DERIVATIONS OF NEGATIVE DEGREE ON QUASIHOMOGENEOUS
ISOLATED COMPLETE INTERSECTION SINGULARITIES

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Abstract. J. Wahl conjectured that every quasihomogeneous isolated normal singularity admits a positive grading for which there are no derivations of negative weighted degree. We confirm his conjecture for quasihomogeneous isolated complete intersection singularities of either order at least 3 or embedding dimension at most 5. For each embedding dimension larger than 5 (and each dimension larger than 3), we give a counter-example to Wahl’s Conjecture.

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Introduction

By a singularity we mean a quotient \( A \) of a convergent power series ring over a valued field \( K \) of characteristic zero (see §1). We use the acronym negative derivation for a derivation of negative weighted degree on a quasihomogeneous singularity. The question of existence of such negative derivations has important consequences in rational homotopy theory (see [Mei82, Thm. A]) and in deformation theory (see [Wah82, Thm. 3.8]).

By a result of Kantor [Kan79], quasihomogeneous curve and hypersurface singularities do not admit any negative derivations. J. Wahl [Wah82, Thm. 2.4, Prop. 2.8] reached the same conclusion in (the much deeper) case of quasihomogeneous normal surface singularities. Motivated by his cohomological characterization of projective space in [Wah83a], he formulates the following conjecture in [Wah83b, Conj. 1.4].

Conjecture (Wahl). Let \( R \) be a normal graded ring, with isolated singularity. Then there is a normal graded \( \bar{R} \), with \( \hat{R} \cong \hat{\bar{R}} \), so that \( \bar{R} \) has no derivations of negative weight.

In case \( R \) is a graded normal locally complete intersection with isolated singularity, \( \bar{R} \) becomes a quasihomogeneous normal isolated complete intersection singularity (ICIS) and Wahl’s conjecture can be rephrased as follows (see Lemma 5 and Remark 7).

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Conjecture (Wahl, ICIS case). Any quasihomogeneous normal ICIS has no negative derivations with respect to some positive grading.

For quasihomogeneous normal ICIS, there is an explicit description of all derivations due to Kersken [Ker84]. Based on this description, we prove our main

Theorem 1. For any quasihomogeneous normal ICIS of order at least 3 there are no negative derivations with respect to any positive grading.

Proof. This follows from Corollary 12 and Proposition 16.

Our investigations lead to a family of counter-examples to Wahl’s Conjecture. In order to describe it, we fix our notation. A quasihomogeneous singularity can be represented as

\[ A = P/\mathfrak{a}, \quad \mathfrak{a} = \langle g_1, \ldots, g_t \rangle \subseteq K\langle \langle x_1, \ldots, x_n \rangle \rangle =: P \]

where \( g_1, \ldots, g_t \) are homogeneous polynomials of degree \( p_i := \deg(g_i) \) with respect to weights \( w_1, \ldots, w_n \in \mathbb{Z}_+ \) on the variables \( x_1, \ldots, x_n \) (see §1). We order these weights and degrees decreasingly as

\[ w_1 \geq \cdots \geq w_n > 0, \quad p_1 \geq \cdots \geq p_t. \]

Example 2. Let \( n \geq 6 \) and pick \( c_7, \ldots, c_n \in K \setminus \{1\} \) pairwise different such that \( c_i^9 + 1 \neq 0 \) for all \( i \). Assigning weights 8, 8, 5, 2, \ldots, 2 to the variables \( x_1, \ldots, x_n \), the equations

\[ g_1 := x_1x_4 + x_2x_3 + x_3^2 - x_4^5 + \sum_{i=7}^{n} x_i^5 \]
\[ g_2 := x_1x_5 + x_2x_6 + x_3^2 + x_6^5 + \sum_{i=7}^{n} c_ix_i^5 \]

define a quasihomogeneous complete intersection \( A \) as in (0.1) with isolated singularity.

On \( A \) there is a derivation

\[ \eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4x_6 - x_5^2)\partial_3 \]

of degree \(-1\). We work out the details of this example in §4.

We show that Example 2.8 gives a counter-example to the ICIS case of Wahl’s conjecture of minimal embedding dimension \( n = 6 \).

Theorem 3. Exactly up to embedding dimension 5, all quasihomogeneous ICIS have no negative derivations with respect to some positive grading.

Proof. This follows from Kantor [Kan79], [Wah82, Thm. 2.4, Prop. 2.8], Proposition 18, Example 2 and Corollary 12.

