J. reine angew. Math. **648** (2010), 201—244
DOI 10.1515/CRELLE.2010.084

Journal für die reine und angewandte Mathematik
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Berlin · New York 2010

Smooth toric Deligne-Mumford stacks

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Abstract. We give a geometric definition of smooth toric Deligne-Mumford stacks using the action of a "torus". We show that our definition is equivalent to the one of Borisov, Chen and Smith in terms of stacky fans. In particular, we give a geometric interpretation of the combinatorial data contained in a stacky fan. We also give a bottom up classification in terms of simplicial toric varieties and fiber products of root stacks.

Contents

Introduction

A toric variety is a normal, separated variety X with an open embedding $T \hookrightarrow X$ of a torus such that the action of the torus on itself extends to an action on X. To a toric variety one can associate a fan, a collection of cones in the lattice of one-parameter subgroups of T. Toric varieties are very important in algebraic geometry, since algebro-geometric properties of a toric variety translate in combinatorial properties of the fan, allowing to test conjectures and produce interesting examples.

In [10] Borisov, Chen and Smith define toric Deligne-Mumford stacks as explicit global quotient (smooth) stacks, associated to combinatorial data called stacky fans. Later, Iwanari proposed in [22] a definition of toric triple as an orbifold with a torus action having a dense orbit isomorphic to the torus[1)] and he proved that the 2-category of toric triples is equivalent to the 2-category of "toric stacks" (We refer to [22] for the definition of "toric stacks"). Nevertheless, it is clear that not all toric Deligne-Mumford stacks are toric triples, since some of them are not orbifolds.

Then the generalization of the Δ-collections defined for toric varieties by Cox in [14] was done by Iwanari in [23] in the orbifold case and by Perroni in [31] in the general case.

In this paper, we define a Deligne-Mumford torus $\mathcal{T}$ as a Picard stack isomorphic to $T \times \mathcal{B}G$, where T is a torus, and G is a finite abelian group; we then define a smooth toric Deligne-Mumford stack as a smooth separated Deligne-Mumford stack with the action of a Deligne-Mumford torus $\mathcal{T}$ having an open dense orbit isomorphic to $\mathcal{T}$. We prove a classification theorem for smooth toric Deligne-Mumford stacks and show that they coincide with those defined by [10].

The first main result of this paper is a bottom-up description of smooth toric Deligne-Mumford stacks, as follows: the structure morphism $\mathcal{X} \to X$ to the coarse moduli space factors canonically via the toric morphisms

$$\mathcal{X} \to \mathcal{X}^{\mathrm{rig}} \to \mathcal{X}^{\mathrm{can}} \to X$$

where $\mathcal{X} \to \mathcal{X}^{\mathrm{rig}}$ is an abelian gerbe over $\mathcal{X}^{\mathrm{rig}}$; $\mathcal{X}^{\mathrm{rig}} \to \mathcal{X}^{\mathrm{can}}$ is a fibered product of roots of toric divisors; and $\mathcal{X}^{\mathrm{can}} \to X$ is the minimal orbifold having X as coarse moduli space. Here X is a simplicial toric variety, and $\mathcal{X}^{\mathrm{rig}}$ and $\mathcal{X}^{\mathrm{can}}$ are smooth toric Deligne-Mumford stacks. More precisely, this bottom up construction can be stated as follows.

Theorem I. *Let $\mathcal{X}$ be a smooth toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times \mathcal{B}G$. Denote by X the coarse moduli space of $\mathcal{X}$. Denote by n the number of rays of the fan of X.*

1) For the meaning of orbifold in this paper, see §1.2.

(1) *There exist unique* $(a_1, \dots, a_n) \in (\mathbb{N}_{>0})^n$ *such that the stack* $\mathscr{X}^{\mathrm{rig}}$ *is isomorphic, as toric Deligne-Mumford stack, to*

$$\sqrt[a_1]{D_1^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}} \times_{\mathscr{X}^{\mathrm{can}}} \cdots \times_{\mathscr{X}^{\mathrm{can}}} \sqrt[a_n]{D_n^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}},$$

where D_i^{can} *is the divisor corresponding to the ray* ρ_i.

(2) *Given* $G = \prod_{j=1}^{\ell} \mu_{b_j}$. *There exist* $L_1, \dots, L_\ell$ *in* $\mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ *such that* $\mathscr{X}$ *is isomorphic, as toric Deligne-Mumford stack, to*

$$\sqrt[b_1]{L_1/\mathscr{X}^{\mathrm{rig}}} \times_{\mathscr{X}^{\mathrm{rig}}} \cdots \times_{\mathscr{X}^{\mathrm{rig}}} \sqrt[b_\ell]{L_\ell/\mathscr{X}^{\mathrm{rig}}}.$$

Moreover, for any $j \in \{1, \dots, \ell\}$, *the class* $[L_j]$ *in* $\mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})/b_j \, \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ *is unique.*

In the process, we get a description of the Picard group of smooth toric Deligne-Mumford stacks, which allows us to characterize weighted projective stacks as complete toric orbifolds with cyclic Picard group (cf. Proposition 7.28). Moreover, we classify all complete toric orbifolds of dimension 1 (cf. Example 7.31). We also show that the natural map from the Brauer group of a smooth toric Deligne-Mumford stack with trivial generic stabilizer to its open dense torus is injective (cf. Theorem 6.11).

The second main result of this article is to give an explicit relation between the smooth toric Deligne-Mumford stacks and the stacky fans.

Theorem II. *Let* $\mathscr{X}$ *be a smooth toric Deligne-Mumford stack with coarse moduli space the toric variety denoted by* X. *Let* Σ *be a fan of* X *in* $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. *Assume that the rays of* Σ *generate* $N_{\mathbb{Q}}$. *There exists a stacky fan such that* $\mathscr{X}$ *is isomorphic, as toric Deligne-Mumford stack, to the smooth Deligne-Mumford stack associated to the stacky fan. Moreover, if* $\mathscr{X}$ *has a trivial generic stabilizer then the stacky fan is unique.*

When the smooth toric Deligne-Mumford stack $\mathscr{X}$ has a generic stabilizer the non-uniqueness of the stacky fan comes from three different choices. We refer to Remark 7.26 for a more precise statement. This result gives a geometrical interpretation of the combinatorial data of the stacky fan. In fact, the stacky fan can be read off the geometry of the smooth toric Deligne-Mumford stack just like the fan can be read off the geometry of the toric variety. Notice that one can deduce the above theorem when $\mathscr{X}$ is an orbifold from [31], Theorem 2.5, and [23], Theorem 1.4, and the geometric characterization of [24], Theorem 1.3.

In the first part of this article, we fix the conventions and collect some results on smooth Deligne-Mumford stacks, root constructions, rigidification, toric varieties, Picard stacks and the action of a Picard stack. In Section 2, we define Deligne-Mumford tori. Section 3 contains the definition of smooth toric Deligne-Mumford stacks. In Section 4, we first define canonical smooth Deligne-Mumford stacks and then we show that the canonical stack associated to a simplicial toric variety is a smooth toric Deligne-Mumford stack (cf. Theorem 4.11). In Section 5, we prove the first part of Theorem I. In Section 6, we first prove in Proposition 6.9 that the essentially trivial banded gerbes over $\mathscr{X}$ are in bijection with finite extensions of the Picard group of $\mathcal{X}$; then, we show that the natural map from the Brauer group of a smooth toric Deligne-Mumford stack with trivial generic stabilizer to its open dense torus is injective (cf. Theorem 6.11). Finally, we prove the second statement of Theorem I. In Section 7, we prove Theorem II and give some explicit examples. In Appendix B, we have put some details about the action of a Picard stack.

Acknowledgments. The authors would like to acknowledge support from IHP, Mittag-Leffler Institut, SNS where part of this work was carried out, as well as the European projects MISGAM and ENIGMA. We would like to thank Ettore Aldrovandi, Lev Borisov, Jean-Louis Colliot-Thélène, Andrew Kresch, Fabio Perroni, Ilya Tyomkin and Angelo Vistoli for helpful discussions; in particular Aldrovandi for explanations about group-stacks and reference [11], Borisov for pointing out a mistake in a preliminary version, Colliot-Thélène for [20], §6, Tyomkin for [9] and Vistoli for useful information about the classification of gerbes.

1. Notations and background

1.1. Conventions and notations. A scheme will be a separated scheme of finite type over $\mathbb{C}$. A variety will be a reduced, irreducible scheme. A point will be a $\mathbb{C}$-valued point. The smooth locus of a variety X will be denoted by X_{sm}.

We work in the étale topology. For an algebraic stack $\mathcal{X}$, we will write that x is a point of $\mathcal{X}$ or just $x \in \mathcal{X}$ to mean that x is an object in $\mathcal{X}(\mathbb{C})$; we denote by $\mathrm{Aut}(x)$ the automorphism group of the point x. We will say that a morphism between stacks is unique if it is unique up to a unique 2-arrow. As usual, we denote $\mathbb{G}_m$ the sheaf of invertible sections in $\mathcal{O}_{\mathcal{X}}$ on the étale site of $\mathcal{X}$.

1.2. Smooth Deligne-Mumford stacks and orbifolds. A Deligne-Mumford stack will be a separated Deligne-Mumford stack of finite type over $\mathbb{C}$; we will always assume that its coarse moduli space is a scheme. An *orbifold* will be a smooth Deligne-Mumford stack with trivial generic stabilizer. For a smooth Deligne-Mumford stack $\mathcal{X}$, we denote by $\varepsilon_{\mathcal{X}}$ or just ε the natural morphism from $\mathcal{X}$ to its coarse moduli space X, which is a variety with finite quotient singularities.

Let $\imath : \mathcal{U} \to \mathcal{X}$ be an open embedding of irreducible smooth Deligne-Mumford stacks with complement of codimension at least 2. We have that:

- The natural map $\imath^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{U})$ is an isomorphism.

- For any line bundle $L \in \mathrm{Pic}(\mathcal{X})$, the natural morphism $\imath^* : H^0(\mathcal{X}, L) \to H^0(\mathcal{U}, \imath^* L)$ is also an isomorphism.

The *inertia stack*, denoted by $I(\mathcal{X})$, is defined to be the fibered product $I(\mathcal{X}) := \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$. A point of $I(\mathcal{X})$ is a pair (x, g) with $x \in \mathcal{X}$ and $g \in \mathrm{Aut}(x)$. The inertia stack of a smooth Deligne-Mumford stack is smooth but different components will in general have different dimensions. The natural morphism $I(\mathcal{X}) \to \mathcal{X}$ is representable, unramified, proper and a relative group scheme. The identity section gives an irreducible component canonically isomorphic to $\mathcal{X}$; all other components are called *twisted sectors*.

A smooth Deligne-Mumford stack of dimension d is an orbifold if and only if all the twisted sectors have dimension $\leqq d-1$, and is *canonical* if and only if all twisted sectors have dimension $\leqq d-2$.

Remark 1.1 (sheaves on global quotients). According to [33], Appendix, a coherent sheaf on a Deligne-Mumford stack $[Z/G]$ is a G-equivariant sheaf on Z, i.e., the data of a coherent sheaf L_Z on Z and for every $g \in G$ an isomorphism $\varphi_g : L_Z \to g^* L_Z$ such that $\varphi_{gh} = h^* \varphi_g \circ \varphi_h$.

Notice that if Z is a subvariety of $\mathbb{C}^n$ of codimension higher or equal than two then an invertible sheaf on $[Z/G]$ is the structure sheaf $\mathcal{O}_Z$ and a one dimensional representation of G, i.e., $\chi : G \to \mathbb{C}^*$. A global section of such an invertible sheaf on $[Z/G]$ is a χ-equivariant global section of $\mathcal{O}_Z$.

We end this subsection with a proposition extending to stacks a property of separated schemes. We will prove it in Appendix A.

Proposition 1.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Deligne-Mumford stacks. Assume that $\mathcal{X}$ is normal and $\mathcal{Y}$ is separated. Let $\iota : \mathcal{U} \hookrightarrow \mathcal{X}$ be a dominant open immersion of the Deligne-Mumford stack $\mathcal{U}$. If $F, G : \mathcal{X} \to \mathcal{Y}$ are two morphisms of stacks such that there exits a 2-arrow $F \circ \iota \overset{\beta}{\Rightarrow} G \circ \iota$ then there exists a unique 2-arrow $\alpha : F \Rightarrow G$ such that $\alpha * \mathrm{id}_\iota = \beta$.*

The previous proposition is well known for $\mathcal{X}$ a reduced scheme and $\mathcal{Y}$ a separated scheme. Nevertheless, if $\mathcal{X}$ is not a normal stack we have the following counter-example: Let $\mathcal{Y}$ be $\mathcal{B}\mu_2$. Let $\mathcal{X}$ be a rational curve with one node. Let $F_1 : \mathcal{X} \to \mathcal{Y}$ (resp. F_2) be a stack morphism given by a non-trivial (resp. trivial) double cover of $\mathcal{X}$. Putting $\mathcal{U} = \mathcal{X} \backslash \{\text{node}\}$, the proposition is false.

1.3. Root constructions. For this subsection we refer to the paper of Cadman [12] (see also [2], Appendix B). In this part $\mathcal{X}$ will be a Deligne-Mumford stack over $\mathbb{C}$ (it is enough to assume that $\mathcal{X}$ is Artin.)

1.3.a. Root of an invertible sheaf. This part follows closely [2], Appendix B. Let L be an invertible sheaf on the Deligne-Mumford stack $\mathcal{X}$. Let b be a positive integer. We denote by $\sqrt[b]{L/\mathcal{X}}$ the following fiber product

$$\begin{array}{ccc} \sqrt[b]{L/\mathcal{X}} & \longrightarrow & \mathcal{B}\mathbb{C}^* \\ \Big\downarrow & \square & \Big\downarrow{\scriptstyle \wedge b} \\ \mathcal{X} & \xrightarrow{\;L\;} & \mathcal{B}\mathbb{C}^* \end{array}$$

where $\wedge b : \mathcal{B}\mathbb{C}^* \to \mathcal{B}\mathbb{C}^*$ sends an invertible sheaf M over a scheme S to $M^{\otimes b}$. More explicitly, an object of $\sqrt[b]{L/\mathcal{X}}$ over $f : S \to \mathcal{X}$ is a couple (M, φ) where M is an invertible sheaf M on the scheme S and $\varphi : M^{\otimes b} \xrightarrow{\sim} f^* L$ is an isomorphism. The arrows are defined in an obvious way.

The morphism $\sqrt[b]{L/\mathcal{X}} \to B\mathbb{C}^*$ corresponds to an invertible sheaf, denoted by $L^{1/b}$ in [8], on $\sqrt[b]{L/\mathcal{X}}$ whose b-th power is isomorphic to the pullback of L.

The stack $\sqrt[b]{L/\mathscr{X}}$ is a μ_b-banded gerbe over $\mathscr{X}$ (see the second paragraph of Subsection 6.1 below). The Kummer exact sequence

$$1 \to \mu_b \to \mathbb{G}_m \xrightarrow{\wedge b} \mathbb{G}_m \to 1$$

induces the boundary morphism $\partial : H^1_{\text{ét}}(\mathscr{X}, \mathbb{G}_m) \to H^2_{\text{ét}}(\mathscr{X}, \mu_b)$. The cohomology class of the μ_b-banded gerbe $\sqrt[b]{L/\mathscr{X}}$ in $H^2_{\text{ét}}(\mathscr{X}, \mu_b)$ is the image by ∂ of the class $[L] \in H^1_{\text{ét}}(\mathscr{X}, \mathbb{G}_m)$.

The gerbe is trivial if and only if the invertible sheaf L has a b-th root in $\operatorname{Pic}(\mathscr{X})$. More generally, the gerbe $\sqrt[b]{L/\mathscr{X}}$ is isomorphic, as a μ_b-banded gerbe, to $\sqrt[b]{L'/\mathscr{X}}$ if and only if $[L] = [L']$ in $\operatorname{Pic}(\mathscr{X})/b\operatorname{Pic}(\mathscr{X})$. We have the following morphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times b} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/b\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Pic}(\mathscr{X}) & \longrightarrow & \operatorname{Pic}(\sqrt[b]{L/\mathscr{X}}) & \longrightarrow & \mathbb{Z}/b\mathbb{Z} & \longrightarrow & 0 \end{array} \tag{1.3}$$

where the first and second vertical morphisms are defined by $1 \mapsto L$ and $1 \mapsto L^{1/b}$, respectively.

1.3.b. Roots of effective Cartier divisors. In the articles [12] and [2], the authors define the notion of root of an invertible sheaf with a section on an algebraic stack: here, we only consider roots of effective Cartier divisors on a smooth algebraic stack, since this is what we will use.

Let n be a positive integer. Consider the quotient stack $[\mathbb{A}^n/(\mathbb{C}^*)^n]$ where the action of $(\mathbb{C}^*)^n$ is given multiplication coordinates by coordinates. Notice that $[\mathbb{A}^n/(\mathbb{C}^*)^n]$ is the moduli stack of n line bundles with n global sections. Let $\boldsymbol{a} := (a_1, \dots, a_n) \in (\mathbb{N}_{>0})^n$ be an n-tuple. Denote by $\wedge\boldsymbol{a} : [\mathbb{A}^n/(\mathbb{C}^*)^n] \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$ the stack morphism defined by sending $x_i \mapsto x_i^{a_i}$ and $\lambda_i \mapsto \lambda_i^{a_i}$ where x_i (resp. λ_i) are coordinates of $\mathbb{A}^n$ (resp. $(\mathbb{C}^*)^n$).

Let $\mathscr{X}$ be a smooth algebraic stack. Let $\boldsymbol{D} := (D_1, \dots, D_n)$ be n effective Cartier divisors. The $\boldsymbol{a}$-th root of $(\mathscr{X}, \boldsymbol{D})$ is the fiber product

$$\begin{array}{ccc} \sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathscr{X}} & \longrightarrow & [\mathbb{A}^n/(\mathbb{C}^*)^n] \\ {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \wedge\boldsymbol{a}} \\ \mathscr{X} & \xrightarrow{\boldsymbol{D}} & [\mathbb{A}^n/(\mathbb{C}^*)^n]. \end{array}$$

The morphism $\sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathscr{X}} \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$ corresponds to the effective Cartier divisors $\tilde{\boldsymbol{D}} := (\tilde{D}_1, \dots, \tilde{D}_n)$, where $\tilde{D}_i$ is the reduced closed substack $\pi^{-1}(D_i)_{\text{red}}$. More explicitly, an object of $\sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathscr{X}}$ over a scheme S is a couple $(f, (\tilde{D}_1, \dots, \tilde{D}_n))$ where $f : S \to \mathscr{X}$ is a morphism and for any i, D_i is an effective divisor on S such that $a_i\tilde{D}_i = f^*D_i$.

We have the following properties:

(1) The fiber product of $\sqrt[a_i]{D_i/\mathscr{X}}$ over $\mathscr{X}$ is isomorphic to $\sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathscr{X}}$ (cf. [12], Remark 2.2.5).

(2) The canonical morphism $\sqrt[a]{\boldsymbol{D}/\mathscr{X}} \to \mathscr{X}$ is an isomorphism over $\mathscr{X} \setminus \bigcup_i D_i$.

(3) If $\mathscr{X}$ is smooth, each D_i is smooth and the D_i have simple normal crossing then $\sqrt[a]{\boldsymbol{D}/\mathscr{X}}$ is smooth (cf. Section 2.1 of [8]) and $\tilde{D}_i$ have simple normal crossing.

