow C^{\mathrm{Sing}}(X)$ induces an isomorphism in homology, by small chains, Theorem 6.75, hence the five-lemma implies that the right hand vertical morphism also induces an isomorphism in homology, as required. $\square$

6.14. **The long exact sequence associated to a morphism.** Recall from Section 4.11 the *mapping cylinder* M_f and the *mapping cone* C_f of a continuous map $f : X \to Y$.

The mapping cylinder provides a *homotopically good* factorization

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[rr, bend left, "f"] & M_f \arrow[r, "\simeq"'] & Y
\end{tikzcd}$$

of f as a *good inclusion* (in the language of a homotopy theory, a *cofibration*) followed by a homotopy equivalence.

The mapping cone C_f is the quotient of M_f obtained by collapsing the free end X of the cylinder to a point. However, the mapping cylinder is also homeomorphic to the identification space:

$$C_f \cong CX \cup_X M_f,$$

where the base of the cone is glued to the end of the cylinder.

This gives rise to a commutative diagram of inclusions:

$$\begin{tikzcd}
X \arrow[r, tail] \arrow[d, tail] & M_f \arrow[d, tail] \\
CX \arrow[r, tail] & C_f
\end{tikzcd}$$

and, in particular, a morphism of pairs:

$$(M_f, X) \to (C_f, CX).$$

FIGURE 5. The mapping cylinder of $f : X \to Y$

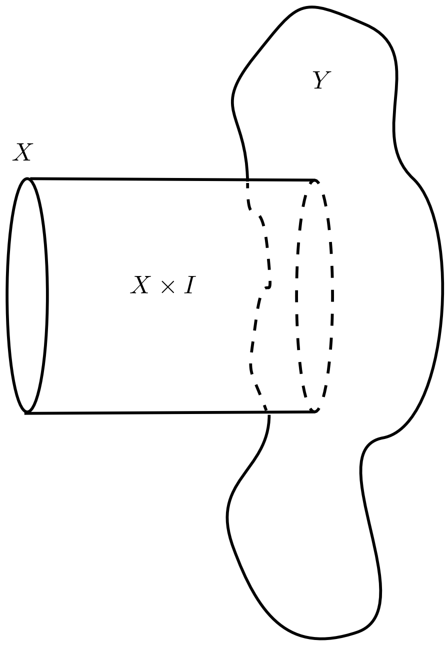
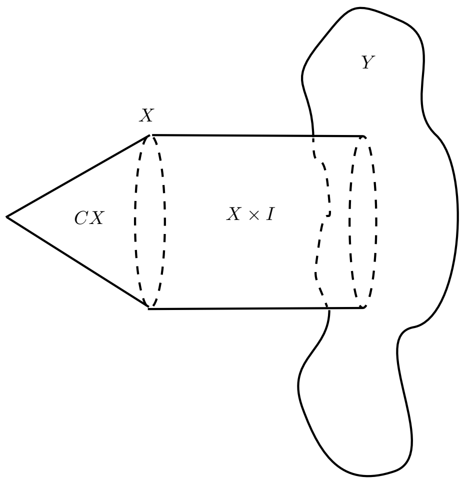


FIGURE 6. The mapping cone C_f of $f : X \to Y$



Theorem 6.92. *For $f : X \to Y$ a continuous map, there is a natural isomorphism of relative homology groups $H_*(M_f, X) \cong H_*(C_f, CX)$, hence the long exact in homology of the pair (M_f, X) identifies with:*

$$\dots H_n(X) \xrightarrow{H_n(f)} H_n(Y) \to \tilde{H}_n(C_f) \to H_{n-1}(X) \to \dots .$$

Proof. The cone CX is contractible, hence a standard argument using the five-lemma (Proposition 6.90) implies that the morphism of pairs $(C_f, *) \to (C_f, CX)$ induces an isomorphism of relative homology groups. (Exercise: provide the details.)

Hence, it suffices to establish the isomorphism $H_*(M_f, X) \cong H_*(C_f, CX)$, which follows by excision (Theorem 6.91). Namely, taking the 'half cone' $U := C_{\frac{1}{2}}X \subset CX$, excision provides an isomorphism $H_*(C_f \backslash U, CX \backslash U) \cong H_*(C_f, CX)$. By construction, $C_f \backslash U \simeq M_f$ and $CX \backslash U \simeq X$ via the inclusions. Homotopy invariance of homology shows that $H_*(C_f \backslash U, CX \backslash U) \cong H_*(M_f, X)$ as required. (Exercise: prove this affirmation.) □

Example 6.93. For $f : X = S^n \to Y$ a continuous map consider the mapping cone C_f, which is homeomorphic to the space $Y \cup_f e^{n+1}$ obtained by gluing an $n+1$-dimensional cell e^{n+1} along its boundary $\partial e^{n+1} \cong S^n$ via f.

