VIII L'application de Gauss en coordonnées locales

Soit S une surface régulière de $\mathbb{R}^3$ orienté

Soit $X : U \subset \mathbb{R}^2 \longrightarrow S$ un système de coordonnées sur un ouvert de S

$$(u,v) \longmapsto X(u,v)$$

Supposons que l'orientation $N : S \longrightarrow S^2$ est telle que

$$N \circ X(u,v) = \frac{\frac{\partial X}{\partial u}(u,v) \wedge \frac{\partial X}{\partial v}(u,v)}{\left\| \frac{\partial X}{\partial u}(u,v) \wedge \frac{\partial X}{\partial v}(u,v) \right\|} \quad \text{pour tout } (u,v) \in U$$

c à d coïncide avec l'orientation canonique sur $X(U)$.

SERT ?

Soit $p \in X(U)$.

Soit $\vec{w} \in T_p(S)$. Alors $\vec{w} = \dfrac{d(X \circ \gamma)}{dt}(0)$

où $\gamma :]-\varepsilon, +\varepsilon[\longrightarrow U$

$$t \longmapsto (u(t), v(t)) \quad tq \quad X \circ \gamma(0) = p$$

$$\vec{w} = \frac{\partial X}{\partial u}(p)\, u'(0) + \frac{\partial X}{\partial v}(p)\, v'(0)$$

$$dN(\vec{w}) \stackrel{def}{=} \frac{d(N \circ X \circ \gamma)}{dt}(0) = \frac{\partial N}{\partial u}(p)\, u'(0) + \frac{\partial N}{\partial v}(p)\, v'(0)$$

$\dfrac{\partial N}{\partial u}(p) \in T_p S$ car $\vec{N}^2 = 1 \Longrightarrow \dfrac{\partial N}{\partial u}(p) \cdot \vec{N}(p) = 0$

donc $(*)\ \dfrac{\partial N}{\partial u}(p) = a_{11} \dfrac{\partial X}{\partial u}(p) + a_{21} \dfrac{\partial X}{\partial v}(p)$

coordonnées dans la base $\left(\frac{\partial X}{\partial u}(p), \frac{\partial X}{\partial v}(p)\right)$ de $T_p S$.

$$(**)\ \frac{\partial N}{\partial v}(p) = a_{12}\frac{\partial X}{\partial u}(p) + a_{22}\frac{\partial X}{\partial v}(p)$$

donc $$dN_p(\vec{w}) = \left(a_{11}u'(0) + a_{12}v'(0)\right)\frac{\partial X}{\partial u}(p)$$

$$+ \left(a_{21}u'(0) + a_{22}v'(0)\right)\frac{\partial X}{\partial v}(p)$$

on a redémontré que dN_p est linéaire

vecteur coordonnées de $dN_p(\vec{w})$ dans la base $\left(\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\right)$ $= \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\text{matrice de } dN_p \text{ relativement à la base } \frac{\partial X}{\partial u}(p),\ \frac{\partial X}{\partial v}(p)} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}$

$$II_p(\vec{w}) = -dN_p(\vec{w}) \cdot \vec{w}$$

$$= -\left(\frac{\partial N}{\partial u}u'(0) + \frac{\partial N}{\partial v}v'(0)\right) \cdot \left(\frac{\partial X}{\partial u}u'(0) + \frac{\partial X}{\partial v}v'(0)\right)$$

$$= -\frac{\partial N}{\partial u}\cdot\frac{\partial X}{\partial u}u'(0)^2 - \left(\frac{\partial N}{\partial v}\cdot\frac{\partial X}{\partial u} + \frac{\partial N}{\partial u}\cdot\frac{\partial X}{\partial v}\right)u'(0)v'(0)$$

$$- \frac{\partial N}{\partial v}\cdot\frac{\partial X}{\partial v}v'(0)^2$$

Comme $N \cdot \frac{\partial X}{\partial u} = 0$ et $N \cdot \frac{\partial X}{\partial v} = 0$,

en dérivant par rapport à u et à v

$$\Pi_p\left(\frac{\partial X}{\partial u}(p)\, u'(0) + \frac{\partial X}{\partial v}(p)\, v'(0)\right)$$

$$= e\, u'(0)^2 + 2 f\, u'(0) v'(0) + g\, v'(0)^2$$

avec $e = -\frac{\partial N}{\partial u}(p) \cdot \frac{\partial X}{\partial u}(p) = N(p) \cdot \frac{\partial^2 X}{\partial u^2}(p)$

$$f = -\frac{\partial N}{\partial v}(p) \cdot \frac{\partial X}{\partial u}(p) = N(p) \cdot \frac{\partial^2 X}{\partial v \partial u}(p) = -\frac{\partial N}{\partial u}(p) \cdot \frac{\partial X}{\partial v}(p)$$

$$g = -\frac{\partial N}{\partial v}(p) \cdot \frac{\partial X}{\partial v}(p) = N(p) \cdot \frac{\partial^2 X}{\partial v^2}(p)$$

En utilisant *) et **)

$$-f = \frac{\partial N}{\partial u}(p) \cdot \frac{\partial X}{\partial v}(p) = a_{11} \underbrace{\frac{\partial X}{\partial u}(p) \cdot \frac{\partial X}{\partial v}(p)}_{F} + a_{21} \underbrace{\frac{\partial X}{\partial v}(p) \cdot \frac{\partial X}{\partial v}(p)}_{G}$$

$$-f = \frac{\partial N}{\partial v}(p) \cdot \frac{\partial X}{\partial u}(p) = a_{12} E + a_{22} F$$

$$-e = \frac{\partial N}{\partial u}(p) \cdot \frac{\partial X}{\partial u}(p) = a_{11} E + a_{21} F$$

$$-g = \frac{\partial N}{\partial v}(p) \cdot \frac{\partial X}{\partial v}(p) = a_{12} F + a_{22} G$$

c à d sous forme matricielle

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

donc

$$(***)\quad \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

or $$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{\det}\, {}^t\mathrm{comm}$$

$$= \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$\Rightarrow \quad a_{11} = \frac{fF-eG}{EG-F^2} \qquad a_{21} = \frac{eF-fE}{EG-F^2}$$

$$a_{12} = \frac{gF-fG}{EG-F^2} \qquad a_{22} = \frac{fF-gE}{EG-F^2}$$

equations de Weingarten

d'après $(***)$

$$K = \det dN_p = \det a_{ij} = (-1)^2\frac{eg-f^2}{EG-F^2} = k_1 k_2$$

$$H = -\frac{\mathrm{tr}\, dN_p}{2} = -\frac{1}{2}(a_{11}+a_{22}) = \frac{1}{2}\,\frac{eG-2fF+gE}{EG-F^2}$$

$$= \frac{k_1+k_2}{2}$$

$-k_1$ et $-k_2$ sont racines du polynôme caractéristiques

$$\det(dN_p - X\,Id) = X^2 - \mathrm{tr}\, dN_p\, X + \det dN_p$$

c à d k_1 et k_2 sont racine de

$$k^2 + \mathrm{tr}\, dN_p\, k + \det dN_p$$

$$k^2 - 2H\, k + K = 0$$

donc $k_1 = H + \sqrt{H^2 - K} \geqslant k_2 = H - \sqrt{H^2 - K}$