As a consequence of our arguments we obtain a simple special case of the following conjecture due to S. Halperin.

Conjecture (Halperin). On any graded zero-dimensional complete intersection there are no negative derivations.

The following result bounds the degree of negative derivations (see also [Ale91, Prop.]). The bound does not require a complete intersection hypothesis and it is independent of further hypotheses as for instance in [Hau02, Thm. 2].
Proposition 4. For any quasihomogeneous zero-dimensional singularity \( A \) as in (0.1) there are no derivations of degree strictly less than \( p_n - p_1 \). In particular, Halperin’s conjecture holds true if \( p_1 = p_n \).

Proof. As \( A \) is assumed to be zero-dimensional, condition \( \mathcal{A}(k) \) on page 8 must hold true for all \( k = 1, \ldots, n \). Then the claim follows from Remark 14 and Lemma 15.

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1. Graded analytic algebras

Consider a (local) analytic algebra \( A = (A, m_A) \) over a (possibly trivially) valued field \( K \) of characteristic zero. We assume in addition that \( A \) is non-regular and can be represented as a quotient \( A = \frac{P}{\mathfrak{a}} \) of a convergent power series ring \( P := K\langle\langle x_1, \ldots, x_n \rangle\rangle \supseteq \mathfrak{a} \). In the sequel such an \( A \) will be referred to as a singularity. We choose \( n \) minimal such that \( n = \text{embdim} A \) and set \( d := \text{dim} A \).

A \( K_+ \)-grading on \( A \) is given by a diagonalizable derivation \( \chi \in \text{Der}_K A =: \Theta_A \) which means that \( m_A \) is generated by eigenvectors \( x_1, \ldots, x_n \) (see [SW73, (2.2),(2.3)]). Such a derivation is also called an Euler derivation. We refer to \( w_1, \ldots, w_n \) defined by \( w_i := \chi(x_i)/x_i \) as the eigenvalues of \( \chi \). More generally, we call \( \chi \)-eigenvectors \( f \in A \) homogeneous and define their degree to be the corresponding eigenvalue denoted by \( \text{deg}(f) := \chi(f)/f \in k \). We denote by \( A_a \) the \( K \)-vector space of all such eigenvector \( f \in A \) with \( \text{deg}(f) = a \). This defines a \( K \)-subalgebra

\[
\tilde{A} := \bigoplus_{a \in K} A_a \subset A \subset \hat{A}.
\]

The derivation \( \chi \in \Theta_A \) lifts to \( \chi \in \Theta_P := \text{Der}_K P \) (see [SW73, (2.1)]). In particular, \( P \) is \( K_+ \)-graded and \( \mathfrak{a} \subseteq P \) is a \( \chi \)-invariant ideal and hence homogeneous (see [SW73, (2.4)]). Pick homogeneous \( g_1, \ldots, g_t \in \mathfrak{a} \) inducing a \( K \)-vector space basis of \( \mathfrak{a}/m_A \mathfrak{a} \). Then \( \mathfrak{a} = \langle g_1, \ldots, g_t \rangle \) by Nakayama’s Lemma. We set \( p_i := \text{deg}(g_i) \) ordered as in (0.2). To summarize, we can write \( A \) as in (0.1).

A \( K_+ \)-grading is called a positive grading if \( w_i \in \mathbb{Z}_+ \) for all \( i = 1, \ldots, n \) (see [SW73, §3, Def.]). We call \( A \) quasihomogeneous if it admits a positive grading. In this case, we shall always normalize \( \chi \) to make the \( w_i \) coprime and order the variables according to \( (0.2) \). Positivity of weights enforces \( g_i \in \bar{P} = K[x_1, \ldots, x_n] \) and that

\[
\tilde{A} = \bigoplus_{i \geq 0} A_i = \bar{P}/\bar{\mathfrak{a}}, \quad \bar{\mathfrak{a}} = \langle g_1, \ldots, g_t \rangle \subseteq K[x_1, \ldots, x_n] = \bar{P},
\]

is a (positively) graded-local \( k \)-algebra with completion

\[
\hat{A} = \hat{\tilde{A}}
\]

and graded maximal ideal \( m_{\tilde{A}} = m_A \). The preceding discussion enables us to reformulate Wahl’s Conjecture in the language of Scheja and Wiebe [SW73, §2].