(4) We have the following morphism of short exact sequences (cf. [12], Corollary 3.1.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\times \boldsymbol{a}} & \mathbb{Z}^n & \longrightarrow & \prod_{i=1}^n \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Pic}(\mathscr{X}) & \xrightarrow{\pi^*} & \operatorname{Pic}(\sqrt[a]{\boldsymbol{D}/\mathscr{X}}) & \xrightarrow{q} & \prod_{i=1}^n \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0 \end{array} \tag{1.4}$$

where the first and second vertical morphisms are defined by $e_i \mapsto \mathscr{O}(D_i)$ and $e_i \mapsto \mathscr{O}(\tilde{D}_i)$, respectively. Every invertible sheaf $L \in \operatorname{Pic}(\sqrt[a]{\boldsymbol{D}/\mathscr{X}})$ can be written in a unique way as $L \cong \pi^* M \otimes \prod_{i=1}^n \mathscr{O}(k_i \tilde{D}_i)$ where $M \in \operatorname{Pic}(\mathscr{X})$ and $0 \leqq k_i < a_i$; the morphism q maps L to $(k_1, \dots, k_n)$.

We finish this section with the following observation. Let D_1 and D_2 be two effective Cartier divisors on $\mathscr{X}$ such that $D_1 \cap D_2 \neq \emptyset$. The stacks $\sqrt[a]{D_1 \cup D_2/\mathscr{X}}$ and $\sqrt[(a,a)]{(D_1, D_2)/\mathscr{X}}$ are not isomorphic. Indeed, the stabilizer group at any point in the preimage of $x \in D_1 \cap D_2$ in $\sqrt[a]{D_1 \cup D_2/\mathscr{X}}$ (resp. $\sqrt[(a,a)]{(D_1, D_2)/\mathscr{X}}$) is μ_a (resp. $\mu_a \times \mu_a$).

1.4. Rigidification. In this subsection, we sum up some results on the rigidification of an irreducible d-dimensional smooth Deligne-Mumford stack $\mathscr{X}$. Intuitively, the rigidification of $\mathscr{X}$ by a central subgroup G of the *generic stabilizer* is constructed as follows: first, one constructs a prestack where the objects are the same and the automorphism groups of each object x are the quotient $\operatorname{Aut}_{\mathscr{X}}(x)/G$; then the rigidification $\mathscr{X} /\!\!/ G$ is the stackification of this prestack. For the most general construction we refer to [3], Appendix A (see also [1], Section 5.1, [32] and [2], Appendix C).

We consider the union $I^{\text{gen}}(\mathscr{X}) \subset I(\mathscr{X})$ of all d-dimensional components of $I(\mathscr{X})$; it is a subsheaf of groups of $I(\mathscr{X})$ over $\mathscr{X}$ which is called the generic stabilizer. Most of the time in this article, we will rigidify by the generic stabilizer. In this case, we write $\mathscr{X}^{\text{rig}}$ in order to mean $\mathscr{X} /\!\!/ I^{\text{gen}}(\mathscr{X})$ and call it *the* rigidification.

The rigidification $r : \mathscr{X} \to \mathscr{X}^{\text{rig}}$ has the following properties:

(1) The coarse moduli space of $\mathscr{X}^{\text{rig}}$ is the coarse moduli space of $\mathscr{X}$.

(2) $\mathscr{X}^{\text{rig}}$ is an orbifold.

(3) If $\mathscr{X}$ is an orbifold then $\mathscr{X}^{\text{rig}}$ is $\mathscr{X}$.

(4) The morphism r makes $\mathscr{X}$ into a gerbe over $\mathscr{X}^{\text{rig}}$.

We refer to [1], Theorem 5.1.5(2), for the proof of the following proposition.

Proposition 1.5 (universal property of the rigidification). *Let $\mathcal{X}$ be a smooth Deligne-Mumford stack. Let $\mathcal{Y}$ be an orbifold. Let $f : \mathcal{X} \to \mathcal{Y}$ be a dominant stack morphism. Then there exists $g : \mathcal{X}^{\mathrm{rig}} \to \mathcal{Y}$ and a 2-morphism $\alpha : g \circ r \Rightarrow f$ such that the following is 2-commutative*:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ r\ } & \mathcal{X}^{\mathrm{rig}} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \exists g} \\ & & \mathcal{Y}. \end{array}$$

If there exists $g' : \mathcal{X}^{\mathrm{rig}} \to \mathcal{Y}$ and a 2-morphism $\alpha' : g' \circ r \Rightarrow f$ satisfying the same property then there exists a unique $\gamma : g' \Rightarrow g$ such that $\alpha \circ (\gamma * \mathrm{id}_r) = \alpha'$.

1.5. Diagonalizable group schemes. In this short subsection, we recall some results on diagonalizable groups.

Definition 1.6. A group scheme G over $\operatorname{Spec} \mathbb{C}$ will be called *diagonalizable* if it is isomorphic to the product of a torus and a finite abelian group.

We use multiplicative notation for diagonalizable group. For any diagonalizable group G, its character group $G^\vee := \operatorname{Hom}(G, \mathbb{C}^*)$ is a finitely generated abelian group (or coherent $\mathbb{Z}$-module). The duality contravariant functor $G \mapsto G^\vee$ induces an equivalence of categories from diagonalizable to coherent $\mathbb{Z}$-module. Its inverse functor is given by $F \mapsto G_F := \operatorname{Hom}(F, \mathbb{C}^*)$. Both $G \mapsto G^\vee$ and $F \mapsto G_F$ are contravariant and exact.

1.6. Toric varieties. We recall some results on toric varieties that can be found in [17] (see also [15]). The principal construction used in this paper is the description of toric varieties as global quotients found by Cox (see [13]).

We fix a torus T, and denote by $M = T^\vee$ the lattice of characters and by $N := \operatorname{Hom}(M, \mathbb{Z})$ the lattice of one-parameter subgroups. A toric variety X with torus T corresponds to a fan $\Sigma(X)$, or just Σ, in $N_{\mathbb{Q}} := \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{Q}$, which we will always assume to be simplicial.

Let $\rho_1, \dots, \rho_n$ be the one-dimensional cones, called rays, of Σ. For any ray ρ_i, denote by v_i the unique generator of $\rho_i \cap N$. For any i in $\{1, \dots, n\}$, we denote by D_i the irreducible T-invariant Weil divisor defined by the ray ρ_i. The free abelian group of T-invariant Weil divisor is denoted by L.

Let $\imath : M \to L$ be the morphism that sends $m \mapsto \sum_{i=1}^{n} m(v_i)$. If the rays span $\mathbb{N}_{\mathbb{Q}}$ (which is not a strong assumption[2)]), the morphism $\imath$ is injective, and fits into an exact sequence in $\operatorname{Coh}(\mathbb{Z})$

$$0 \to M \xrightarrow{\imath} L \to A \to 0, \tag{1.7}$$

[2)] Indeed, if the rays do not span $N_{\mathbb{Q}}$ then X is isomorphic to the product of a torus and a toric variety $\tilde{X}$ whose rays span $\tilde{N}_{\mathbb{Q}}$.

where A is the class group of X (i.e., the Chow group $A^1(X)$). We deduce that the short exact sequence of diagonalizable groups

$$1 \to G_A \to G_L \to T \to 1. \tag{1.8}$$

Let $Z_\Sigma \subset \mathbb{C}^n$ be the $G_L = (\mathbb{C}^*)^n$-invariant open subset defined as $Z_\Sigma := \bigcup_{\sigma \in \Sigma} Z_\sigma$, where $Z_\sigma := \{x \mid x_i \neq 0 \text{ if } \rho_i \notin \sigma\}$. The induced action of G_A on Z_Σ has finite stabilizers (by the simpliciality assumption) and X is the geometric quotient Z_Σ/G_A, with torus $(\mathbb{C}^*)^n/G_A$ (see [13], Theorem 2.1). For any $i \in \{1, \dots, n\}$, the T-invariant Weil divisor $D_i \subset X$ is the geometric quotient

$$(\{x_i = 0\} \cap Z_\Sigma)/G_A. \tag{1.9}$$

If X is smooth then the natural morphism $L \to \mathrm{Pic}(X)$ given by $e_i \mapsto \mathcal{O}_X(D_i)$ is surjective and has kernel M; in other words, it induces a natural isomorphism $A \to \mathrm{Pic}(X)$.

If X is a d-dimensional toric variety, we will write X^0 for the union of the orbits of dimension $\geqq d-1$; in other words, X^0 is the toric variety associated to the fan $\Sigma_{\leqq 1} := \{\sigma \in \Sigma \mid \dim \sigma \leqq 1\}$. The toric variety X^0 is always smooth and the toric divisors D^0_ρ are smooth, disjoint, and homogeneous under the T-action (with stabilizer the one-dimensional subgroup which is the image of ρ).

1.7. Picard stacks and action of a Picard stack. Deligne defined Picard stacks in [7], Exposé XVIII, as stacks analogous to sheaves of abelian groups. For the reader's convenience, we collect here a sketch of the definition and the main properties we need; details can be found in [7], Exposé XVIII, and also in [26], Section 14.

Here we summarize the definition of a Picard stack. For the details we refer to Definition B.1.

Definition 1.10. Let $\mathcal{G}$ be a stack over a base scheme S. A *Picard stack* $\mathcal{G}$ over S is given by the following set of data:

- a multiplication stack morphism $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, also denoted by
$$m(g_1, g_2) = g_1 \cdot g_2;$$
- an associativity 2-arrow $(g_1 \cdot g_2) \cdot g_3 \Rightarrow g_1 \cdot (g_2 \cdot g_3)$;
- a commutativity 2-arrow $g_1 \cdot g_2 \Rightarrow g_2 \cdot g_1$.

These data satisfy some compatibility relations, which we list in B.1.

The definition implies that there also exists an identity $e : S \to \mathcal{G}$ and an inverse $i : \mathcal{G} \to \mathcal{G}$ with the obvious properties; in particular, a 2-arrow $\varepsilon : (e \cdot g) \Rightarrow g$.

Definition 1.11 (see [7], Section 1.4.6). Let $\mathcal{G}$, $\mathcal{G}'$ be two Picard stacks. A *morphism of Picard stacks* $F : \mathcal{G} \to \mathcal{G}'$ is a morphism of stacks and a 2-arrow α such that for any two objects g_1, g_2 in $\mathcal{G}$, we have

$$F(g_1 \cdot g_2) \overset{\alpha}{\Rightarrow} F(g_1) \cdot F(g_2).$$

Again we refer to Appendix B for the list of compatibilities satisfied by α. The Picard stacks over S form a category where the objects are Picard stacks and morphisms are equivalence classes of morphisms of Picard stacks.

Remark 1.12. To any complex $G^\bullet := [G^{-1} \to G^0]$ of sheaves of abelian groups, we can associate a Picard stack $\mathscr{G}$. In this paper, $G^\bullet$ will be a complex of diagonalizable groups and the associated Picard stack is the quotient stack $[G^{-1}/G^0]$.

Proposition 1.13 (see [7], Proposition 1.4.15). *The functor that associates to a length* 1 *complex of sheaves of abelian groups a Picard stack induces an equivalence of categories between the derived category, denoted by* $D^{[-1,0]}(S, \mathbb{Z})$, *of length* 1 *complexes of sheaves of abelian groups and the category of Picard stacks.*

In particular, if G is any sheaf of abelian groups on the base scheme S, the quotient $[S/G]$, i.e. the gerbe $\mathscr{B}G$, is naturally a Picard stack.

We finish this section with a sketch of the definition of an action of a Picard stack on a stack. This is a generalization of the action of a group scheme on a stack defined by Romagny in [32]. We refer to Definition B.12 for the details.

Definition 1.14 (action of a Picard stack). Let $\mathscr{G}$ be a Picard stack. Denote by e the neutral section and by ϵ the corresponding 2-arrow. Let $\mathscr{X}$ be a stack. An *action* of $\mathscr{G}$ on $\mathscr{X}$ is the following data:

- a stack morphism $a : \mathscr{G} \times \mathscr{X} \to \mathscr{X}$, also denoted by $a(g, x) = g \times x$;
- a 2-arrow $e \times x \Rightarrow x$;
- an associativity 2-arrow $(g_1 \cdot g_2) \times x \Rightarrow g_1 \times (g_2 \times x)$.

These data satisfy some compatibility relations, which we list in Appendix B.

2. Deligne-Mumford tori

In this section we define Deligne-Mumford tori which will play the role of the torus for a toric variety.

We start with a technical lemma.

Lemma 2.1. *Let* $\phi : A^0 \to A^1$ *be a morphism of finitely generated abelian groups such that* $\ker\phi$ *is free. In the derived category of complexes of finitely generated abelian groups of length* 1, *the complex* $[A^0 \to A^1]$ *is isomorphic to* $[\ker\phi \xrightarrow{0} \operatorname{coker}\phi]$.

Proof. We have a morphism of complexes

$$[A^0 \xrightarrow{\phi} A^1] \to [A^0/A^0_{\mathrm{tor}} \xrightarrow{\tilde{\phi}} A^1/A^0_{\mathrm{tor}}]$$

induced by the quotient morphisms. As $\ker\phi$ is free, we deduce after a diagram chasing that this morphism is a quasi-isomorphism of complexes. In the derived category, we replace

A^1/A^0_{tor} with a projective resolution $[\mathbb{Z}^\ell \xrightarrow{Q} \mathbb{Z}^{d+\ell}]$. Then the mapping cone of the morphism of complexes $[0 \to A^0/A^0_{\mathrm{tor}}] \to [Q : \mathbb{Z}^\ell \to \mathbb{Z}^{d+\ell}]$ is $[[\tilde{\phi} Q] : A^0/A^0_{\mathrm{tor}} \times \mathbb{Z}^\ell \to \mathbb{Z}^{d+\ell}]$ which is quasi-isomorphic to $[A^0/A^0_{\mathrm{tor}} \xrightarrow{\phi} A^1_0/A^0_{\mathrm{tor}}]$. A morphism of free abelian groups f is quasi-isomorphic to the complex $[\ker f \to \operatorname{coker} f]$ and this finishes the proof. □

The reader who is familiar with the article [10] has probably recognized part of the construction of the stack associated to a stacky fan.

Remark 2.2. Let $\phi : A^0 \to A^1$ be a morphism of finitely generated abelian groups as in the above lemma. Applying the contravariant functor $\operatorname{Hom}(\cdot, \mathbb{C}^*)$ of Section 1.5 to the complex $A^0 \to A^1$, we get a length 1 complex of diagonalizable groups $[G_{A^1} \xrightarrow{G_\phi} G_{A^0}]$. According to Remark 1.12, the associated Picard stack $[G_{A^0}/G_{A^1}]$ is a Deligne-Mumford stack if and only if the cokernel of ϕ is finite.

Example 2.3. Let $w_0, \dots, w_n$ be in $\mathbb{N}_{>0}$. Let $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}$ that sends $(a_0, \dots, a_n)$ to $\sum w_i a_i$. We have that $\ker \phi = \mathbb{Z}^n$ and $\operatorname{coker} \phi = \mathbb{Z}/d\mathbb{Z}$ where $d := \gcd(w_0, \dots, w_n)$. Hence, the associated Picard stack is $(\mathbb{C}^*)^n \times \mathscr{B}\mu_d$.

Definition 2.4. A *Deligne-Mumford torus* is a Picard stack over $\operatorname{Spec} \mathbb{C}$ which is obtained as a quotient $[G_{A^0}/G_{A^1}]$, where $\phi : A^0 \to A^1$ is a morphism of finitely generated abelian groups such that $\ker \phi$ is free and $\operatorname{coker} \phi$ is finite.

Let G be a finite abelian group. Notice that $\mathscr{B}G$ is a Deligne-Mumford torus. Recall that by Proposition 1.13, $T \times \mathscr{B}G$ has a natural structure of Picard stack.

Definition 2.5. A short exact sequence of Picard S-stacks is the sequence of morphisms of Picard S-stacks associated to a distinguished triangle in $D^{[-1,0]}(S)$.

Proposition 2.6. *Any Deligne-Mumford torus $\mathscr{T}$ is isomorphic as Picard stack to $T \times \mathscr{B}G$ where T is a torus and G is a finite abelian group.*

Proof. Let $\mathscr{T} = [G_{A^0}/G_{A^1}]$ with $\phi : A^0 \to A^1$ as above. The distinguished triangle $[\ker G_\phi \to 0] \to [G_{A^1} \xrightarrow{G_\phi} G_{A^0}] \to [0 \to \operatorname{coker} G_\phi]$ in the derived category $D^{[-1,0]}(\operatorname{Spec} \mathbb{C})$ induces an exact sequence of Picard stacks $1 \to \mathscr{B}G \to \mathscr{T} \to T \to 1$ where $T := G_{A^0}/G_{A^1}$. Proposition 1.13 and Lemma 2.1 imply that there is a non-canonical isomorphism of Picard stacks $\mathscr{T} \equiv \mathscr{B}G \times T$. □

Note that the scheme T in the previous proof is the coarse moduli space of $\mathscr{T}$.

3. Definition of toric Deligne-Mumford stacks

Definition 3.1. A *smooth toric Deligne-Mumford stack* is a smooth separated Deligne-Mumford stack $\mathscr{X}$ together with an open immersion of a Deligne-Mumford torus $\iota : \mathscr{T} \hookrightarrow \mathscr{X}$ with dense image such that the action of $\mathscr{T}$ on itself extends to an action $a : \mathscr{T} \times \mathscr{X} \to \mathscr{X}$.

As in this paper all toric Deligne-Mumford stacks are smooth, we will write toric Deligne-Mumford stack instead of smooth toric Deligne-Mumford stack. We will see later in Theorem 7.24 that our definition a posteriori coincides with that in [10] via stacky fans. It seems natural to define a toric Deligne-Mumford stack by replacing smooth with normal in the above definition. All the definitions and results in this section apply also in this case, with the exception of Proposition 3.6 and Lemma 3.8. Ilya Tyomkin is currently working on this. A *toric orbifold* is a toric Deligne-Mumford stack with generically trivial stabilizer. A toric Deligne-Mumford stack is a toric orbifold if and only if its Deligne-Mumford torus is an ordinary torus. Hence, the notion of toric orbifold is the same as the one used in [22], Theorem 1.3.

Remark 3.2. (1) Separatedness of $\mathcal{X}$ and Proposition 1.2 imply that the action of $\mathcal{T}$ on $\mathcal{X}$ is uniquely determined by ι.

(2) Notice that we have assumed in Section 1.2 that the coarse moduli space is a scheme. Without this assumption, if the coarse moduli space X of a toric Deligne-Mumford stack is a smooth and complete algebraic space then the main theorem of Bialynicki-Birula in [9] implies that X is a scheme. We don't know whether such an assumption is necessary in general.

(3) A toric variety admits a structure of toric Deligne-Mumford stack if and only if it is smooth.

Proposition 3.3. *Let $\mathcal{X}$ be a smooth Deligne-Mumford stack together with an open dense immersion of a Deligne-Mumford torus $\iota : \mathcal{T} \hookrightarrow \mathcal{X}$ such that the action of $\mathcal{T}$ on itself extends to a stack morphism $a : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$. Then the stack morphism a induces naturally an action of $\mathcal{T}$ on $\mathcal{X}$.*

Proof. We will define a 2-arrow $\eta : a \circ (e, \mathrm{id}_{\mathcal{X}}) \Rightarrow \mathrm{id}_{\mathcal{X}}$ and a 2-arrow

$$\sigma : a \circ (m, \mathrm{id}_{\mathcal{X}}) \Rightarrow a \circ (\mathrm{id}_{\mathcal{X}}, a)$$

such that they verify conditions (1) and (2) of Definition B.12. We will only prove the existence of η because the existence of σ and the relations (1) and (2) follow with a similar argument.