The associated long exact sequence in homology is

$$\ldots \to H_t(S^n) \stackrel{H_t(f)}{\to} H_t(Y) \to \tilde{H}_t(Y \cup_f e^{n+1}) \to H_{t-1}(S^n) \to \ldots.$$

If $n > 0$, then $H_0(S^n) \to H_0(Y)$ is a split monomorphism, determined by the basepoint component of X to which f maps, so one gets an isomorphism $\tilde{H}_0(Y) \stackrel{\cong}{\to} \tilde{H}_0(C_f)$.

The only other homology groups for which $H_t(Y)$ are $H_t(Y \cup_f e^{n+1})$ are not isomorphic occur in the exact sequence

$$0 \to H_{n+1}(Y) \to H_{n+1}(Y \cup_f e^{n+1}) \to \mathbb{Z} \stackrel{H_n(f)}{\to} H_n(Y) \to H_n(Y \cup_f e^{n+1}) \to 0.$$

This implies that $H_n(Y \cup_f e^{n+1}) \cong H_n(Y)/\mathrm{image}(f)$. Moreover, if $H_n(f)$ is injective then $H_{n+1}(Y) \cong H_{n+1}(Y \cup_f e^{n+1})$; otherwise $\ker(H_n(f))$ is a non-trivial subgroup of $\mathbb{Z}$, hence is a free abelian group. In this case, one gets an isomorphism

$$H_{n+1}(Y \cup_f e^{n+1}) \cong H_{n+1}(Y) \oplus \mathbb{Z}.$$

(Exercise: provide the details.)

Example 6.94. For $0 \neq d \in \mathbb{Z}$, there is a based continuous map $S^1 \stackrel{d}{\to} S^1$ which induces multiplication by d on $\pi_1(S^1, *) \cong \mathbb{Z}$. (It is a basic exercise to show that $H_1(f) : H_1(S^1) \to H_1(S^1)$ also corresponds to multiplication by d on $H_1(S^1) \cong \mathbb{Z}$.

The mapping cone C_d has homology

$$H_*(C_d) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/d & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

This construction generalizes to higher dimensions, by replacing the circle by S^n and using a *degree* d map.

Exercise 6.95. Show that for any graded abelian group A (concentrated in degrees $0 < n \in \mathbb{N}$) which is finitely-generated in each degree, there exists a topological space X_A such that the reduced homology realizes A:

$$\tilde{H}_*(X_A) \cong A.$$

(X_A is far from being unique!)

Example 6.96. The complex projective plane $\mathbb{C}P^2$ has the homotopy type of the mapping cone C_η of the Hopf map $\eta : S^3 \to S^2$. For degree reasons (exercise: make these explicit!), $H_*(\eta) = 0$, so is not detected by homology. It follows that

$$H_*(\mathbb{C}P^2) \cong \begin{cases} \mathbb{Z} & * \in \{0, 2, 4\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathbb{C}P^2$ has the same homology groups as $S^2 \vee S^4$, although these spaces do *not* have the same homotopy type.

To see the difference, one requires to look at more structure. For example, one of the following suffice:

(1) pass to *cohomology* and use the *cup product*;
(2) use *cohomology* or *homology* operations.

6.15. **Hurewicz.** The homotopy and homology groups of a space are related by the Hurewicz morphism.

Theorem 6.97. *Let X be a path connected topological space. For $0 < n \in \mathbb{N}$, there is a natural morphism (the* Hurewicz morphism*) of groups*

$$\pi_n(X, x) \to H_n(X)$$

which sends the homotopy class $[f]$ of a continuous (based) map $f : S^n \to X$ to the image $H_n(f)(1_n)$ of a generator $1_n \in H_n(S^n) \cong \mathbb{Z}$.

For $n = 1$, this induces an isomorphism

$$\pi_1(X, x)_{ab} \overset{\cong}{\to} H_1(X)$$

from the abelianization of $\pi_1(X, x)$ to $H_1(X)$.