Lemma 5. The following supplementary structures on a singularity \( A \) are equivalent:

1. an Euler derivation \( \chi \) on \( A \) with positive eigenvalues,
2. a positive grading on \( A \),
3. a positive grading on \( \tilde{A} \),
4. a (positively) graded \( K \)-algebra \( \tilde{A} \) such that \( \hat{A} = \hat{\tilde{A}} \).
Proof. The equivalences of (1), (2), and (3) are due to Scheja and Wiebe (see [SW73, (2.2), (2.3)] and [SW77, (1.6)]). For the equivalence with (4), note that the obvious Euler derivation on a graded $K$-algebra $A$ lifts to an Euler derivation on the completion $\hat{A}$. The converse follows from from (1.1), (1.2) and (1.3). □

Let us assume now that $A$ is an isolated complete intersection singularity (ICIS). We may then take $g_1, \ldots, g_t$ to be a regular sequence and $d + t = n$. The isolated singularity hypothesis can be expressed in terms of the Jacobian ideal

$$J_A := \left\langle \frac{\partial g}{\partial x_i} \mid |\nu| = t \right\rangle \subseteq A$$

of $A$ as follows.

Proposition 6. A complete intersection singularity $A$ is isolated if and only if $J_A$ is $m_A$-primary. An analogous statement holds for $A$.

Proof. We denote by $\Omega^1_{A/k}$ the universally finite module of differentials of $A$ over $k$. By the standard sequence

$$a/a^2 \longrightarrow A \otimes_P \Omega^1_{P/k} \longrightarrow \Omega^1_{A/k} \longrightarrow 0,$$

the Jacobian ideal $J_A$ is the 0th Fitting ideal $F^0_{A/\Omega^1_{A/k}}$. By [SS72, (6.4), (6.9)], reducedness of $A$ is equivalent to $\text{rk} \Omega^1_{A/k} = d$ and $A_P$ is regular if and only if $\Omega^1_{A_P/k}$ is free. Hence, $A_p$ being regular is equivalent to $p \not\supset F^1_{A/\Omega^1_{A/k}} = J_A$ by [BH93, Lem. 1.4.9]. In particular, $A$ having an isolated singularity means exactly that $A/J_A$ is supported at $m_A$ and hence that $J_A$ is $m_A$-primary as claimed. The analogous statement for $\hat{A}$ is proved similarly. □

Remark 7. Let $A$ be a quasihomogeneous singularity. By (1.2),

$$J_A := \hat{J}_A = \left\langle \frac{\partial g}{\partial x_i} \mid |\nu| = t \right\rangle \subseteq \hat{A}$$

is the Jacobian ideal of $\hat{A}$ defined analogous to (1.4). By (1.3), $A$ is a complete intersection if and only if $\hat{A}$ is locally a complete intersection (see [BH93, Def. 2.3.1, Ex. 2.3.21.(c)]). By Proposition 6, $A$ is an ICIS if and only if $J_A$ is $m_A$-primary. This is equivalent to $J_{\hat{A}}$ being $m_{\hat{A}}$-primary. The latter is then equivalent to $\hat{A}$ being locally a complete intersection with isolated singularity by (1.5) and Proposition 6. Complete intersections are Cohen–Macaulay and hence $(S_2)$ so normality is equivalent to $(R_1)$ by Serre’s Criterion (see [BH93, §2.3, Thm. 2.2.22]). Since $d = \dim A = \dim \hat{A}$ by (1.3) (see [BH93, Cor. 2.1.8]), normality for both $A$ and $\hat{A}$ reduces to $d \geq 2$.

Scheja and Wiebe [SW77, (3.1)] (see also [Sai71, Satz 1.3]) proved that any $K_+$ graded ICIS is quasihomogeneous unless $t = 1$ and $g_i \notin m^3_P$. The following result gives numerical constraints for $A$ to be a quasihomogeneous ICIS.

Lemma 8. If $A$ is a quasihomogeneous ICIS then

$$p_1 + \cdots + p_j \geq w_1 + \cdots + w_j + j$$

for all $j = 1, \ldots, t$.