Denote by $e : \operatorname{Spec} \mathbb{C} \to \mathcal{T}$ the neutral element of $\mathcal{T}$ and by $m : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ the multiplication on $\mathcal{T}$. Denote by ε the 2-arrow $m \circ (e, \mathrm{id}_{\mathcal{T}}) \Rightarrow \mathrm{id}_{\mathcal{T}}$. As the stack morphism a extends m, we have a 2-arrow $\alpha : a \circ (\mathrm{id}_{\mathcal{T}}, \iota) \Rightarrow \iota \circ m$. Denote by β the 2-arrow $(e, \mathrm{id}_{\mathcal{X}}) \circ \iota \Rightarrow (\mathrm{id}_{\mathcal{T}}, \iota) \circ (e, \mathrm{id}_{\mathcal{T}})$. Consider the two stack morphisms:

$$\mathcal{T} \xhookrightarrow{\ \iota\ } \mathcal{X} \underset{a \circ (e, \mathrm{id}_{\mathcal{X}})}{\overset{\mathrm{id}_{\mathcal{X}}}{\rightrightarrows}} \mathcal{X}.$$

Applying Proposition 1.2 with the composition of the following 2-arrows

$$a \circ (e, \mathrm{id}_{\mathcal{X}}) \circ \iota \xRightarrow{\mathrm{id}_a * \beta} a \circ (\mathrm{id}_{\mathcal{T}}, \iota) \circ (e, \mathrm{id}_{\mathcal{T}}) \xRightarrow{\alpha * \mathrm{id}_{(e, \mathrm{id}_{\mathcal{T}})}} \iota \circ m \circ (e, \mathrm{id}_{\mathcal{T}}) \xRightarrow{\mathrm{id}_\iota * \varepsilon} \iota \circ \mathrm{id}_{\mathcal{T}} = \mathrm{id}_{\mathcal{X}} \circ \iota,$$

we deduce the existence of $\eta : a \circ (e, \mathrm{id}_{\mathcal{X}}) \Rightarrow \mathrm{id}_{\mathcal{X}}$. □

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Definition 3.4. Let $\mathcal{X}$ (resp. $\mathcal{X}'$) be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T}$ (resp. $\mathcal{T}'$). A *morphism of toric Deligne-Mumford stacks* $F : \mathcal{X} \to \mathcal{X}'$ is a morphism of stacks between $\mathcal{X}$ and $\mathcal{X}'$ which extends a morphism of Deligne-Mumford tori $\mathcal{T} \to \mathcal{T}'$:

Remark 3.5. The extended morphism F in the previous definition is unique by Proposition 1.2. Moreover the definition of morphism between Picard stacks and Proposition 1.2 provide us the following 2-cartesian diagram:

$$\begin{array}{ccc} \mathcal{X} \times \mathcal{T} & \xrightarrow{(F, F|_{\mathcal{T}})} & \mathcal{X}' \times \mathcal{T}' \\ {\scriptstyle a}\downarrow & \square & \downarrow{\scriptstyle a'} \\ \mathcal{X} & \xrightarrow{\quad F \quad} & \mathcal{X}'. \end{array}$$

Proposition 3.6. *Let $\mathcal{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T}$. Let X (resp. T) be the coarse moduli space of $\mathcal{X}$ (resp. $\mathcal{T}$). Then X has a structure of simplicial toric variety with torus T where the open dense immersion $\bar{\imath} : T \hookrightarrow X$ and the action $\bar{a} : T \times X \to X$ is induced respectively by $\imath : \mathcal{T} \hookrightarrow \mathcal{X}$ and $a : \mathcal{T} \times \mathcal{X} \to \mathcal{X}$.*

Proof. The morphisms $\imath$ and a induce morphisms on the coarse moduli spaces $\bar{\imath} : T \to X$ and $\bar{a} : T \times X \to X$, by the universal property of the coarse moduli space. It is immediate to verify that $\bar{\imath}$ is an open embedding with dense image and $\bar{a}$ is an action, extending the action of T on itself. On the other hand, since X is the coarse moduli space of $\mathcal{X}$, it is a normal separated variety with finite quotient singularities. Therefore X is a toric variety, and it is simplicial by [21], §7.6, p. 121 (see also [15], Theorem 3.1, p. 28). $\square$

Remark 3.7 (divisor multiplicities). According to [26], Corollary 5.6.1, the structure morphism $\varepsilon : \mathcal{X} \to X$ induces a bijection on reduced closed substacks. For each $i = 1, \dots, n$, denote by $\mathcal{D}_i \subset \mathcal{X}$ the reduced closed substack with support $\varepsilon^{-1}(D_i)$. Since $D_i \cap X_{\mathrm{sm}}$ is a Cartier divisor, there exists a unique positive integer a_i such that $\varepsilon^{-1}(D_i \cap X_{\mathrm{sm}}) = a_i\big(\mathcal{D}_i \cap \varepsilon^{-1}(X_{\mathrm{sm}})\big)$. We call $a = (a_1, \dots, a_n)$ *the divisor multiplicities of* $\mathcal{X}$.

Let $\mathcal{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T} = T \times \mathcal{B}G$. By Appendix B, we have that $\mathcal{B}G$ acts on $\mathcal{X}$. Proposition B.15 implies that we have an étale morphism $j : G \times \mathcal{X} \to I^{\mathrm{gen}}(\mathcal{X})$.

Lemma 3.8. *Let $\mathcal{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathcal{T} = T \times \mathcal{B}G$. The morphism $j : G \times \mathcal{X} \to I^{\mathrm{gen}}(\mathcal{X})$ is an isomorphism.*

Proof. As the stack $\mathcal{X}$ is separated, we have that the natural morphism $I(\mathcal{X}) \to \mathcal{X}$ is proper. As the projection $G \times \mathcal{X} \to \mathcal{X}$ is a proper morphism, the morphism j is also a proper morphism. Its image contains the substack $I(\mathcal{T}) = I^{\mathrm{gen}}(\mathcal{T})$ which is open and dense in $I^{\mathrm{gen}}(\mathcal{X})$. We deduce that the morphism j is birational. As the morphism j is étale, it is quasi-finite (cf. [19], Exposé I, §3). The morphism j is proper hence closed and as its image contains the open dense torus, j is surjective. The morphism j is a representable, birational, surjective and quasi-finite morphism to the smooth Deligne-Mumford stack $\mathcal{X}$. The stacky Zariski's main theorem C.1 finishes the proof. $\square$

4. Canonical toric Deligne-Mumford stacks

In §4.1 we define the canonical smooth Deligne-Mumford stack associated to a variety with finite quotient singularities and we show that a canonical smooth Deligne-Mumford stack satisfies a universal property (Theorem 4.6). This should be well known, but we include it for the reader's convenience.

In §4.2, we characterize the canonical toric Deligne-Mumford stack via its coarse moduli space.

4.1. Canonical smooth Deligne-Mumford stacks. In this subsection, we do not assume that smooth Deligne-Mumford stacks are toric. First, we define canonical smooth Deligne-Mumford stacks and then we prove their universal property.

We recall a classical result.

Lemma 4.1. *Let S be a smooth variety, and T be an affine scheme. Let $S' \subset S$ be an open subvariety such that the complement has codimension at least 2 in S. Let $f : S' \to T$ be a morphism. Then the morphism f extends uniquely to a morphism $S \to T$.*

Proof. The morphism f corresponds to an algebra homomorphism

$$K[T] \to \Gamma(S', \mathcal{O}_{S'}).$$

Since the complement has codimension 2, the restriction map $\Gamma(S, \mathcal{O}_S) \to \Gamma(S', \mathcal{O}_{S'})$ is an isomorphism. □

Definition 4.2. (1) A dominant morphism $f : V \to W$ of irreducible varieties is called *codimension preserving* if, for any irreducible closed subvariety Z of W and every irreducible component Z_V of $f^{-1}(Z)$, one has $\operatorname{codim}_V Z_V = \operatorname{codim}_W Z$.

(2) A dominant morphism of orbifolds is called *codimension preserving* if the induced morphism on every irreducible component of the coarse moduli spaces is codimension preserving.

Remark 4.3. For any Deligne-Mumford stack, the structure morphism to the coarse moduli space is codimension preserving. Every flat morphism and in particular every smooth and étale morphism is codimension preserving. A composition of codimension preserving morphisms is codimension preserving.

Definition 4.4. Let $\mathcal{X}$ be an irreducible d-dimensional smooth Deligne-Mumford stack. Let $\varepsilon : \mathcal{X} \to X$ be the structure morphism to the coarse moduli space. The Deligne-Mumford stack $\mathcal{X}$ will be called *canonical* if the locus where ε is not an isomorphism has dimension $\leqq d - 2$.

Remark 4.5. Let $\mathcal{X}$ be a smooth canonical stack

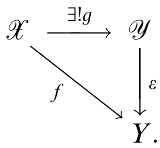
(1) The locus where the structure map to the coarse moduli space $\varepsilon : \mathcal{X} \to X$ is an isomorphism is precisely $\varepsilon^{-1}(X_{\mathrm{sm}})$, where X_{sm} is the smooth locus of X.

(2) The composition of the following isomorphisms

$$A^1(X) \xrightarrow{\simeq} A^1(X_{\mathrm{sm}}) \xrightarrow{\simeq} \mathrm{Pic}(X_{\mathrm{sm}}) \xrightarrow{\simeq} \mathrm{Pic}\big(\varepsilon^{-1}(X_{\mathrm{sm}})\big) \xrightarrow{\simeq} \mathrm{Pic}(\mathscr{X})$$

is the map sending $[D]$ to $\mathcal{O}\big(\varepsilon^{-1}(D)\big)$.

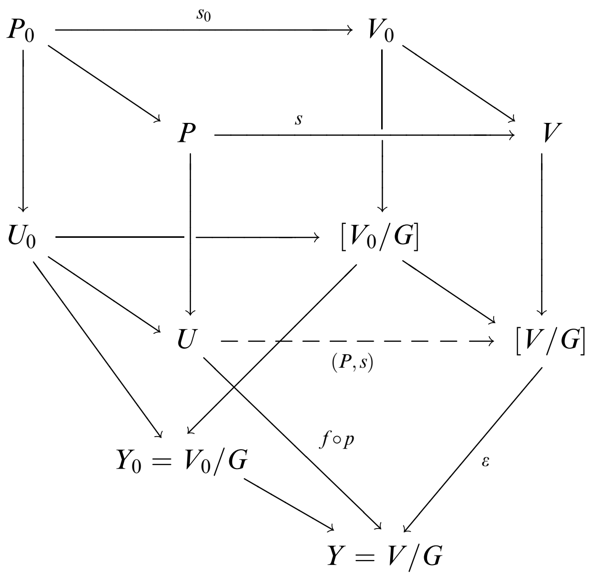
Theorem 4.6 (universal property of canonical smooth Deligne-Mumford stacks). *Let $\mathscr{Y}$ be a canonical smooth Deligne-Mumford stack, $\varepsilon : \mathscr{Y} \to Y$ the morphism to the coarse moduli space, and $f : \mathscr{X} \to Y$ a dominant codimension preserving morphism with $\mathscr{X}$ an orbifold. Then there exists a unique, up to a unique 2-arrow, $g : \mathscr{X} \to \mathscr{Y}$ such that the following diagram is commutative*:



Proof. We first prove uniqueness. Any two morphisms g, $\bar{g}$ making the diagram commute must agree on the open dense subscheme $f^{-1}(Y_{\mathrm{sm}})$. Put $\iota : f^{-1}(Y_{\mathrm{sm}}) \hookrightarrow \mathscr{X}$. Since $\mathscr{Y}$ is assumed to be separated, by Proposition 1.2, there exists a unique $\alpha : g \to \bar{g}$ such that $\alpha * \mathrm{id}_\iota = \mathrm{id}$.

By uniqueness, it is enough to prove the result étale locally in $\mathscr{Y}$, so we can assume that $\mathscr{Y} = [V/G]$ where V is a smooth affine variety and G a finite group acting on V without pseudo-reflections. It is enough to show that there exists an étale surjective morphism $p : U \to \mathscr{X}$ with U a smooth variety and a morphism $\bar{g} : U \to \mathscr{Y}$ such that $f \circ p = \varepsilon \circ \bar{g}$. In fact, g is defined from $\bar{g}$ by descent, with the appropriate compatibility conditions being taken care of by the uniqueness part. In this case $Y = V/G$, and $Y_0 := V_0/G$ where $V_0 \subset V$ is the open locus where G acts freely. Let $U_0 := (f \circ p)^{-1}(Y_0)$. As $[V_0/G]$ is isomorphic to Y_0, we have a natural morphism $U_0 \to [V_0/G]$. This morphism defines a principal G-bundle P_0 on U_0 and a G-equivariant morphism $s_0 : P_0 \to V_0$.

(4.7)



Since the $U \backslash U_0$ has codimension $\geqq 2$, the principal G-bundle P_0 extends uniquely to a principal G-bundle P over U, and by Lemma 4.1 (since V is affine) the G-equivariant morphism $s_0 : P_0 \to V_0$ extends to a morphism $s : P \to V$ which is again G-equivariant, yielding a morphism $\bar{g} : U \to [V/G]$. The construction above is summarized in the 2-commutative diagram (4.7) where the squares are 2-cartesian. □

Corollary 4.8. *Let $\mathscr{X}$ (resp. $\mathscr{Y}$) be a canonical smooth Deligne-Mumford stack with coarse moduli space X (resp. Y). Let $\bar{f} : X \to Y$ be an isomorphism. Then there is a unique isomorphism $f : \mathscr{X} \to \mathscr{Y}$ inducing $\bar{f}$.*

Proof. It is enough to apply the theorem twice, reversing the role of $\mathscr{X}$ and $\mathscr{Y}$. □

Remark 4.9. One can use the corollary to prove the classical fact that every variety Y with finite quotient singularities is the coarse moduli space of a canonical smooth Deligne-Mumford stack unique up to rigid isomorphism, which we denote by $\mathscr{Y}^{\text{can}}$ (do it étale locally and then glue). If Y is the geometric quotient Z/G where Z is a smooth variety and G is a group without pseudo-reflections acting with finite stabilizers, then $\mathscr{Y}^{\text{can}} = [Z/G]$. Notice that this is the case of simplicial toric varieties (cf. Section 1.6).

We finish this section with a corollary that will play an important role.

Corollary 4.10. *Let $\mathscr{X}$ be a smooth Deligne-Mumford stack with coarse moduli space $\varepsilon : \mathscr{X} \to X$. There is a unique morphism $\mathscr{X} \to \mathscr{X}^{\text{can}}$ through which ε factors.*

Proof. Apply the theorem with $Y = X$, $\mathscr{Y} = \mathscr{X}^{\text{can}}$ and $f = \varepsilon$. □

4.2. The canonical stack of a simplicial toric variety. In this section, we study the canonical stack associated to a simplicial toric variety.

The main result of this section is the following theorem.

Theorem 4.11. *Let X be a simplicial toric variety with torus T. Its canonical stack $\mathscr{X}^{\text{can}}$ has a natural structure of toric orbifold such that the action $a^{\text{can}} : T \times \mathscr{X}^{\text{can}} \to \mathscr{X}^{\text{can}}$ lifts the action $\bar{a} : T \times X \to X$.*

Proof. Denote by Σ the fan in $N \otimes_{\mathbb{Z}} \mathbb{Q}$ of the toric variety X. Without loss of generality, we can assume that the rays of Σ generate $N \otimes_{\mathbb{Z}} \mathbb{Q}$, so that $X = Z_\Sigma / G_A$ (cf. §1.6). The subvariety of points where G_A acts with non-trivial stabilizers has codimension $\geqq 2$. Remark 4.9 implies that the canonical stack $\mathscr{X}^{\text{can}}$ is isomorphic to $[Z_\Sigma / G_A]$. Let $T := (\mathbb{C}^*)^n / G_A$ be the torus of the toric variety X. Notice that $\mathscr{T}^{\text{can}} = [(\mathbb{C}^*)^n / G_A]$ is open dense and isomorphic via $\varepsilon|_{\mathscr{T}^{\text{can}}}$ to T. Proposition 3.3 and the universal property (see Theorem 4.6) of the canonical stack imply that the action of T on X lifts to an action of T on $\mathscr{X}^{\text{can}}$. □

Remark 4.12. (1) Under the hypothesis of Theorem 4.11, we have that the restriction of the structure morphism $\varepsilon : \mathscr{X}^{\text{can}} \to X$ to $\mathscr{T}^{\text{can}}$ is an isomorphism with T.

(2) Let $\mathscr{X}$ be a canonical toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T} = T$ with coarse moduli space the simplicial toric variety X. Denote by

$\Sigma \subset N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ the fan of X. Assume that the rays of Σ generate $N_{\mathbb{Q}}$. The proof above shows that $\mathscr{X} = [Z_\Sigma/G_A]$ where $G_A = \mathrm{Hom}(A^1(X), \mathbb{C}^*) = \mathrm{Hom}(\mathrm{Pic}(\mathscr{X}), \mathbb{C}^*)$ (cf. Remark 4.5(2)).

Corollary 4.13. *Let $\mathscr{X}$ be a canonical toric Deligne-Mumford stack with torus $\mathscr{T} = T$ and coarse moduli space the simplicial toric variety X. Denote $\Sigma \subset N_{\mathbb{Q}}$ the fan of X.*

(1) *The boundary divisor $\mathscr{X}\backslash T$ is a simple normal crossing divisor, with irreducible components, denoted by $\mathscr{D}_i$. Moreover, if the rays of Σ generate $N_{\mathbb{Q}}$, then the divisor $\mathscr{D}_i$ is isomorphic to $[Z_i/G_A]$ where $Z_i = \{x_i = 0\} \cap Z_\Sigma$.*

(2) *The composition morphism $L \to A^1(X) \xrightarrow{\varepsilon^*} \mathrm{Pic}(\mathscr{X})$ sends e_i to $\mathscr{O}_{\mathscr{X}}(\mathscr{D}_i)$.*

Proof. The first point of the corollary follows from the fact that the inverse image inside Z_Σ of the torus $T = (\mathbb{C}^*)^n/G_A$ is $(\mathbb{C}^*)^n$.

The second part of the corollary follows from the exact sequence (1.7) and Remark 4.5(2). □

Remark 4.14. Let $\mathscr{X}$ be a canonical toric Deligne-Mumford stack with coarse moduli space X.

(1) Denote by Σ the fan of X in $N_{\mathbb{Q}}$. If the rays of Σ span $N_{\mathbb{Q}}$, from the corollary and the exact sequence (1.7), we have the exact sequence

$$0 \to M \to L \to \mathrm{Pic}(\mathscr{X}) \to 0.$$

(2) For any $i \in \{1, \dots, n\}$, the divisor $\mathscr{D}_i$ is Cartier. Hence it corresponds to the invertible sheaf $\mathscr{O}(\mathscr{D}_i)$ with the canonical section s_i. Using Remark 1.1, the invertible sheaf $\mathscr{O}(\mathscr{D}_i)$ is associated to the representation $G_A \to G_L = (\mathbb{C}^*)^n \xrightarrow{p_i} \mathbb{C}^*$ where p_i is the i-th projection. Moreover, the canonical section s_i is the i-th coordinate of Z_Σ.

(3) Let $\mathscr{X}$ be a canonical toric Deligne-Mumford stack, then all divisor multiplicities of $\mathscr{X}$ are equal to 1 (for the definition of divisor multiplicity see Remark 3.7).

5. Toric orbifolds

In this section, we only consider toric Deligne-Mumford stacks with trivial generic stabilizer that is toric orbifolds.

Let $\mathscr{X}$ be a smooth Deligne-Mumford stack with coarse moduli space X. By Proposition 3.6 and Theorem 4.11, the canonical stack $\mathscr{X}^{\mathrm{can}}$ has an induced structure of toric orbifold. Denote by $\varepsilon_{\mathscr{X}} : \mathscr{X} \to X$ (resp. $\varepsilon_{\mathscr{X}^{\mathrm{can}}} : \mathscr{X}^{\mathrm{can}} \to X$) the morphism to the coarse moduli space. Theorem 4.6 implies that there exists a unique $f : \mathscr{X} \to \mathscr{X}^{\mathrm{can}}$ such that $\varepsilon_{\mathscr{X}^{\mathrm{can}}} \circ f = \varepsilon_{\mathscr{X}}$.