If $n > 1$ and $\pi_t(X, x) = \{e\}$ for $t < n$, then $\pi_n(X, x) \overset{\cong}{\to} H_n(X)$ is an isomorphism.

Proof. (Indications.) It is a basic exercise to show that the construction defines a morphism of groups, using the definition of the group structure of $\pi_n(X, x)$.

For $n = 1$, since $H_1(X)$ is an abelian group, the morphism necessarily factors across the *abelianization*

$$\pi_1(X, x) \twoheadrightarrow \pi_1(X, x)_{ab}.$$

(Recall that the abelianization of a group G is the quotient $G/[G, G]$ of G by its commutator subgroup.) The proof that $\pi_1(X, x)_{ab} \to H_1(X)$ is surjective is straightforward; injectivity requires slightly more work.

The case $n > 1$ is beyond the scope of this course. □

Remark 6.98. The *Hurewicz theorem* above is one of the foundational results of algebraic topology. It can be proved using the *Freudenthal suspension theorem*, which gives an understanding of the relationship between the homotopy groups of a pointed space and of its suspension (in low dimensions). This is *much* more complicated for homotopy groups than for homology (recall that, by Proposition 6.89, the homology groups of the spheres are known; this is *far* from being the case for homotopy groups).

7. Omissions

The previous chapters have only covered the *basics*. There are a number of notable omissions, some of which are indicated below.

7.1. **Coefficients for homology.** In this section, $\mathscr{A}$ denotes the category of modules over a fixed commutative ring R. (For applications to homology of topological spaces with coefficients, $R = \mathbb{Z}$).

Definition 7.1. For (C, d) a chain complex in $\mathfrak{Ch}(\mathscr{A})$ and M an R-module, the chain complex $(C, d) \otimes_R M \in \mathrm{Ob}\, \mathfrak{Ch}(\mathscr{A})$ is given by

$$\begin{aligned} (C \otimes_R M)_n &:= C_n \otimes_R M \\ d_{C \otimes_R M} : C_n \otimes_R M &\stackrel{d^C \otimes \mathrm{id}}{\longrightarrow} C_{n-1} \otimes_R M. \end{aligned}$$

This construction defines a functor $- \otimes_R M : \mathfrak{Ch}(\mathscr{A}) \to \mathfrak{Ch}(\mathscr{A})$.

Remark 7.2.

(1) When $R = \mathbb{Z}$, an R-module is simply an abelian group.
(2) The above is a special case of the construction of the *tensor product of chain complexes*.

Example 7.3. For M an abelian group and X a topological space, the chain complex $C^{\mathrm{Sing}}(X) \otimes M$ has terms

$$(C^{\mathrm{Sing}}(X) \otimes M)_n = \mathbb{Z}[\mathfrak{Sing}_n(X)] \otimes M \cong \bigoplus_{f \in \mathfrak{Sing}_n(X)} M.$$

Definition 7.4. For M an abelian group and X a topological space, the *homology with coefficients in M* of X is

$$H_*(X; M) := H_*(C^{\mathrm{Sing}}(X) \otimes M).$$

Homology with coefficients defines a functor $H_*(-; M) : \mathfrak{Top} \to \mathfrak{Ab}^{\mathbb{N}}$ with values in $\mathbb{N}$-graded abelian groups.

Exercise 7.5. Check that homology with coefficients is functorial in the coefficients, hence corresponds to a functor:

$$H_*(-; -) : \mathfrak{Top} \times \mathfrak{Ab} \to \mathfrak{Ab}^{\mathbb{N}}.$$

Remark 7.6. One of the advantages of using coefficients is that a judicious choice of coefficients can *simplify* the calculation of homology whilst retaining the information required. Moreover, sometimes additional structure is available when working with coefficients.

For example, if M is taken to be a field $\mathbb{K}$, one has a *Künneth formula* for the homology of a product of topological spaces:

$$H_*(X \times Y; \mathbb{K}) \cong H_*(X; \mathbb{K}) \otimes_{\mathbb{K}} H_*(Y; \mathbb{K}),$$

where the right hand side is the tensor product of *graded* $\mathbb{K}$-vector spaces. In particular, this implies that the diagonal map $X \to X \times X$ induces a *coproduct* on $H_*(X; \mathbb{K})$ which has the structure of a *coalgebra*.

7.2. **Cohomology.** Homology with coefficients was introduced by using the functor $- \otimes_R M$ on the category of R-modules. Cohomology corresponds to using the *contravariant* functor $\mathrm{Hom}_R(-, M)$.