Proof. We proceed by induction on $j$. Assume that $p_1 + \cdots + p_{j-1} \geq w_1 + \cdots + w_{j-1} + j - 1$ but $p_1 + \cdots + p_j \leq w_1 + \cdots + w_j + j - 1$. Then $p_j \leq w_j$ and hence $g_i = g_i(x_{j+1}, \ldots, x_n)$ for all $i = j, \ldots, n$. Then $J_A$ maps to zero in

$$A/(x_{j+1}, \ldots, x_n) = K((x_1, \ldots, x_j))/\langle g_1, \ldots, g_{j-1} \rangle$$

and hence $J_A$ cannot be $m_A$-primary. □
2. Negative derivations

Let $A$ be a quasihomogeneous singularity as in §1. The target of our investigations is the positively graded $A$-module $\Theta_A = \text{Der}_K A$ of $K$-linear derivations on $A$. More precisely, we are concerned with the question whether its negative part

$$\Theta_{A,<0} = \Theta_{A,<0} = \bigoplus_{i<0} \Theta_{A,i}$$

is trivial. A priori this condition depends on the choice of a grading. In Proposition 9 below, we shall prove the independence of this choice for a general singularity under a strong hypothesis satisfied in the ICIS case (see Corollary 12). To this end, we write (see [SW73, (2.1)])

$$\Theta_A = \Theta_{a \subset P} / a \Theta_P$$

as a quotient of a $(k, P)$-Lie algebra

$$\Theta_{a \subset P} := \{ \delta \in \Theta_P \mid \delta a \subset a \} \supseteq a \Theta_P$$

of logarithmic derivations along $a$ by the $(k, P)$-Lie ideal $a \Theta_P$.

**Proposition 9.** Let $A$ be a quasihomogeneous singularity with positive grading given by $\chi$ and assume that

$$\Theta_{a \subset P} = P\chi + \Theta'_P + a \Theta_P,$$

$$\Theta'_P \subseteq m_P^2 \Theta_P.$$

Then the condition $\Theta_{A,<0} = 0$ and the $p_1, \ldots, p_t$ in (0.2) are independent of the chosen positive grading.

**Proof.** Consider a second positive grading with corresponding Euler derivation $\chi'$ (see Lemma 5). By (2.1) and (2.2), any $\delta \in \Theta_A$ lifts to an element of $\Theta_P$ of the form

$$\delta = c \chi + \delta_+, \quad \delta_+ = a \chi + \eta, \quad c \in K, \quad a \in m_P, \quad \eta \in \Theta'_P.$$

By (2.3) and the Leibniz rule,

$$\chi m^k_P \subseteq m^k_P, \quad \delta_+ m^k_P \subseteq m^{k+1}_P$$

for all $k \geq 1$. This implies first that $\chi_+ = 0$ and $\chi' = c \chi$ on $m_A/m^2_A = m_P/m^2_P$, and hence $c = 1$ by the definition of a positive grading and our normalization of weights.

Using (2.1), we equip $\Theta_A$ with the decreasing $m_P$-adic filtration $F^\bullet$ induced from $\Theta_P$ which is defined as follows

$$F^k \Theta_A = (\Theta_{a \subset P} \cap m^k_P \Theta_P) / (a \Theta_P \cap m^k_P \Theta_P).$$

Due to (2.4) and (2.5) this is a filtration $(k, P)$-Lie ideals and

$$\delta_+ F^k \Theta_A \subseteq F^{k+1} \Theta_A.$$

Therefore, for any $k \geq 1$, the adjoint action of $\chi' = \chi + \chi_+$ on the truncation

$$F^{\leq k} \Theta_A := \Theta_A / F^{k+1} \Theta_A$$

is triangularizable with semisimple part equal to that of $\chi$. Thus, $\chi'$ and $\chi$ have the same eigenvalues on $F^{\leq k} \Theta_A$ for any $k \geq 1$. The first claim then follows by choosing $k$ sufficiently large. A similar argument yields the second claim. \qed
For a Gorenstein singularity $A$, there is a natural way to produce elements of $\Theta_A$. The $A$-submodule $\Theta'_A \subset \Theta_A$ of is by definition the image of the inclusion

$$\Omega^{d-1}_{A/K} \hookrightarrow \omega^{d-1}_{A/K} = \text{Hom}_A(\Omega^1_{A/K}, \omega^d_{A/K}) = \text{Der}_K A \otimes_A \omega^d_{A/K} \cong \Theta_A.$$  

We return to the case of an ICIS singularity $A$. For $1 \leq \nu_0 < \cdots < \nu_t \leq n$ with complementary indices $1 \leq \mu_1 < \cdots < \mu_{d-1} \leq n$, the lift to $P$ of the image of $dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_{d-1}}$ can be written (up to sign) explicitly as

$$\delta_{\nu} := \begin{vmatrix}
\partial_{\nu_0} & \cdots & \partial_{\nu_t} \\
\partial_{\nu_0} g_1 & \cdots & \partial_{\nu_t} g_1 \\
\vdots & \ddots & \vdots \\
\partial_{\nu_t} g_1 & \cdots & \partial_{\nu_t} g_1 
\end{vmatrix}.$$  