Proposition 5.1. *Let $\mathscr{X}$ be a toric orbifold with torus T and coarse moduli space X. The canonical morphism $f : \mathscr{X} \to \mathscr{X}^{\mathrm{can}}$ is a morphism of toric Deligne-Mumford stacks where $\mathscr{X}^{\mathrm{can}}$ is endowed with the induced structure of toric orbifold.*

Proof. The universal property of the canonical stack (cf. Theorem 4.6) applied to $\mathrm{id} : T \to T$ implies that $f|_T : T \to \mathscr{T}^{\mathrm{can}}$. $\square$

Notice that the morphism $f|_T : T \to \mathscr{T}^{\mathrm{can}}$ in the proof above is an isomorphism because $\mathscr{X}$ is a toric orbifold.

Denote $\boldsymbol{D}^{\mathrm{can}} := (D_1^{\mathrm{can}}, \dots, D_n^{\mathrm{can}})$ (cf. Section 1.3.b).

Theorem 5.2. (1) *Let X be a simplicial toric variety with torus T. Denote by Σ a fan of X. For each ray ρ_i of Σ, choose a_i in $\mathbb{N}_{>0}$. Denote $\boldsymbol{a} := (a_1, \dots, a_n) \in (\mathbb{N}_{>0})^n$. Then $\sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ has a unique structure of toric orbifold with torus T such that the canonical morphism $\pi : \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}} \to \mathscr{X}^{\mathrm{can}}$ is a morphism of toric Deligne-Mumford stacks with divisor multiplicities $\boldsymbol{a}$.*

(2) *Let $\mathscr{X}$ be a toric orbifold with coarse moduli space X. Let $\boldsymbol{a} := (a_1, \dots, a_n)$ be its divisors multiplicities. Then $\mathscr{X}$ is naturally isomorphic as toric Deligne-Mumford stack to $\sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ defined in* (1).

Proof. (1) Let $\mathscr{T}^{\mathrm{can}} \subseteq \mathscr{X}^{\mathrm{can}}$ be the inverse image of T (which is isomorphic to T). Note that $\pi^{-1}(T) \subseteq \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ is isomorphic to $\mathscr{T}^{\mathrm{can}}$ by property (2) of Section 1.3.b. Let $j : T \to \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ be the dominant open embedding. We need to prove that T acts on $\sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ compatibly with j. We know that T acts on $\mathscr{X}^{\mathrm{can}}$. To define $T \times \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}} \to \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ we use the universal property and the fact that $\boldsymbol{D}^{\mathrm{can}} \subseteq \mathscr{X}^{\mathrm{can}}$ is T-invariant.

(2) For any $i \in \{1, \dots, n\}$, denote by D_i, D_i^{can}, $\mathscr{D}_i(\mathscr{X})$ the divisor corresponding to the ray ρ_i in respectively X, $\mathscr{X}^{\mathrm{can}}$ and $\mathscr{X}$. Theorem 4.11 implies there exists a unique morphism $f : \mathscr{X} \to \mathscr{X}^{\mathrm{can}}$ such that $\varepsilon_{\mathscr{X}^{\mathrm{can}}} \circ f = \varepsilon_{\mathscr{X}}$. By definition of the divisors multiplicities, for any ray ρ_i, we have $f^{-1}D_i^{\mathrm{can}} = a_i \mathscr{D}_i(\mathscr{X})$. The Cartier divisors $\mathscr{D}(\mathscr{X}) := (\mathscr{D}_1(\mathscr{X}), \dots, \mathscr{D}_n(\mathscr{X}))$ define a morphism $\mathscr{X} \to [\mathbb{A}^n/(\mathbb{C}^*)^n]$ such that the following diagram is 2-commutative:

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{\mathscr{D}(\mathscr{X})} & [\mathbb{A}^n/(\mathbb{C}^*)^n] \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \wedge \boldsymbol{a}} \\
\mathscr{X}^{\mathrm{can}} & \xrightarrow{\boldsymbol{D}^{\mathrm{can}}} & [\mathbb{A}^n/(\mathbb{C}^*)^n]
\end{array}
\tag{5.3}
$$

where the morphism $\wedge \boldsymbol{a}$ is defined in Section 1.3.b. By the universal property of fiber product, we deduce a unique morphism $g : \mathscr{X} \to \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ such that the following diagram is strictly commutative:

$$
\begin{array}{ccc}
\mathscr{X} & \xrightarrow{g} & \sqrt[\boldsymbol{a}]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}} \\
 & {\scriptstyle f}\searrow & \big\downarrow{\scriptstyle \pi} \\
 & & \mathscr{X}^{\mathrm{can}}.
\end{array}
$$

We will use the Zariski's main theorem (cf. Theorem C.1) to prove that g is an isomorphism. We first notice that $\sqrt[a]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}$ is smooth for property (3) in §1.3.b. As, the restriction of g over $\mathscr{X}^{\mathrm{can}} - \bigcup_{i,j} D_i^{\mathrm{can}} \cap D_j^{\mathrm{can}}$ is an isomorphism, the morphism g is birational. Notice that $\bigcup_{i,j} D_i^{\mathrm{can}} \cap D_j^{\mathrm{can}}$ is a subset of codimension $\geqq 2$. The morphism g is proper, hence closed, so we deduce that g is also surjective because its image contains the dense torus. Let us show that g is representable and étale. Let S be a scheme. Consider the following 2-cartesian diagram:

$$\begin{array}{ccc} \mathscr{Y} & \xrightarrow{\quad \bar{g} \quad} & S \\ \downarrow & \square & \downarrow \\ \mathscr{X} & \xrightarrow{\quad g \quad} & \sqrt[a]{\boldsymbol{D}^{\mathrm{can}}/\mathscr{X}^{\mathrm{can}}}. \end{array}$$

Let $U \to \mathscr{Y}$ be an étale atlas of $\mathscr{Y}$. First we observe that the morphism $U \to S$, denote it by $\tilde{g}$, must be flat, so that the morphism g is flat too. To verify this we can apply [27], Thm. 23.1, using that both S and U are smooth and the dimension of the fibers of $\tilde{g}$ is constantly zero. To prove that the dimension of the fibers is zero we just need to observe that both π and f are quasi-finite, since they are morphisms from a stack to its coarse moduli space, and f factors through g so that it must be quasi-finite too. We now note that the morphism $U \to S$ is étale away from a codimension $\geqq 2$ subset, so we can apply the theorem of purity of branch locus (cf. [5], Theorem 6.8, p. 125) and deduce that $U \to S$ is étale, i.e., $\bar{g} : \mathscr{Y} \to S$ is étale. Without loss of generality we can assume that S is actually an atlas; we assume that $\mathscr{Y}$ is a stack and we prove that it must be actually a scheme. First of all we observe that it cannot have generically non-trivial stabilizer, since the morphism $\mathscr{Y} \to \mathscr{X}$ is representable it must induce an injection of the stabilizer at each geometric point [4], but $\mathscr{X}$ is an orbifold so that $\mathscr{Y}$ must be an orbifold too. There exists an étale representable map $[V/K] \to \mathscr{Y}$ where V is a smooth variety and K is a finite group. Hence the induced map $V \to S$ is étale. By the universal property, it factors via the coarse moduli space V/K, and the map $V \to V/K$ is not injective on tangent vectors unless K is acting freely, hence $V \to V/K$ cannot be étale unless $\mathscr{Y}$ has trivial stabilizers everywhere. We now observe that the morphism $V/K \to S$ is still birational surjective and quasi-finite, using Zariski's main theorem for schemes we can deduce that it is an isomorphism, in particular it is étale and this implies that $V \to V/K$ must be étale. We conclude that $\mathscr{Y}$ is a scheme, i.e., g is representable and étale. So it is also quasi-finite (cf. [19], Exposé I, §3).

As the morphism g is representable, surjective, birational and quasi-finite, the stacky Zariski's main theorem C.1 implies that g is an isomorphism. □

The following corollary is a consequence of property (3) of Section 1.3.b and Theorem 5.2.

Corollary 5.4. *Let $\mathscr{X}$ be a toric orbifold with coarse moduli space X. The reduced closed substack $\mathscr{X} \backslash \mathscr{T}$ is a simple normal crossing divisor.*

Remark 5.5. Let $\mathscr{X}$ be a toric orbifold with coarse moduli space X. Diagram (1.4) and Theorem 5.2 imply that we have the following morphism of exact sequences:

$$
(5.6)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\times \boldsymbol{a}} & \mathbb{Z}^n & \longrightarrow & \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{Pic}(\mathscr{X}^{\mathrm{can}}) & \xrightarrow{f^*} & \operatorname{Pic}(\mathscr{X}) & \xrightarrow{q} & \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

where the vertical morphisms send $1 \mapsto \mathcal{O}(D_i^{\mathrm{can}})$ and $1 \mapsto \mathcal{O}(\mathscr{D}_i)$.

6. Toric Deligne-Mumford stacks

In this section we will show that each toric Deligne-Mumford stack is isomorphic to a fibered product of root stacks on its rigidification. To prove this theorem, we will recall in Section 6.1 the relation between banded gerbes and root constructions. Then we will show in Theorem 6.11 that any toric Deligne-Mumford stack is an essentially trivial gerbe on its rigidification. In Section 6.3, we will prove the main result in Theorem 6.25.

6.1. Gerbes and root constructions. First, we recall some general notion on banded gerbes (*gerbes liées*). We refer to [18], chapter IV.2, for a complete treatment and to [16], Section 3, for a shorter reference. Let $\mathscr{X}$ be a smooth Deligne-Mumford stack. Let G be an abelian sheaf of groups[3] and $\mathscr{G} \to \mathscr{X}$ a gerbe. For every étale chart U of $\mathscr{X}$ and every object $x \in \mathscr{G}(U)$ let $\alpha_x : G|_U \to \mathrm{Aut}_U(x)$ be an isomorphism of sheaves of groups such that the natural compatibilities coming from the fibered structure of the gerbe are satisfied. The collection of these isomorphisms is called a *G-banding*. A *G-banded* gerbe is the data of a gerbe and a G-banding. Two G-banded gerbes are said to be *G-equivalent* if they are isomorphic as stacks and the isomorphism makes the two bandings compatible in the natural way. Giraud proved in [18] (Chapter IV, 3.4) that the group $H^2_{\text{ét}}(\mathscr{X}, G)$ classifies equivalence classes of G-banded gerbes.

Remark 6.1. We anticipate some observations about the banding which will be useful in the following:

(1) The b-th root of a line bundle on $\mathscr{X}$ is a gerbe which is banded in a natural way by the constant sheaf μ_b; the banding is the canonical isomorphism between the group of automorphisms of any object and μ_b.

(2) Given $\mathscr{G} \to \mathscr{X}$ a G-banded gerbe, every rigidification of $\mathscr{G}$ by a subgroup H of G inherits a (G/H)-banding from the G-banding of $\mathscr{G}$.

Here we introduce the concept of an essentially trivial gerbe which will play an important role in this section. The Kummer sequence

$$1 \to \mu_b \xrightarrow{\iota} \mathbb{G}_m \xrightarrow{\wedge b} \mathbb{G}_m \to 1$$

[3] The non-abelian case has a richer structure but for the sake of simplicity we just skip all these additional features and refer the interested reader to [18].

induces the long exact sequence

$$\cdots \to H^1_{\text{ét}}(\mathscr{X}, \mathbb{G}_m) \xrightarrow{\partial} H^2_{\text{ét}}(\mathscr{X}, \mu_b) \xrightarrow{\iota_*} H^2(\mathscr{X}, \mathbb{G}_m) \to \cdots. \tag{6.2}$$

Definition 6.3. A μ_b-banded gerbe in $H^2_{\text{ét}}(\mathscr{X}, \mu_b)$ is *essentially trivial* if its image by ι_* is the trivial gerbe in $H^2_{\text{ét}}(\mathscr{X}, \mathbb{G}_m)$.

Remark 6.4. (1) It follows from Section 1.3.a that a μ_b-banded gerbe is essentially trivial if and only if it is a b-th root of an invertible sheaf on $\mathscr{X}$.

(2) As the μ_b-banded gerbe $\sqrt[b]{L \otimes M^{\otimes b}/\mathscr{X}}$ is isomorphic to $\sqrt[b]{L/\mathscr{X}}$, we deduce a bijection between essentially trivial μ_b-banded gerbes and $\operatorname{Pic}(\mathscr{X})/b\operatorname{Pic}(\mathscr{X})$.

Lemma 6.5. *There is a natural bijection between essentially trivial gerbes in $H^2_{\text{ét}}(\mathscr{X}, \mu_b)$ and elements in $\operatorname{Ext}^1(\mathbb{Z}/b\mathbb{Z}, \operatorname{Pic}(\mathscr{X}))$.*

Proof. By Remark 6.4(2), it is enough to show that $\operatorname{Ext}^1(\mathbb{Z}/b\mathbb{Z}, \operatorname{Pic}(\mathscr{X}))$ is isomorphic to $\operatorname{Pic}(\mathscr{X})/b\operatorname{Pic}(\mathscr{X})$. This follows from the exact sequence

$$\operatorname{Hom}(\mathbb{Z}, \operatorname{Pic}(\mathscr{X})) \xrightarrow{\wedge b} \operatorname{Hom}(\mathbb{Z}, \operatorname{Pic}(\mathscr{X})) \to \operatorname{Ext}^1(\mathbb{Z}/b, \operatorname{Pic}(\mathscr{X})) \to 0. \quad \square$$

Let G be a finite abelian group. Fix a decomposition $G = \prod_{j=1}^{\ell} \mu_{b_j}$. We deduce an isomorphism

$$\begin{aligned} H^2_{\text{ét}}(\mathscr{X}, G) &\to \bigoplus_{j=1}^{\ell} H^2_{\text{ét}}(\mathscr{X}, \mu_{b_j}), \\ \alpha &\mapsto (\alpha_1, \ldots, \alpha_\ell). \end{aligned} \tag{6.6}$$

Definition 6.7. Let G be a finite abelian group. A G-banded gerbe associated to $\alpha \in H^2(\mathscr{X}, G)$ is *essentially trivial* if there is a decomposition of $G = \prod_{j=1}^{\ell} \mu_{b_j}$ such that for any $j \in \{1, \ldots, \ell\}$, the μ_{b_j}-banded gerbe α_j is essentially trivial.

Remark 6.8. Being essentially trivial does not depend on the choice of a decomposition of G.

Proposition 6.9. *Let G be a finite abelian group. Fix a decomposition of $G = \prod_{j=1}^{\ell} \mu_{b_j}$. There are bijections between*

$$\begin{aligned} &\left\{ \textit{Essentially trivial gerbes in } \bigoplus_{j=1}^{\ell} H^2_{\text{ét}}(\mathscr{X}, \mu_{b_j}) \right\} \\ &\quad \overset{1:1}{\leftrightarrow} \{ \textit{Fibered products over } \mathscr{X} \textit{ of } b_j\textit{-th roots of invertible sheaves} \} \\ &\quad \overset{1:1}{\leftrightarrow} \prod_{j=1}^{\ell} \operatorname{Pic}(\mathscr{X})/b_j \operatorname{Pic}(\mathscr{X}) \overset{1:1}{\leftrightarrow} \prod_{j=1}^{\ell} \operatorname{Ext}^1(\mathbb{Z}/b_j\mathbb{Z}, \operatorname{Pic}(\mathscr{X})). \end{aligned}$$

Remark 6.10. To be more concrete, let us explicitly describe the last bijection. For the sake of simplicity, we consider the case $j = 1$. To the class $[L_0]$ in $\mathrm{Pic}(\mathscr{X})/b\,\mathrm{Pic}(\mathscr{X})$, we associate the extension

$$0 \to \mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X}) \times_{\mathrm{Pic}(\mathscr{X})/b\,\mathrm{Pic}(\mathscr{X})} \mathbb{Z}/b\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z} \to 0$$

where the fiber product is given by the standard projection $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X})/b\,\mathrm{Pic}(\mathscr{X})$ and the morphism $\mathbb{Z}/b\mathbb{Z} \to \mathrm{Pic}(\mathscr{X})$ that sends the class of 1 to the class $[L_0]$. The first morphism in the extension sends the invertible sheaf L to $(L^{\otimes b}, 0)$.

Let $0 \to \mathrm{Pic}(\mathscr{X}) \to A \to \mathbb{Z}/b \to 0$ be an extension. We consider the projective resolution $0 \to \mathbb{Z} \xrightarrow{\times b} \mathbb{Z} \to \mathbb{Z}/b \to 0$. There exists f and $\tilde{f}$ such that the following diagram is a morphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/b & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \tilde{f}} & & \big\downarrow{\scriptstyle f} & & \big\| & & \\
0 & \longrightarrow & \mathrm{Pic}(\mathscr{X}) & \longrightarrow & A & \longrightarrow & \mathbb{Z}/b & \longrightarrow & 0.
\end{array}$$

The class $[\tilde{f}(1)]$ in $\mathrm{Pic}(\mathscr{X})/b\,\mathrm{Pic}(\mathscr{X})$ is the element that corresponds to the above extension. Notice that different liftings $f, \tilde{f}$ lead to different elements in $\mathrm{Pic}(\mathscr{X})$ with the same class in $\mathrm{Pic}(\mathscr{X})/b\,\mathrm{Pic}(\mathscr{X})$.

The two maps defined above are inverse to each other.

Proof of Proposition 6.9. Most of the proposition is a direct consequence of Remark 6.4 and Lemma 6.5. The only non-trivial fact to prove is that an essentially trivial gerbe defined by $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \bigoplus_{j=1}^{\ell} H^2_{\text{ét}}(\mathscr{X}, \mu_{b_j})$ is given by a fiber product of the gerbes defined by the α_j's. Without loss of generality, we can assume that $\alpha = (\alpha_1, \alpha_2)$; the general case is proved by induction. The gerbe defined by α_1 (resp. α_2) is isomorphic to the rigidification $\mathscr{G} /\!\!/ \mu_{b_2}$ (resp. $\mathscr{G} /\!\!/ \mu_{b_1}$). Hence we have the following 2-commutative diagram:

$$\begin{array}{ccccc}
 & & \mathscr{G} /\!\!/ \mu_{b_2} & & \\
 & \nearrow & & \searrow & \\
\mathscr{G} & & \longrightarrow & & \mathscr{X} \\
 & \searrow & & \nearrow & \\
 & & \mathscr{G} /\!\!/ \mu_{b_1}. & &
\end{array}$$

Remark 6.1(2) implies that $\mathscr{G} \to \mathscr{G} /\!\!/ \mu_{b_1}$ (resp. $\mathscr{G} \to \mathscr{G} /\!\!/ \mu_{b_2}$) is a μ_{b_2}-banded gerbe (resp. μ_{b_1}-banded). By the universal property of the fiber product we are given a morphism $\mathscr{G} \to \mathscr{G} /\!\!/ \mu_{b_1} \times_{\mathscr{X}} \mathscr{G} /\!\!/ \mu_{b_1}$. Two gerbes banded by the same group over the same base $\mathscr{X}$ are either isomorphic as stacks or they have no morphisms at all; this completes the proof. $\square$

6.2. Gerbes on toric orbifolds.

Theorem 6.11. *Let $\mathcal{X}$ be a toric orbifold with torus T. Denote by $\imath : T \hookrightarrow \mathcal{X}$ the immersion of the torus. Then the morphism*

$$\imath^* : H^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m) \to H^2_{\text{ét}}(T, \mathbb{G}_m)$$

is injective.

Notice that in the following proof we will use that a toric orbifold is a global quotient $[Z_\Sigma/G_{\mathcal{X}}]$ where $G_{\mathcal{X}} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(\mathcal{X}), \mathbb{C}^*)$. This will be proved in Theorem 7.7 and does not depend on the results of this subsection.

We first proof some preliminary results.