Remark 7.7. If one applies $\mathrm{Hom}_R(-, M)$ to $d : C_n \to C_{n-1}$, one obtains a morphism of R-modules:

$$\mathrm{Hom}_R(C_{n-1}, M) \to \mathrm{Hom}_R(C_n, M)$$

from a term indexed by $n-1$ to one indexed by n; that is, the degree *increases*. To stay within chain complexes as defined here, one uses the standard trick of declaring $\mathrm{Hom}_R(C_n, M)$ to be indexed by $-n$.

This avoids the confusion inherent in talking about *cochain complexes*, since no change of variance is involved, so that the <u>co</u> in cochain is a misnomer.

Definition 7.8. For (C, d) a chain complex in $\mathfrak{Ch}(\mathscr{A})$ and M an R-module, the chain complex $\mathrm{Hom}_R(C, M) \in \mathrm{Ob}\,\mathfrak{Ch}(\mathscr{A})$ is given by

$$\begin{aligned} \mathrm{Hom}_R(C, M)_{-n} &:= \mathrm{Hom}_R(C_n, M) \\ d_{\mathrm{Hom}_R(C,M)} : \mathrm{Hom}_R(C_{n-1}, M) &\xrightarrow{\mathrm{Hom}(d^C, \mathrm{id})} \mathrm{Hom}_R(C_n, M). \end{aligned}$$

This construction defines a functor $\mathrm{Hom}_R(-, M) : \mathfrak{Ch}(\mathscr{A})^{\mathrm{op}} \to \mathfrak{Ch}(\mathscr{A})$.

Remark 7.9. With the above definition, if $(C, d) \in \mathrm{Ob}\,\mathfrak{Ch}_{\geq 0}(\mathscr{A})$, then $\mathrm{Hom}_R(C, M)$ has non-zero terms concentrated in degrees ≤ 0.

Definition 7.10. For M an abelian group and X a topological space, the nth singular *cohomology of X with coefficients in M* is defined by

$$H^n(X; M) := H_{-n}(\mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{Sing}}(X), M)),$$

so that singular cohomology with coefficients in M defines a functor $H^*(-; M) : \mathfrak{Top}^{\mathrm{op}} \to \mathfrak{Ab}^{\mathbb{N}}$ with values in $\mathbb{N}$-graded abelian groups.

A fundamental result is the existence of the *cup product*; this is induced by the *diagonal map* but, unlike for the coproduct for homology, exists without having to take coefficients in a field.

Theorem 7.11. *For R a commutative ring and X a topological space, the diagonal map $X \to X \times X$ induces the cup product (for $p, q \in \mathbb{N}$):*

$$H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R)$$

which gives $H^*(X; R)$ the structure of a graded commutative R-algebra.

Proof. The key point is the construction of the *external cup product* for topological spaces X, Y:

$$H^p(X; R) \otimes H^q(Y; R) \to H^{p+q}(X \times Y; R)$$

which involves comparing the chain complex $C^{\mathrm{Sing}}(X \times Y)$ with $C^{\mathrm{Sing}}(X) \otimes C^{\mathrm{Sing}}(Y)$ (using the tensor product of chain complexes which has not been introduced here!).

The proof of these fundamental results is not particularly difficult, using the material covered in this course. □

APPENDIX A. DEFINITIONS FROM CATEGORY THEORY

A.1. **Categories.**

Definition A.1. A category $\mathscr{C}$ is a class of objects $\mathrm{Ob}\mathscr{C}$ equipped with a set of morphisms $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ (sometimes written $\mathscr{C}(X,Y)$) for each pair of objects X,Y, together with

- ▷ an *identity* $\mathrm{Id}_X \in \mathrm{Hom}_{\mathscr{C}}(X,X)$, $\forall X \in \mathrm{Ob}\,\mathscr{C}$;
- ▷ a *composition law* $\circ : \mathrm{Hom}_{\mathscr{C}}(Y,Z)\times\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$, $\forall X,Y,Z \in \mathrm{Ob}\,\mathscr{C}$,

which satisfy the following axioms:

- ▷ *identity axiom*: $\forall f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$;
- ▷ *associativity*: $h\circ(g\circ f) = (h\circ g)\circ f$ whenever the composites are defined.

The category $\mathscr{C}$ is *small* if $\mathrm{Ob}\,\mathscr{C}$ is a set; in this case, the morphisms of $\mathscr{C}$ also form a set, denoted $\mathrm{Mor}\mathscr{C}$.