Note that

$$\deg g_\nu = p_1 + \cdots + p_t - w_{\nu_0} - \cdots - w_{\nu_t},$$

$$\delta_{\nu} g_j = 0$$

for all $j = 1, \ldots, t$ and $\nu$. The lift of $\Theta'_A$ to $P$,

$$\Theta'_P := \langle \delta_{\nu} | 1 \leq \nu_0 < \cdots < \nu_t \leq n \rangle_P \subset \Theta_P,$$

is called the module of trivial derivations. The key to our investigations is the following result due to Kersken [Ker84, (5.2)]. From now on we assume in addition that $A$ is quasihomogeneous and normal, that is, $\dim A \geq 2$.

**Theorem 10** (Kersken). Let $A$ be a quasihomogeneous normal ICIS. Then the module $\Theta_A$ of $K$-linear derivations on $A$ is generated by the Euler derivation $\chi$ and the trivial derivations $\Theta'_A$.

Although Kersken only states that $\Theta'_A$ is minimally generated by the $\delta_\nu$ in (2.7), his arguments show that together with $\chi$ they form a minimal set of generators of $\Theta_A$.

**Corollary 11.** Let $A$ be quasihomogeneous normal ICIS. Then $\Theta_A$ is minimally generated by the Euler derivation $\chi$ and the trivial derivations $\delta_\nu$ in (2.7). In particular,

$$\mu(\Theta_A) = \binom{n}{t+1} + 1.$$  

**Proof.** Since the case $d = 2$ is covered by [Wah87, Prop. 1.12], we may assume that $d \geq 3$. In this case, the inclusion (2.6) fits into the following commutative diagramm with exact
rows and columns (see [Ker84, Proof of (4.8)] or [Wah87, Prop. 1.7]).

\[
\begin{array}{cccc}
0 & 0 & \chi & 0 \\
H^1_{m_A}(\Omega^d_{A/K}) & \chi & H^1_{m_A}(\Omega^{d-1}_{A/K}) & 0 \\
\omega^d_{A/K} & \chi & \omega^{d-1}_{A/K} & \chi & \omega^{d-2}_{A/K} \\
\Omega^d_{A/K} & \chi & \Omega^{d-1}_{A/K} & \chi & \Omega^{d-2}_{A/K} \\
0 & 0 & \Omega^d_{A/K} & \chi & \Omega^{d-1}_{A/K} & 0 \\
\end{array}
\]

It follows that

\[
\chi(\omega^{d-1}_{A/K}) \cong \chi(\Omega^{d-1}_{A/K}) \cong \Omega^{d-1}_{A/K}/\chi(\Omega^d_{A/K})
\]

where \(\chi(\Omega^d_{A/K}) \subset m_A \Omega^{d-1}_{A/K}\) and hence

\[
\mu(\chi(\Omega^d_{A/K})) = \mu(\Omega^{d-1}_{A/K}) = \mu(\Theta'_A).
\]

Now the middle row of (2.11) yields an exact sequence

\[
0 \longrightarrow A \longrightarrow \chi \Theta_A \longrightarrow \chi(\Omega^d_{A/K}) \otimes (\omega^d_{A/K})^{-1} \longrightarrow 0
\]

Since \(\chi \notin m_A \Theta_A\), the claim follows. \(\square\)

Note that \(\Theta'_P\) in (2.10) satisfies (2.3) due to (2.7) unless \(t = 1\) and \(g_1 \notin m^3_P\). As a consequence of Proposition 9 and Theorem 10 we therefore obtain the following result.

It is crucial for Example 2 to be a counter-example to Wahl’s Conjecture.

Corollary 12. Let \(A\) be a quasihomogeneous normal ICIS. Unless \(t = 1\) and \(g_1 \notin m^3_P\), the condition \(\Theta_{A,0} = 0\) and the \(p_1, \ldots, p_t\) in (0.2) are independent of the choice of a positive grading. \(\square\)

We shall now derive numerical constraints for minimal negative trivial derivations. To this end, suppose that \(0 \neq \eta \in \Theta_{A,0}\). For reasons of degree (see (0.2)), \(\eta\) can be written as

\[
(2.12) \quad \eta = q_1 \partial_1 + \cdots + q_n \partial_n, \quad q_i = q_i(x_{i+1}, \ldots, x_n)
\]