Lemma 6.12 (Artin). *Let S be a smooth quasi-projective variety, $S_2 \subseteqq S$ a closed subscheme of codimension $\geqq 2$. Then the natural map $H^i_{\text{ét}}(S, \mathbb{G}_m) \to H^i_{\text{ét}}(S \backslash S_2, \mathbb{G}_m)$ is an isomorphism for all i.*

Proof. The statement is obvious if we replace sheaf cohomology with Čech cohomology. To prove the lemma, we just apply [6], Corollary 4.2, p. 295 (see also [28], Theorem 2.17, p. 104). □

Lemma 6.13 (Olsson). *Let $\mathcal{X}$ be an Artin stack and X_0 an atlas. Denote by $X_p = X_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X_0$. Let $\mathcal{F}$ be an abelian sheaf of groups on $\mathcal{X}$ and $\mathcal{F}_p$ its restriction to X_p. There is a spectral sequence with $E_1^{pq}(\mathcal{X}) := H^q_{\text{ét}}(X_p, \mathcal{F}_p)$ that abuts to $H^{p+q}_{\text{ét}}(\mathcal{X}, \mathcal{F})$.*

Proof. This lemma follows immediately from [30], Corollary 2.7, p. 4 and Theorem 4.7, p. 13. □

Proof of Theorem 6.11. Let $\mathcal{X}$ be a toric orbifold with coarse moduli space a simplicial toric variety X. Denote by $\Sigma \subset N_{\mathbb{Q}}$ the fan of X. Without lost of generality, we can assume that the rays of Σ generate $N_{\mathbb{Q}}$.

By Theorem 5.2 in the case of orbifolds and Lemma 7.1, we have that $\mathcal{X} = [Z_\Sigma/G_{\mathcal{X}}]$ where $G_{\mathcal{X}} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}(\mathcal{X}), \mathbb{C}^*)$. Denote by n the number of rays of the fan Σ. Put

$$Z_2 := \left\{ z \in Z_\Sigma \subset \mathbb{C}^n \mid \forall i \in \{1, \ldots, n\}, \prod_{j \neq i} z_j = 0 \right\}$$

the union of T-orbits in Z_Σ of codimension $\geqq 2$. The closed subscheme Z_2 of Z_Σ is of codimension 2. Hence the quotient stack $[(Z_\Sigma \backslash Z_2)/G_{\mathcal{X}}]$ is a closed substack of codimension 2 of $\mathcal{X}$. For any $i \in \{1, \ldots, n\}$, put

$$U_i := \{ z \in Z_\Sigma \subset \mathbb{C}^n \mid \forall j \in \{1, \ldots, n\} \backslash \{i\}, z_j \neq 0 \}.$$

We have that U_i is isomorphic to $\mathbb{A}^1 \times (\mathbb{C}^*)^{n-1}$ and that the natural morphism

$$\coprod_{i \in \{1, \ldots, n\}} U_i \to Z_\Sigma \backslash Z_2$$

is étale and surjective. We deduce that $\coprod_{i\in\{1,\dots,n\}}[U_i/G_{\mathscr{X}}]\to[(Z_\Sigma\backslash Z)/G_{\mathscr{X}}]$ is étale and surjective. Put $X_0:=\coprod_{i\in\{1,\dots,n\}}U_i$. The natural morphism $X_0\to[(Z_\Sigma\backslash Z_2)/G_{\mathscr{X}}]$ is an étale atlas.

Denote $X_p=X_0\times_{\mathscr{X}}\cdots\times_{\mathscr{X}}X_0$. From Lemma 6.13 we have a spectral sequence $E_1^{pq}(\mathscr{X}):=H^q_{\text{ét}}(X_p,\mathbb{G}_m|_{X_p})$ abutting to $H^{p+q}_{\text{ét}}\big([(Z_\Sigma\backslash Z_2)/G_{\mathscr{X}}],\mathbb{G}_m\big)$. Using this spectral sequence and Lemma 6.12 we obtain that the natural morphism

$$H^i_{\text{ét}}(\mathscr{X},\mathbb{G}_m)=H^i_{\text{ét}}\big([(Z_\Sigma\backslash Z_2)/G_{\mathscr{X}}],\mathbb{G}_m\big)$$

is an isomorphism for $i=(0,1,2)$. Finally, the theorem follows from Lemmas 6.14 and 6.15. □

Lemma 6.14. *We have the following morphism of short exact sequences*:

$$\begin{array}{ccccccccc}0&\longrightarrow&E_4^{20}(\mathscr{X})&\longrightarrow&H^2_{\text{ét}}(\mathscr{X},\mathbb{G}_m)&\longrightarrow&E_2^{02}(\mathscr{X})&\longrightarrow&0\\&&\downarrow{\scriptstyle\alpha}&&\downarrow{\scriptstyle j^*}&&\downarrow{\scriptstyle\beta}&&\\0&\longrightarrow&E_4^{20}(\mathscr{T})&\longrightarrow&H^2_{\text{ét}}(\mathscr{T},\mathbb{G}_m)&\longrightarrow&E_2^{02}(\mathscr{T})&\longrightarrow&0.\end{array}$$

Lemma 6.15. *The vertical maps* $\alpha:E_4^{20}(\mathscr{X})\to E_4^{20}(\mathscr{T})$ *and* $\beta:E_2^{02}(\mathscr{X})\to E_2^{02}(\mathscr{T})$ *are injective.*

Proof of Lemma 6.14. To prove the lemma, we are just interested in $E_\infty^{pq}(\mathscr{X})$ for $p+q=2$. We start by proving that we have

$$0\to E_\infty^{20}(\mathscr{X})\to H^2_{\text{ét}}(\mathscr{X},\mathbb{G}_m)\to E_\infty^{02}(\mathscr{X})\to 0. \tag{6.16}$$

Hilbert's Theorem 90 (cf. [28], Proposition 4.9) implies that

$$H^1_{\text{ét}}(X_p,\mathbb{G}_m)=H^1_{\text{Zariski}}(X_p,\mathscr{O}^*_{X_p})=\operatorname{Pic}(X_p).$$

Using the notation of the proof of Theorem 6.11, for any ray $i\in\{1,\dots,n\}$, we have that $[U_i/G_{\mathscr{X}}]$ is isomorphic to $[\mathbb{A}^1/\mu_{a_i}]\times(\mathbb{C}^*)^{n-1}$ where a_i is the multiplicity along the divisor $\mathscr{D}_i$ (cf. Remark 3.7). Hence, we have that $X_p=\coprod_{i_0,\dots,i_p\in\{1,\dots,n\}}U_{i_0\cdots i_p}$ where

$$U_{i_0\cdots i_p}=\begin{cases}U_{i_0}\times\mu_{a_0}^{p+1} & \text{if } i_0=\cdots=i_p,\\ \mathscr{T} & \text{otherwise.}\end{cases}$$

Hence, for any p we have that $E_1^{p1}(\mathscr{X})=E_\infty^{p1}(\mathscr{X})=H^1_{\text{ét}}(X_p,\mathbb{G}_m)=0$. We deduce the exact sequence (6.16).

We now show that $E_\infty^{20}(\mathscr{X})=E_4^{20}(\mathscr{X})$ and $E_\infty^{02}(\mathscr{X})=E_2^{02}(\mathscr{X})$.

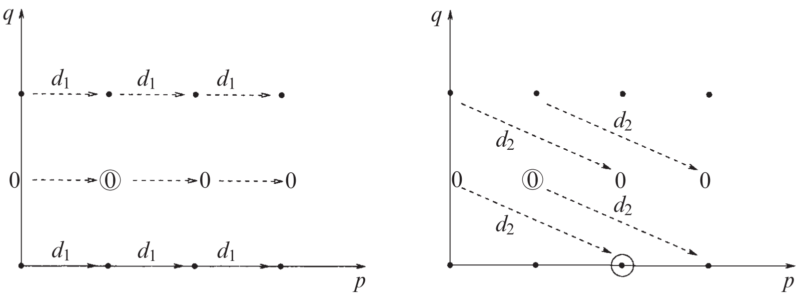


Figure 1. Terms $E_1^{pq}(\mathscr{X})$ and $E_2^{pq}(\mathscr{X})$

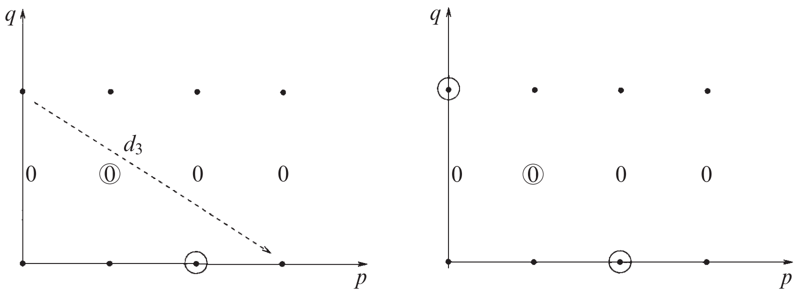


Figure 2. Terms $E_3^{pq}(\mathscr{X})$ and $E_4^{pq}(\mathscr{X})$

In Figures 1 and 2, the circled terms mean that they will stay constant that is they are equal to $E_\infty^{pq}(\mathscr{X})$. We deduce that $E_\infty^{20}(\mathscr{X}) = E_2^{20}(\mathscr{X})$ and $E_\infty^{02}(\mathscr{X}) = E_4^{02}(\mathscr{X})$.

The same argument for T proves the lemma. $\square$

Proof of Lemma 6.15. First, we show that the morphism $\alpha : E_4^{20}(\mathscr{X}) \to E_4^{20}(T)$ is injective. From Figures 1 and 2, we have that

$$E_4^{20}(\mathscr{X}) = \ker\big(d_3 : E_2^{20}(\mathscr{X}) \to E_2^{03}(\mathscr{X})\big), \tag{6.17}$$

$$E_4^{20}(T) = \ker\big(d_3 : E_2^{20}(T) \to E_2^{03}(T)\big). \tag{6.18}$$

Moreover, we have that

$$E_2^{20}(\mathscr{X}) = \ker\big(d_1 : H^2_{\text{ét}}(X_0, \mathbb{G}_m) \to H^2_{\text{ét}}(X_1, \mathbb{G}_m)\big), \tag{6.19}$$

$$E_2^{20}(T) = \ker\big(d_1 : H^2_{\text{ét}}(T_0, \mathbb{G}_m) \to H^2_{\text{ét}}(T_1, \mathbb{G}_m)\big). \tag{6.20}$$

Recall that $U_i \simeq \mathbb{A}^1 \times (\mathbb{C}^*)^{n-1}$ and $T_0 = (\mathbb{C}^*)^n$. By Grothendieck's *Exposés* on the Brauer group [20], §6, p. 133, we have the following long exact sequence:

$$\cdots \to H^2_{X_0 - T_0}(X_0, \mathbb{G}_m) \to H^2_{\text{ét}}(X_0, \mathbb{G}_m) \to H^2_{\text{ét}}(T_0, \mathbb{G}_m) \to \cdots. \tag{6.21}$$

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Moreover, we have that:

- The spectral sequence $F_2^{pq} := H^p\big((X_0\backslash T_0), \underline{H^q_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m)}\big)$ converges to $H_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m)$.
- $H^0_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m) = H^2_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m) = 0$ and $H^1_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m) = \mathbb{Z}$.

This implies that $F_2^{20} = F_2^{02} = 0$. As $X_0\backslash T_0 = (\mathbb{C}^*)^{n-1}$, we have that

$$F_2^{11} = H^1(X_0\backslash T_0, \mathbb{Z}) = 0.$$

The spectral sequence F_2^{pq} implies $H^2_{(X_0\backslash T_0)}(X_0, \mathbb{G}_m) = 0$. Hence, sequence (6.21) and equalities (6.17), (6.18), (6.19) and (6.20), imply that α is injective.

Let us prove that $\beta : E_2^{02}(\mathscr{X}) \to E_2^{02}(T)$ is injective. Recall that $E_2^{02}(\mathscr{X}) = \ker d_2/\operatorname{Im} d_1$ and $E_2^{02}(T) = \ker \tilde{d}_2/\operatorname{Im} \tilde{d}_1$. We have the following commutative diagram:

$$\begin{array}{ccccc}
H^0(X_1, \mathbb{G}_m) & \xrightarrow{\delta_1} & H^0(X_2, \mathbb{G}_m) & \xrightarrow{\delta_2} & H^0(X_3, \mathbb{G}_m) \\
\big\downarrow\cap & & \big\downarrow\cap & & \big\downarrow\cap \\
H^0(T_1, \mathbb{G}_m) & \xrightarrow{\tilde{\delta}_1} & H^0(T_2, \mathbb{G}_m) & \xrightarrow{\tilde{\delta}_2} & H^0(T_3, \mathbb{G}_m).
\end{array}$$

As $T_{ii'} \hookrightarrow U_{ii'}$ is open and dense, the vertical maps are injective. Notice that these maps are isomorphisms except on U_{ii} and U_{iii}. Let $\tilde{y} \in H^0(T_{ii}, \mathbb{G}_m)$ such that there exists $x \in H^0(U_{iii}, \mathbb{G}_m)$ that lifts $\tilde{\delta}_1(\tilde{y})$, i.e., we have the following diagram:

$$\begin{array}{ccc}
 & & x \\
 & & \big\downarrow \\
\tilde{y} & \longmapsto & \tilde{\delta}_1(\tilde{y}).
\end{array}$$

The morphism $\tilde{\delta}_2|_{T_{ii}} : H^0(T_{ii}, \mathbb{G}_m) \to H^0(T_{iii}, \mathbb{G}_m)$ is defined, for any $\tilde{y} \in H^0(T_{ii}, \mathbb{G}_m)$ and any $t, g, h \in T_{iii} = T_i \times \mu_{a_i} \times \mu_{a_i}$, by

$$\tilde{\delta}_2|_{T_{ii}}(\tilde{y})(t, g, h) = \tilde{y}(ht, g)\tilde{y}(t, h)/\tilde{y}(t, gh).$$

The divisor $U_i\backslash T_i$ is a principal divisor associate to the rational function φ. For any $g \in \mu_{a_i}$, the function $\tilde{y}$ is rational on $U_{ii}|_g = U_i \times \{g\}$. Hence there exists a unique $n(g)$ in $\mathbb{N}^*$ such that $\tilde{y}\varphi^{n(g)}$ is a regular function on $U_i \times \{g\}$. As $\tilde{y}(ht, g)\tilde{y}(t, h)/\tilde{y}(t, gh)$ is a regular function, we deduce that $\varphi^{n(g)+n(h)-n(gh)} = 1$. Hence, the function $n : \mu_{a_i} \to \mathbb{Z}$ is a group homomorphism, therefore $n(g) = 1$ for every g. We deduce that $\tilde{y}$ is a regular function on U_i which implies that the morphism $\beta : E_2^{02}(\mathscr{X}) \to E_2^{02}(\mathscr{T})$ is injective. $\square$

6.3. Characterization of a toric Deligne-Mumford stack as a gerbe over its rigidification. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}$ isomorphic to $T \times BG$ and coarse moduli space X. Denote by $\mathscr{X}^{\mathrm{rig}}$ the rigidification of $\mathscr{X}$ (cf. Section 1.4) which is by definition an orbifold with coarse moduli space X. The universal property of the rigidification and of the canonical stack (see Proposition 1.5 and Corollary 4.10) imply that we have the following strictly commutative diagram:

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\ r\ } & \mathscr{X}^{\mathrm{rig}} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^{\mathrm{rig}}} & \\ \mathscr{X}^{\mathrm{can}}. & & \end{array} \tag{6.22}$$

Section 1.4 and Lemma 3.8 imply that we can define $\mathscr{X}/\!\!/G$.

Lemma 6.23. *Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}$ isomorphic to $T \times BG$.*

(1) *The orbifold $\mathscr{X}^{\mathrm{rig}}$ is canonically isomorphic to $\mathscr{X}/\!\!/G$.*

(2) *There is a unique structure of toric orbifold on $\mathscr{X}^{\mathrm{rig}}$ with torus T such that the morphism $r : \mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is a morphism of toric Deligne-Mumford stacks induced by $\mathscr{T} \to T$.*

Remark 6.24. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}$ isomorphic to $T \times G$ and coarse moduli space X.

(1) Proposition 5.1 implies that the morphism $f^{\mathrm{rig}} : \mathscr{X}^{\mathrm{rig}} \to \mathscr{X}^{\mathrm{can}}$ is a morphism of toric Deligne-Mumford stacks. Hence we deduce that the commutative diagram (6.22) is a commutative diagram of morphisms of toric Deligne-Mumford stacks.

(2) Let H be a subgroup of G. The stack $\mathscr{X}/\!\!/H$ is a toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $\mathscr{T}/\!\!/H \simeq T \times \mathscr{B}(G/H)$. Moreover, the natural morphisms $\mathscr{X} \to \mathscr{X}/\!\!/H$ and $\mathscr{X}/\!\!/H \to \mathscr{X}/\!\!/G$ are morphisms of toric Deligne-Mumford stacks.

(3) Note that we did not use the non-canonical isomorphism $\mathscr{T} \equiv T \times \mathscr{B}G$ but only the short exact sequence of Picard stacks $1 \to \mathscr{B}G \to \mathscr{T} \to T \to 1$.

Proof of Lemma 6.23. (1) As $\mathscr{T}/\!\!/G$ is isomorphic to the scheme T which is open and dense in $\mathscr{X}/\!\!/G$, the stack $\mathscr{X}/\!\!/G$ is an orbifold which is canonically isomorphic to $\mathscr{X}^{\mathrm{rig}}$.

(2) The morphisms $\imath : \mathscr{T} \hookrightarrow \mathscr{X}$ and $a : \mathscr{T} \times \mathscr{X} \to \mathscr{X}$ induce morphisms on the rigidifications $\imath^{\mathrm{rig}} : \mathscr{T}/\!\!/G \simeq T \to \mathscr{X}^{\mathrm{rig}}$ and $a^{\mathrm{rig}} : T \times \mathscr{X}^{\mathrm{rig}} \to \mathscr{X}^{\mathrm{rig}}$, by the universal property of the rigidification (see Proposition 1.5). It is immediate to verify that a^{rig} is an action, extending the action of T on itself. As $r^{-1}(T)$ is isomorphic to $\mathscr{T}$, we deduce that this is the only toric structure on $\mathscr{X}^{\mathrm{rig}}$ which is compatible with the morphism r. $\square$

Since the morphism $r : \mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is étale, the divisor multiplicities of $\mathscr{X}$ and $\mathscr{X}^{\mathrm{rig}}$ are the same.

Theorem 6.25. (1) *Let $\mathscr{Y}$ be a toric orbifold with Deligne-Mumford torus T. Let $\mathscr{X} \to \mathscr{Y}$ be an essentially trivial G-gerbe. Then $\mathscr{X}$ has a unique structure of toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times \mathscr{B}G$ such that the morphism $\mathscr{X} \to \mathscr{Y}$ is a morphism of toric Deligne-Mumford stacks.*

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(2) *Conversely, let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T} \simeq T \times \mathscr{B}G$. Then $\mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is an essentially trivial G-gerbe.*

Proof. (1) The inverse image of T in $\mathscr{X}$, denoted by $\mathscr{T}$, is open dense. The restriction of the essentially trivial G-banded gerbe $\mathscr{X} \to \mathscr{Y}$ to T is the essentially trivial G-banded gerbe $\mathscr{T} \to T$. Remark 6.4(1) implies that the gerbe $\mathscr{T} \to T$ is trivial. The action of T on $\mathscr{Y}$ induces by pullback an action of $\mathscr{T}$ on $\mathscr{X}$. This is the only structure of toric Deligne-Mumford stack on $\mathscr{X}$ compatible with the morphism $\mathscr{X} \to \mathscr{Y}$.