Example A.2. For K a set, the discrete category associated to K has objects $\mathrm{Ob}\,K = K$ and $\mathrm{Hom}_K(X,Y) = \emptyset$ if $X \neq Y \in K$, whereas $\mathrm{Hom}_K(X,X) = \{\mathrm{Id}_X\}$ (the only morphisms are the identity morphisms). This category is small.

Definition A.3. A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ is a category $\mathscr{D}$ such that $\mathrm{Ob}\,\mathscr{D} \subset \mathrm{Ob}\,\mathscr{C}$ (sub-class) and, $\forall X,Y \in \mathrm{Ob}\,\mathscr{D}$, $\mathrm{Hom}_{\mathscr{D}}(X,Y) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ so that the identity elements and composition law are compatible.

A subcategory $\mathscr{D}$ of $\mathscr{C}$ is *full* if $\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$; a full subcategory is therefore defined by its class of objects.

Definition A.4. For $\mathscr{C}$ a category, the *opposite category* $\mathscr{C}^{\mathrm{op}}$ is the category with $\mathrm{Ob}\,\mathscr{C}^{\mathrm{op}} = \mathrm{Ob}\,\mathscr{C}$,

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X,Y) := \mathrm{Hom}_{\mathscr{C}}(Y,X)$$

and with composition law and identity elements induced from $\mathscr{C}$. (This is the category obtained from $\mathscr{C}$ by *reversing* the morphisms.)

Definition A.5. Let $\mathscr{C}$ be a category.

1. (1) A morphism $f : X \to Y$ admits an *inverse* (or is *inversible*) if $\exists\, g : Y \to X$ such $f\circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.
2. (2) The category $\mathscr{C}$ is a *groupoid* if every morphism admits an inverse.

Example A.6. A discrete group G can be condidered as a groupoid with $\mathrm{Ob}\,G = \{*\}$ and $\mathrm{Hom}_G(*,*) = G$.

A.2. **Functors.** A *functor* is a morphism between categories.

Definition A.7. For categories $\mathscr{C},\mathscr{D}$, a *functor* $F : \mathscr{C} \to \mathscr{D}$ assigns to each $X \in \mathrm{Ob}\,\mathscr{C}$ an object $F(X) \in \mathrm{Ob}\,\mathscr{D}$, together with a map of sets $\forall X,Y \in \mathrm{Ob}\,\mathscr{C}$:

$$\begin{array}{rcl}\mathrm{Hom}_{\mathscr{C}}(X,Y) & \to & \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)) \\ f & \mapsto & F(f)\end{array}$$

such that

- ▷ $\forall X \in \mathrm{Ob}\,\mathscr{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$;
- ▷ $F(g\circ f) = F(g)\circ F(f)$, for all composable morphisms $f,g \in \mathrm{Mor}\mathscr{C}$.

A *contravariant functor* from $\mathscr{C}$ to $\mathscr{D}$ is a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$.

Exercise A.8.

1. (1) For $\mathscr{C}$ a category, describe the *identity functor* $\mathrm{Id}\mathscr{C} : \mathscr{C} \to \mathscr{C}$.
2. (2) For functors $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{E}$, show that there is a composite functor $G\circ F : \mathscr{C} \to \mathscr{E}$ such that
   - ▷ $G\circ F(X) = G(F(X))$;

▷ $G \circ F(f) = G(F(f))$.

Definition A.9. A functor $F : \mathscr{C} \to \mathscr{D}$ is

▷ *faithful* if $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is an inclusion $\forall X, Y \in \mathrm{Ob}\,\mathscr{C}$;

▷ *fully faithful* if $F : \mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is a bijection $\forall X, Y \in \mathrm{Ob}\,\mathscr{C}$.

Example A.10. A subcategory $\mathscr{D} \subset \mathscr{C}$ induces an *inclusion functor* $\mathscr{D} \to \mathscr{C}$ which is *faithful*; it is *fully faithful* if and only if $\mathscr{D}$ is a full subcategory.

Proposition A.11. *The class of small categories and functors form a category (denoted $\mathfrak{CAT}$).*

Proof. Exercise (the restriction to *small* categories is necessary so that the functors beween two categories form a set). □

A.3. **Natural transformations.** Natural transformations formalize the notion of *natural* relations between constructions. They can also be understood as being *morphisms* between functors.