By Theorem 10, we may assume that \(\eta = \delta_{\nu} \neq 0\) is a trivial derivation as in (2.7). By (0.2) and (2.8), we may further assume that \(\nu_i = i+1\) for \(i = 0, \ldots, t\). Explicitly, we may write

\[
(2.13) \quad q_i = (-1)^{-1} \begin{vmatrix}
\partial_1 g_1 & \cdots & \partial_1 g_1 & \cdots & \partial_{t+1} g_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\partial_t g_t & \cdots & \partial_t g_t & \cdots & \partial_{t+1} g_t
\end{vmatrix}, \quad q_{t+2} = \cdots = q_n = 0.
\]

Now (2.8) and (2.9) specialize to the following simple
Lemma 13. For \( \eta \) as in (2.12) with (2.13), we have
\[
(2.14) \quad \eta g_j = 0
\]
for all \( j = 1, \ldots, t \). If \( \Theta_{A,<0} \neq 0 \) for a quasihomogeneous normal ICIS then
\[
(2.15) \quad p_1 + \cdots + p_t < w_1 + \cdots + w_{t+1}. \quad \square
\]

Remark 14. For degree reasons (see (0.2)), the identity (2.14) holds true for any \( \eta \in \Theta_{A,<p_t-p_i} \) and any quasihomogeneous singularity \( A \) as in (0.1).

Following Scheja and Wiebe [SW77, §2] or Saito [Sai71, Lem. 1.5], \( A \) being an ICIS implies, by Proposition 6, that for each \( k = 1, \ldots, n \) one of the following two conditions must hold true.

\( \mathfrak{A}(k) \) For some \( m \geq 2 \) and \( j \), \( x_k^m \) occurs in \( g_j \).

\( \mathfrak{B}(k) \) For some \( \nu_1, \ldots, \nu_t \) each \( g_j \) contains \( x_k^{m_j} x_{\nu(j)} \) for some \( m_j \geq 1 \).

Lemma 15. Assume that the identity (2.14) holds true for all \( j = 1, \ldots, t \). Then \( \mathfrak{A}(k) \) implies \( q_k = 0 \) in (2.12) for a suitable choice of coordinates.

Proof. Pick \( k \in \{1, \ldots, t+1\} \) such that \( \mathfrak{A}(k) \) holds. Then some \( g_j \) contains \( x_k^m \), \( m > 1 \), and all other monomials in \( g_j \) contain only strictly lower powers of \( x_k \) by homogeneity.

Let \( t_{k,j} = t_{k,j}(x_1, \ldots, x_k, \ldots, x_n) \) denote the coefficient of \( x_k^{m-1} \) in \( g_j \), and assume, without loss of generality, that the coefficient of \( x_k^m \) is \( \frac{1}{m} \). Note that \( t_{k,j} \) is independent of variables of weight larger than \( w_k \). Expanding (2.14) with respect to the variable \( x_k \) and taking the terms involving \( x_k^{m-1} \) gives
\[
q_k x_k^{m-1} = q_k \partial_k \left( \frac{1}{m} x_k^m \right) = - \sum_{i \neq k} q_i \partial_i (t_{k,j} x_k^{m-1}) = - \sum_{i \neq k} q_i \partial_i (t_{k,j}) x_k^{m-1}
\]
and hence
\[
(2.16) \quad \eta = \sum_{i \neq k} q_i (\partial_i - \partial_i (t_{k,j}) \partial_k).
\]

The \( \chi \)-homogeneous coordinate change
\[
x_k \mapsto x_k + t_{k,j}, \quad x_i \mapsto x_i \text{ for } i \neq k,
\]
replaces \( \partial_i - \partial_i (t_{k,j}) \partial_k \) in (2.16) by \( \partial_i \), reducing the number of terms in \( \eta \). Iterating this process yields the claim. \( \square \)

Our main technical result is the following

Proposition 16. Let \( A \) be a quasihomogeneous normal ICIS such that \( \Theta_{A,<0} \neq 0 \). Then
\( \mathfrak{B}(k) \) holds for at least two indices \( k \leq t+1 \). Each such \( k \) satisfies \( k \geq t - d + 2 \) and
\( g_k, \ldots, g_t \notin \mathfrak{m}^2 \).

Proof. By hypothesis and Lemma 15, \( \mathfrak{B}(k) \) holds for some \( k \leq t+1 \) with \( q_k \neq 0 \). Assuming that \( k \) is unique, (2.9) reads \( q_k \partial_k g_j = 0 \) which would imply that \( g_j \) is independent of \( x_k \) for all \( j = 1, \ldots, t \). By the isolated singularity hypothesis, this is impossible.