(2) Denote by $\alpha \in H^2_{\text{ét}}(\mathscr{X}^{\mathrm{rig}}, G)$ the G-banded gerbe $\mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$. By Proposition 2.6, the restriction of α on the Deligne-Mumford torus $\mathscr{T}$ is the trivial G-banded gerbe in $H^2_{\text{ét}}(T, G)$. Fix a cyclic decomposition of $G = \prod_{j=1}^{\ell} \mu_{b_j}$. By the isomorphism (6.6), the class α is sent to $(\alpha_1, \dots, \alpha_\ell) \in \bigoplus_{j=1}^{\ell} H^2_{\text{ét}}(\mathscr{X}^{\mathrm{rig}}, \mu_{b_j})$. We have that for any $j \in \{1, \dots, \ell\}$, the class of α_j restricts to the trivial class in $H^2_{\text{ét}}(T, \mu_{b_j})$. Theorem 6.11 states the injectivity of ι^* in the following diagram:

$$\begin{array}{ccccc}
H^1_{\text{ét}}(\mathscr{X}^{\mathrm{rig}}, \mathbb{G}_m) & \xrightarrow{\sqrt[b_j]{\cdot/\mathscr{X}^{\mathrm{rig}}}} & H^2_{\text{ét}}(\mathscr{X}^{\mathrm{rig}}, \mu_{b_j}) & \longrightarrow & H^2_{\text{ét}}(\mathscr{X}^{\mathrm{rig}}, \mathbb{G}_m) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \iota^* \\
1 & \longrightarrow & H^2_{\text{ét}}(T, \mu_{b_j}) & \longrightarrow & H^2_{\text{ét}}(T, \mathbb{G}_m).
\end{array}$$

A simple diagram chasing finishes the proof. □

Corollary 6.26. *Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}$ isomorphic to $T \times \mathscr{B}G$.*

(1) *Given $G = \prod_{j=1}^{\ell} \mu_{b_j}$. There exists L_j in* $\mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ *such that $\mathscr{X}$ is isomorphic as G-banded gerbe over $\mathscr{X}^{\mathrm{rig}}$ to*

$$\sqrt[b_1]{L_1/\mathscr{X}^{\mathrm{rig}}} \times_{\mathscr{X}^{\mathrm{rig}}} \cdots \times_{\mathscr{X}^{\mathrm{rig}}} \sqrt[b_\ell]{L_\ell/\mathscr{X}^{\mathrm{rig}}}.$$

Moreover, the classes $([L_1], \dots, [L_\ell])$ in $\prod_{j=1}^{\ell} \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})/b_j \, \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ are unique.

(2) *The reduced closed substack $\mathscr{X} \backslash \mathscr{T}$ is a simple normal crossing divisor.*

The first part of the corollary is very similar to [31], Proposition 2.5.

Remark 6.27. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}$ isomorphic to $T \times \mathscr{B}G$ and $G = \prod_{j=1}^{\ell} \mu_{b_j}$. Diagram (1.3) and the corollary above imply that we have the following morphism of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\ell} & \xrightarrow{\times \boldsymbol{b}} & \mathbb{Z}^{\ell} & \longrightarrow & \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}}) & \xrightarrow{r^*} & \mathrm{Pic}(\mathscr{X}) & \longrightarrow & \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z} & \longrightarrow & 0
\end{array}
\tag{6.28}
$$

where the vertical morphisms sends $e_j \mapsto L_j$ and $e_j \mapsto L_j^{1/b_j}$.

Proof of Corollary 6.26. Theorem 6.25(2) implies that $\mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is an essentially trivial G-banded gerbe. The first statement follows from Proposition 6.9.

By Corollary 5.4, we have that the reduced closed substack $\mathscr{X}^{\mathrm{rig}} \backslash \mathscr{T}^{\mathrm{rig}}$ is a simple normal crossing divisor. As the morphism $\mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is étale, we deduce the second statement of the corollary. □

7. Toric Deligne-Mumford stacks versus stacky fans

In this section, we will show that the toric Deligne-Mumford stacks that we have defined correspond exactly with those of [10].

In the first subsection, we show that our toric Deligne-Mumford stacks with a spanning condition are global quotients. The second subsection makes the correspondence with the article of [10].

7.1. Toric Deligne-Mumford stacks as global quotients. Let Z be a subvariety in $\mathbb{C}^n$ of codimension equal or higher than two. Let G be an abelian group scheme over $\mathbb{C}$ that acts on Z such that $[Z/G]$ is a Deligne-Mumford stack. According to Remark 1.1, a line bundle on $[Z/G]$ is given by a character χ of G. Hence the data of an invertible sheaf L with a global section s on $[Z/G]$ give a morphism of groupoids between $[Z/G]$ and $[\mathbb{A}^1/\mathbb{C}^*]$. Explicitly, this morphism is given by $(s, \chi) : Z \times G \to \mathbb{A}^1 \times \mathbb{C}^*$ and $s : Z \to \mathbb{A}^1$.

In the following lemma, we use a slightly more general notion of a root of Cartier divisors that is a root of invertible sheaves with global sections. All the properties of Section 1.3.b are still true (see [12] or [2]).

Lemma 7.1. *Let Z be a scheme. Let G be an abelian group scheme over $\mathbb{C}$ that acts on Z such that $[Z/G]$ is a Deligne-Mumford stack. Let $(\boldsymbol{L}, \boldsymbol{s}) := \big((L_1, s_1), \dots, (L_k, s_k)\big)$ be k invertible sheaves with global sections over the quotient stack $[Z/G]$. Denote by $\boldsymbol{\chi} := (\chi_1, \dots, \chi_k)$ the representations associated to the invertible sheaves $\boldsymbol{L}$. Let $\boldsymbol{d} := (d_1, \dots, d_k)$ be in $(\mathbb{N}_{>0})^k$.*

(1) *We have that $\sqrt[\boldsymbol{d}]{(\boldsymbol{L}, \boldsymbol{s})/[Z/G]}$ is isomorphic to $[\tilde{Z}/\tilde{G}]$ where $\tilde{Z}$ and $\tilde{G}$ are defined by the following cartesian diagrams*:

$$
\begin{array}{ccc}
\tilde{Z} & \longrightarrow & \mathbb{A}^k \\
\downarrow & \square & \downarrow {\scriptstyle \wedge \boldsymbol{d}} \\
Z & \xrightarrow{\boldsymbol{s}} & \mathbb{A}^k,
\end{array}
\qquad
\begin{array}{ccc}
\tilde{G} & \longrightarrow & \mathbb{G}_m^k \\
\downarrow {\scriptstyle \varphi} & \square & \downarrow {\scriptstyle \wedge \boldsymbol{d}} \\
G & \xrightarrow{\boldsymbol{\chi}} & \mathbb{G}_m^k.
\end{array}
$$

The action of $\tilde{G}$ *on* $\tilde{Z}$ *is given by*

$$\big(g,(\lambda_1,\ldots,\lambda_k)\cdot\big(z,(x_1,\ldots,x_k)\big)\big)=\big(gz,(\lambda_1x_1,\ldots,\lambda_kx_k)\big)$$

for any $\big(g,(\lambda_1,\ldots,\lambda_k)\big)\in\tilde{G}$ *and* $\big(z,(x_1,\ldots,x_k)\big)\in\tilde{Z}$.

(2) *We have that* $\sqrt[d]{\boldsymbol{L}/[Z/G]}$ *is isomorphic to* $[Z/\tilde{G}]$ *where* $\tilde{G}$ *is defined above. The action of* $\tilde{G}$ *on* Z *is given via* φ.

Proof. It is a straightforward computation on fibered products of groupoids. □

Remark 7.2. (1) We have that $\ker\varphi$ is isomorphic to $\prod_{i=1}^{k}\mu_{d_i}$. Notice that the action of $\tilde{G}$ on Z in the second part of the proposition above implies that the kernel of φ acts trivially on Z. Hence, $[Z/\tilde{G}]$ is a $\prod_{i=1}^{k}\mu_{d_i}$-banded gerbe over $[Z/G]$.

(2) In both cases we have that $\tilde{G}\in\operatorname{Ext}^1\left(G,\prod_{i=1}^{k}\mu_{d_i}\right)$.

Lemma 7.3. *Let A be an abelian group of finite type. Let E in* $\operatorname{Ext}^1\left(\bigoplus_{i=1}^{k}\mathbb{Z}/d_i\mathbb{Z},A\right)$. *If we have a morphism of short exact sequences*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{(d_1,\ldots,d_k)} & \mathbb{Z}^k & \longrightarrow & \bigoplus_{i=1}^{k}\mathbb{Z}/d_i\mathbb{Z} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \bigoplus_{i=1}^{k}\mathbb{Z}/d_i\mathbb{Z} & \longrightarrow & 0
\end{array}$$

then the left square is cocartesian.

Remark 7.4. Diagrams (5.6) and (6.28) imply that we have the following cocartesian diagrams:

$$\begin{array}{ccccccc}
\mathbb{Z}^\ell & \xrightarrow{\times\boldsymbol{b}} & \mathbb{Z}^\ell & & \mathbb{Z}^n & \xrightarrow{\times\boldsymbol{a}} & \mathbb{Z}^n\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\operatorname{Pic}(\mathscr{X}) & \longrightarrow & \operatorname{Pic}(\sqrt[\boldsymbol{b}]{\boldsymbol{L}/\mathscr{X}}), & & \operatorname{Pic}(\mathscr{X}) & \longrightarrow & \operatorname{Pic}(\sqrt[\boldsymbol{a}]{\boldsymbol{D}/\mathscr{X}}).
\end{array}\tag{7.5}$$

Proof of Lemma 7.3. Denote by P the push-out of $\mathbb{Z}^k\to\mathbb{Z}^k$ and $\mathbb{Z}^k\to A$. Using the universal property of co-cartesian diagrams we deduce a morphism f from P to E and the following morphisms of extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^k & \xrightarrow{\times(d_1,\ldots,d_k)} & \mathbb{Z}^k & \longrightarrow & \bigoplus_{i=1}^{k}\mathbb{Z}/d_i\mathbb{Z} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow\alpha & & \\
0 & \longrightarrow & A & \xrightarrow{q} & P & \longrightarrow & \operatorname{coker}(q) & \longrightarrow & 0\\
 & & \| & & \downarrow f & & \downarrow\beta & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & \bigoplus_{i=1}^{k}\mathbb{Z}/d_i\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Notice that the composition $\beta \circ \alpha$ is the isomorphism in Lemma 7.3. By simple diagram chasing, we deduce that f is an isomorphism. $\square$

Remark 7.6. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with coarse moduli space X. Proposition 3.6 implies that X is a simplicial toric variety. Denote by Σ a fan of X. Assume that the rays of Σ generate $N_{\mathbb{Q}}$. As explained in Section 1.6, we have that X is the geometric quotient Z_Σ / G_A where $G_A := \mathrm{Hom}\big(A^1(X), \mathbb{C}^*\big)$. Put $G_{\mathscr{X}} := \mathrm{Hom}\big(\mathrm{Pic}(\mathscr{X}), \mathbb{C}^*\big)$. Notice that $G_{\mathscr{X}^{\mathrm{rig}}}$ acts on Z_Σ via the dual (in the sense of Section 1.5) of the morphism $\mathbb{Z}^n \to \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$. The group $G_{\mathscr{X}}$ acts on Z_Σ via the dual of the morphism $\mathrm{Pic}(\mathscr{X}^{\mathrm{rig}}) \to \mathrm{Pic}(\mathscr{X})$. Consider the quotient stack $[Z_\Sigma / G_{\mathscr{X}}]$. The quotient stack $[(\mathbb{C}^*)^n / G_{\mathscr{X}}]$ is a Deligne-Mumford torus which is open and dense in $[Z_\Sigma / G_{\mathscr{X}}]$. As the natural action of $(\mathbb{C}^*)^n$ on Z_Σ extends the action of $(\mathbb{C}^*)^n$ on itself, we deduce a stack morphism $a : [(\mathbb{C}^*)^n / G_{\mathscr{X}}] \times [Z_\Sigma / G_{\mathscr{X}}] \to [Z_\Sigma / G_{\mathscr{X}}]$ that extends the action of $[(\mathbb{C}^*)^n / G_{\mathscr{X}}]$ on itself. Proposition 3.3 implies that the stack morphism a induces a natural action of the Deligne-Mumford torus on $[Z_\Sigma / G_{\mathscr{X}}]$ that is $[Z_\Sigma / G_{\mathscr{X}}]$ is a toric Deligne-Mumford stack.

Theorem 7.7. *Let $\mathscr{X}$ be a toric Deligne-Mumford stack with coarse moduli space X. Denote by Σ the fan associated to X. Assume that the rays of Σ generate $N \otimes \mathbb{Q}$. Then $\mathscr{X}$ is naturally isomorphic, as a toric stack, to $[Z_\Sigma / G_{\mathscr{X}}]$ where $G_{\mathscr{X}} := \mathrm{Hom}\big(\mathrm{Pic}(\mathscr{X}), \mathbb{C}^*\big)$.*

Remark 7.8. Removing the spanning condition of the rays gives the following result. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with torus $\mathscr{T}$ (isomorphic to $T \times \mathscr{B}G$) and with coarse moduli space the simplicial toric variety X. Denote by $\Sigma \subset N_{\mathbb{Q}}$ the fan of X. From the footnote 2 of Section 1.6, we deduce that the toric variety X is isomorphic to $\tilde{X} \times \tilde{T}$ where $\tilde{X}$ is a simplicial toric variety whose the rays of its fan $\tilde{\Sigma}$ span $\tilde{N}_{\mathbb{Q}}$. Notice that the dimension of $\tilde{T}$ is $\mathrm{rk}(N_{\mathbb{Q}}) - \mathrm{rk}(\tilde{N}_{\mathbb{Q}})$. The previous theorem implies that $\mathscr{X}$ is isomorphic, as toric stacks, to $[Z_{\tilde{\Sigma}} / G_{\tilde{\mathscr{X}}}] \times (\tilde{T} \times \mathscr{B}G)$.

Proof of Theorem 7.7. If $\mathscr{X}$ is $\mathscr{X}^{\mathrm{can}}$, the theorem follows from Remark 4.12(2). If $\mathscr{X}$ is $\mathscr{X}^{\mathrm{rig}}$, the theorem follows from the right cocartesian square of diagram (7.5) and Lemma 7.1(1). For a general $\mathscr{X}$, it follows from the left cocartesian square of diagram (7.5) and Lemma 7.1(2). $\square$

7.2. Toric Deligne-Mumford stacks and stacky fans. First we recall the definition of a stacky fan from [10].

Definition 7.9. A *stacky fan* is a triple $\mathbf{\Sigma} := (N, \Sigma, \beta)$ where N is a finitely generated abelian group, Σ is a rational simplicial fan in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ with n rays, denoted by $\rho_1, \ldots, \rho_n$, and a morphism of groups $\beta : \mathbb{Z}^n \to N$ such that:

(1) The rays span $N_{\mathbb{Q}}$.

(2) For any $i \in \{1, \ldots, n\}$, the element $\overline{\beta(e_i)}$ in $N_{\mathbb{Q}}$ is on the ray ρ_i where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{Z}^n$ and the natural map $N \to N_{\mathbb{Q}}$ sends $m \mapsto m$.

Remark 7.10. Let $\mathbf{\Sigma} := (N, \Sigma, \beta)$ be a stacky fan.

(1) As the rays span $N_{\mathbb{Q}}$, we have that β has finite cokernel.

(2) For any $i \in \{1, \dots, n\}$, denote by v_i the unique generator of $\rho_i \cap (N/N_{\mathrm{tor}})$ where N_{tor} is the torsion part part of N. Denote by β^{rig} the composition of β followed by the quotient morphism $N \to N/N_{\mathrm{tor}}$. There exists a unique $a_i \in \mathbb{N}_{>0}$ such that $\beta^{\mathrm{rig}}(e_i) = a_i v_i$. Denote $\mathbf{\Sigma}^{\mathrm{rig}} := (N/N_{\mathrm{tor}}, \Sigma, \beta^{\mathrm{rig}})$. There exists a unique group homomorphism $\beta^{\mathrm{can}} : \mathbb{Z}^n \to N/N_{\mathrm{tor}}$ such that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\beta} & N \\ {\scriptstyle \mathrm{diag}(a_1,\dots,a_n)}\downarrow & \searrow^{\beta^{\mathrm{rig}}} & \downarrow \\ \mathbb{Z}^n & \xrightarrow{\beta^{\mathrm{can}}} & N/N_{\mathrm{tor}}. \end{array} \tag{7.11}$$

Denote $\mathbf{\Sigma}^{\mathrm{can}} := (N/N_{\mathrm{tor}}, \Sigma, \beta^{\mathrm{can}})$.

In [10], Remark 4.5, the authors define the notion of morphism of stacky fans. The commutative diagram (7.11) provides us the morphisms of stacky fans $\mathbf{\Sigma} \to \mathbf{\Sigma}^{\mathrm{rig}} \to \mathbf{\Sigma}^{\mathrm{can}}$.

(3) To the fan Σ, we can associate canonically the stacky fan $\mathbf{\Sigma}^{\mathrm{can}}$.

Construction 7.12 (construction of the Deligne-Mumford stack associated to the stacky fan $\mathbf{\Sigma}$). Now we explain how to associate a Deligne-Mumford stack $\mathscr{X}(\mathbf{\Sigma})$ to a stacky fan $\mathbf{\Sigma}$ following [10], Sections 2 and 3. Denote by d the rank of N. Choose a projective resolution of N with two terms that is

$$0 \to \mathbb{Z}^{\ell} \xrightarrow{Q} \mathbb{Z}^{d+\ell} \to N \to 0.$$

Choose a map $B : \mathbb{Z}^n \to \mathbb{Z}^{d+\ell}$ lifting the map $\beta : \mathbb{Z}^n \to N$. Consider the morphism $[BQ] : \mathbb{Z}^{n+\ell} \to \mathbb{Z}^{d+\ell}$. Denote $DG(\beta) := \mathrm{coker}([BQ]^*)$. Denote by $\beta^{\vee} : (\mathbb{Z}^n)^* \to DG(\beta)$ the group morphism that makes the following diagram commute:

$$\begin{array}{ccc} (\mathbb{Z}^n)^* & \hookrightarrow & (\mathbb{Z}^{n+\ell})^* \\ & \searrow^{\beta^{\vee}} & \downarrow \\ & & DG(\beta) := \mathrm{coker}[BQ]^*. \end{array}$$

Let Z_Σ be the quasi-affine variety associated to the fan Σ (see Section 1.6). Define the action of $G_{\mathbf{\Sigma}} := \mathrm{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*)$ on Z_Σ as follows. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the morphism $\beta^{\vee} : (\mathbb{Z}^n)^* \to DG(\beta)$, we get a group morphism $G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$. Via the natural action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$, we define an action of $G_{\mathbf{\Sigma}}$ on Z_Σ. Finally, the stack associated to the stacky fan $\mathbf{\Sigma} := (N, \Sigma, \beta)$ is the quotient stack $\mathscr{X}(\mathbf{\Sigma}) := [Z_\Sigma / G_{\mathbf{\Sigma}}]$.

Notation. We will later see that the group $G_{\mathbf{\Sigma}}$ is isomorphic to

$$G_{\mathscr{X}} := \mathrm{Hom}(\mathrm{Pic}(\mathscr{X}), \mathbb{C}^*).$$

By [10], Proposition 3.2, we have that $[Z_\Sigma / G_{\mathbf{\Sigma}}]$ is a smooth Deligne-Mumford stack.

Remark 7.13. In [24], Iwanari defined a smooth toric Artin stack over any scheme associated to a stacky fan $\mathbf{\Sigma}^{\mathrm{rig}}$.

Remark 7.14. As it was observed in [10], Section 4, the condition that the rays span $N_{\mathbb{Q}}$ in Definition 7.9 is not natural. Indeed a Deligne-Mumford torus $(\mathbb{C}^*)^d \times \mathscr{B}G$ where G is a finite abelian group can not be produced as a stack $\mathscr{X}(\mathbf{\Sigma})$ for $\mathbf{\Sigma}$ a stacky fan with the condition that the rays span $N_{\mathbb{Q}}$. Nevertheless, it is not really true to say that toric Deligne-Mumford stacks are a "generalization" of the stacks $\mathscr{X}(\mathbf{\Sigma})$. Indeed, as for toric variety, we will see that a toric Deligne-Mumford stack is a product of a $\mathscr{X}(\mathbf{\Sigma})$ by a Deligne-Mumford torus.