Definition A.12. A *natural transformation* $\eta : F \to G$ between functors $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$ is a collection of morphisms $\eta_X : F(X) \to G(X)$ in $\mathscr{D}$, for each $X \in \mathrm{Ob}\,\mathscr{C}$, such that, $\forall$ morphism of $\mathscr{C}$, $f : X \to Y$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y).
\end{tikzcd}$$

The identity morphisms $\mathrm{Id}_{F(X)}$ define the *identity* natural transformation Id_F.

Lemma A.13. *For natural transformations $\eta : F \to G$ and $\zeta : G \to H$, where F, G, H are functors from $\mathscr{C}$ to $\mathscr{D}$, the composite morphisms $\zeta_X \circ \eta_X : F(X) \to H(X)$ define a natural transformation $\zeta \circ \eta : F \to H$.*

Proof. Exercise. □

Definition A.14. A natural transformation $\eta : F \to G$ is a *natural equivalence* if it admits an inverse natural transformation $\eta^{-1} : G \to F$ such that $\eta^{-1} \circ \eta = \mathrm{Id}_F$ and $\eta \circ \eta^{-1} = \mathrm{Id}_G$. This is equivalent to the condition that $\eta_X : F(X) \to G(X)$ is an isomorphism in $\mathscr{D}$, for each $X \in \mathrm{Ob}\,\mathscr{C}$. (Exercise: check this affirmation.)

Using the notion of natural equivalence, one has the associated notion of *equivalence of categories*; this is weaker than the obvious notion of *isomorphism* of categories (and much more useful).

Definition A.15. Categories $\mathscr{C}, \mathscr{D}$ are *equivalent* if there exists functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and natural equivalences $FG \overset{\cong}{\Rightarrow} \mathrm{Id}_{\mathscr{D}}$ and $GF \overset{\cong}{\Rightarrow} \mathrm{Id}_{\mathscr{C}}$.

Remark A.16. Hidden in the above is the fact that $\mathfrak{CAT}$ has the structure of a 2-*category*. As with the consideration of homotopies between continuous maps, it can be useful to represent a natural transformation $\eta : F \to G$ by a diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathscr{D}. \arrow[from=U, to=D, Rightarrow, "\eta"]
\end{tikzcd}$$

The direction of $\Rightarrow$ is important; in general there is no natural transformation in the opposite direction!

A.4. **Yoneda's lemma.** The Yoneda lemma is a fundamental result in category theory.

Notation A.17. For $\mathscr{C}$ a small category, $\mathfrak{Set}^{\mathscr{C}}$ denotes the category of functors from $\mathscr{C}$ to $\mathfrak{Set}$, with natural transformations as morphisms.

Example A.18. For $X \in \mathrm{Ob}\,\mathscr{C}$ an object of $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, -)$ is an object of $\mathfrak{Set}^{\mathscr{C}}$, the functor *represented* by X.

The Yoneda lemma is the following result:

Proposition A.19. *For $\mathscr{C}$ a small category, $F \in \mathrm{Ob}\,\mathfrak{Set}^{\mathscr{C}}$ and X an object of $\mathscr{C}$, there is a natural bijection:*

$$\mathfrak{Y} : \mathrm{Hom}_{\mathfrak{Set}^{\mathscr{C}}}(\mathrm{Hom}_{\mathscr{C}}(X, -), F) \stackrel{\cong}{\to} F(X).$$

Proof. A natural transformation $\eta \in \mathrm{Hom}_{\mathfrak{Set}^{\mathscr{C}}}(\mathrm{Hom}_{\mathscr{C}}(X, -), F)$ determines, in particular, a map of sets $\mathrm{Hom}_{\mathscr{C}}(X, X) \stackrel{\eta_X}{\to} F(X)$. Define $\mathfrak{Y}(\eta) := \eta_X(1_X)$.

Consider a morphism $f : X \to Y$ of $\mathscr{C}$; a natural transformation η must provide set maps η_X, η_Y which make the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(X, X) \arrow[r, "\eta_X"] \arrow[d, "\mathrm{Hom}(X,f)"'] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{\mathscr{C}}(X, Y) \arrow[r, "\eta_Y"'] & F(Y).
\end{tikzcd}$$

Since $\mathrm{Hom}(X, f)(1_X) = f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, this means that $\eta_Y(f) = F(f)\big(\eta_X(1_X)\big)$. Hence, η is uniquely determined by $\mathfrak{Y}(\eta)$.

Conversely, an element of $x \in F(X)$ induces a natural transformation by this recipe. $\square$

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