Combining (1.6) and (2.15), we obtain
\[
(2.17) \quad p_j + \cdots + p_t + j \leq w_j + \cdots + w_{t+1}
\]
for all \(j = 1, \ldots, t\). Using (0.2), \(\mathcal{B}(k)\) and (2.17) for \(j = k\), we compute
\[
    m_kw_k + \cdots + m_tw_t \leq (m_k + \cdots + m_t)w_k \\
    = \deg(\partial_{\nu_k}g_k \cdots \partial_{\nu_t}g_t) \\
    = p_k + \cdots + p_t - w_{\nu_k} - \cdots - w_{\nu_t} \\
    \leq w_k + \cdots + w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}.
\]
and hence
\[
    (m_k - 1)w_k + \cdots + (m_t - 1)w_t \leq w_{t+1} - k - w_{\nu_k} - \cdots - w_{\nu_t}.
\]
By (0.2), this forces
\[
    m_k = \cdots = m_t = 1, \\
    w_{t+1} \geq w_{\nu_k} + \cdots + w_{\nu_t} + k.
\]
In particular,
\[
    \nu_k, \ldots, \nu_t \geq t + 2
\]
and hence \(k \geq t - d + 2\). \(\square\)

3. ICIS of embedding dimension 5

**Lemma 17.** Let \(A\) be a quasihomogeneous normal ICIS such that \(\Theta_{A, < 0} \neq 0\). Then \(\mathfrak{A}(k_1)\) and \(\mathfrak{B}(k_2)\) for \(\{k_1, k_2\} = \{1, 2\}\) is impossible.

**Proof.** Assuming the contrary, one of the \(g_j\) has a monomial divisible by \(x_{k_1}^2\) by \(\mathfrak{A}(k_1)\) and each of the \(g_j\) has a monomial divisible by \(x_{k_2}\) by \(\mathfrak{B}(k_2)\). In particular,
\[
    p_1 + \cdots + p_t \geq 2w_{k_1} + (t - 1)w_{k_2} \geq w_1 + \cdots + w_{t+1}
\]
contradicting (2.15). \(\square\)

**Proposition 18.** For any quasihomogeneous ICIS \(A\) as in (0.1) with \(n = 5\) and \(t = 2\), we have \(\Theta_{A, < 0} = 0\).

**Proof.** Assume that \(\Theta_{A, < 0} \neq 0\). By Proposition 15 and Lemma 17, we must have \(\mathfrak{B}(1)\) and \(\mathfrak{B}(2)\). Using (0.2), (2.18), and (2.19), we may write
\[
    g_1 = x_1x_4 + c_1x_2^3x_1 + \cdots \\
    g_2 = x_1x_5 + c_2x_2^3x_2 + \cdots
\]
with \(\{k_1, k_2\} = \{4, 5\}\) and \(c_1, c_2 \in K^*\). As in the proof of Lemma 17, the inequality (2.15) can only hold true if \(j = 1\). In this case,
\[
    A/(J_A + \langle x_3, \ldots, x_n \rangle) = K/\langle x_1, x_2 \rangle/\left\langle \frac{\partial g}{\partial (x_4, x_5)} \right\rangle.
\]
for degree reasons (see (0.2)), and hence \(J_A\) is not \(m_A\)-primary. This contradicts to the isolated singularity hypothesis. \(\square\)
Proof of Example 2. The sequence \( g \) is clearly regular and defines a complete intersection as in (0.1). Note that \( \eta \) in (0.4) agrees with \( \eta = \delta_{1,2,3} \) in (2.12). Since \( \deg(g_1) = 10 = \deg(g_2) \), (2.9) shows that \( \eta \) has negative degree \( \deg \eta = -1 \).

It remains to check that \( A \) has an isolated singularity, that is, the Jacobian ideal \( J_A \) from (1.4) is \( m_A \)-primary. To this end, we may assume that \( K = K \) which enables us to argue geometrically on the variety

\[
\tilde{X} := \text{Spec} \tilde{A} \subset K^n
\]

with \( \tilde{A} \) as in (1.2) using the Nullstellensatz.