Lemma 7.15. *Let $\mathbf{\Sigma} := (N, \Sigma, \beta)$ be a stacky fan.*

(1) *The stack $\mathscr{X}(\mathbf{\Sigma})$ is a toric Deligne-Mumford stack.*

(2) *The stack $\mathscr{X}(\mathbf{\Sigma})$ is a toric orbifold if and only if the finitely generated abelian group N is free.*

(3) *The stack $\mathscr{X}(\mathbf{\Sigma})$ is canonical if and only if $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathrm{can}}$.*

Proof. (1) The group morphism $G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$ defined in Construction 7.12 defines the quotient stack $[(\mathbb{C}^*)^n / G_{\mathbf{\Sigma}}]$ which is by definition a Deligne-Mumford torus. As the open dense immersion $(\mathbb{C}^*)^n \hookrightarrow Z_{\Sigma}$ is $G_{\mathbf{\Sigma}}$-equivariant, we have that the stack morphism $[(\mathbb{C}^*)^n / G_{\mathbf{\Sigma}}] \to [Z_{\Sigma} / G_{\mathbf{\Sigma}}]$ is an open dense immersion. Using the same arguments of Remark 7.6, we have that the action of the Deligne-Mumford torus $[(\mathbb{C}^*)^n / G_{\mathbf{\Sigma}}]$ on itself extends to an action on $[Z_{\Sigma} / G_{\mathbf{\Sigma}}]$. That is $\mathscr{X}(\mathbf{\Sigma})$ is a toric Deligne-Mumford stack.

(2) The stack $\mathscr{X}(\mathbf{\Sigma})$ is a toric orbifold if and only if $G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$ is injective, if and only if $\beta^{\vee}$ is surjective, if and only if N is free.

(3) Assume that $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathrm{can}}$. As the coarse moduli space X of $\mathscr{X}(\mathbf{\Sigma})$ is the geometrical quotient $Z_{\Sigma} / G_{A^1(X)}$ where $G_{A^1(X)} := \mathrm{Hom}(A^1(X), \mathbb{C}^*)$, we have that $\mathscr{X}^{\mathrm{can}} = [Z_{\Sigma} / G_{A^1(X)}]$. Construction 7.12 implies that $G_{\mathbf{\Sigma}}$ is $G_{A^1(X)}$. Conversely, if $\mathbf{\Sigma} \neq \mathbf{\Sigma}^{\mathrm{can}}$ then either N has torsion (i.e., $\mathscr{X}(\mathbf{\Sigma})$ is a gerbe) or there exists a divisor D associated to a ray such that any geometric point of D has a non-trivial stabilizer. □

Remark 7.16. Let $\mathscr{X}(\mathbf{\Sigma})$ be a canonical stack (i.e., $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathrm{can}}$). The proof of the third statement of Lemma 7.15 implies that $DG(\beta^{\mathrm{can}}) = \mathrm{Pic}(\mathscr{X}(\mathbf{\Sigma}))$.

Theorem 7.17. *Let $\mathscr{X}$ be a toric orbifold with coarse moduli space X. Denote by Σ a fan of X in $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume that the rays of Σ span $N_{\mathbb{Q}}$. Then there is a unique $\beta : \mathbb{Z}^n \to N$ such that the stack associated to the stacky fan (N, Σ, β) is isomorphic as toric orbifold to $\mathscr{X}$.*

Remark 7.18. An arbitrary toric orbifold is isomorphic to a product $\mathscr{X}(\mathbf{\Sigma}) \times (\mathbb{C}^*)^k$.

Proof of Theorem 7.17. Denote by $\boldsymbol{a} := (a_1, \ldots, a_n)$ the divisor multiplicities of $\mathscr{X}$. We define the morphism of groups $\beta : \mathbb{Z}^n \to N$ by sending $e_i \mapsto a_i v_i$ where v_i is the generator of the semi-group $\rho_i \cap N$. Denote by $\mathbf{\Sigma}$ the stacky fan (N, Σ, β).

Theorem 7.7 states that $\mathscr{X}$ is isomorphic to $[Z_\Sigma/G_\mathscr{X}]$. In order to prove that the two stacks are isomorphic, we will show that $G_\mathscr{X}$ is isomorphic to $G_\mathbf{\Sigma}$ such that the two actions on Z_Σ are compatible. From diagram (7.11), we deduce a morphism of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}^n)^* & \longrightarrow & (\mathbb{Z}^n)^* & \longrightarrow & \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Pic}(\mathscr{X}^{\mathrm{can}}) & \longrightarrow & DG(\beta) & \longrightarrow & \bigoplus_{i=1}^{n} \mathbb{Z}/a_i\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

The right cocartesian square of diagram (7.5) implies that $G_\mathbf{\Sigma}$ is isomorphic to $G_\mathscr{X}$ such that the actions of $G_\mathbf{\Sigma}$ and $G_\mathscr{X}$ on Z_Σ are compatible.

The uniqueness of β follows from the geometrical interpretation of the divisor multiplicities. □

Remark 7.19. (1) The proof shows also that $\mathrm{Pic}(\mathscr{X})$ is isomorphic to $DG(\beta)$.

(2) Marking a point $a_i v_i$ on the ray $\rho_i \cap N$ corresponds geometrically to putting a generic stabilizer μ_{a_i} on the divisor $\mathscr{D}_i$ associated to the ray ρ_i.

Proposition 7.20. *Let $\mathbf{\Sigma} := (N, \Sigma, \beta)$ be a stacky fan. There is a unique α in $\mathrm{Ext}^1\big(N_{\mathrm{tor}}, \mathrm{Pic}\big(\mathscr{X}(\mathbf{\Sigma}^{\mathrm{rig}})\big)\big)$ such that the essentially trivial $\mathrm{Hom}(N_{\mathrm{tor}}, \mathbb{C}^*)$-banded gerbe over $\mathscr{X}(\mathbf{\Sigma}^{\mathrm{rig}})$ associated to α is isomorphic as banded gerbe to $\mathscr{X}(\mathbf{\Sigma})$.*

Proof. Fix a decomposition $N = \mathbb{Z}^d \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z}$. It follows from Construction 7.12 that we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & DG(\beta^{\mathrm{rig}}) & \longrightarrow & DG(\beta) & \longrightarrow & \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (\beta^{\mathrm{rig}})^\vee} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\mathbb{Z}^n)^* & \longrightarrow & (\mathbb{Z}^{n+\ell})^* & \longrightarrow & (\mathbb{Z}^\ell)^* & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle [BQ]^*} & & \uparrow{\scriptstyle \times(b_1,\dots,b_\ell)} & & \\
0 & \longrightarrow & (\mathbb{Z}^d)^* & \longrightarrow & (\mathbb{Z}^{d+\ell})^* & \longrightarrow & (\mathbb{Z}^\ell)^* & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{7.21}$$

From Remark 7.19, we have that $\mathrm{Pic}\big(X(\mathbf{\Sigma}^{\mathrm{rig}})\big)$ is isomorphic to $DG(\beta^{\mathrm{rig}})$. The first line of diagram (7.21) is an element $\alpha \in \mathrm{Ext}^1\big(N_{\mathrm{tor}}, \mathrm{Pic}\big(\mathscr{X}(\mathbf{\Sigma}^{\mathrm{rig}})\big)\big)$. By Proposition 6.9, the element α induces an element $([L_1], \dots, [L_\ell]) \in \prod_{j=1}^{\ell} \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})/b_j \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$. The last row of the diagram above is a projective resolution of $\bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j$. Hence, we deduce that there exists a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}^{\ell})^* & \xrightarrow{\times \boldsymbol{b}} & (\mathbb{Z}^{\ell})^* & \longrightarrow & \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow \tilde{f} & & \downarrow f & & \| & & \\ 0 & \longrightarrow & \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}}) & \xrightarrow{r^*} & DG(\beta) & \longrightarrow & \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z} & \longrightarrow & 0. \end{array} \tag{7.22}$$

The morphism $\tilde{f}$ is the same as the choice of $L_1, \dots, L_\ell$ in $\mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ in the classes $[L_1], \dots, [L_\ell]$. By the left cocartesian square of diagram (7.5), we deduce that $G_{\boldsymbol{\Sigma}}$ is isomorphic to $G_{\mathscr{X}}$. We conclude that $\mathscr{X}$ is isomorphic to $\mathscr{X}(\boldsymbol{\Sigma})$. The uniqueness of α follows from Proposition 6.9. □

Remark 7.23. Denote by $\mathscr{X}_1$ and by $\mathscr{X}_2$ respectively the stacks associated to stacky fans (Σ, N, β_1) and (Σ, N, β_2). The stacks $\mathscr{X}_1$ and $\mathscr{X}_2$ are isomorphic, as toric Deligne-Mumford stack, if and only if the extensions defined in diagram (7.21) in $\mathrm{Ext}^1\big(N_{\mathrm{tor}}, \mathrm{Pic}\big(\mathscr{X}(\boldsymbol{\Sigma}^{\mathrm{rig}})\big)\big)$ are isomorphic.

Theorem 7.24. *Let $\mathscr{X}$ be a toric Deligne-Mumford stack with coarse moduli space X. Denote by Σ a fan of X in $N_{\mathbb{Q}}$. Assume that the rays of Σ span $N_{\mathbb{Q}}$. There exist N and $\beta : \mathbb{Z}^n \to N$ such that the stack associated to the stacky fan (N, Σ, β) is isomorphic as toric Deligne-Mumford stacks to $\mathscr{X}$.*

Remark 7.25. Let $\boldsymbol{\Sigma}$ be a stacky fan. Corollary 6.26 and the theorem above imply that $\mathscr{X}(\boldsymbol{\Sigma})$ is isomorphic to a product of root stacks over its rigidification. This result was discovered independently by Perroni (cf. [31], Proposition 3.2) and by Jiang and Tseng (cf. [25], Remark 2.10).

Proof of Theorem 7.24. If $\mathscr{X}$ is a toric orbifold then the statement was already proved in Theorem 7.17.

Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times \mathscr{B}G$. By Theorem 7.7, we have that $\mathscr{X}$ is isomorphic to $[Z_\Sigma/G_{\mathscr{X}}]$. By Theorem 7.17, there exists a unique stacky fan $\boldsymbol{\Sigma}^{\mathrm{rig}} = (\Sigma, \mathbb{Z}^d, \beta^{\mathrm{rig}})$ where $d := \dim \mathscr{X}$ such that $\mathscr{X}^{\mathrm{rig}}$ is isomorphic to $\mathscr{X}(\boldsymbol{\Sigma}^{\mathrm{rig}})$.

There exist $(b_1, \dots, b_\ell) \in (\mathbb{N}_{>0})^\ell$ such that $G = \prod_{j=1}^{\ell} \mu_{b_j}$. Put $N := \mathbb{Z}^d \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z}$. Corollary 6.26 gives us ℓ invertible sheaves $L_1, \dots, L_\ell$ on $\mathscr{X}^{\mathrm{rig}}$. For any j, choose $c_{1j}, \dots, c_{nj} \in \mathbb{Z}$ such that $L_j = \bigotimes_{i=1}^{n} \mathscr{O}(\mathscr{D}_i^{\mathrm{rig}})^{c_{ij}}$ where $\mathscr{D}_i^{\mathrm{rig}}$ is the Cartier divisor associated to the ray ρ_i. Put

$$\beta : \mathbb{Z}^n \to \mathbb{Z}^d \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}/b_j\mathbb{Z},$$

$$e_i \mapsto \big(\beta^{\mathrm{rig}}(e_i), [c_{i1}], \dots, [c_{i\ell}]\big),$$

where $[c_{ij}]$ is the class of c_{ij} modulo b_j. It is straightforward to check that $X(\boldsymbol{\Sigma})$ is isomorphic to $\mathscr{X}$. □

Remark 7.26. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with Deligne-Mumford torus isomorphic to $T \times \mathscr{B}G$. The non-uniqueness of N and β comes from three different kinds:

(1) the decomposition of G in product of cyclic groups, i.e., $G = \prod_{j=1}^{\ell} \mu_{b_j}$,

(2) the choice of the lift for the class $[L_j] \in \operatorname{Pic}(\mathscr{X}^{\mathrm{rig}})/b_j \operatorname{Pic}(\mathscr{X}^{\mathrm{rig}})$ for $j = 1, \dots, \ell$, and

(3) the choice of the decomposition $L_j = \bigotimes_{i=1}^{n} \mathscr{O}(\mathscr{D}_i^{\mathrm{rig}})^{c_{ij}}$ (see Example 7.29 for such an example).

7.3. Examples.

Example 7.27 (weighted projective spaces). Let $w_0, \dots, w_n$ be in $\mathbb{N}_{>0}$. Denote by $\mathbb{P}(w)$ the quotient stack $[\mathbb{C}^{n+1} \backslash \{0\}/\mathbb{C}^*]$ where the action of $\mathbb{C}^*$ is defined by $\lambda(x_0, \dots, x_n) = (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n)$ for any $\lambda \in \mathbb{C}^*$ and any $(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \backslash \{0\}$. The stack $\mathbb{P}(w)$ is a complete toric Deligne-Mumford stack with Deligne-Mumford torus $[(\mathbb{C}^*)^{n+1}/\mathbb{C}^*] \simeq \mathbb{C}^n \times B\mu_d$ where $d := \gcd(w_0, \dots, w_n)$ (cf. Example 2.3).

We have that:

(1) The stack $\mathbb{P}(w)$ is canonical if and only if for any $i \in \{0, \dots, n\}$, we have that $\gcd(w_0, \dots, \hat{w}_i, \dots, w_n) = 1$ (e.g., the weights are well-formed).

(2) The stack $\mathbb{P}(w)$ is an orbifold if and only if $\gcd(w_0, \dots, w_n) = 1$.

(3) The Picard group of $\mathbb{P}(w)$ is cyclic. More precisely, we have

$$\operatorname{Pic}(\mathbb{P}(w)) = \begin{cases} \mathbb{Z} & \text{if } \dim \mathbb{P}(w) \geqq 1, \\ \mathbb{Z}/w_0\mathbb{Z} & \text{if } \mathbb{P}(w) = \mathbb{P}(w_0). \end{cases}$$

Proposition 7.28. *Let $\mathscr{X}$ be a complete toric Deligne-Mumford stack of dimension n such that its Picard group is cyclic. Then there exists unique up to order $(w_0, \dots, w_n)$ in $(\mathbb{N}_{>0})^{n+1}$ such that $\mathscr{X}$ is isomorphic to $\mathbb{P}(w_0, \dots, w_n)$.*

Proof. Denote by X the coarse moduli space of $\mathscr{X}$. Denote by Σ a fan of X. If the Picard group is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ then Theorem 7.7 implies that $\mathscr{X} = [Z_\Sigma/\mu_d]$ with $Z_\Sigma \subset \mathbb{C}^n$. Hence, the fan Σ has n rays. In this case, $\mathscr{X}$ is complete if and only if $n = 0$. We deduce that $\mathscr{X} = \mathscr{B}\mu_d \simeq \mathbb{P}(d)$.

If the Picard group is $\mathbb{Z}$, Theorem 7.7 implies that $\mathscr{X} = [Z_\Sigma/\mathbb{C}^*]$ with $Z_\Sigma \subset \mathbb{C}^{n+1}$. As X is complete, the fan Σ is complete. We deduce that $Z_\Sigma = \mathbb{C}^{n+1} \backslash \{0\}$. The Deligne-Mumford torus is isomorphic to $[(\mathbb{C}^*)^{n+1}/\mathbb{C}^*]$. The action of $\mathbb{C}^*$ is given by the morphism $\mathbb{C}^* \to (\mathbb{C}^*)^{n+1}$ that sends $\lambda \mapsto (\lambda^{w_0}, \dots, \lambda^{w_n})$ with $w_i \in \mathbb{Z} \backslash \{0\}$. Notice that if the w_i's do not

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have the same sign then $\mathcal{X}$ is not separated. If the w_i's are all negative then replacing λ by λ^{-1} induces an isomorphism with a weighted projective space. □

Example 7.29. In this example, we give two isomorphic stacky fans for $\mathbb{P}(6,4)$ which was considered in [10], Example 3.5. As we have seen in Section 7.2, N and Σ are fixed whereas β is not unique. Let N be $\mathbb{Z} \times \mathbb{Z}/2$. Let Σ be the fan in $N_{\mathbb{Q}} = \mathbb{Q}$ where the cones are 0, $\mathbb{Q}_{\geqq 0}$, $\mathbb{Q}_{\leqq 0}$. Put

$$\text{(7.30)} \qquad \begin{aligned} &\beta_1 : \mathbb{Z}^2 \to \mathbb{Z} \times \mathbb{Z}/2, &&\beta_2 : \mathbb{Z}^2 \to \mathbb{Z} \times \mathbb{Z}/2, \\ &e_1 \mapsto (2,1), &&e_1 \mapsto (2,1), \\ &e_2 \mapsto (-3,0), &&e_2 \mapsto (-3,1). \end{aligned}$$

One can check that the stack associated to (N, Σ, β_1) and (N, Σ, β_2) is $\mathbb{P}(6,4)$.

Let us explicit the bottom up construction in this case. Its coarse moduli space is $\mathbb{P}^1$. The rigidification of $\mathbb{P}(6,4)$ is $\mathbb{P}(3,2)$. Denote by x_1, x_2 the homogeneous coordinates of $\mathbb{P}^1$. We have that $\mathbb{P}(3,2) = \sqrt[(2,3)]{(D_1, D_2)/\mathbb{P}^1}$ where D_i is the Cartier divisor $\big(\mathcal{O}_{\mathbb{P}^1}(1), x_i\big)$. We have that $\mathcal{O}_{\mathbb{P}(3,2)}(\mathcal{D}_1) = \mathcal{O}_{\mathbb{P}(3,2)}(3)$, $\mathcal{O}_{\mathbb{P}(3,2)}(\mathcal{D}_2) = \mathcal{O}_{\mathbb{P}(3,2)}(2)$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}(3,2)}(6)$ where $\pi : \mathbb{P}(3,2) \to \mathbb{P}^1$ is the structure morphism. The stack $r : \mathbb{P}(6,4) \to \mathbb{P}(3,2)$ is a μ_2-banded gerbe isomorphic to $\sqrt[2]{\mathcal{O}_{\mathbb{P}(3,2)}(1)/\mathbb{P}(3,2)}$. In $\operatorname{Pic}\big(\mathbb{P}(3,2)\big)/2\operatorname{Pic}\big(\mathbb{P}(3,2)\big)$, the class of $\mathcal{O}_{\mathbb{P}(3,2)}(1)$ is also the class of $\mathcal{O}_{\mathbb{P}(3,2)}(\mathcal{D}_1)$ or the class of $\mathcal{O}_{\mathbb{P}(3,2)}(\mathcal{D}_1) \otimes \mathcal{O}_{\mathbb{P}(3,2)}(\mathcal{D}_2)$. These different choices lead to the two isomorphic stacky fans in (7.30).

Example 7.31 (complete toric lines). Here, we explicitly describe all complete toric orbifolds of dimension 1. Notice that the coarse moduli space of a complete toric line is $\mathbb{P}^1$. Denote by x_1, x_2 the homogeneous coordinates. Let D_i be the Cartier divisor $\big(\mathcal{O}(1), x_i\big)$. Let a_1, a_2 in $\mathbb{N}_{>0}$. Denote by d (resp. m) the greatest common divisor (resp. the lowest common multiple) of a_1, a_2. The Picard group of the root stack $\sqrt[(a_1,a_2)]{(D_1, D_2)/\mathbb{P}^1}$ is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/d\mathbb{Z})$. Notice that it is not a weighted projective space in general. As a global quotient, the stack $\sqrt[(a_1,a_2)]{(D_1, D_2)/\mathbb{P}^1}$ is $[(\mathbb{C}^2 \backslash \{0\})/(\mathbb{C}^* \times \mu_d)]$ where the action is given by

$$\begin{aligned} &\mathbb{C}^* \times \mu_d \times (\mathbb{C}^2 \backslash \{0\}) \to (\mathbb{C}^2 \backslash \{0\}), \\ &\qquad \big((\lambda, t), (x_1, x_2)\big) \mapsto (\lambda^{m/a_1} t^{k_2} x_1, \lambda^{m/a_2} t^{-k_1} x_2) \end{aligned}$$

where k_1, k_2 are integers such that $\dfrac{k_1}{a} + \dfrac{k_2}{b} = \dfrac{1}{m}$.