The ideal \( J_A \) is the image in \( A \) of the Jacobian ideal \( \tilde{J}_g \leq \tilde{P} \) of \( g \) generated by the

\[
M \quad \text{setting} \quad c
\]

Due to the \( 2 \times 2 \)-minors of \( J \) of the Jacobian matrix of \( g \) with \( \bar{c} \) as in (0.1). Note that

\[
\text{deg}(\eta) \quad \text{in (0.4)} \quad \text{agrees with} \quad \text{deg}(\eta) = 7.
\]

Counter-examples

Proof of Example 2. The sequence \( g \) is clearly regular and defines a complete intersection

as in (0.1). Note that \( \eta \) in (0.4) agrees with \( \eta = \delta_{1,2,3} \) in (2.12). Since \( \deg(g_1) = 10 = \deg(g_2) \), (2.9) shows that \( \eta \) has negative degree \( \deg \eta = -1 \).

It remains to check that \( A \) has an isolated singularity, that is, the Jacobian ideal \( J_A \) from (1.4) is \( m_A \)-primary. To this end, we may assume that \( K = K \) which enables us to argue geometrically on the variety

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\tilde{X} := \text{Spec} \tilde{A} \subset K^n
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with \( \tilde{A} \) as in (1.2) using the Nullstellensatz.

The ideal \( J_A \) is the image in \( A \) of the Jacobian ideal \( \tilde{J}_g \leq \tilde{P} \) of \( g \) generated by the

\[
M \quad \text{setting} \quad c
\]

Due to the \( 2 \times 2 \)-minors of \( \partial g/\partial x \) involving only the columns 3, 7, 8, 9, \ldots, \( n \), only one of

components \( x_3, x_7, x_8, x_9, \ldots, x_n \) of any \( x \in \text{Sing} \tilde{X} \) can be non-zero. We may therefore

reduce to the case \( n \leq 7 \).

Because of the 3rd column of \( \partial g/\partial x \), we have \( \tilde{J}_g \cap K[x_1, \ldots, x_6] \supset x_3 I \) where

\[
I := \langle x_4 - x_5, x_5 - x_6, x_1 - x_2, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.
\]

Note that \( V(I) \) is the \( x_3 \)-axis which is not contained in \( V(g) \). It follows that \( \text{Sing} \tilde{X} \cap V(x_3) \)

is contained in the hyperplane \( V(x_3) \). Similarly because of the 7th column of \( \partial g/\partial x \) and

setting \( c := c_7 \), we have \( \tilde{J}_g \cap K[x_1, \ldots, x_6, x_7] \supset x_7 I' \) where

\[
I' := \langle cx_4 - x_5, cx_5 - x_6, cx_2 - x_1, x_1 - 5x_4^4, x_2 + 5x_6^4 \rangle.
\]

Using \( c^9 + 1 \neq 0 \), we find that \( V(I') \) is the \( x_7 \)-axis and conclude \( \text{Sing} \tilde{X} \cap V(x_3) \subset V(x_7) \)

as before. Summarizing the two cases, \( \text{Sing} \tilde{X} \) is in fact contained in \( V(x_3, x_7) \).

Fix a point \( (x_1, x_2, 0, x_4, x_5, x_6, 0) \in \text{Sing} \tilde{X} \). Successively using the the equations

\[
M_{1,2} = x_4 x_6 - x_5^2 = 0, \quad M_{2,5} = x_1 x_5 - x_2 x_6 = 0, \quad g_2 = x_1 x_5 + x_2 x_6 + x_6^5 = 0, \quad M_{4,5} = x_1 (x_1 - 5x_4^4) = 0, \quad M_{5,6} = x_2 (x_2 + 5x_6^4) = 0,
\]

we derive

\[
x_4 = 0 \Rightarrow x_5 = 0 \Rightarrow x_2 x_6 = 0 \Rightarrow x_6 = 0 \Rightarrow x_1 = x_2 = 0.
\]

Similarly \( x_6 = 0 \) leaves no possibility except \( x = 0 \) and \( x_5 = 0 \) reduces to one of these

two cases by \( M_{1,2} = 0 \).

Assume now that \( x_4, x_5, x_6 \) are all non zero. Then the minors \( M_{1,5}, M_{2,4}, M_{2,5}, M_{2,6} \)

give equations

\[
x_1 x_4 = x_2 x_5, \quad x_1 = 5x_4^4, \quad x_1 x_5 = x_2 x_6, \quad x_2 = -5x_6^4.
\]
Substituting into and \( g \), we obtain
\[
g_1 = 2x_1x_4 - x_4^2 = 9x_4^5, \quad g_2 = 2x_2x_6 + x_6^5 = -9x_6^5
\]
and hence \( x_4 = x_6 = 0 \) contradicting our assumption. \( \square \)

References


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