Appendix A. Uniqueness of morphisms to separated stacks

We prove Proposition 1.2.

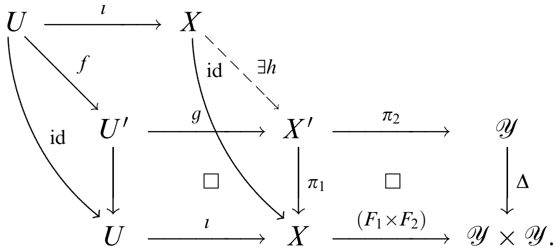
Proposition A.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Deligne-Mumford stacks. Assume that $\mathcal{X}$ is normal and $\mathcal{Y}$ is separated. Let $\imath : \mathcal{U} \hookrightarrow \mathcal{X}$ be a dominant open immersion. If $F_1, F_2 : \mathcal{X} \to \mathcal{Y}$ are two morphisms of stacks such that there exits a 2-arrow $\beta : F_1 \circ \imath \Rightarrow F_2 \circ \imath$ then there exists a unique 2-arrow $\alpha : F_1 \Rightarrow F_2$ such that $\alpha * \mathrm{id}_\imath = \beta$.*

Proof. *Uniqueness*: We first assume that $\mathscr{X}$ is a scheme, denoted by X, and $\mathscr{Y}$ is a global quotient $[V/G]$ where G is a separated group scheme. Denote by U the scheme $\mathscr{U}$, open dense in X. For i in $\{1,2\}$, the morphism F_i is given by an object x_i which is a G-torsor $\pi_i : P_i \to X$ and a G-equivariant morphism $P_i \to V$. Let $\alpha, \alpha' : P_1 \to P_2$ be morphisms between the objects x_1 and x_2 such that $\alpha|_{\pi_1^{-1}(U)} = \alpha'|_{\pi_2^{-1}(U)}$. As G is separated, we have that π_i is separated. We deduce that $\alpha = \alpha'$.

Now we prove the uniqueness of the proposition in the case where $\mathscr{Y} = [V/G]$. Let X be an étale atlas of $\mathscr{X}$. By the previous point, we deduce that $\alpha|_X = \alpha'|_X$. As $\mathrm{Mor}(F_1, F_2)$ is a sheaf on $\mathscr{X}$, we conclude that $\alpha = \alpha'$.

For the general case, we reduce to the previous by covering $\mathscr{Y}$ by global quotients and then we use that $\mathrm{Mor}(F_1, F_2)$ is a sheaf on $\mathscr{X}$.

Existence: It is enough to do it for an étale affine chart of $\mathscr{X}$. By hypothesis, this chart is a disjoint union of affine irreducible normal varieties. Hence, we can assume that $\mathscr{X}$ is an affine irreducible normal variety, denoted by X. Denote by U the scheme $\mathscr{U}$ open dense in X. The morphism $F_1 \circ \imath : U \to \mathscr{Y}$, the 2-arrow β and the universal property of the strict fiber product give a morphism $f : U \to U'$. The existence of α is equivalent to the existence of a morphism $h : X \dashrightarrow X'$ such that $\pi_1 \circ h = \mathrm{id}$ and $h \circ \imath = g \circ f$. Denote by $\Delta : \mathscr{Y} \to \mathscr{Y} \times \mathscr{Y}$ the diagonal. We can sum up the informations in the following diagram:



By definition of the separatedness of $\mathscr{Y}$, we have the Δ is proper. By [26], Lemma 4.2, we have that Δ is finite and X' is a scheme. We deduce that $\pi_1 : X' \to X$ is finite. The morphism $g \circ f : U \to X'$ is a section of π_1. By Lemma A.2, we deduce a morphism $h : X \to X'$ such that $h \circ \imath = g \circ f$. This completes the proof. □

Lemma A.2. *Let X' be a scheme and X be an irreducible normal variety. Let $\pi : X' \to X$ be a finite morphism. Let $U \hookrightarrow X$ be an open dense immersion. Let $s : U \to X'$ be a section of π. Then the section s extends to a section $\tilde{s} : X \to X'$.*

Proof. Denote by U_0 the closure of the $s(U)$ in the fiber product $U' := U \times_X X'$. Denote by $p : U' \to U$ and $q : U' \to X'$ the morphisms induced by the fiber product U'. Looking at the fractional fields, we deduce that the morphisms $s : U \to U_0$ and $p|_{U_0} : U_0 \to U$ are birational morphisms. Denote by X_0 the closure of U_0 in X'. As the morphism q is an open embedding, we have that $q|_{U_0}$ is dominant. We deduce that $\pi|_{X_0} : X_0 \to X$ is birational and quasi-finite. As X would be an irreducible normal variety, the Zariski main theorem implies that $\pi|_{X_0}$ is an isomorphism. Its inverse is the wanted section of π. □

Appendix B. Action of a Picard stack

In this appendix, we recall the definition of a Picard stack. Then we define the action of a Picard stack on a stack which extends the definition of Romagny in [32]. In [11], Definition 6.1, Breen defines the notion of a $\mathscr{G}$-torsor over a stack where $\mathscr{G}$ is a Picard stack. Our definition of the action is actually already included in that definition.

To define the notion of Picard stacks, we do not need the stacks to be algebraic.

Definition B.1 (Picard stacks [7], Exp. XVIII). Let S be a base scheme. A Picard S-stack $\mathscr{G}$ is an S-stack with the following data:

- (multiplication) a morphism of S-stacks:

$$\mathscr{G} \times_S \mathscr{G} \xrightarrow{m} \mathscr{G},$$
$$(g_1, g_2) \mapsto g_1 \cdot g_2,$$

- (2-associativity) a 2-arrow θ implementing the associativity law:

$$\theta_{g_1,g_2,g_3} : (g_1 \cdot g_2) \cdot g_3 \Rightarrow g_1 \cdot (g_2 \cdot g_3), \tag{B.2}$$

- (2-commutativity) a 2-arrow τ implementing commutativity:

$$\tau_{g_1,g_2} : g_1 \cdot g_2 \Rightarrow g_2 \cdot g_1. \tag{B.3}$$

These data must satisfy the following conditions:

(1) For every chart U and every object $g \in \mathscr{G}(U)$ the map $m_g : \mathscr{G} \to \mathscr{G}$ which multiplies every object by g and every arrow by id_g is an isomorphism of stacks.

(2) (*Pentagon relation*) For every chart U and 4-tuples of objects $g_i \in \mathscr{G}(U)$, we have

$$(\mathrm{id}_{g_1} \cdot \theta_{g_2,g_3,g_4}) \circ \theta_{g_1,g_2 \cdot g_3,g_4} \circ (\theta_{g_1,g_2,g_3} \cdot \mathrm{id}_{g_4}) = \theta_{g_1,g_2,g_3 \cdot g_4} \circ \theta_{g_1 \cdot g_2,g_3,g_4}. \tag{B.4}$$

(3) For every chart U and every object $g \in \mathscr{G}(U)$, we have $\tau_{g,g} = \mathrm{id}_{g \cdot g}$.

(4) For every chart U and every objects $g_1, g_2 \in \mathscr{G}(U)$, we have $\tau_{g_1,g_2} \circ \tau_{g_2,g_1} = \mathrm{id}_{g_2 \cdot g_1}$.

(5) (*Hexagon relation*) For every chart U and every triple of objects g_1, g_2, g_3 in $\mathscr{G}(U)$, we have

$$\theta_{g_1,g_2,g_3} \circ \tau_{g_3,g_1 \cdot g_2} \circ \theta_{g_3,g_1,g_2} = (\mathrm{id}_{g_1} \cdot \tau_{g_3,g_2}) \circ \theta_{g_1,g_3,g_2} \circ (\tau_{g_3,g_1} \cdot \mathrm{id}_{g_2}). \tag{B.5}$$

Remark B.6. The pentagon relation establishes the compatibility law between 2-arrows θ when expressing the associativity with 4 objects.

The third condition means that every object strictly commutes with itself.

The last condition states compatibility between the 2-arrow of associativity and the 2-arrow of commutativity.

Remark B.7. It can be proved, see [7], Exp. XVIII, 1.4.4, that the previous definition is enough to guarantee the existence of a neutral element in the group stack. More precisely it is a couple (e, ϵ) where $e : S \to \mathscr{G}$ is a section and $\epsilon : e \cdot e \Rightarrow e$. A neutral element is unique up to a unique isomorphism.

Definition B.8 (morphisms of Picard stacks [7], Exp. XVIII). Let $(\mathscr{G}, \theta, \tau)$ and $(\mathscr{H}, \psi, \rho)$ be two Picard S-stacks. A morphism of Picard S-stacks is a morphism of S-stacks $F : \mathscr{G} \to \mathscr{H}$ with a 2-arrow $\phi_{g_1, g_2} : F(g_1 \cdot g_2) \Rightarrow F(g_1) \cdot F(g_2)$ for any g_1, g_2 objects of $\mathscr{G}$ satisfying the following compatibility conditions:

- For every chart U and every couple of objects $g_1, g_2 \in \mathscr{G}(U)$ we have

$$\rho_{F(g_1), F(g_2)} \circ \phi_{g_1, g_2} = \phi_{g_2, g_1} \circ F(\tau_{g_1, g_2}). \tag{B.9}$$

- For every chart U and every triple of objects $g_1, g_2, g_3 \in \mathscr{G}(U)$ we have

$$\begin{aligned} &\phi_{g_1, g_2 \cdot g_3} \circ (\mathrm{id}_{F(g_1)} \cdot \phi_{g_2, g_3}) \circ F(\theta_{g_1, g_2, g_3}) \\ &\quad = \psi_{F(g_1), F(g_2), F(g_3)} \circ (\phi_{g_1, g_2} \cdot \mathrm{id}_{F(g_3)}) \circ \phi_{g_1 \cdot g_2, g_3}. \end{aligned} \tag{B.10}$$

Remark B.11. (1) It should be observed that the morphism F maps the pentagon relation (resp. the hexagon relation) for the Picard stack $\mathscr{G}$ to the pentagon relation (resp. the hexagon relation) for $\mathscr{H}$.

(2) Denote by $(e_{\mathscr{G}}, \epsilon_{\mathscr{G}})$ a neutral element of $\mathscr{G}$ and $(e_{\mathscr{H}}, \epsilon_{\mathscr{H}})$ a neutral element of $\mathscr{H}$. The couple $\big(F(e_{\mathscr{G}}), F(\epsilon_{\mathscr{G}}) \circ \phi^{-1}_{e_{\mathscr{G}}, e_{\mathscr{G}}}\big)$ is a neutral element of $\mathscr{H}$. By Remark B.7 there exists a unique 2-arrow $\lambda : F(e_{\mathscr{G}}) \Rightarrow e_{\mathscr{H}}$ such that $\lambda \circ F(\epsilon_{\mathscr{G}}) \circ \phi^{-1}_{e_{\mathscr{G}}, e_{\mathscr{G}}}) = \epsilon_{\mathscr{H}} \circ \lambda^2$.

(3) It can be useful to notice that given $\alpha : g_1 \Rightarrow g_2$ and $\beta : g_3 \Rightarrow g_4$ morphisms in $\mathscr{G}(U)$ the following identities involving morphisms holds:

$$F(\alpha \cdot \beta) = \phi^{-1}_{g_2, g_4} \circ \big(F(\alpha) \cdot F(\beta)\big) \circ \phi_{g_1, g_3}.$$

Definition B.12 (action of a Picard stack). Let $(\mathscr{G}, \tau, \theta)$ be a Picard S-stack. Denote by e the neutral section and by ϵ the corresponding 2-arrow. Let $\mathscr{X}$ be an S-stack. An action of $\mathscr{G}$ on $\mathscr{X}$ is the following data:

- a morphism of S-stack:

$$\mathscr{G} \times_S \mathscr{X} \xrightarrow{a} \mathscr{X},$$

$$g, x \mapsto g \times x,$$

- a 2-arrow η:

$$\eta_x : e \times x \Rightarrow x,$$

- a 2-arrow σ:

$$\sigma_{g_1,g_2,x} : (g_1 \cdot g_2) \times x \Rightarrow g_1 \times (g_2 \times x).$$

These data must satisfy the following conditions:

(1) (*Pentagon*) For every chart U, every objects $g_1, g_2, g_3 \in \mathscr{G}(U)$ and every object $x \in \mathscr{X}(U)$, we have

$$(\mathrm{id}_{g_1} \times \sigma_{g_2,g_3,x}) \circ \sigma_{g_1,g_2 \cdot g_3,x} \circ (\theta_{g_1,g_2,g_3} \times \mathrm{id}_x) = \sigma_{g_1,g_2,g_3 \times x} \circ \sigma_{g_1 \cdot g_2,g_3,x}.$$

(2) For any chart U and any object $x \in \mathscr{X}(U)$, we have

$$(\mathrm{id}_e \times \eta_x) \circ \sigma_{e,e,x} = (\epsilon \times \mathrm{id}_x).$$

Remark B.13. (1) If the Picard stack is a group-scheme then our definition of the action is compatible with the one given by Romagny in [32].

(2) Let $(\mathscr{G}, m, \theta, \tau)$ be a Picard S-stack. The multiplication m defines an action of $\mathscr{G}$ on itself.

Proposition B.14. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two Picard S-stacks. Let $F : \mathscr{G}_1 \to \mathscr{G}_2$ be a morphism of Picard stacks with the 2-arrow $\phi_{g_1,g_2} : F(g_1 \cdot g_2) \Rightarrow F(g_1) \cdot F(g_2)$. Let $\mathscr{X}$ be an S-stack with an action of $\mathscr{G}_2$ given by (a, η, σ). Then the morphism F induces a natural action of $\mathscr{G}_1$ on $\mathscr{X}$.*

Proof. The natural action is given by $(\tilde{a}, \tilde{\eta}, \tilde{\sigma})$ where we put:

- $\tilde{a} := a \circ F$.

- For every object x in $\mathscr{X}$, $\tilde{\eta}_x := \big(\eta_x \circ (\lambda \times \mathrm{id}_x)\big)$ where λ is the 2-arrow defined in Remark B.11.

- For every couple (g_1, g_2) of objects of $\mathscr{G}_1$ and every x object of $\mathscr{X}$, $\tilde{\sigma}_{g_1,g_2,x} := \sigma_{F(g_1),F(g_2),x} \circ (\phi_{g_1,g_2} \times \mathrm{id}_x)$.

It is straightforward but tedious to check that the triple so defined satisfies all the properties in Definition B.12. □

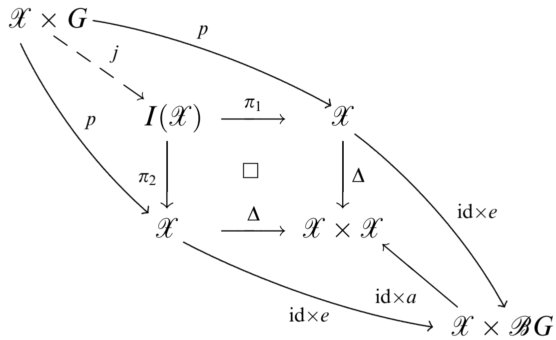
We finish this section with a proposition about actions on algebraic stacks. We refer to [26], Definition 12.1, for the notion of étale site of a Deligne-Mumford stack.

Proposition B.15. *Let $\mathscr{X}$ be a smooth Deligne-Mumford stack and G a finite abelian group. An action of $\mathscr{B}G$ on $\mathscr{X}$ induces a morphism of sheaves of groups $j : G \times \mathscr{X} \to I^{\mathrm{gen}}(\mathscr{X})$ on the étale site of $\mathscr{X}$. Moreover, as morphism of stacks, j is étale.*

Proof. We may assume $\mathscr{X}$ to be irreducible and d-dimensional. First we produce a stack morphism $j : \mathscr{X} \times G \to I^{\mathrm{gen}}(\mathscr{X})$ and we prove that j is étale. Denote by $e : \operatorname{Spec} \mathbb{C} \to \mathscr{B}G$ the neutral section. Denote by $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ the diagonal morphism.

Denote by $a : \mathscr{B}G \times \mathscr{X} \to \mathscr{X}$ the action. Using the universal property of the fibered product, we have the following 2-commutative diagram:

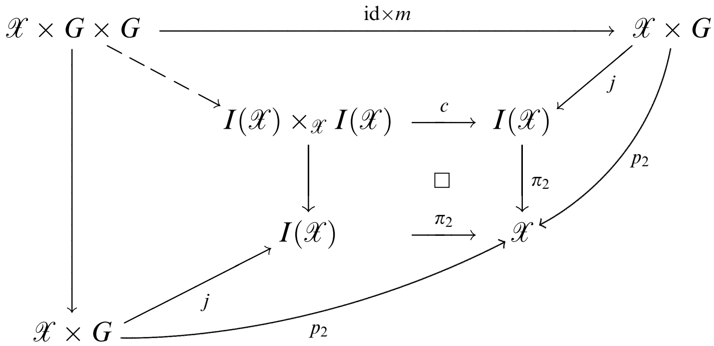
(B.16)



where $p : \mathscr{X} \times G \to \mathscr{X}$ is the projection. The stack morphism j must be unramified since it is a factor of the étale morphism $p : \mathscr{X} \times G \to \mathscr{X}$. Since every component of $I(\mathscr{X})$ has dimension at most d, the stack morphism j is actually étale and its image is contained in $I^{\mathrm{gen}}(\mathscr{X})$.

Now, it remains to prove that $j : \mathscr{X} \to I^{\mathrm{gen}}(\mathscr{X})$ is a morphism of sheaves of groups on the étale site of $\mathscr{X}$. The two upper triangles of diagram (B.16) are strictly commutative since $I(\mathscr{X})$ is the strict fibered product. This implies that j is a morphism of sheaves of sets over $\mathscr{X}$. Notice that on the étale site, the sheaf $I(\mathscr{X})$ is $I^{\mathrm{gen}}(\mathscr{X})$.

To finish the proof, we need to show that j is a morphism of sheaves of groups. Let us check the compatibility between the composition law in $I(\mathscr{X})$ and the multiplication of G. This compatibility follows from the existence of a dashed arrow such that the upper square in the following diagram is strictly commutative:



where the stack morphism c is the composition law of the inertia stack. The external square of the diagram above is 2-cartesian and the stack morphism $\mathrm{id} \times m : \mathscr{X} \times G \times G \to \mathscr{X} \times G$ is the identity on $\mathscr{X}$ and the multiplication in G. By the universal property of the strict fiber product, we deduce the dashed arrow such that the upper square is strictly commutative. This ends the proof. □

Appendix C. Stacky version of Zariski's Main Theorem

Here, we prove a stacky version of Zariski's Main Theorem. We did not find any reference in the literature for this version.

Theorem C.1 (Zariski's Main Theorem for stacks). *Let $\mathscr{X}$, $\mathscr{Y}$ be smooth Deligne-Mumford stacks. Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable, birational, quasi-finite and surjective morphism. Then f is an isomorphism.*

Proof. Let $Y \to \mathscr{Y}$ be an étale atlas. Consider the following fiber product:

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & Y \\ \big\downarrow & \square & \big\downarrow \\ \mathscr{X} & \xrightarrow{f} & \mathscr{Y}. \end{array}$$

The morphism $f : X \to Y$ is proper, birational, surjective and quasi-finite between smooth varieties. Hence, the Zariski Main Theorem (see for example [29], p. 209) implies that f is an isomorphism. This implies that $f : \mathscr{X} \to \mathscr{Y}$ is an isomorphism. $\square$

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Eingegangen 21. Januar 2009, in revidierter Fassung 19. September 2009